Supplement to: Analysis and Comparison of Queues with Different Levels of Delay Information

August 22, 2003; last revised January 7, 2006

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1. Notation

Here is some of the notation used in the paper.

- λ = arrival rate of potential customers
- μ = service rate
- W = waiting time in queue
- θ = customer-type parameter, indicating the importance of time, $\theta \in [0, 1]$
- H = cumulative distribution function of θ, assumed continuous on $[0, 1]$, with density h
- $c(w)$ = basic cost to wait time w, a positive, increasing, unbounded, continuous function
- r = reward to the customer for receiving service, $r > 0$.
- u = average utility

2. Solutions

2.1 No Information

Assume that $E[c(W|-)]$ is finite for any $\lambda_- < \mu$.

Proposition 1 *For no information, there exists a unique equilibrium arrival rate λ_-.*

Proof. By assumption, c is increasing and continuous. Thus, $\lambda/E[c(W|-)]$ is a decreasing, continuous function of λ_- mapping the interval $[0, \min\{\lambda, \mu\}]$ into itself. Thus, (1) has a unique solution. ■

2.2 Partial Information

Verification of formula (4) for uniform H and linear c: Observe that

$$\begin{aligned} \frac{d}{d\lambda}\gamma(\mu,\lambda) &= \lambda^{\mu-1}e^{-\lambda} = \frac{d}{d\lambda}\left(\lambda^{\mu-1}e^{-\lambda}\sum_{n=1}^{\infty}\frac{\Gamma(\mu)}{\Gamma(\mu+n)}\lambda^n\right) \\ &= (\mu-1-\lambda)(\lambda^{\mu-2}e^{-\lambda})\sum_{n=1}^{\infty}\frac{\Gamma(\mu)}{\Gamma(\mu+n)}\lambda^n + (\lambda^{\mu-1}e^{-\lambda})\sum_{n=1}^{\infty}\frac{\Gamma(\mu)}{\Gamma(\mu+n)}n\lambda^{n-1} \\ &= (\mu-1-\lambda)\lambda^{\mu-2}e^{-\lambda}\Theta + (\lambda^{\mu-2}e^{-\lambda}/p_0)E\left[N\right]. \end{aligned}$$

Thus,

$$\begin{aligned} \lambda &= (\mu-1-\lambda)\Theta + E\left[N\right]/p_0 \\ &= (\mu-1-\lambda)(1-p_0)/p_0 + E\left[N\right]/p_0, \end{aligned}$$

or

$$E\left[N\right] = \lambda - (\mu-1)\left(1-p_0\right).$$

3. Comparisons

3.1 Cost-Scale Distributions

We verify the assertions about beta distributions. One can directly compute

$$-\frac{\theta h'(\theta)}{h(\theta)} = -(\alpha-1) + (\beta-1)\frac{\theta}{1-\theta}.$$

Condition 1 stipulates that this quantity be no more than 2 for all θ. This is clearly true, if and only if $\beta \le 1$.

The other condition is more intricate. In general,

$$(J\circ H^{-1})''(\phi) = \frac{\phi h^2(\phi) - [2h(\phi)+\phi h'(\phi)][H(\phi)-J(\phi)]}{\phi^2 h^3(\phi)}.$$

Condition 2 means that the numerator be nonnegative. For a beta distribution, the numerator can be written as

$$\frac{\phi^{\alpha-2}(1-\phi)^{\beta-2}}{B(\alpha,\beta)}\left\{\frac{\phi^{\alpha+1}(1-\phi)^{\beta}}{B(\alpha,\beta)}-[(\alpha+1)-(\alpha+\beta)\phi]\frac{\alpha}{\alpha+\beta}H(\phi;\alpha+1,\beta)\right\},$$

where $H(\phi;\alpha+1,\beta)$ denotes the cdf but with parameter $\alpha+1$ instead of α. Using a standard series representation of H, this becomes

$$\frac{\phi^{2\alpha-1}(1-\phi)^{2\beta-2}}{(\alpha+1)B^2(\alpha,\beta)}\left\{(\alpha+1)-[(\alpha+1)-(\alpha+\beta)\phi]\left[1+\frac{B(\alpha+2,1)}{B(\alpha+\beta+1,1)}\phi+...\right]\right\}.$$

For $\beta=1$

$$1+\frac{B(\alpha+2,1)}{B(\alpha+\beta+1,1)}\phi+...=\frac{1}{1-\phi}$$

and

$$[(\alpha+1)-(\alpha+\beta)\phi]=(\alpha+1)(1-\phi),$$

so the numerator reduces to 0. For $\beta<1$ each coefficient in the power series above is >1, so

$$1+\frac{B(\alpha+2,1)}{B(\alpha+\beta+1,1)}\phi+...>\frac{1}{1-\phi}.$$

Also,

$$[(\alpha+1)-(\alpha+\beta)\phi]>(\alpha+1)(1-\phi).$$

Thus, $(J\circ H^{-1})''(\phi)<0$. Similarly, for $\beta>1$, $(J\circ H^{-1})''(\phi)>0$. Thus, the condition holds for $\beta\geq 1$ but not for $\beta<1$.

The function J plays an important role in the following. Observe that, since H is increasing, so is J, and hence $J(1/x)$ is decreasing in x. Also,

$$\begin{aligned}\frac{d^2}{dx^2}J(1/x) &= -(1/x^2)H(1/x)+(1/x^2)H(1/x)+(1/x^3)h(1/x)\\ &= (1/x^3)h(1/x)>0.\end{aligned}$$

Thus, $J(1/x)$ is strictly convex in $x\geq 1$.

3.2 No Information and Partial Information

We start with two preliminary results.

Lemma 1 *p_n^{part} is log-concave in n.*

Proof.

$$\frac{p_{n+1}^{part}}{p_n^{part}} = (\lambda/\mu)H(\theta_n),$$

which is decreasing in n. ■

This implies that $\bar{P}_n^{part}$ (the complementary cdf of N^{part}) is also log-concave. (See, e.g., Karlin 1968.) This property means precisely that N^{part} has increasing failure rate. Also, since p_n^{no} and $\bar{P}_n^{no}$ are geometric (log-linear) sequences, the ratios p_n^{part}/p_n^{no} and $\bar{P}_n^{part}/\bar{P}_n^{no}$ are also log-concave. In particular, these ratios are unimodal.

Lemma 2 *If $p_0^{part} \geq p_0^{no}$, then $N^{part} \preceq_{st} N^{no}$.*

Proof. The sequence p_n^{part}/p_n^{no} is log-concave and hence unimodal. If it starts above 1, then it must cross 1 exactly once, by normalization. It follows that $\bar{P}_n^{part} \leq \bar{P}_n^{no}$ for all n. That is, $N^{part} \preceq_{st} N^{no}$. ■

Now we prove the first main result.

Proposition 2 *If $p_0^{part} \geq p_0^{no}$, then $u^{part} > u^{no}$.*

Proof. In this case, $N^{part} \preceq_{st} N^{no}$. Since c_n is increasing, $E[c_{N^{part}}] \leq E[c_{N^{no}}]$. Since $J(1/x)$ is decreasing and strictly convex, by Jensen's inequality,

$$\begin{aligned} u^{no} &= J(\theta_-) = J(1/E[c_{N^{no}}]) \\ &\leq J(1/E[c_{N^{part}}]) \\ &< E[J(1/c_{N^{part}})] \\ &= E[J(\theta_{N^{part}})] = u^{part}. \end{aligned}$$

■

Here is the proof of the next result.

Proposition 3 *Under Condition 1 [$H(1/x)$ is convex], $p_0^{part} \leq p_0^{no}$.*

Proof. Suppose to the contrary that $p_0^{part} > p_0^{no}$. Then, $N^{part} \preceq_{st} N^{no}$. Thus,

$$\begin{aligned} H(\theta_-) &= H(1/E[c_{N^{no}}]) \\ &\leq H(1/E[c_{N^{part}}]) \\ &\leq E[H(1/c_{N^{part}})] \\ &= E[H(\theta_{N^{part}})]. \end{aligned}$$

(The second inequality uses Jensen's inequality and the convexity of $H(1/x)$.) Therefore,

$$\begin{aligned} p_0^{no} &= 1-(\lambda/\mu)H(\theta_-) \\ &\geq 1-(\lambda/\mu)E[H(\theta_{N^{part}})] = p_0^{part}, \end{aligned}$$

a contradiction. ■

Next, we prove the result comparing utilities.

Proposition 4 *Under Condition 2 [$J \circ H^{-1}$ is convex], $u^{part} > u^{no}$. Moreover, if $p_0^{part} < p_0^{no}$, then*

$$\frac{u^{part}}{u^{no}} \geq \frac{1-p_0^{part}}{1-p_0^{no}}.$$

Proof. We already know that $u^{part} > u^{no}$ for the case $p_0^{part} \geq p_0^{no}$. Otherwise, if $p_0^{part} < p_0^{no}$, let

$$\tau = \frac{1-p_0^{part}}{1-p_0^{no}} = \frac{E[H(\theta_{N^{part}})]}{H(\theta_-)}.$$

We have $\tau > 1$, and so

$$\begin{aligned} \tau u^{no} &= \tau J(\theta_-) = \tau J \circ H^{-1}[H(\theta_-)] \\ &\leq J \circ H^{-1}(\tau H(\theta_-)) \\ &= J \circ H^{-1}(E[H(\theta_{N^{part}})]) \\ &\leq E[J \circ H^{-1}(H(\theta_{N^{part}}))] \\ &= E[J(\theta_{N^{part}})] = u^{part}. \end{aligned}$$

(The first inequality follows from the fact that $J \circ H^{-1}$ is increasing and the second from Jensen's inequality.) ■

Finally, we prove the result about $E[N]$ for a special case.

Proposition 5 *For uniform H and linear cost, the relation between $E[N^{no}]$ and $E[N^{part}]$ is the same as that between μ and 1. That is, they are equal for $\mu = 1$, $E[N^{no}] > E[N^{part}]$ for $\mu > 1$, and $E[N^{no}] < E[N^{part}]$ for $\mu < 1$.*

Proof. From (4),

$$E\left[N^{part}\right] = \lambda - (\mu - 1)\left(1 - p_0^{part}\right).$$

For $\mu = 1$, therefore, $E[N^{part}] = \lambda = E[N^{no}]$.

So, assume $\mu \neq 1$. We have

$$E[N^{no}] = \frac{\rho^{no}}{1-\rho^{no}},$$

or

$$\rho^{no} = \frac{E[N^{no}]}{E[N^{no}]+1}.$$

We know that ρ^{no} satisfies (3). Thus,

$$(1-\mu)[E[N^{no}]]^2 + (\mu+\lambda)E[N^{no}][E[N^{no}]+1] - \lambda[E[N^{no}]+1]^2 = 0,$$

or

$$[E[N^{no}]]^2 + E[N^{no}](\mu-\lambda) - \lambda = 0. \tag{7}$$

Since $1-p_0^{part} = \rho^{part} > \rho^{no}$, we have $(1-\mu)(\rho^{part})^2 + (\mu+\lambda)\rho^{part} - \lambda > 0$. Thus, again using (4),

$$\left[E\left[N^{part}\right]\right]^2(1-\mu) + E\left[N^{part}\right](1-\mu)(\mu-\lambda) - \lambda(1-\mu) > 0.$$

For $\mu < 1$ this reduces to

$$\left[E\left[N^{part}\right]\right]^2 + E\left[N^{part}\right](\mu-\lambda) - \lambda > 0.$$

Comparing this to (7), we see that $E[N^{no}] < E[N^{part}]$. Similarly, the opposite conclusion holds for $\mu > 1$. ■

3.3 No Information and Full Information

Observe that

$$\frac{d\ln f^{full}(v)}{dv} = \lambda H\left[\frac{1}{c(v)}\right] - \mu,$$

which is decreasing in v. Thus, f^{full} is log-concave, and so the ratio f^{full}/f^{no} is too. Also, $f^{full}(0^+) = \lambda p_0^{full}$ and

$$\begin{aligned} f^{no}(0^+) &= \rho_-(1-\rho_-)\mu = \lambda\theta_-(1-\rho_-) \\ &\leq \lambda(1-\rho_-) = \lambda p_0^{no}. \end{aligned}$$

So, if $p_0^{full} \geq p_0^{no}$, then $f^{full}(0^+) \geq f^{no}(0^+)$, and so $V^{full} \preceq_{st} V^{no}$.

With this key fact established, the results are entirely analogous to those above. We provide a complete proof only for the first one, which asserts that, if $p_0^{full} \geq p_0^{no}$, then $u^{full} > u^{no}$.

Proof. As above, $p_0^{full} \geq p_0^{no}$ implies $V^{full} \preceq_{st} V^{no}$. Therefore, $E[c(V^{full})] \leq E[c(V^{no})]$. Also, the equilibrium condition can be expressed as $\theta_- = 1/E[c(V^{no})]$. Thus,

$$\begin{aligned} u^{no} &= J(\theta_-) = J(1/E[c(V^{no})]) \\ &\leq J(1/E[c(V^{full})]) \\ &< E[J(1/c(V^{full}))] \\ &= E[J(\theta_{V^{full}})] = u^{full}. \end{aligned}$$

■

3.4 Partial Information and Full Information

It appears harder to compare partial and full information. In general f^{full}/f^{part} is not log-concave nor even unimodal. (The case of two types of customers provides a counterexample.) We have the analogue to only one of the results above.

First, we establish some preliminary results. Denote $a = \lambda/\mu$, and define

$$\Upsilon(a) = \frac{1}{p_0^{full}} - 1 = a\mu \int_0^\infty e^{-\mu v} \exp\left[a\mu C(v)\right] dv.$$

We analyze the power-series reprentation of $\Upsilon(a)$. We have

$$\begin{aligned} \Upsilon(a) &= a\mu \int_0^\infty e^{-\mu v} \sum_{n=0}^\infty \frac{[a\mu C(v)]^n}{n!} dv \\ &= \sum_{n=0}^\infty \frac{a^{n+1}}{n!} \int_0^\infty \mu e^{-\mu v} [\mu C(v)]^n dv \\ &= \sum_{n=1}^\infty \frac{a^n}{n!} n \int_0^\infty \mu e^{-\mu v} [\mu C(v)]^{n-1} dv. \end{aligned}$$

Thus, $\Upsilon(0) = 0$, and for $n > 0$,

$$\Upsilon^{(n)}(0) = n \int_0^\infty \mu e^{-\mu v} \left[\mu C(v)\right]^{n-1} dv = nE[\{\mu C(S)\}^{n-1}]. \tag{8}$$

In particular, $\Upsilon'(0) = 1$.

The following is an alternative representation of the higher coefficients.

Lemma 3 *For $n > 1$*

$$\frac{\Upsilon^{(n)}(0)}{n!} = E\left[C'(S^{(1)})C'(S^{(2)}) \cdots C'(S^{(n-1)})\right].$$

Proof. Integrate (8) by parts to obtain

$$\begin{aligned}\frac{\Upsilon^{(n)}(0)}{n} &= \left\{(-e^{-\mu v})\left[\mu C(v)\right]^{n-1}\right\}_0^\infty + \int_0^\infty e^{-\mu v}(n-1)\mu C'(v)\left[\mu C(v)\right]^{n-2} dv \\ &= (n-1)\int_0^\infty \mu e^{-\mu v} C'(v)\left[\mu C(v)\right]^{n-2} dv,\end{aligned}$$

or

$$\frac{\Upsilon^{(n)}(0)}{n(n-1)} = E\left[C'(S)\left[\mu C(S)\right]^{n-2}\right].$$

In particular,

$$\frac{\Upsilon^{(2)}(0)}{2!} = E\left[C'(S)\right].$$

Thus, the result holds for $n = 2$.

Next, for $n > 2$,

$$\begin{aligned}\frac{\Upsilon^{(n)}(0)}{n(n-1)} &= \int_0^\infty \mu e^{-\mu v} C'(v)\left[\mu C(v)\right]^{n-2} dv \\ &= \int_0^\infty \mu e^{-\mu v} C'(v)\left[\mu C(v)\right]^{n-3} \mu \int_0^v C'(t_1) dt_1 dv \\ &= \int_0^\infty \mu C'(t_1) \int_{t_1}^\infty \mu e^{-\mu v} C'(v)\left[\mu C(v)\right]^{n-3} dv dt_1 \\ &= \int_0^\infty \mu e^{-\mu t_1} C'(t_1) \int_0^\infty \mu e^{-\mu t_2} C'(t_1+t_2)\left[\mu C(t_1+t_2)\right]^{n-3} dt_2 dt_1,\end{aligned}$$

or

$$\frac{\Upsilon^{(n)}(0)}{n(n-1)} = E\left[C'(S^{(1)})C'(S^{(2)})\left[\mu C(S^{(2)})\right]^{n-3}\right].$$

In particular,

$$\frac{\Upsilon^{(3)}(0)}{3!} = E\left[C'(S^{(1)})C'(S^{(2)})\right],$$

which is the result for $n = 3$.

Now, for $n > 3$,

$$\begin{aligned}
\frac{\Upsilon^{(n)}(0)}{n(n-1)} &= \int_0^\infty \mu e^{-\mu t_1} C'(t_1) \int_0^\infty \mu e^{-\mu t_2} C'(t_1+t_2) \left[\mu C(t_1+t_2)\right]^{n-3} dt_2 dt_1 \\
&= \int_0^\infty \mu e^{-\mu t_1} C'(t_1) \int_0^\infty \mu e^{-\mu t_2} C'(t_1+t_2) \left[\mu C(t_1+t_2)\right]^{n-4} \int_0^{t_1+t_2} \mu C'(v) dv dt_2 dt_1 \\
&= \int_0^\infty \mu e^{-\mu t_1} C'(t_1) \int_0^\infty \mu e^{-\mu t_2} C'(t_1+t_2) \left[\mu C(t_1+t_2)\right]^{n-4} \\
&\quad \bullet \left(\int_0^{t_1} \mu C'(v) dv + \int_{t_1}^{t_1+t_2} \mu C'(v) dv \right) dt_2 dt_1 \\
&= \int_0^\infty \mu C'(v) \int_v^\infty \mu e^{-\mu t_1} C'(t_1) \int_0^\infty \mu e^{-\mu t_2} C'(t_1+t_2) \left[\mu C(t_1+t_2)\right]^{n-4} dt_2 dt_1 dv \\
&\quad + \int_0^\infty \mu e^{-\mu t_1} C'(t_1) \int_{t_1}^\infty \mu C'(v) \int_{v-t_1}^\infty \mu e^{-\mu t_2} C'(t_1+t_2) \left[\mu C(t_1+t_2)\right]^{n-4} dt_2 dv dt_1 \\
&= 2 \int_0^\infty \mu e^{-\mu s_1} C'(s_1) \int_0^\infty \mu e^{-\mu s_2} C'(s_1+s_2) \\
&\quad \bullet \int_0^\infty \mu e^{-\mu s_3} C'(s_1+s_2+s_3) \left[\mu C(s_1+s_2+s_3)\right]^{n-4} ds_3 ds_2 ds_1 \\
&= 2E\left[C'(S^{(1)}) C'(S^{(2)}) C'(S^{(3)}) \left[\mu C(S^{(3)})\right]^{n-4} \right].
\end{aligned}$$

Continuing in this manner, we obtain for $n > k \geq 3$

$$\frac{\Upsilon^{(n)}(0)}{n(n-1)} = (k-1)! E\left[C'(S^{(1)}) \cdots C'(S^{(k)}) \left[\mu C(S^{(k)})\right]^{n-k-1} \right]$$

In particular, for $k = n-1$,

$$\frac{\Upsilon^{(n)}(0)}{n!} = E\left[C'(S^{(1)}) \cdots C'(S^{(n-1)}) \right],$$

as asserted. ■

Lemma 4 *For $n > 1$,*

$$\begin{aligned}
& E\left[C'\left(S^{(1)}\right) C'\left(S^{(2)}\right) \cdots C'\left(S^{(n-1)}\right) \right] \\
\geq\ & E\left[C'\left(S^{(1)}\right) \right] E\left[C'\left(S^{(2)}\right) \right] \cdots E\left[C'\left(S^{(n-1)}\right) \right].
\end{aligned}$$

Proof. The two expressions are identical for $n = 2$. Consider the case $n = 3$.

$$E\left[C'\left(S^{(1)}\right) C'\left(S^{(2)}\right) \right] = E_{S_1}\left[C'\left(S_1\right) E_{S_2}\left[C'\left(S_1 + S_2\right) \right] \right].$$

Both $C'(S_1)$ and $E_{S_2}\left[C'(S_1+S_2)\right]$ are decreasing as functions of S_1. These two random variables are therefore positively correlated (e.g., Casella and Berger, 2001). Thus,

$$\begin{aligned}
E\left[C'\left(S^{(1)}\right) C'\left(S^{(2)}\right) \right] &\geq E_{S_1}\left[C'\left(S_1\right) \right] E_{S_1}\left[E_{S_2}\left[C'\left(S_1 + S_2\right) \right] \right] \\
&= E\left[C'\left(S^{(1)}\right) \right] E\left[C'\left(S^{(2)}\right) \right].
\end{aligned}$$

The general case follows similarly. ■

Now we are ready to prove

Proposition 6 *Under Condition 1 [$H(1/x)$ is convex], $p_0^{full} \leq p_0^{part}$.*

Proof. We have

$$\frac{\Upsilon^{(n)}(0)}{n!} = E\left[C'\left(S^{(1)}\right)C'\left(S^{(2)}\right)\cdots C'\left(S^{(n-1)}\right)\right], n > 1.$$

By Lemma (4),

$$\frac{\Upsilon^{(n)}(0)}{n!} \geq E\left[C'\left(S^{(1)}\right)\right]E\left[C'\left(S^{(2)}\right)\right]\cdots E\left[C'\left(S^{(n-1)}\right)\right].$$

By Jensen's inequality and the convexity of $H(1/x)$, each

$$E\left[C'\left(S^{(m)}\right)\right] = E\left[H\left\{\frac{1}{c\left(S^{(m)}\right)}\right\}\right] \geq H\left\{\frac{1}{E[c\left(S^{(m)}\right)]}\right\} = H\left(\frac{1}{c_m}\right).$$

Thus,

$$\frac{\Upsilon^{(n)}(0)}{n!} \geq H\left(\frac{1}{c_1}\right)H\left(\frac{1}{c_2}\right)\cdots H\left(\frac{1}{c_{n-1}}\right) = \Theta_n, n > 1.$$

This is the corresponding factor in Θ for partial information. ■

4. Extension

For the case $c(0) < 1$, average utility and throughput are no longer proportional, because J is no longer proportional to H over their extended domain. Unlike H, J is not constant for $\theta \geq 1$.

The proofs of Proposition 2 and the preceding lemmas go through as is.

Proposition 4 requires more care. The case $p_0^{part} \geq p_0^{no}$ is fine, as before. For the other case, $p_0^{part} < p_0^{no}$, H^{-1} exists for the original H, but not the extended one, so the proof needs to be modified.

First, assume $\eta = h(1) > 0$. Define

$$\hat{H}(\theta) = \begin{cases} H(\theta) & , \quad \theta \leq 1 \\ 1 + \eta[1 - (1/\theta)] & , \quad \theta > 1. \end{cases}$$

This function is strictly increasing everywhere, and so has an inverse, $\hat{H}^{-1}$. Also, $\hat{H}$ is continuously differentiable at $\theta = 1$, and so is $\hat{H}^{-1}$. By Condition 2, $J \circ \hat{H}^{-1}(\phi)$ is convex for $\phi < 1$. For $\phi > 1$,

$$J \circ \hat{H}^{-1}(\phi) = 1 - [1 - J(1)][1 - (1/\eta)(\phi - 1)].$$

This is linear, and so convex. Finally, $J \circ \hat{H}^{-1}$ too is continuously differentiable at $\phi = 1$. So, $J \circ \hat{H}^{-1}$ is convex overall. Now, define

$$\hat{\tau} = \frac{E[\hat{H}(\theta_{N^{part}})]}{\hat{H}(\theta_-)}.$$

A proof just like that of Proposition 4, with $\hat{H}$ replacing H, shows that $u^{part}/u^{no} \geq \hat{\tau}$. Moreover, $\hat{H}(\theta) \geq H(\theta)$, and $p_0^{part} < p_0^{no}$ implies $\theta_- \leq 1$, so $\hat{H}(\theta_-) = H(\theta_-)$. Therefore,

$$\hat{\tau} \geq \frac{E[H(\theta_{N^{part}})]}{H(\theta_-)} = \frac{1 - p_0^{part}}{1 - p_0^{no}}.$$

This completes the proof, assuming $h(1) > 0$.

For the case $h(1) = 0$, we can use a limit argument. Construct a sequence of H's, each with $h(1) > 0$, which converges to the original H. Apply the argument above to each item in the sequence. Then use continuity.

As for Proposition 3, the crucial step is the use of Condition 1 and Jensen's inequality to get

$$H(1/E[c_{N^{part}}]) \leq E[H(1/c_{N^{part}})].$$

For large λ, $c_{N^{part}}$ is nearly always > 1. It spends most of its time in the region where $H(1/x)$ is convex. Moreover, since $H(1/x)$ decreases from 1 to 0 and never becomes negative, there must be places where it's strictly convex or kinked. By combining these facts carefully, one can verify the inequality above.

References

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