Information Structures with Unawareness†

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Abstract

I construct a multi-agent state space model with unawareness following Aumann (1976). Dekel, Lipman and Rustichini (1998) show that standard state space models are incapable of representing unawareness. The model circumvents the impossibility result by endowing the agent a subjective state space that differs from the full state space when he has the unawareness problem. Information is modeled as a pair, consisting of both factual information and *awareness information*. The model exhibits nice properties parallel to those in the standard information partition model.

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"There are things we know that we know. There are known unknowns - that is to say, there are things that we now know we don't know. But there are also unknown unknowns. There are things we do not know we don't know."

Donald Rumsfeld, the U.S. Secretary of Defense

1 Introduction

A person is unaware of an event if he does not know it, and he does not know that he does not know it, and so on *ad infinitum*. In real life, *formulating* a decision problem, including recognizing all relevant uncertainties and available options, is at least as important as finding the solution to the formulated problem. Being unaware of some aspects of the situation is a common problem at this stage. For example, prior to the 911 attacks, most of us did not know that terrorists might use civilian aircraft as a weapon, and more importantly, we did not know that we did not know this. We were simply unaware of this possibility.

Unawareness plays an important role in economic life, especially through recognition of the possibility of being unaware of *something*. One may not wish to commit to a seemingly attractive position in an alien environment due to the vast unknown unknowns. Contractual parties may opt to write incomplete contracts in order to preserve some flexibility to deal with contingencies of which they were unaware at the contractual date. Realizing the opponents may be unaware of some aspects of the game, players may want to play otherwise inferior actions in order to affect the opponents' perceptions of the game.

These issues cannot be analyzed using the standard tools of economics, which is perhaps the main reason why there is little research on these obviously important issues. The prevailing model of uncertainty in economics is the standard information partition model: uncertainties are represented by a state space; information takes the form of a partition over the state space; at each state, the agent is informed of the corresponding partition element. But then the agent cannot be unaware of anything: having an information partition is equivalent to having a knowledge hierarchy in which whenever the agent doesn't know something, he knows he doesn't know it (Bacharach 1985).

Geanakoplos (1990) explores non-partitional information structures. But in such a model one has nonsensical knowledge such as "I know that I don't know that I don't know the event." In fact, Dekel, Lipman and Rustichini (henceforth DLR) show that the problem is at a more fundamental level: *any* model that uses the standard state space specification necessarily imposes either full awareness or full unawareness (Dekel, Lipman and Rustichini 1998).

On the other hand, there are fruitful results in modeling unawareness using syntactic models, for example, Fagin and Halpern (1988), Halpern (2001), Modica and Rustichini (1994, 1999), to name a few. More recently, Feinberg (2004, 2005) studies games where players may be unaware of some actions or players by introducing a language incorporating important aspects of unawareness.

While this research has greatly improved our understanding of unawareness, the tools developed along this line are rather extraneous to many economists. Given the central role of decision-making under uncertainty in economics, a model that uses the standard language, i.e. semantics, and highlights how unawareness fits in the standard model of information is much desired.

In this paper, I generalize the standard information partition model to allow for nontrivial unawareness in a way that highlights the implications of unawareness in information processing in both single-agent and multi-agent environments.[1] The idea is the following. Fix the set of payoff-relevant uncertainties. One can think of them as a set of relevant questions. If the agent is unaware of a question, then a message reminding the agent of the question itself must be informative.[2] Such information is fundamentally different from the type of information in the standard models, which is necessarily factual. Therefore, to model unawareness is equivalent to modeling such *awareness information*. The second observation is that one can only reason about things of which one is aware. Fixing a full state space and a full information partition that capture all payoff-relevant uncertainties and the factual signal structure, if the agent is unaware of some uncertainties, his mind-set of reasoning must be represented by a less model, in which the uncertainties of which he is unaware are lacking. Therefore, I allow the agent to have (full-)state-contingent subjective models, each being essentially a standard information partition model.

As an illustration, consider the following episode: Sherlock Holmes and Watson are investigating a crime. A horse has been stolen and the keeper was killed. From the narration of the local police, Holmes notices the dog in the stable did not bark that night and hence concludes that there was no intruder in the stable. Watson, on the other hand, although he also knows the dog did not bark – he himself mentioned this fact to Holmes – somehow does not come up with the inference that there was no intruder.

The feature I would like to capture in this story is the following. Watson is unaware of the possibility that there was no intruder, and hence fails to recognize the factual information "there was no intruder" contained in the message "the dog did not bark." Had someone asked Watson, "Could there have been an intruder in the stable that night?" He would have recognized his negligence and replied, "Of course not, the

[1]In an independently conceived work, Heifetz, Meier and Schipper (2004) propose a set-theoretic model by exploring a complete lattice of state spaces, ordered by "expressive power." They impose a natural restriction on the class of events one can meaningfully investigate in their set-up, and show the information has properties analogous to those of an information partition. This paper focuses more heavily on understanding the effects of introducing unawareness on the standard representation of uncertainty and information.

[2]This is without loss of generality. One could imagine situations where the agent has "partial awareness," that is, he is aware of the question but unaware of some answers. This model is capable of handling such situations, via proper rephrasing of the questions and answers. Li (2006b) discusses the issue in detail.

dog did not bark!"

The relevant question in this example is whether there was an intruder to the stable that night. Let $a = (a', \Delta)$, $b = (b', \Delta)$, where a' stands for "there was an intruder," b' stands for "there was no intruder" and Δ stands for "*cogito ergo sum.*" The full state space is $\{a, b\}$. The dog barked in a and did not bark in b, inducing the full information partition $\{\{a\}, \{b\}\}$. However, in b, Watson is unaware of the possibility of no intruder, and hence a and b are beyond his mind-set of reasoning. In other words, Watson only has the awareness information of Δ but not $\{a', b'\}$. Therefore I let Watson's mind-set be represented by the subjective state space $\{\Delta\}$, containing only his awareness information. Since the question of an intruder never occur to Watson, the full factual signal "the dog did not bark," or $\{\{a\}, \{b\}\}$, does not "ring a bell" in his mind. To Watson, the information he has is just the trivial partition of the subjective state space $\{\{\Delta\}\}$. As a consequence, Watson does not know $\{a, b\}$, and does not know that he does not know it.[3]

The question "Could there have been an intruder in the stable that night?" reveals the awareness information $\{a', b'\}$ to Watson. Now Watson adds the specification of whether there was an intruder to his subjective state space, updates it to the full state space $\{a, b\}$, and hence recognizes the information partition $\{\{a\}, \{b\}\}$, obtaining the knowledge "there was no intruder" as a result of simply being asked a question.

The model is a natural generalization of Aumann (1976) and exhibits nice properties parallel to those in the standard information partition model. A full state specifies what the agent is or is not aware of, as well as the resolution of external uncertainties and the agent's knowledge. Information takes the form of a pair, consisting of awareness information, represented by essentially a subjective state space, and factual information, represented by the familiar information partition. The role of awareness in knowledge becomes explicit: the agent is said to know an event E if and only if, he is aware of E, and there is no uncertainty regarding E given his factual signal. Each subjective state corresponds to an event in the full state space. Thus unawareness results in the agent's inability to imagine any scenario precisely. The subjective state space is incomplete with respect to the full state space in the sense of omitting "dimensions," not omitting "points," which suggests a clear distinction between probability zero events and unawareness in this framework: if the agent is unaware of an event, then the event is beyond the agent's probability space and he is unable to assign any probability to it.

In a multi-agent environment, agents could reason about what others are aware of as well as what they know, within the confines of their own mind-sets. I also allow the possibility that an agent could be unaware of another agent's unawareness. The resulting interactive knowledge hierarchies have rich but tractable patterns. In partic-

[3]Ely (1998) proposes a similar framework in the context of the Watson example. The observation is that unawareness of uncertainties causes unawareness of signals, thus the same signal structure induces different information partitions under different awareness. Thus Ely considers an information structure that takes the form of state-contingent *partition* of the state space. This approach is the closest in the literature to the one adopted in this paper. Li (2006b) has more details.

ular, unawareness has profound implications for common knowledge. I provide bounds for common knowledge in this environment under general conditions; and characterize common knowledge when there is no unawareness of interactive unawareness and agents receive "nice" factual signals. The characterization is a natural generalization of the classic characterization of Aumann's.

The rest of the paper is organized as follows: Section 2 reviews the possibility correspondence models and DLR's impossibility results, highlighting the implicit assumption of full-awareness in the standard model and pointing to the parallel structure with the current model. Readers familiar with these models could proceed directly to the next section. Section 3 presents the model of unawareness, which I dub "the product model" for the use of the product structure of the state space. Section 4 characterizes the knowledge hierarchy with nontrivial unawareness. Section 5 extends the product model to multi-agent environment. Section 6 concludes. Proofs not found in the text are collected in the Appendix.

2 A Review of the Standard Model

The standard model, also known as the possibility correspondence model, consists of a state space Ω and a possibility correspondence $P : \Omega \to 2^{\Omega} \setminus \{\emptyset\}$. Each state $\omega \in \Omega$ completely specifies the resolution of all relevant uncertainties. An "event" in the ordinary usage of the term corresponds to a set of states in the model. For instance, the informal idea of the event that "there was an intruder" is formally taken to be the set of states where there was an intruder.

With this formulation, one can identify logical relations with set operations: set inclusion "$\subseteq$", set intersection "$\cap$", set union "$\cup$" and set complement (with respect to the state space) "$\setminus$" correspond to logical consequence "$\rightarrow$", conjunction "$\wedge$", disjunction "$\vee$" and negation "$\neg$" respectively.[4]

The agent's information structure is represented by a possibility correspondence $P : \Omega \to 2^{\Omega} \setminus \{\emptyset\}$. P associates each state ω with a nonempty event $P(\omega)$, which is interpreted as the agent's information at ω. The idea is, at ω, the agent considers $P(\omega)$ to be the set of possible states. Note that such information is factual information since it concerns only the resolution of uncertainties.

Definition 1 *P induces an* **information partition** *over the state space if (1) for any $\omega \in \Omega$, $\omega \in P(\omega)$; and (2) for any $\omega, \omega' \in \Omega$, $\omega' \in P(\omega)$ implies $P(\omega') = P(\omega)$.*

If P induces an information partition, then (Ω, P) is an *information partition model.* Otherwise, it is a *non-partitional* model.

Knowledge is characterized as "truth in all possible states." Intuitively, if there was no intruder in every state Holmes considers possible, then for Holmes, there is no

[4] Aumann (1976), Fagin, Halpern, Moses and Vardi (1995), Geanakoplos (1990, 1992) and Rubinstein (1998) are all excellent references on the set-theoretic approach to modelling knowledge.

uncertainty left regarding this event, i.e., he knows "there was no intruder." Formally, for any event $E \subseteq \Omega$, define the knowledge operator $K : 2^\Omega \to 2^\Omega$ by

$$K(E) = \{\omega : P(\omega) \subseteq E\}$$

$K(E)$ is the set of states in which the agent knows E, and hence is interpreted as the event "the agent knows E." To see that it makes sense to interpret K as knowledge, consider the following properties. For any $E, F \subseteq \Omega$,

K1 *Necessitation*: $K(\Omega) = \Omega$

K2 *Monotonicity*: $E \subseteq F \Rightarrow K(E) \subseteq K(F)$

K3 *Conjunction*: $K(E) \cap K(F) = K(E \cap F)$[5]

K4 *The axiom of knowledge*: $K(E) \subseteq E$

K5 *The axiom of transparency*: $K(E) \subseteq KK(E)$[6]

K6 *The axiom of wisdom*: $\neg K(E) \subseteq K \neg K(E)$

A statement like "A is A" is universally true and the agent should know this. Indeed, a tautology is represented by the universal event Ω, and hence necessitation corresponds to knowledge of tautologies. Monotonicity says the agent is able to perform logical deductions. If E implies F and the agent knows E, then he knows F. Conjunction says the agent knows the events E and F if and only if he knows the event "E and F." K1-3 are basic properties of knowledge that make the characterization sensible. Without them, it is not clear what it means "to know" something. In contrast, the next three axioms K4-6 reflect the agent's *rationality* in information processing. The axiom of knowledge says the agent cannot know anything false. The axiom of transparency says whenever the agent knows something, he knows that he knows it. The axiom of wisdom says if the agent does not know something, he knows that he does not know it.

Theorem 1 *(Geanakoplos (1990)) In the possibility correspondence model* (Ω, P)*, knowledge satisfies K1-3. It satisfies K4-6 if* P *induces an information partition.*

It is obvious that the axiom of wisdom prevents an information partition model from having nontrivial unawareness. However, it is less obvious that the implicit assumption of full-awareness lies on the structure of the standard state space specification instead of the partitional information structure (Dekel et al. 1998). DLR introduce an unawareness operator: $U : 2^\Omega \to 2^\Omega$, where $U(E)$ is the set of states where the agent is unaware of E, and hence is interpreted as the event "the agent is unaware of E." They consider three intuitive properties of unawareness: for any event $E \subseteq \Omega$,

[5]Note that conjunction implies monotonicity.

[6]In places where there is no risk of confusion, I omit the parentheses when applying the operators.

U1 *Plausibility*: $U(E) \subseteq \neg K(E) \cap \neg K \neg K(E)$

U2 *AU introspection*: $U(E) \subseteq UU(E)$

U3 *KU introspection*: $KU(E) = \emptyset$

Plausibility says if one is unaware of something, then one does not know it, and does not know that one does not know it. AU introspection says if one is unaware of something, then one must be unaware of the possibility of being unaware of it. KU introspection says under no circumstances can one know exactly what one is unaware of.

DLR show that the combination of these three axioms implies one critical property of unawareness: whenever the agent is unaware of something, he must not know the state space. That is, U1-3 imply $U(E) \subseteq \neg K(\Omega)$ for any $E \subseteq \Omega$. But then adding necessitation or monotonicity eliminates nontrivial unawareness.

3 The Product Model

3.1 The primitives

I explore a product structure on the full state space. Intuitively, one can think of the set of payoff-relevant uncertainties as a set of questions, and each state specifies a complete collection of resolutions to these uncertainties, or answers to the questions, one for each. Therefore, without loss of generality, one can write the state space as the Cartesian product of the sets of answers.

Let $\mathcal{D}^* = \{D_i\}_{i \in A}$,[7] where each D_i is the set of answers to question i and A is an arbitrary index set for all relevant questions. Without loss of generality, I assume D_i is non-empty for all $i \in A$. The collection $\mathcal{D}^*$ represents full awareness.

The full state space Ω^* is defined as the Cartesian product of all sets in the collection of full awareness information:[8,9]

$$\Omega^* = \times_{i \in A} D_i \equiv \times \mathcal{D}^*$$

The *general information structure* is a pair (W^*, P^*), representing the awareness information and factual information respectively. The novel component $W^* : \Omega^* \to 2^{\mathcal{D}^*}$,

[7] To differentiate the "full model" from the potentially incomplete subjective models, I add a $*$ to elements in the full model for emphasis wherever I can.

[8] Despite of the syntactic flavor, the product structure does not impose real limitations. Li (2006b) presents an alternative construction of the unawareness model based on an arbitrary full state space, which, under some regularity conditions, is shown to be equivalent to the product model.

[9] The product structure may impose impossible full states in the sense of being logically inconsistent. For example, suppose there are two relevant uncertainties, whether it rains and whether there is a hurricane. But obviously there could never be a hurricane without raining. I discard all such states. Therefore the full state space is in fact typically a subset of Ω^*, only containing those logically consistent states. With this understanding, I still use Ω^* to denote the full state space.

dubbed *the awareness function*, associates each full state with a subset of $\mathcal{D}^*$. The interpretation is that at ω^*, the agent is aware of the uncertainties contained in $W^*(\omega^*)$.

The second component of the generalized information structure is the full possibility correspondence $P^* : \Omega^* \to 2^{\Omega^*} \setminus \{\emptyset\}$, defined over the full state space and interpreted as the factual signal structure.

I model the agent's state-dependent mind-sets using subjective state spaces and subjective factual information. At ω^*, the agent's subjective state space only contains those uncertainties of which the agent is aware, i.e. contained in $W^*(\omega^*)$, and hence is naturally defined as the Cartesian product of all sets in his awareness information:

$$\Omega(\omega^*) = \times W^*(\omega^*) \tag{3.1}$$

Every subjective state $\omega \in \Omega(\omega^*)$ leaves some questions (those corresponding to sets of answers not included in $W^*(\omega^*)$) unanswered, and thus is a "blurry" picture of the environment. Different subjective state spaces blur the full state space in different ways. Let $\mathcal{S} = \{\Omega = \times\mathcal{D} : \mathcal{D} \subseteq \mathcal{D}^*\}$ be the collection of all possible subjective state spaces. For any $\Omega \in \mathcal{S}$ such that $\Omega = \times\mathcal{D}$, let $\mathbb{P}^{\Omega}$ be the projection operator that yields the projection of some subsets or elements of $\times\mathcal{D}'$ such that $\mathcal{D} \subseteq \mathcal{D}' \subseteq \mathcal{D}^*$ on Ω.

What the agent subjectively knows are what he can recognize, given his awareness constraints, from his factual signal. This is naturally modeled as the projection of the full factual information onto the corresponding subjective state space. For every $\omega^* \in \Omega^*$, let the subjective possibility correspondence $P_{\omega^*} : \Omega(\omega^*) \to 2^{\Omega(\omega^*)} \setminus \{\emptyset\}$ be defined by:[10,11]

$$P_{\omega^*}(\omega) = \mathbb{P}^{\Omega(\omega^*)}(P^*(\omega^*)) \quad \text{for all} \quad \omega \in \mathbb{P}^{\Omega(\omega^*)}(P^*(\omega^*)) \tag{3.2}$$

Let $s(\omega^*) = \mathbb{P}^{\Omega(\omega^*)}(\omega^*)$ denote the "true state" in the agent's mind-set. At ω^*, the agent's subjective factual information is $P_{\omega^*}(s(\omega^*)) = \mathbb{P}^{\Omega(\omega^*)}(P^*(\omega^*))$, which equals to $P^*(\omega^*)$ only if the agent is fully aware.

The triple (Ω^*, W^*, P^*) consists of the primitives of the product model of unawareness.[12] The model extends the information structure from a possibility correspondence defined over the full state space (P^*) to a pair (W^*, P^*), from which state-dependent subjective models characterizing the agent's mind-sets are derived. If the agent has non-trivial unawareness at ω^*, then the subjective model $(\Omega(\omega^*), P_{\omega^*})$ is a coarser version of the "full model" (Ω^*, P^*).

I consider some natural requirements on the generalized information structure. Aumann (1987) points out that in the standard model, it is a tautology to say the agent

[10] The image of P_{ω^*} at subjective states the agent knows to be false is irrelevant for the knowledge hierarchy in single-agent environment, hence I leave it unspecified for now. For a full specification of P_{ω^*}, see Section 5.

[11] The specification of P_{ω^*} implicitly assumes P^* induces an information partition over Ω^*. However, I note here that in general, modeling unawareness alone neither requires, nor implies a partitional structure of P^*. The details of a more general model allowing for non-partitional P^* is available upon request.

[12] Alternatively, one may think of $(\mathcal{D}^*, W^*, P^*)$ as the primitives of the model, since the state space is in fact a derived concept.

knows his own information structure, as the model itself is merely a coding system. To invoke the same argument in this environment, i.e. to validate that it is a tautology to say that the agent knows his own information structure within the confines of his own awareness information, especially that the agent knows his own *awareness information*, one may need to include the specification of awareness information in the full state space. This imposes some obvious restrictions on the awareness function.

For any $D \in \mathcal{D}^*$, let $\tilde{D} = \left\{A^D, U^D\right\}$, where A^D, U^D represent "the agent is aware of D" and "the agent is unaware of D" respectively.

Definition 2 *Suppose $\tilde{D} \in \mathcal{D}^*$. Then, W^* is* **consistent** *if it satisfies:*[13] *for any $\omega^* \in \Omega^*$,*

1. $\mathbb{P}^{\left\{A^D, U^D\right\}}(\omega^*) = A^D \Rightarrow D \in W^*(\omega^*)$;
2. $\mathbb{P}^{\left\{A^D, U^D\right\}}(\omega^*) = U^D \Rightarrow D \notin W^*(\omega^*)$;
3. $\tilde{D} \in W^*(\omega^*) \Rightarrow D \in W^*(\omega^*)$.

The first two conditions require W^* be consistent with the specification of each full state; while the last condition says the agent cannot be aware that he might be unaware of D without being aware of D itself.

Next consider the following generalization of information partition in the standard model to this environment:

Definition 3 *The generalized information structure* (W^*, P^*) *is* **rational** *if it satisfies: for all $\omega_1^*, \omega_2^* \in \Omega^*$, $\omega_1^* \in P^*(\omega_2^*) \Rightarrow (W^*(\omega_1^*), P^*(\omega_1^*)) = (W^*(\omega_2^*), P^*(\omega_2^*))$.*

The rationality condition reflects the idea that a rational agent should be able to exclude states in which he receives different information. To understand this condition, I break it into the following two components:

1. *Factual partition*: P^* induces an information partition over Ω^*;
2. *Rational awareness*: $\omega_1^* \in P^*(\omega_2^*) \Rightarrow W^*(\omega_1^*) = W^*(\omega_2^*)$.

Clearly, factual partition ensures the agent excludes all states in which he receives different factual information, while the additional condition "rational awareness" ensures the agent excludes all states in which he has different awareness. The combination of the two conditions ensures the agent has "partitional" subjective factual information, in the sense that at any full state, the agent excludes all *subjective* states in which he receives different *subjective* factual information.

[13]Since logically inconsistent states are not included in Ω^* (see footnote 9), consistency is well-defined.

Example 1. Suppose Charlie has an episodic hearing problem that causes him to hear a lot of noise when he experiences the problem, which prevents him from telling whether it rains outside. Suppose Charlie is never aware of the hearing problem.

This is modeled as follows. Let r, nr, p, np denote "it is raining," "it is not raining," "experiencing the hearing problem," "not experiencing the hearing problem" respectively.

$\mathcal{D}^* = \{\{r, nr\}, \{p, np\}\}$
$\Omega^* = \times\mathcal{D}^* = \{(r,p), (r,np), (nr,p), (nr,np)\}$
$W^*(\omega^*) = \{\{r, nr\}\}$ for all $\omega^* \in \Omega^*$;
P^* induces the full information partition $\{\{(r,p),(nr,p)\}, \{(r,np)\}, \{(nr,np)\}\}$

At (r,p), Charlie's full factual signal is the event $\{(r,p),(nr,p)\}$: the fact he cannot tell whether it rains indicates he has a hearing problem. However, being unaware of the hearing problem, Charlie only recognizes that he does not know whether it rains. This is reflected in his subjective model at (r,p):

$\Omega((r,p)) = \times\{\{r,nr\}\} = \{r, nr\}$
$P_{(r,p)}(s(r,p)) = P_{(r,p)}(r) = \mathbb{P}^{\{r,nr\}}(\{(r,p),(nr,p)\}) = \{r, nr\}$

Example 2. Suppose Alice moves first, choosing between a_1 and a_2. Bob does not observe Alice's choice. If Alice chooses a_1, then Bob chooses between b_1 and b_2; but if Alice chooses a_2, then Bob is only aware of b_1. I am interested in Bob's information structure.

Case 1. Suppose Bob is aware that he could be unaware of b_2 if Alice plays a_1. Let A^b, U^b represents "Bob is aware of b_2," "Bob is unaware of b_2" respectively.[14]

$\mathcal{D}^* = \left\{\{a_1, a_2\}, \{b_1, b_2\}, \left\{A^b, U^b\right\}\right\}$
$\Omega^* = \left\{a_1b_1A^b, a_2b_1A^b, a_1b_1U^b, a_2b_1U^b, a_1b_2A^b, a_2b_2A^b\right\}$
$W^*(\omega^*) = \mathcal{D}^*$ for all ω^* such that $\mathbb{P}^{\left\{A^b,\, U^b\right\}}(\omega^*) = A^b$, and $= \{\{a_1, a_2\}\}$ otherwise;
P^* induces the full information partition:

$$\left\{\left\{a_1b_1A^b, a_2b_1A^b\right\}, \left\{a_1b_1U^b, a_2b_1U^b\right\}, \left\{a_1b_2A^b, a_2b_2A^b\right\}\right\}$$

At $a_2b_1U^b$, Bob's subjective model is:

$\Omega(a_2b_1U^b) = \times\{\{a_1, a_2\}\} = \{a_1, a_2\}$
$P_{a_2b_1U^b}(s(a_2b_1U^b)) = P_{a_2b_1U^b}(a_2) = \mathbb{P}^{\{a_1,a_2\}}(\left\{a_1b_1U^b, a_2b_1U^b\right\}) = \{a_1, a_2\}$

On the other hand, at $a_1b_1A^b$ and $a_1b_2A^b$, since Bob is fully aware, his subjective model is identical to the full model, thus he is able to exclude states in which Alice plays

[14]Notice that b_2U^b states are logically inconsistent states – Bob cannot play b_2 while being unaware of it – and hence these are discarded from the full state space.

a_1 on the basis of his own awareness of b_2.

Case 2. Alternatively, suppose Bob is never aware that he could be unaware of b_2. To model this situation, let the full model (Ω^*, P^*) be the same as above, but the awareness function be specified as follows:

$W^*(\omega^*) = \{\{a_1, a_2\}, \{b_1, b_2\}\}$ for all ω^* such that $\mathbb{P}^{\{A^b,\, U^b\}}(\omega^*) = A^b$, and $= \{\{a_1, a_2\}\}$ otherwise.

Now at $a_1 b_1 A^b$, Bob's subjective model is:

$$\Omega(a_1 b_1 A^b) = \times \{\{a_1, a_2\}, \{b_1, b_2\}\} = \{a_1 b_1, a_2 b_1, a_1 b_2, a_2 b_2\}$$
$$P_{a_1 b_1 A^b}(s(a_1 b_1 A^b)) = P_{a_1 b_1 A^b}(a_1 b_1) = \mathbb{P}^{\Omega(a_1 b_1 A^b)}(\{a_1 b_1 A^b, a_2 b_1 A^b\}) = \{a_1 b_1, a_2 b_1\}$$

Since Bob is unaware that he could be unaware of b_2, he cannot use his awareness of b_2 at $a_1 b_1 A^b$ to exclude (subjective) states in which Alice plays a_2.

3.2 The events

In the standard model where all information is factual, events only differ in the facts they convey. With the generalized information structure, events can differ in awareness, in facts, or in both. For instance, in the hearing problem example, the events "it rains, and there is a possibility that Charlie has a hearing problem" and "it rains" are different events because they involve different levels of awareness. By construction, every subjective state has awareness built into it, and hence can be conveniently used to represent the enriched set of events.

Let E be a nonempty subset of some subjective state space. By construction, one can identify its space and hence its awareness information. Let it be denoted $\mathcal{D}_E$, i.e. $E \subseteq \times \mathcal{D}_E$. But the empty set, being a subset of any space, creates both technical difficulties and conceptual inconsistency. The empty set represents an impossible event or a logical contradiction. Intuitively, there are many different contradictions that involve different sets of questions. For example, the statements "it rains and it does not rain" and "it rains, it does not rain, and I have a hearing problem" are both logical contradictions, yet the former contains less awareness than the latter. Presumably an agent who is unaware of the possibility of the hearing problem cannot perceive the latter.

In light of this, for any state space $\Omega \in \mathcal{S}$, let $\emptyset_\Omega$ denote the empty set associated with Ω. Intuitively, this object is the empty set tagged with the awareness information. It behaves in the same way as the usual empty set, except that it is confined to its state space. To incorporate this new object in set operations, I extend the usual set inclusion, intersection, union and complement notions (for notational ease, I use the conventional symbols for these operations): For any state space Ω and any sets $E, F \neq \emptyset_\Omega,\ E, F \subseteq \Omega$, the set inclusion, intersection, union and complement notions are defined in the usual way, except that for disjoint E and F, $E \cap F = \emptyset_\Omega$ instead of $\emptyset$. In addition, for any $E \subseteq \Omega$, $\emptyset_\Omega \cup E = E$, $\emptyset_\Omega \cap E = \emptyset_\Omega$, $E \setminus \emptyset_\Omega = E$, and for any E such that $E \neq \emptyset$ and $E \subseteq \Omega$, $\emptyset_\Omega \subseteq E$.[15] Finally, for convenience, I rule out the empty state space by requiring $\{\Delta\} \in \mathcal{D}^*$ and $\{\Delta\} \in W^*(\omega^*)$ for all $\omega^* \in \Omega^*$.[16]

The collection of events in this model is:

$$\mathcal{E}^p = \{E \neq \emptyset : E \subseteq \Omega \text{ for some } \Omega \in \mathcal{S}\}$$

This collection consists of all perceivable events given the full state space. Note $\{\emptyset_\Omega\}_{\Omega \in \mathcal{S}}$ are elements of $\mathcal{E}^p$.

Events that convey the same facts but contain different awareness play a critical role in an environment with unawareness.

Definition 4 *F is said to be an* **elaboration** *of E and E is said to be a* **reduction** *of F, if*

$$\mathcal{D}_F \supseteq \mathcal{D}_E \text{ and } F = \left\{\omega \in \times\mathcal{D}_F : \mathbb{P}^{\times\mathcal{D}_E}(\omega) \in E\right\}$$

For example, $\{(r,p),(r,np)\}$ is an elaboration of $\{r\}$. For any $E \in \mathcal{E}^p$ and Ω satisfying $\mathcal{D}_E \subseteq \mathcal{D}_\Omega$, let E_Ω denote the elaboration of E in Ω. That is, E_Ω is the unique event satisfying $E_\Omega \subseteq \Omega$ and $E \leq E_\Omega$. In the above example, $\{(r,p),(r,np)\} = \{r\}_{\Omega^*}$.

Since the logical relations between events concern only facts, they are preserved by elaborations. Thus one can deal with logical relations between arbitrary subjective events E and F using their minimal elaborations that live in the same space, i.e. the space defined by $\times(\mathcal{D}_E \cup \mathcal{D}_F)$. This observation suggests the following definitions of *extended set relations and operations* on elements of $\mathcal{E}^p$ and their connections to logical relations:

Definition 5 Extended set relations and operations[17]

1. *Extended set inclusion: E is an extended weak subset of F, denoted by $E \subseteq_* F$ provided $E_{\Omega^*} \subseteq F_{\Omega^*}$; In this case, F is a logical consequence of E;*

2. *Extended set intersection, denoted by $\cap_*$: $E \cap_* F \equiv E_{\times(\mathcal{D}_E \cup \mathcal{D}_F)} \cap F_{\times(\mathcal{D}_E \cup \mathcal{D}_F)}$ is the conjunction between E and F;*

3. *Extended set union, denoted by $\cup_*$: $E \cup_* F \equiv E_{\times(\mathcal{D}_E \cup \mathcal{D}_F)} \cup F_{\times(\mathcal{D}_E \cup \mathcal{D}_F)}$ is the disjunction of E and F.*

[15] Another way to think about the multiple empty sets is that an event E in the model is actually a pair $(\times\mathcal{D}_E, E)$ where $E \subseteq \times\mathcal{D}_E$. The first object in the pair, $\times\mathcal{D}_E$, specifies the awareness in E while the second object represents the involved facts. In particular, $\emptyset_\Omega$ should be interpreted as the pair $(\Omega, \emptyset)$. Then the usual set inclusion can be extended to this space by letting $(\Omega, E) \subseteq (\Omega', F)$ iff $\Omega = \Omega'$ and $E \subseteq F$, and similarly for other set operations. For $E \neq \emptyset$, $\mathcal{D}_E$ is uniquely identified from it and hence is redundant.

[16] For simplicity, I omit Δ when there is no risk of confusion.

[17] These relations and operations reduce to the usual ones for events from the same space.

Finally, *Negation* of a subjective event involves the same amount of awareness but the facts are reversed. It is naturally identified with the set complement operation with respect to the corresponding subjective state space:

$$\neg E = \times \mathcal{D}_E \setminus E$$

3.3 The knowledge and unawareness operators

One can only reason about things of which one is aware. For any event E, only if there is some version of E – either E itself or some elaboration of it – in the agent's mind-set, can he reason about it. In addition, the agent's mind-set also constrains his reasoning about his own knowledge. Recall the hearing problem example. Charlie knows it rains only if it rains and he does not experience the hearing problem. Therefore, the event "*Charlie* knows it rains" (perhaps from the modeler's perspective) should be the singleton set $\{(r, np)\}$. On the other hand, at (r, np), since Charlie himself is unaware of the hearing problem, *from his perspective*, the event "*I* know it rains" is represented by the singleton set $\{r\}$ in the subjective state space. Therefore, there are two versions of knowledge in this model: *objective knowledge*, for example, $\{(r, np)\}$; and the corresponding *subjective knowledge*, in this case, $\{r\}$. A crucial observation is that, when dealing with higher-order knowledge such as "Charlie knows that *he knows it rains*," the event "he knows it rains" should refer to the subjective knowledge. To obtain an objective account for Charlie's knowledge hierarchy, one needs to examine his subjective knowledge hierarchies *based on his own (full-)state-dependent mind-sets.*

Fix a full state ω^*, the agent's mind-set is modeled by $(\Omega(\omega^*), P_{\omega^*})$. Notice this is a standard information partition model when restricted to subjective states contained in $P_{\omega^*}(s(\omega^*))$. So subjective knowledge can be computed just as in the standard model. For notational ease, let $\emptyset_E \equiv \emptyset_{\times \mathcal{D}_E}$ denote the empty set that is associated with the state space of E. The *subjective knowledge operator* at ω^*, denoted by $\tilde{K}_{\omega^*}$, mapping $\mathcal{E}^p$ to the set of subjective events in $\Omega(\omega^*)$ and all impossible events, is defined by:

$$\tilde{K}_{\omega^*}(E) = \begin{cases} \{\omega \in \Omega(\omega^*) : P_{\omega^*}(\omega) \subseteq E_{\Omega(\omega^*)}\} & \text{if } \mathcal{D}_E \subseteq W^*(\omega^*) \\ \emptyset_E & \text{if } \mathcal{D}_E \not\subseteq W^*(\omega^*) \end{cases} \tag{3.3}$$

In words, the agent knows E if he is aware of E and E is true in all *subjective* states he considers possible. Here $E_{\Omega(\omega^*)}$ is the version of E in the agent's mind-set provided he is aware of E. If he is unaware of E, i.e. $\mathcal{D}_E \not\subseteq W^*(\omega^*)$, the event $E_{\Omega(\omega^*)}$ is not defined and $\tilde{K}_{\omega^*}(E)$ is empty. To preserve the awareness information in E, I denote it by the empty set that contains the same awareness as E.

The subjective event $\tilde{K}_{\omega^*}(E)$ is interpreted as "knowledge of E in the agent's mind-set." Similar to the standard model, iteration of the subjective knowledge operator yields *subjective higher-order knowledge* in the agent's mind-set: "I know that I know $\cdots$ I know E."

$$\tilde{K}^n_{\omega^*}(E) = \tilde{K}_{\omega^*}\tilde{K}^{n-1}_{\omega^*}(E) \tag{3.4}$$

The *objective n-th order knowledge* is then obtained by putting together the relevant pieces of the subjective knowledge hierarchies.

$$K^n(E) = \left\{\omega^* \in \Omega^* : s(\omega^*) \in \tilde{K}^n_{\omega^*}(E)\right\} \tag{3.5}$$

Clearly, in the n-th order knowledge, the second- and higher-order knowledge are subjective knowledge. But to be consistent with the literature, I use the customary notation K^n.[18]

When $n = 1$, equation (3.5) reduces to:

$$K(E) \;=\; \{\omega^* \in \Omega^* : \mathcal{D}_E \subseteq W^*(\omega^*),\ P^*(\omega^*) \subseteq E_{\Omega^*}\} \tag{3.6}$$

(3.6) has a straightforward interpretation: the agent knows E if and only if he is aware of E and the factual signal implies E. Note (3.6) says the first-order knowledge can be directly derived from the generalized information structure without referring to the subjective models.

Similar to the objective knowledge, the event "the agent is unaware of E" should be understood as an objective event from the modeler's perspective. Since awareness information is built into every event, it is easy to define the *unawareness operator*:

$$U(E) = \{\omega^* \in \Omega^* : \mathcal{D}_E \nsubseteq W^*(\omega^*)\} \tag{3.7}$$

In words, the agent is unaware of E if and only if E contains awareness information the agent does not have.

Since awareness itself could be part of the specification of the full state space, for any $E \in \mathcal{E}^p$, the subjective event $\underset{\left\{D \in \mathcal{D}_E : \tilde{D} \in \mathcal{D}^*\right\}}{\cup} \left\{U^D\right\}$ also represents "the agent is unaware of E." Then an obvious question is whether they coincide. The following result says they do, as long as the awareness function is consistent. For notational ease, let $\tilde{\mathcal{D}}_E = \left\{D : D \in \mathcal{D}_E,\ \tilde{D} \in \mathcal{D}^*\right\}$.

Proposition 2 *Let (Ω^*, W^*, P^*) be consistent. Then for all $E \in \mathcal{E}^p$,*

$$\underset{D \in \tilde{\mathcal{D}}_E}{\cup} \left\{U^D\right\} \subseteq_* U(E)$$

Moreover, if $\tilde{\mathcal{D}}_E = \mathcal{D}_E$ then the "$\subseteq$" in the above expression can be replaced by "$=$."

One may wish to talk about the agent's knowledge about his own (un)awareness. For example, "Charlie knows he is aware of E." By the same argument leading to the objective knowledge hierarchy, the event "he is aware of E" should refer to the subjective

[18] Although $K(E)$ is a legitimate event in this model, it is by definition not an event the agent can conceive, and hence "the agent knows $K(E)$" is not a well-defined event.

event $\bigcap_{D \in \mathcal{D}_E} \{A^D\}$ instead of the objective event $\neg U(E)$. Again, to be consistent with the literature and for notational ease, I use $K\neg U(E)$ to represent the event "Charlie knows he is aware of E."

Finally, to see the connection with the standard model, notice that if the agent is fully aware in all full states, then he has the same subjective model – the full model – in all full states. The subjective knowledge operator $\tilde{K}$ then reduces to the usual K in the standard model and all subjective knowledge hierarchies become identical and coincide with the objective knowledge hierarchy. In that case the product model simply reduces to the standard model.

4 The Knowledge Hierarchy with Unawareness

Recall that in the standard information partition model (Ω, P), the agent's knowledge hierarchy is completely characterized at the first level: for any $E \subseteq \Omega$,

1. $K(E) = KK(E)$;
2. $\neg K(E) = K\neg K(E)$.

In words, for any event, the agent either knows it or does not know it, and he always knows whether he knows it. Natural generalization of the above characterization obtains in the product model.

Theorem 3 *In the product model (Ω^*, W^*, P^*), let (W^*, P^*) be consistent and rational. Then the agent's knowledge hierarchy satisfies: for any $E \in \mathcal{E}^p$,*

1. $U(E) = \neg K(E) \cap \neg K\neg K(E)$;
2. $K(E) = KK(E)$;
3. $\neg K(E) \cap \neg U(E) = K\neg K(E)$.

In words, for any event, the agent is either unaware of it, in which case he does not know it, and he does not know that he does not know it, and so on; or he is aware of it, in which case he either knows it or does not know it, and he always knows whether he knows it. Analogous to the standard model, the knowledge hierarchy is completely pinned down at the first level. In particular, this means as long as (W^*, P^*) is consistent and rational, the entire knowledge hierarchy can be derived from the pair (W^*, P^*) directly.

I prove theorem 3 via two Lemmas. The first lemma deals with the basic properties of the product model without imposing rationality of (W^*, P^*). Consider the following basic desiderata of unawareness and knowledge: for all $E,\ F \in \mathcal{E}^p$,

U0* *Symmetry*: $U(E) = U(\neg E)$

U1′ *Strong plausibility*: $U(E) \subseteq \bigcap_{n=1}^{\infty}(\neg K)^n(E)$

U2* *AU introspection*: $U(E) \subseteq UU(E)$

U3′ *Weak KU introspection*: $U(E) \cap KU(E) = \emptyset_{\Omega^*}$

K1* *Subjective necessitation*: $\omega^* \in K(\Omega(\omega^*))$ for all $\omega^* \in \Omega^*$

K2* *Generalized monotonicity*:[19] $E \subseteq_* F,\ \mathcal{D}_E \supseteq \mathcal{D}_F \Rightarrow K(E) \subseteq K(F)$

K3* *Conjunction*:[20] $K(E) \cap K(F) = K(E \cap_* F)$

Symmetry is proposed by Modica and Rustichini (1999). It says that one is unaware of an event if and only if one is unaware of the negation of it. The other three unawareness properties correspond to the three axioms proposed by DLR, with slight modifications of plausibility and KU introspection. Strong plausibility strengthens DLR's plausibility axiom. Plausibility requires that whenever one is unaware of something, one does not know it and does not know that one does not know it. I require such a lack of knowledge to be extended to an arbitrarily high order.[21] While KU introspection requires the agent never know exactly what he is unaware of, the weak KU introspection slightly weakens it by allowing the agent to have false knowledge of his being unaware of a particular event.

$K1^* - 3^*$ are natural analogues of K1-3 in the context of nontrivial unawareness. Recall that necessitation says the agent knows all tautological statements: $K(\Omega^*) = \Omega^*$. However, while all theorems are tautologies, arguably Newton does not know the theory of general relativity because he is unaware of it. This is reflected in subjective necessitation, which says the agent knows all tautological statements *of which he is aware*.[22]

The essence of monotonicity is the intuitive notion that knowledge should be monotonic with respect to the information content of events. In the standard model, an event is more informative than another if and only if it conveys more facts. In the product model, an event is more informative than another if and only if it contains both more facts and *more awareness*. Alternatively, note monotonicity means the agent knows the logical consequences of his knowledge, while generalized monotonicity, which explicitly takes into account that the agent may not be fully aware, says the agent knows those logical consequences of his knowledge *of which he is aware*.

[19] This property implies the agent is no longer logically omniscient: he knows the logical consequences of his knowledge *only if he is aware of them.*

[20] Parallel to the standard model, conjunction implies generalized monotonicity.

[21] DLR consider the weaker property plausibility, which is sufficient for the negative result in which they are interested. However, when it comes to providing *positive* results in a model that deals with unawareness, strong plausibility, or an even stronger property that equates unawareness with the lack of knowledge of all orders which I discuss shortly, seems to be more interesting.

[22] Subjective necessitation is equivalent to the "weak necessitation" property DLR discussed in the context of propositional models.

Lemma 4 *The product model $\{\Omega^*, W^*, P^*\}$ satisfies U0*, U1$'$, U2*, U3$'$ and K1* − 3* provided it is consistent.*

To see the connection between lemma 4 and DLR's impossibility results, first notice that by (3.6),

$$K(\Omega^*) = \{\omega^* \in \Omega^* : \mathcal{D}^* \subseteq W^*(\omega^*), P^*(\omega^*) \subseteq \Omega^*\} = \{\omega^* \in \Omega^* : \mathcal{D}^* = W^*(\omega^*)\}$$

That is, $K(\Omega^*) = \Omega^* \Leftrightarrow W^*(\omega^*) = \mathcal{D}^*$ for all $\omega^* \in \Omega^*$. In other words, necessitation holds if and only if the agent is fully aware in every full state. But this implies $\mathcal{D}_E \subseteq W^*(\omega^*)$ for all $\omega^* \in \Omega^*$ and $E \in \mathcal{E}$, hence $U(E) = \emptyset_{\Omega^*}$, which is DLR's first impossibility result.

Secondly, observe that generalized monotonicity and monotonicity differ in that generalized monotonicity does not require knowledge to be monotonic when $\mathcal{D}_E \nsubseteq \mathcal{D}_F$, while in the standard model, one necessarily has $\mathcal{D}_E = \mathcal{D}_F$. Let $E \subseteq_* F$ and $K(E) \subseteq K(F)$. By (3.6), $\{\omega^* \in \Omega^* : \mathcal{D}_E \subseteq W^*(\omega^*)\} \subseteq \{\omega^* \in \Omega^* : \mathcal{D}_F \subseteq W^*(\omega^*)\}$. This implies $U(F) \subseteq U(E)$, which says whenever the agent is unaware of F, he is unaware of E. By (3.7), for any G such that $\mathcal{D}_G = \mathcal{D}_E$,

$$U(F) \subseteq U(G) = U(E)$$

That is, whenever the agent is unaware of F, he is unaware of any event that contains the same awareness information as E. By strong plausibility, he cannot know any of them, which is DLR's second impossibility result.

Lemma 4 does not require (W^*, P^*) to be rational. This is because analogous to the standard model, rationality in information processing mainly has implications for higher-order knowledge, while none of the above properties involves higher-order knowledge. It is the generalization of information structure from merely facts to both awareness and facts, rather than specific structure of the possibility correspondence or awareness function other than consistency that captures the essence of nontrivial unawareness.

Consider the following stronger properties of unawareness and knowledge:

U1* *UUU (Unawareness = unknown unknowns)*: $U(E) = \bigcap_{n=1}^{\infty} (\neg K)^n (E)$

U3* *KU introspection*: $KU(E) = \emptyset_{\Omega^*}$

K4* *The axiom of knowledge*: $K(E) \subseteq_* E$

K5* *The axiom of transparency*: $K(E) \subseteq KK(E)$

K6* *The axiom of limited wisdom*: $\neg K(E) \cap \neg U(E) \subseteq K \neg K(E)$

Lemma 5 *The product model (Ω^*, W^*, P^*) satisfies U1*, U3* and K4* − 6* if the generalized information structure (W^*, P^*) is consistent and rational.*

The axiom of limited wisdom extends the axiom of wisdom to an environment with unawareness by only requiring the agent to know that he does not know when he *is aware of the involved event*. UUU says the agent is unaware of an event *if and only if* he does not know it, he does not know that he does not know it, and so on. The extra strength added to strong plausibility is due to the axiom of limited wisdom: if the agent always knows that he does not know if he is aware of the event, then obviously the only circumstance where he does not know that he does not know is that he is unaware of it. Lastly, the axiom of knowledge says the agent can never have false knowledge, which, combined with weak KU introspection, yields KU introspection.

Remark. Information has more dramatic effects on the agent's knowledge hierarchy when the agent has nontrivial unawareness than when he does not. Upon receipt of new information, the agent updates his subjective state space as well as his subjective factual information. Formally, given ω^*, let the agent's initial information be $(W_0^*(\omega^*), P_0^*(\omega^*))$. The agent has subjective factual information:

$$\mathbb{P}^{\times W_0^*(\omega^*)}[P_0^*(\omega^*)]$$

Upon receipt of new information $(W_1^*(\omega^*), P_1^*(\omega^*))$, the agent updates his subjective factual information to

$$\mathbb{P}^{\times [W_0^*(\omega^*) \cup W_1^*(\omega^*)]}[P_0^*(\omega^*) \cap P_1^*(\omega^*)]$$

As long as $W_1^*(\omega^*) \setminus W_0^*(\omega^*) \neq \emptyset$, the agent gains new knowledge.

In particular, if $P_0^*(\omega^*)$ is not an elaboration of $\mathbb{P}^{\times W_0^*(\omega^*)}[P_0^*(\omega^*)]$, that is, if it contains factual information about uncertainties beyond $W_0^*(\omega^*)$, then the agent could learn new facts from introspection of the first-period factual signal along. This aspect of the model clearly differentiates probability zero events from unawareness in dynamic environments: upon receipt of new awareness information, the agent does not update his beliefs about probability zero events, while he could assign any probability to an event of which he was previously unaware.

For a concrete example, recall the Watson story from the introduction. The true full state is "there was no intruder"$=b$. Watson's initial information is the pair $(W_0^*(b), P_0^*(b)) = (\{\Delta\}, \{b\})$. His subjective factual information is $\mathbb{P}^{\{\Delta\}}\{b\} = \{\Delta\}$. The question "Could there have been an intruder?" is represented by the generalized information $(W_1^*(b), P_1^*(b)) = (\{\{a', b'\}\}, \{a, b\})$. Upon being asked the question, Watson updates his subjective state space and recognizes the factual information he has had all along but has been neglected by him: $\mathbb{P}^{\times(\{\Delta\} \cup \{a', b'\})}(\{b\} \cap \{a, b\}) = \{b\}$. On the other hand, had Watson been aware of the state b but simply assigned zero probability to it, the question should not cause him to update this belief.

5 The Multi-agent Model

In multi-agent environment, agents reason about each other's awareness as well as knowledge, within the confines of their own awareness. Thus one needs to model interactive mind-sets: how i reasons about j's mind-set given his own mind-set. For example, recall the Watson story. Suppose Holmes is fully aware, in addition, he is aware that Watson is unaware of the possibility that there might not have been an intruder. Then Holmes knows Watson has a subjective state space $\{\Delta\}$. Consider the event "there was no intruder." Watson does not know it since he is unaware of this event. Does Holmes know Watson doesn't know this? The event "Watson doesn't know there was no intruder" is represented by $\{a, b\}$ in the full state space, which is Holmes' state space, and Holmes receives the information $\{b\}$ at b, therefore Holmes knows it. In fact, in Holmes' mind-set, Watson's information is represented by the product model developed earlier. Therefore, to model interactive awareness, I allow the agents to have *subjective product models*.

Let $N = \{1, \cdots, n\}$ be the set of agents, and $\mathbf{W}^* = (W_1^*, \cdots, W_n^*)$ and $\mathbf{P}^* = (P_1^*, \cdots, P_n^*)$ denote the vector of awareness function and full possibility correspondences. Below I construct the subjective interactive models from the full model $(\Omega^*, \mathbf{W}^*, \mathbf{P}^*)$.

5.1 First-order interactive model.

Let $W_j(\cdot|i_{\omega^*})$ denote *j's subjective awareness function* i ascribes to j in i's mind-set at ω^*. The symbol i_{ω^*} is to be understood as the pair (ω^*, i). For any $\omega \in \Omega_i(\omega^*)$,

$$W_j(\omega|i_{\omega^*}) = W_i^*(\omega^*) \cap [\bigcup_{\{\omega_1^*:\, \mathbb{P}^{\Omega_i(\omega^*)}(\omega_1^*)=\omega\}} W_j^*(\omega_1^*)] \tag{5.1}$$

The intersection captures the idea that i cannot reason beyond his own mind-set, while the union captures the fact that i cannot distinguish awareness information in full states if they are beyond his own subjective state space. Note that $W_i(\cdot|i_{\omega^*})$ is simply i's understanding of his own awareness information at ω^*. Rational awareness ensures $W_i(\omega|i_{\omega^*}) = W_i^*(\omega^*)$ for all $\omega \in P_{\omega^*}(s(\omega^*))$, that is, i knows his own awareness. On the other hand, i can be aware that he is unaware of some uncertainties in a subjective state he knows to be false. For example, in a, Watson could be aware that he would be unaware of the possibility of no intruder in b.

The interactive factual information is naturally modeled as the projection of the full possibility correspondence on i's subjective state space. But what if unawareness affects what the agent *would have known* regarding things of which he is aware? Recall the hearing problem example (Example 1). At (r, np), Charlie knows it rains. It seems plausible to say that since he is unaware of the hearing problem which could cause him to be ignorant about the weather condition, in his mind-set he should consider that he would have known it does not rain had it not rained, the projection of the full factual signal he receives at (nr, np).

Thus, a plausible specification of the subjective factual information at subjective states the agent considers impossible is to take the projection of the partition elements containing the full states with the resolution of the uncertainties of which the agent is unaware fixed at the same values as the true full state. Formally, for any $\omega^* \in \Omega^*$, let $u_i(\omega^*) = \mathbb{P}^{\times(\mathcal{D}^* \setminus W_i^*(\omega^*))}(\omega^*)$ denote the "default" resolution of the uncertainties of which i is unaware in his mind-set at ω^*. For any $\omega \in \Omega_i^*(\omega^*)$, let $\omega \times u_i(\omega^*)$ denote the full state where the uncertainties of which i is aware are resolved as in the subjective state ω, and those uncertainties of which i is unaware are resolved according to the "default."

Let $P_j(\cdot|i_{\omega^*}) : \Omega_i(\omega^*) \to 2^{\Omega_i(\omega^*)} \setminus \{\emptyset\}$ denote j's factual information structure in i's mind-set. Define:

$$P_j(\omega|i_{\omega^*}) = \begin{cases} \mathbb{P}^{\Omega_i(\omega^*)} P_j^*(\omega^*) & \text{for } \omega \in \mathbb{P}^{\Omega_i(\omega^*)} P_j^*(\omega^*) \\ \mathbb{P}^{\Omega_i(\omega^*)} P_j^*(\omega \times u_i(\omega^*)) & \text{otherwise.} \end{cases} \tag{5.2}$$

The tuple $(\Omega_i(\omega^*), W_j(\cdot|i_{\omega^*}), P_j(\cdot|i_{\omega^*}))$ describes the first-order interactive model capturing i's reasoning about j in his mind-set at ω^*. It is a product model itself, with $(\Omega_i(\omega^*), P_j(\cdot|i_{\omega^*}))$ being the "full" state space and "full" possibility correspondence. Therefore, at ω^*, agents' mind-sets are described by a collection of multi-agent product models, one for each agent:

$$(\Omega_i(\omega^*), \mathbf{W}(\cdot|i_{\omega^*}), \mathbf{P}(\cdot|i_{\omega^*}))_{i \in N}$$

where $\mathbf{W}(\cdot|i_{\omega^*}) = (W_1(\cdot|i_{\omega^*}), \cdots, W_n(\cdot|i_{\omega^*}))$ and $\mathbf{P}(\cdot|i_{\omega^*}) = (P_1(\cdot|i_{\omega^*}), \cdots, P_n(\cdot|i_{\omega^*}))$.

5.2 Higher-order interactive models.

Now each agent's mind-set is described by a multi-agent product model *of which i is fully aware*, one can construct higher-order interactive models recursively, such as i's reasoning at ω_1 about j's reasoning at ω_2 about k's reasoning at ω_3 etc. For notational ease, let q^n denote the tuple $((\omega_1, i^1), \cdots, (\omega_k, i^k))$, where $i^1, \cdots, i^n \in N$. Intuitively, q^n is the "reasoning string" of i^1's reasoning at ω_1 about i^2's reasoning at ω_2 about $\cdots$ about i^k's subjective model at ω_k. Let $\Omega_i(\omega|q^n)$ denote i's subjective state space at ω, ascribed by i^n at ω_n, ascribed by $\cdots$ by i^1 at ω_1, similarly for $W_i(\cdot|q^n)$ and $P_i(\cdot|q^n)$.

Not all reasoning strings are plausible. An obvious restriction in this environment is that everyone reasons within his own mind-set, and understands everyone else can only reason within his own mind-set. In other words, at $\omega_1 \in \Omega^*$, i's reasoning about j is limited to his own subjective state space, i.e. $\omega_2 \in \Omega_i(\omega_1)$; and i's reasoning at ω_1 about j's reasoning at ω_2 about k is limited to j's subjective state space at ω_2 in i's mind-set at ω^*, i.e. $\omega_3 \in \Omega_j(\omega|i_{\omega^*}) \equiv \times W_j(\omega|i_{\omega^*})$, and so on.

Definition 6 *For any n, q^n is* **permissible** *if $\omega_1 \in \Omega^*$, and for all $m = 2, \cdots, n$, $\omega_m \in \Omega_{i^{m-1}}(\omega_{m-1}|q^{m-2})$.*

Suppose all $n-1$-th order interactive models are defined. The n-th order interactive model is constructed recursively as follows. Let q^n be permissible. The interactive reasoning of interest is then i^1's reasoning at ω_1 of i^2's reasoning at ω_2 of $\cdots$ of i^n's reasoning at ω_n about i^{n+1}. This is described by the product model

$$(\Omega_{i^n}(\omega_n|q^{n-1}), W_{i^{n+1}}(\cdot|q^n), P_{i^{n+1}}(\cdot|q^n))$$

defined by:

$$\Omega_{i^n}(\omega_n|q^{n-1}) = \times W_{i^n}(\omega_n|q^{n-1}) \tag{5.3}$$

and for all $\omega \in \Omega_{i^n}(\omega_n|q^{n-1})$,

$$W_{i^{n+1}}(\omega|q^n) = W_{i^n}(\omega_n|q^{n-1}) \cap [\bigcup_{\left\{\omega' \in \Omega_{i^{n-1}}(\omega_{n-1}|q^{n-2}):\, \mathbb{P}^{\Omega_{i^n}(\omega_n|q^{n-1})}(\omega')=\omega\right\}} W_{i^{n+1}}(\omega'|q^{n-1})] \tag{5.4}$$

$$P_{i^{n+1}}(\omega|q^n) = \begin{cases} \mathbb{P}^{\Omega_{i^n}(\omega_n|q^{n-1})} P_{i^{n+1}}(\omega_n|q^{n-1}) & \text{for } \omega \in \mathbb{P}^{\Omega_{i^n}(\omega_n|q^{n-1})} P_{i^{n+1}}(\omega_n|q^{n-1}) \\ \mathbb{P}^{\Omega_{i^n}(\omega_n|q^{n-1})} P_{i^{n+1}}(\omega \times u_{i^n}(\omega_n|q^{n-1})|q^{n-1}) & \text{otherwise.} \end{cases} \tag{5.5}$$

Where in (5.5), $u_{i^n}(\omega_n|q^{n-1}) = \mathbb{P}^{\times(W_{i^{n-1}}(\omega_{n-1}|q^{n-2}) \setminus W_{i^n}(\omega_n|q^{n-1}))}(\omega_n)$ is the "default" resolution of those uncertainties of which i^n is unaware at ω_n (but i^{n-1} is aware), according to i^1's reasoning at ω_1 about i^2's reasoning at ω_2 about $\cdots$ about i^{n-1}'s reasoning at ω_{n-1} about i^n. Note for any $\omega \in \Omega_{i^n}(\omega_n|q^{n-1})$, $\omega \times u_{i^n}(\omega_n|q^{n-1})$ is the subjective state in $\Omega_{i^{n-1}}(\omega_{n-1}|q^{n-2})$ where the uncertainties of which i^n is aware are resolved as in ω and the uncertainties of which i^n is unaware are resolved according to the "default" value captured in $u_{i^n}(\omega_n|q^{n-1})$.

The tuple $(\Omega^*, \mathbf{W}^*, \mathbf{P}^*)$ along with the construction in (5.3)-(5.5) consists of the multi-agent product model. It is said to be *rational* if (W_i^*, P_i^*) is rational for all $i \in N$.

Since agents' mind-sets may differ from each other and across full states, different subjective models are needed to describe the interactive reasoning under different permissible sequences. This is in contrast to the standard model, where *all* interactive reasoning is described by the same model, i.e. the full model $(\Omega^*, \mathbf{P}^*)$.

5.3 Consistency, richness and rationality conditions.

In the multi-agent environment, a natural extension of the consistency requirement is that one cannot reason about *anyone's* awareness of E without being aware of E himself.[23]

[23] One may wish to model situations where i is aware of j and D but is unaware that j is aware of D. Intuitively, there is a natural asymmetry in treating interactive awareness: while it is plausible to say i is aware that j is unaware of D only if i himself is aware of D, it seems rather awkward to say i is aware of D in a game with j, while being unaware that j is aware of D. Such situations seem more appropriately modeled using beliefs or knowledge, for example, i believes j does not *know* D.

Definition 7 *The vector of awareness function* $\mathbf{W}^*$ *is* **consistent** *if* W_i^* *is consistent for all* $i \in N$*, and in addition, for all* $i, j \in N, \omega^* \in \Omega^*$*,* $\tilde{D}_j \in W_i^*(\omega^*) \Rightarrow D \in W_i^*(\omega^*)$*.*

Arguably interactive reasoning of awareness is one of the most interesting aspects in this environment. In particular, i could be aware of some event E, while being unaware of the possibility that j might be unaware of it. To allow for such situations, the full state space needs to be exhaustive.

Definition 8 *The full state space* Ω^* *is* **rich** *if, for any* $D \in \mathcal{D}^*, i \in N$*,* $\omega_1^*, \omega_2^* \in \Omega^*$*,* $D \in W_i^*(\omega_1^*) \cap [\mathcal{D}^* \setminus W_i^*(\omega_2^*)] \Rightarrow \tilde{D}_i \in \mathcal{D}^*$*.*

In words, richness requires that whenever the full state space contains states in which i's awareness of some relevant D differs, then *every* full state should contain an explicit specification of i's awareness of D. It is worth pointing about that richness is a regularity condition for the full state space, not an assumption on the agents' awareness structure. This condition ensures an "interactive rational awareness" property in this environment.[24]

Before stating the result, some notations are needed. For any q^n and $k < n$, let $q^{n\setminus k}$ denote the tuple $((\omega_1, i^1), \cdots, (\omega_{k-1}, i^{k-1}), (\omega_{k+1}, i^{k+1}), \cdots, (\omega_n, i^n))$. Let $s(\omega|q^n) \equiv \mathbb{P}^{\Omega_{i^n}(\omega_n|q^{n-1})}(\omega)$ denote the projection of ω on the interactive subjective state space $\Omega_{i^n}(\omega_n|q^{n-1})$.

Proposition 6 *Let the multi-agent product model* $(\Omega^*, \mathbf{W}^*, \mathbf{P}^*)$ *be consistent and rich, and* (W_i^*, P_i^*) *satisfy rational awareness for all* $i \in N$*.*

For any n*, and any permissible* q^n*, suppose* $\omega_k \in P_{i^k}(s(\omega_{k-1}|q^{k-1})|q^{k-1})$ *and* $i^k = i^{k-1}$ *for some* $k < n$*. Then for all* m *such that* $k < m \leq n$*,* $q^{m\setminus k}$ *is permissible; moreover,*

$$(\Omega_{i^m}(\omega_m|q^{m-1}), W_{i^{m+1}}(\cdot|q^m), P_{i^{m+1}}(\cdot|q^m)) = (\Omega_{i^m}(\omega_m|q^{(m-1)\setminus k}), W_{i^{m+1}}(\cdot|q^{m\setminus k}), P_{i^{m+1}}(\cdot|q^{m\setminus k}))$$

Proposition 6 says that, if a permissible reasoning string involves an agent i^{k-1} reasoning about his own reasoning at a possible subjective state ω_k, then it leads to the same interactive model as the one without such self-reasoning. In other words, a rich and consistent model yields a tractable hierarchy of interactive awareness under the construction (5.3)-(5.5): although agents may have different subjective models due to different awareness, everybody knows his own subjective model, everybody knows everybody knows his own subjective model, and so on.

On the other hand, given the unawareness constraint, the factual information partition is not preserved in interactive models in general. The following *sufficient* condition guarantees $P_{i^{n+1}}(\cdot|q^n)$ induces an information partition for all permissible q^n.

[24] To see the role of this condition, recall that in standard models, if i is uncertain about j's information structure, one can simply split the state into multiple states where this uncertainty is resolved. Since everybody is fully aware, rational awareness is always satisfied. It is no longer the case in this environment.

Definition 9 *The possibility correspondence* $P^* : \Omega^* \to 2^{\Omega^*} \setminus \{\emptyset\}$ *satisfies* **product factual partition** *if it induces an information partition over* Ω^**, and that for all* $\omega^* \in \Omega^*$*,* $P^*(\omega^*)$ *can be written as a product set.*

This condition requires every full factual signal can be decomposed into the conjunction of distinct signals, one for each relevant question. Note that the decomposition needs not be the same in different full states. All examples considered so far satisfy this condition.[25]

Definition 10 *The multi-agent generalized information structure* $(\mathbf{W}^*, \mathbf{P}^*)$ *is* **interactively rational** *if* (W_i^*, P_i^*) *satisfies product factual partition and rational awareness.*

The following result shows that in a rich and consistent multi-agent product model, if the information structure is interactively rational, then every interactive model is a product model with rational information structure. This is the key result leading to Theorem 8, by ensuring the results in Section 4 apply to all interactive models.

Lemma 7 *Let* $(\Omega^*, \mathbf{W}^*, \mathbf{P}^*)$ *be rich, consistent and interactively rational. Then for any* n*, and any permissible* q^n*,* $(W_{i^{n+1}}(\cdot|q^n), P_{i^{n+1}}(\cdot|q^n))$ *is rational.*

5.4 Interactive knowledge hierarchy.

Analogous to the single-agent model, the key in characterizing interactive knowledge is to keep track on *subjective interactive knowledge*. For example, in the event "i knows j knows E," the event "j knows k" is the subjective event "I know E" in j's mind-set i ascribes to j. Extending the logic leading to equation (3.5) in the single-agent setting, the event "i knows j knows E," denoted by $K(E|(i,j))$, is represented by the set of states where in his own mind-set, i subjective knows that "j *subjectively* knows E," denoted by $\tilde{K}^j(E|i_{\omega^*})$. The above discussion suggests:

$$K(E|(i,j)) = \left\{\omega^* \in \Omega^* : s(\omega^*|i_{\omega^*}) \in \tilde{K}^i_{\omega^*}(\tilde{K}^j(E|i_{\omega^*}))\right\}$$

The question is how to define the subjective interactive knowledge $K^j(E|i_{\omega^*})$. Intuitively, from i's perspective at ω^*, j's information structure is modeled by the product model $(\Omega_i(\omega^*), W_j(\cdot|i_{\omega^*}), P_j(\cdot|i_{\omega^*}))$, of which i is fully aware. Therefore, the interactive knowledge $\tilde{K}^j(E|i_{\omega^*})$ corresponds to the *objective* knowledge in the product model $(\Omega_i(\omega^*), W_j(\cdot|i_{\omega^*}), P_j(\cdot|i_{\omega^*}))$, and hence is defined by:

$$\tilde{K}^j(E|i_{\omega^*}) = \left\{\omega \in \Omega_i(\omega^*) : P_j(\omega|i_{\omega^*}) \subseteq E_{\Omega_i(\omega^*)},\ W_j(\omega|i_{\omega^*}) \supseteq \mathcal{D}_E\right\}$$

In general, fix a permissible sequence q^n, for any $E \in \mathcal{E}^p$, the subjective event "i^n knows E" in i^{n-1}'s mind-set at ω_{n-1} ascribed by i^{n-2} at ω_{n-2} ascribed by $\cdots$ by i^1 at ω_1,

[25] For example, this condition is satisfied in all two-person games with perfect recall.

denoted by $\tilde{K}^{i^n}(E|q^{n-1})$, can be defined analogously as follows. For notational ease, let $E_{n-1} = E_{\Omega_{i^{n-1}}(\omega_{n-1}|q^{n-2})}$ denote the event E in i^{n-1}'s mind-set ascribed by i^{n-2} ascribed by $\cdots$ by i^1 at ω^*.

$$\tilde{K}^{i^n}(E|q^{n-1}) = \left\{\omega \in \Omega_{i^{n-1}}(\omega_{n-1}|q^{n-2}) : P_{i^n}(\omega|q^{n-1}) \subseteq E_{n-1}, W_{i^n}(\omega|q^{n-1}) \supseteq \mathcal{D}_E\right\} \quad (5.6)$$

Let I^n denote the sequence of players $(i^1, \cdots, i^n)$. Recursively define the permissible sequence $I^n(\omega^*)$ as follows:

$I^1(\omega^*) = ((\omega^*, i^1))$

$I^n(\omega^*) = I^{n-1}(\omega^*) \wedge ((s(\omega^*|I^{n-1}(\omega^*)), i^n)$ where $\wedge$ denotes concatenation.

The *objective interactive n-th order knowledge* is the set of full states where the subjective interactive n-th order knowledge obtains, calculated by recursively applying (5.6). The event "i^1 knows that i^2 knows $\cdots$ i^n knows E," denoted by $K(E|I^n)$, is:

$$K(E|I^n) = \left\{\omega^* \in \Omega^* : s(\omega^*|I^1(\omega^*)) \in \tilde{K}^{i^1}_{\omega^*}(\tilde{K}^{i^2}(\cdots(\tilde{K}^{i^n}(E|I^{n-1}(\omega^*)))\cdots|I^1(\omega^*)))\right\} \quad (5.7)$$

If $i^1 = \cdots = i^n = i$, then Proposition 6 ensures that $K(E|I^n)$ reduces to $K_i^n(E)$ in the single-agent model. The higher-order (un)awareness or knowledge of (un)awareness are defined analogously.

For any I^n, let $I^{n\backslash k} = (i^1, \cdots, i^{k-1}, i^{k+1}, i^n)$ denote the reasoning sequence obtained by removing i^k.

Theorem 8 *Let* $(\Omega^*, \mathbf{W}^*, \mathbf{P}^*)$ *be rich, consistent and interactively rational. Then, for any* $E \in \mathcal{E}^p$*, any* I^n*, and any* $i, j \in N$*,*

IK1 $i^k = i^{k-1}$ *for some* $1 < k \leq n \Rightarrow K(E|I^n) = K(E|I^{n\backslash k})$*;*

IK2 $K(E|(i,j)) \subseteq K_j(E) \cup [U_j(E) \cap U_i U_j(E)]$*.*

$IK1$ says every agent knows his own knowledge, and everybody knows that everybody knows his own knowledge, and so on. $IK2$ says if i knows j knows E while j does not know E, then it must be the case that j is unaware of E and i is unaware that j is unaware of E. These two properties ensure the interactive knowledge hierarchy retains the essence of that in the standard multi-agent information partition model, while allowing for interesting new possibilities in this environment: agents know their own knowledge, including the interactive knowledge, and there are higher-order knowledge of this; on the other hand, unlike the standard model, agents could be wrong in their interactive knowledge: i may mistakenly "know" j knows an event E due to his unawareness of j's unawareness of E; moreover, this is the only mistake i could make. In particular, since i could be aware that j may be unaware of E while unaware that k could be unaware of E, there are rich patterns of interactive awareness and knowledge in the model.

5.5 Common knowledge.

An event E is common knowledge if everybody knows it, everybody knows everybody knows it, and so on. Given the critical role of common knowledge in game theory, an important question is when one can achieve common knowledge in an multi-agent environment with unawareness.

First I give a natural definition of common knowledge in this environment. Let $CK(E)$ represents the event "common knowledge of E." The intuitive notion of common knowledge is mathematically represented by the following: for any $i \in N$, let $I_i^m = (i, i^2, \cdots, i^m)$ denote any reasoning sequence of i. Slightly abusing notation, define:

$$CK(E) \equiv \cap_{i=1}^n \cap_{m=1}^\infty K(E|I_i^m) \tag{5.8}$$

The event $\cap_{m=1}^\infty K(E|I_i^m)$ implies "i knows E, i knows everybody knows E, i knows everybody knows everybody knows E, and so on," or common knowledge of E *in i's mind-set*. I call this event i's *subjective common knowledge* of E, and denote it by $CK_i(E)$:

$$CK_i(E) = \cap_{m=1}^\infty K(E|I_i^m)$$

Thus (5.8) simply says E is common knowledge if and only if E is every agent's subjective common knowledge. Notice that in the standard model, by axiom of knowledge, for any I_i^m, $K(E|I_i^m) \subseteq K(E|(i^2, \cdots, i^m))$, and hence $CK_i(E) \subseteq CK(E)$, or $CK(E) = \cap_{m=1}^\infty K(E|I^m) = CK_i(E)$ for all $i \in N$.

Aumann (1976) characterizes common knowledge in the standard model $(\Omega^*, \mathbf{P}^*)$ elegantly, shown later by Bacharach (1985) to capture the intuitive definition of common knowledge: for any $E \subseteq \Omega^*$,

$$CK(E) = \left\{\omega^* \in \Omega^* : \wedge_{j=1}^n P_j^*(\omega^*) \subseteq E\right\} \tag{5.9}$$

where $\wedge_{j=1}^n P_j^*$ denotes the meet of the information partition generated by the agents' possibility correspondences, and $\wedge_{j=1}^n P_j^*(\omega^*)$ represents the partition element containing ω^* in the meet.

This characterization obviously breaks down when unawareness is an issue. The complication comes from two sources. First, i may not know E even though his factual signal indicates E, due to his unawareness of E. Second, as $IK2$ points out, agents may have false knowledge of interactive knowledge due to their unawareness of other agents' unawareness of E. This could work in two opposite directions: if j is unaware of E, while i is unaware that j is unaware of E, then i could mistakenly consider E to be common knowledge. This is particularly relevant in cases involving common factual signals. For example, suppose Holmes is unaware that Watson could be unaware of the possibility of no intruder, then from Holmes' perspective, the event "there was no intruder" is common knowledge. From this aspect, (5.9) is obviously too weak to ensure even mutual knowledge.

On the other hand, suppose j does not know E in a state j excludes but i's factual signal does not. Suppose further that i is unaware of this state. Then although Aumann's condition fails, it is in fact possible for E to be subjective common knowledge for both i and j. For example, recall the hearing problem example. Suppose it is raining, and Charlie does not have the hearing problem. Suppose Dorothy is unaware that Charlie could have hearing problem, and her full factual signal is whether it rains: $\{\{(r,p),(r,np)\},\{(nr,p),(nr,np)\}\}$. Note Dorothy's full factual signal at (r,np) does not exclude (r,p), where Charlie does not know it rains. However, since Dorothy is unaware of (r,p), she is unaware of the possibility that Charlie could fail to recognize it rains, and hence concludes – correctly, – that "it rains" is common knowledge between her and Charlie. In this case, (5.9) appears too strong.

Taking into account all these complications, I set bounds for common knowledge in this environment. It is obvious that common knowledge of E requires mutual knowledge of E. On the other hand, I impose an additional sufficient condition for "common awareness of E" on (5.9), and denote this set by $\underline{CK}(E)$:

$$\underline{CK}(E) = \left\{\omega^* \in \Omega^* : \mathcal{D}_E \subseteq \cap_{i=1}^n \bigcap_{\left\{\omega_1^* :\, \omega_1^* \in \wedge_{j=1}^n P_j^*(\omega^*)\right\}} W_i^*(\omega_1^*),\ \wedge_{j=1}^n P_j^*(\omega^*) \subseteq E_{\Omega^*}\right\} \tag{5.10}$$

Theorem 9 *Let $(\Omega^*, \mathbf{W}^*, \mathbf{P}^*)$ be rich, consistent and interactively rational. Then for all $E \in \mathcal{E}^p$,*

$$\underline{CK}(E) \subseteq CK(E) \subseteq \cap_{i=1}^n K_i(E)$$

In general, mutual knowledge is much weaker than common knowledge. The example suggests that the weak implication in this environment is due to the fact that unawareness of interactive knowledge uncertainty makes it possible to achieve "false" common knowledge. The following conditions identify the driving force behind it.

Definition 11 $(\mathbf{W}^*, \mathbf{P}^*)$ *is* **strongly rational** *if in addition to rational awareness, it also satisfies:*

1. *Cylinder factual partition: for all $i \in N$ and all $\omega^* \in \Omega^*$, $P_i^*(\omega^*)$ is a cylinder event. In other words, for every i, there exists a collection of partitions, one for each $D \in \mathcal{D}^*$, such that $P_i^*(\omega^*)$ is the Cartesian product of the corresponding partition elements of each D;*

2. *Rational interactive awareness: for all $\omega^* \in \Omega^*$ and any $i, j \in N$, $D \in W_i^*(\omega^*), \tilde{D}_j \in \mathcal{D}^* \Rightarrow \tilde{D}_j \in W_i^*(\omega^*)$.*

Obviously cylinder factual signals condition implies product factual partition. Both conditions require the full factual signals be generated by separate signals about each uncertainty, but cylinder factual partition requires such decomposition be independent of full states, while product factual partition does not. The rational interactive awareness condition, on the other hand, requires that for each uncertainty of which i is aware, and of which j could be unaware, i be aware of the possibility that j could be unaware of it. The next result shows these two additional conditions are sufficient to rule out unawareness of interactive knowledge uncertainty.

Theorem 10 *Let* $(\Omega^*, \mathbf{W}^*, \mathbf{P}^*)$ *be rich, consistent and strongly rational. Then for all* $E \in \mathcal{E}^p$,

$$\underline{CK}(E) = CK(E)$$

5.6 An example.

Recall Example 2 from Section 3. Suppose Alice is aware that her actions affect Bob's awareness, but is unaware that Bob could be unaware of this dependence. Players know their own actions and awareness.

To model this, let A^u, U^u represent "Bob is aware that he could be unaware of b_2" and "Bob is unaware that he could be unaware of b_2" respectively.

$\mathcal{D}^* = \left\{\{a_1, a_2\}, \{b_1, b_2\}, \left\{A^b, U^b\right\}, \{A^u, U^u\}\right\}$
$\Omega^* = \{a_1b_1A^bA^u, a_2b_1A^bA^u, a_1b_2A^bA^u, a_2b_2A^bA^u, a_1b_1A^bU^u, a_2b_1A^bU^u, a_1b_2A^bU^u,$
$a_2b_2A^bU^u, a_1b_1U^bU^u, a_2b_1U^bU^u\}$
$W_A^*(\omega^*) = \left\{\{a_1, a_2\}, \{b_1, b_2\}, \left\{A^b, U^b\right\}\right\}, \quad \forall\, \omega^* \in \Omega^*$
$W_B^*(\omega^*) = \mathcal{D}^*$ for all A^bA^u states; $W_B^*(\omega^*) = \{\{a_1, a_2\}, \{b_1, b_2\}\}$ for all A^bU^u states; and $W_B^*(\omega^*) = \{\{a_1, a_2\}\}$ for all U^bU^u states.
P_A^* induces the information partition containing $\{a_1b_1A^bA^u, a_1b_2A^bA^u, a_1b_1A^bU^u, a_1b_2A^bU^u\}$, $\left\{a_2b_1A^bA^u, a_2b_2A^bA^u, a_2b_1A^bU^u, a_2b_2A^bU^u\right\}, \left\{a_1b_1U^bU^u\right\}$ and $\left\{a_2b_1U^bU^u\right\}$;
P_B^* induces the information partition containing $\left\{a_1b_1A^bA^u, a_2b_1A^bA^u\right\}$, $\left\{a_1b_2A^bA^u, a_2b_2A^bA^u\right\}$, $\left\{a_1b_1A^bU^u, a_2b_1A^bU^u\right\}, \left\{a_1b_2A^bU^u, a_2b_2A^bU^u\right\}$ and $\left\{a_1b_1U^bU^u, a_2b_1U^bU^u\right\}$.

First consider the event "Bob chooses b_1," represented by $\{b_1\}$. By Theorem 3, at $a_1b_1A^bU^u$, Bob knows it while Alice does not. Does Bob know that Alice does not know it?

At $a_1b_1A^bU^u$, Bob's first-order subjective model is:

$\Omega_B(a_1b_1A^bU^u) = \{a_1b_1, a_2b_1, a_1b_2, a_2b_2\}$
$W_A(\omega|B_{a_1b_1A^bU^u}) = W_B(\omega|B_{a_1b_1A^bU^u}) = \{\{a_1, a_2\}, \{b_1, b_2\}\}$ for all $\omega \in \Omega_B(a_1b_1A^bU^u)$;
$P_A(\cdot|B_{a_1b_1A^bU^u})$ induces the information partition

$$\{\{a_1b_1, a_1b_2\}, \{a_2b_1, a_2b_2\}\}$$

$P_B(\cdot|B_{a_1b_1A^bU^u})$ induces the information partition

$$\{\{a_1b_1, a_2b_1\}, \{a_1b_2, a_2b_2\}\}$$

This is a standard information partition model: $\times W_i(\omega|B_{a_1b_1A^bU^u}) = \Omega_B(a_1b_1A^bU^u)$ for all $\omega \in \Omega_B(a_1b_1A^bU^u)$, $i = A, B$. Therefore all Bob's higher-order interactive models coincide with the first-order model, and standard results apply to his subjective interactive knowledge hierarchy.

Since $\{b_1\}_{\Omega_B(a_1b_1A^bU^u)} = \{a_1b_1, a_2b_1\}$, We have:

$$\begin{aligned}\tilde{K}^A(\{b_1\}|B_{a_1b_1A^bU^u}) &= \left\{\omega \in \Omega_B(a_1b_1A^bU^u) : P_A(\omega|B_{a_1b_1A^bU^u}) \subseteq \{a_1b_1, a_2b_1\}\right\} \\ &= \emptyset_{\Omega_B(a_1b_1A^bU^u)}\end{aligned}$$

It follows the event "Alice does not know Bob chooses b_1" in Bob's mind-set is represented by the set $\neg\tilde{K}^A(\{b_1\}|B_{a_1b_1A^bU^u}) = \neg\emptyset_{\Omega_B(a_1b_1A^bU^u)} = \Omega_B(a_1b_1A^bU^u)$. It is easy to see Bob subjectively knows this, and he knows that he knows it, and so on.

On the other hand, by Theorem 3, Alice knows she does not know Bob chooses b_1, and so on. In fact, it is common knowledge at $a_1b_1A^bU^u$ that Alice does not know Bob chooses b_1.

An interesting event in this game is "Alice chooses a_1," represented by $\{a_1\}$. At $a_1b_1A^bU^u$, although Bob does not know Alice chooses a_1, this event is in fact common knowledge from Alice's perspective.

Alice's first-order subjective model at $a_1b_1A^bU^u$ is:

$\Omega_A(a_1b_1A^bU^u) = \left\{a_1b_1A^b, a_2b_1A^b, a_1b_1U^b, a_2b_1U^b, a_1b_2A^b, a_2b_2A^b\right\}$
$W_A(\omega|A_{a_1b_1A^bU^u}) = \left\{\{a_1, a_2\}, \{b_1, b_2\}, \left\{A^b, U^b\right\}\right\}$ for all $\omega \in \Omega_A(a_1b_1A^bU^u)$;
$W_B(\omega|A_{a_1b_1A^bU^u}) = \left\{\{a_1, a_2\}, \{b_1, b_2\}, \left\{A^b, U^b\right\}\right\}$ for all A^b states in $\Omega_A(a_1b_1A^bU^u)$, and $= \{\{a_1, a_2\}\}$ for all U^b states in $\Omega_A(a_1b_1A^bU^u)$;
$P_A(\cdot|A_{a_1b_1A^bU^u})$ induces the information partition

$$\left\{\left\{a_1b_1A^b, a_1b_2A^b\right\}, \left\{a_2b_1A^b, a_2b_2A^b\right\}, \left\{a_1b_1U^b\right\}, \left\{a_2b_1U^b\right\}\right\}$$

$P_B(\cdot|A_{a_1b_1A^bU^u})$ induces the information partition

$$\left\{\left\{a_1b_1A^b, a_2b_1A^b\right\}, \left\{a_1b_2A^b, a_2b_2A^b\right\}, \left\{a_1b_1U^b\right\}, \left\{a_2b_1U^b\right\}\right\}$$

In this subjective model, at the "true state" $a_1b_1A^b$, both Alice and Bob are "fully" aware. Consequently, all Alice's higher-order interactive models at the relevant states coincide with Alice's first-order model.

Since whenever Alice plays b_1, Bob is unaware of b_2, the a_2A^b states are not possible in this game. This enables Alice to remove the subjective states $a_2b_1A^b$ and $a_2b_2A^b$ from her subjective state space, which results in a refinement of both Alice's and Bob's information partition.

Since $\{a_1\}_{\Omega_A(a_1b_1A^bU^u)} = \left\{a_1b_1A^b, a_1b_2A^b, a_1b_1U^b, a_1b_2U^b\right\}$, We have:

$$\begin{aligned}\tilde{K}^B(\{a_1\}|A_{a_1b_1A^bU^u}) &= \left\{\omega : P_B(\omega|A_{a_1b_1A^bU^u}) \subseteq \left\{a_1b_1A^b, a_1b_2A^b, a_1b_1U^b, a_1b_2U^b\right\}\right\} \\ &= \left\{a_1b_1A^b, a_1b_2A^b, a_1b_1U^b\right\}\end{aligned}$$

By (3.6) and Theorem 3, the event $\{a_1b_1A^b, a_1b_2A^b, a_1b_1U^b\}$ is common knowledge in Alice's first-order subjective model, that is, Alice considers it common knowledge that Bob knows Alice chooses a_1.

6 Concluding remarks

In this paper, I construct a set-theoretic model of both single-agent and interactive information processing with unawareness. The main idea is that introducing unawareness necessitates examining a new type of information, i.e. awareness information, *in addition to* the usual factual information in the standard model. The resulting model is a natural generalization of the standard information partition model due to Aumann (1976), and hence well connecting to the existing literature. The model also sheds light on formally understanding the difference between unawareness and probability zero events: if the agent is unaware of an event, then the event is beyond the agent's probability space and he is unable to assign any probability to it.

There are a number of exciting topics one can explore using this model. The anticipation of unforeseen contingencies has a significant impact in real-life decision processes. Many people keep some personal funds for unspecified emergencies. At a collective level, the scale can be quite impressive. For instance, "the City of New York's Five-Year Financial Plan" includes a "general reserve for *unforeseen contingencies* of \$42 million in FY 2001 and reserves of \$545 million in FY 2002."[26] However, decision-making under unforeseen contingencies is not well understood. The standard Savage framework assumes away unforeseen contingencies. Research in this field has been focusing on axiomatization of preferences over menus of items, which provides the dynamic structure intrinsically related to unforeseen contingencies. This model is suggestive on how a more direct approach exploring the generalized information structure might work. Contractual incompleteness is a particularly interesting and important economic phenomenon where anticipation of unforeseen contingencies seems to play an important role. It is not clear how to apply research in decision-making with unforeseen contingencies to the contractual environment without an explicit account for parties' information structure in this environment. This model provides a simple tool for that purpose.

7 Appendix.

7.1 Proof of Proposition 2.

Proof. First notice that:

$$[\bigcup_{D\in\tilde{\mathcal{D}}_E}\left\{U^D\right\}]_{\Omega^*} = \bigcup_{D\in\tilde{\mathcal{D}}_E}\left\{\omega^* : \mathbb{P}^{\left\{A^D,\, U^D\right\}}(\omega^*) = U^D\right\}$$

[26] The emphasis is mine. Source: "Review of the Mayor's Executive Budget for Fiscal Year 2002" by H. Carl McCall (State Comptroller), web address: http://www.osc.state.ny.us/osdc/rpt2002.pdf

Let $\omega^* \in \bigcup_{D\in\tilde{\mathcal{D}}_E} \{U^D\}]_{\Omega^*}$. Therefore there exists some some $D \in \tilde{\mathcal{D}}_E \subseteq \mathcal{D}_E$ such that $\mathbb{P}^{\{A^D,U^D\}}(\omega^*) = U^D$. By consistency, $D \notin W^*(\omega^*)$. But then by definition (3.7), $\omega^* \in U(E)$.

On the other hand, let $\omega^* \in U(E)$. Then there exists some $D \in \mathcal{D}_E$ such that $D \notin W^*(\omega^*)$. Since $\tilde{D} \in \mathcal{D}^*$, we must have $\mathbb{P}^{\{A^D,U^D\}}(\omega^*) = U^D$ by consistency, which implies $\omega^* \in [\bigcup_{D\in\tilde{\mathcal{D}}_E} \{U^D\}]_{\Omega^*}$. □

7.2 Proof of Lemma 4.

U0* *Symmetry*: $U(E) = U(\neg E)$

Follows from the fact that $\mathcal{D}_E = \mathcal{D}_{\neg E}$.

U1′ *Strong plausibility*: $U(E) \subseteq \bigcap_{n=1}^{\infty}(\neg K)^n(E)$

Let $\omega^* \in U(E)$. Then $\mathcal{D}_E \nsubseteq W(\omega^*)$. By 3.3, $\tilde{K}_{\omega^*}(E) = \emptyset_E$. By 3.5, $\omega^* \in \neg K(E)$. Note that $\mathcal{D}_{\neg\tilde{K}_{\omega^*}(E)} = \mathcal{D}_{\tilde{K}_{\omega^*}(E)} = \mathcal{D}_{\emptyset_E} = \mathcal{D}_E \nsubseteq W(\omega^*)$, now it follows $\tilde{K}_{\omega^*}\neg\tilde{K}_{\omega^*}(E) = \emptyset_E$ and $\omega^* \in \bar{K}_2(E)$. It is easy to see that $\mathcal{D}_{\neg\tilde{K}_{\omega^*}^{n-1}(E)} = \mathcal{D}_E$ for all n, which implies $\tilde{K}_{\omega^*}(\neg\tilde{K}_{\omega^*}^{n-1}(E)) = \emptyset_E$ for all n, and hence Thus, $\omega^* \in (\neg K)^n(E)$ for all n.

U2*AU *introspection*: $U(E) \subseteq UU(E)$

Consider the event $\bigcup_{D\in\tilde{\mathcal{D}}_E} \{U^D\}$. Let $\omega^* \in U(E)$. It follows that $D \notin W^*(\omega^*)$ for some $D \in \mathcal{D}_E$. The conclusion follows already if all such $D \notin \tilde{\mathcal{D}}_E$. Suppose there exists $D \in \tilde{\mathcal{D}}_E$ and $D \notin W^*(\omega^*)$. By consistency, $\tilde{D} \notin W^*(\omega^*) \Rightarrow \omega^* \in U \bigcup_{D\in\tilde{\mathcal{D}}_E} \{U^D\} = UU(E)$.

U3′ *Weak KU introspection*: $U(E) \cap KU(E) = \emptyset_{\Omega^*}$

Follows from plausibility and AU introspection: $U(E) \subseteq UU(E) \subseteq \neg KU(E) \Rightarrow U(E) \cap KU(E) = \emptyset_{\Omega^*}$.

K1**Subjective necessitation*: for all $\omega^* \in \Omega^*$, $\omega^* \in K(\Omega(\omega^*))$

For any $\omega^* \in \Omega^*$, $P_{\omega^*}(s(\omega^*)) = \mathbb{P}^{\Omega(\omega^*)}(P^*(\omega^*)) \subseteq \Omega(\omega^*)$, which implies $s(\omega^*) \in \tilde{K}_{\omega^*}(\Omega(\omega^*))$, and hence $\omega^* \in K(\Omega(\omega^*))$.

K2**Generalized monotonicity*: $E \subseteq_* F$, $\mathcal{D}_F \subseteq \mathcal{D}_E \Rightarrow K(E) \subseteq K(F)$

This is implied by conjunction. Take E and F such that $E \subseteq_* F$, $\mathcal{D}_F \subseteq \mathcal{D}_E$. By conjunction,

$$\begin{aligned} K(E) \cap K(F) &= K(E \cap_* F) \\ &= K(E \cap F_{\times \mathcal{D}_E}) \\ &= K(E) \end{aligned}$$

It follows $K(E) \subseteq K(F)$.

K3* *Conjunction*: $K(E) \cap K(F) = K(E \cap_* F)$

Let $\omega^* \in K(E) \cap K(F)$. Then $\mathcal{D}_E \subseteq W^*(\omega^*)$, $\mathcal{D}_F \subseteq W^*(\omega^*)$ and $P^*(\omega^*) \subseteq E_{\Omega^*}$, $P^*(\omega^*) \subseteq F_{\Omega^*}$. Note that:

(1) $\mathcal{D}_E \subseteq W^*(\omega^*)$, $\mathcal{D}_F \subseteq W^*(\omega^*)$ if and only if $\mathcal{D}_E \cup \mathcal{D}_F \subseteq W^*(\omega^*)$, which implies $\mathcal{D}_{E \cap_* F} \subseteq W^*(\omega^*)$. Thus, the event $E \cap_* F$ has an elaboration in the space $\Omega(\omega^*)$.

(2) $P^*(\omega^*) \subseteq E_{\Omega^*}$, $P^*(\omega^*) \subseteq F_{\Omega^*}$ if and only if $P^*(\omega^*) \subseteq (E_{\Omega^*} \cap F_{\Omega^*})$;

Using the product structure, one has $E_{\Omega^*} = E_{\times(\mathcal{D}_E \cup \mathcal{D}_F)} \times (\mathcal{D} \setminus (\mathcal{D}_E \cup \mathcal{D}_F))$, $F_{\Omega^*} = F_{\times(\mathcal{D}_E \cup \mathcal{D}_F)} \times (\mathcal{D} \setminus (\mathcal{D}_E \cup \mathcal{D}_F))$. Therefore

$$\begin{aligned} E_{\Omega^*} \cap F_{\Omega^*} &= [E_{\times(\mathcal{D}_E \cup \mathcal{D}_F)} \times (\mathcal{D} \setminus (\mathcal{D}_E \cup \mathcal{D}_F))] \cap [F_{\times(\mathcal{D}_E \cup \mathcal{D}_F)} \times (\mathcal{D} \setminus (\mathcal{D}_E \cup \mathcal{D}_F))] \\ &= [E_{\times(\mathcal{D}_E \cup \mathcal{D}_F)} \cap F_{\times(\mathcal{D}_E \cup \mathcal{D}_F)}] \times (\mathcal{D} \setminus (\mathcal{D}_E \cup \mathcal{D}_F)) \\ &= (E \cap_* F)_{\Omega^*} \end{aligned}$$

Back to (2), one has $P^*(\omega^*) \subseteq (E \cap_* F)_{\Omega^*}$. Using (1), $P_{\omega^*}(s(\omega^*)) = \mathbb{P}^{\Omega(\omega^*)} P^*(\omega^*) \subseteq \mathbb{P}^{\Omega(\omega^*)}((E \cap_* F)_{\Omega^*}) = (E \cap_* F)_{\Omega(\omega^*)}$, hence $s(\omega^*) \in \tilde{K}_{\omega^*}((E \cap_* F)_{\Omega(\omega^*)})$, and hence $\omega^* \in K(E \cap_* F)$ □

7.3 Proof of Lemma 5.

U1* *UUU*: $U(E) = \bigcap_{n=1}^{\infty} (\neg K)^n (E)$

Strong plausibility gives $\Rightarrow$; Applying De Morgan's law on the axiom of limited wisdom gives the other direction.

U3* *KU introspection*: $KU(E) = \emptyset_{\Omega^*}$

By the axiom of knowledge, $K(E) \subseteq_* E$. Thus $KU(E) \subseteq U(E)$, hence $KU(E) \cap U(E) = KU(E)$. The result follows from weak KU introspection.

Observe that when (W^*, P^*) is rational, the subjective model is a standard information partition model at the set of possible subjective states, and hence the standard results carry through.

K4* *The axiom of knowledge*: $K(E) \subseteq_* E$

$\omega^* \in K(E) \Rightarrow P^*(\omega^*) \subseteq E_{\Omega^*}$, by non-delusion, $\omega^* \in P^*(\omega^*) \Rightarrow \omega^* \in E_{\Omega^*}$. The result follows from $K(E) \subseteq E_{\Omega^*}$.

K5* *The axiom of transparency*: $K(E) \subseteq KK(E)$

Let $\omega^* \in K(E)$. It suffices to show $s(\omega^*) \in \tilde{K}_{\omega^*}\tilde{K}_{\omega^*}(E)$.
Since (W^*, P^*) is rational, $P_{\omega^*}(\omega) = P_{\omega^*}(s(\omega^*))$ for all $\omega \in P_{\omega^*}(s(\omega^*))$. Now

$$\begin{aligned}
& s(\omega^*) \in \tilde{K}_{\omega^*}(E) \\
\Rightarrow\ & P_{\omega^*}(s(\omega^*)) \subseteq E_{\Omega(\omega^*)} \\
\Rightarrow\ & P_{\omega^*}(\omega) \subseteq E_{\Omega(\omega^*)} \\
\Rightarrow\ & \omega \in \tilde{K}_{\omega^*}(E) \text{ for all } \omega \in P_{\omega^*}(s(\omega^*)) \\
\Rightarrow\ & P_{\omega^*}(s(\omega^*)) \subseteq \tilde{K}_{\omega^*}(E) \\
\Rightarrow\ & s(\omega^*) \in \tilde{K}^2_{\omega^*}(E)
\end{aligned}$$

K6* *The axiom of limited wisdom*: $\neg U(E) \cap \neg K(E) \subseteq K\neg K(E)$

Let $\omega^* \in \neg U(E) \cap \neg K(E)$. Then $\mathcal{D}_E \subseteq W^*(\omega^*)$, so we only need to show $P_{\omega^*}(s(\omega^*)) \subseteq \neg\tilde{K}_{\omega^*}(E)$. Let $\omega \in P_{\omega^*}(s(\omega^*))$. By (3.2), $P_{\omega^*}(\omega) = P_{\omega^*}(s(\omega^*))$, but $\omega^* \in \neg K(E) \Rightarrow P_{\omega^*}(s(\omega^*)) \nsubseteq E_{\Omega(\omega^*)}$, it follows $P_{\omega^*}(\omega) \nsubseteq E_{\Omega(\omega^*)} \Rightarrow \omega \in \neg\tilde{K}_{\omega^*}(E)$. □

7.4 Proof of Proposition 6.

I first prove two intermediate results.

Lemma 11 *Let $(\Omega^*, \mathbf{W}^*, \mathbf{P}^*)$ be consistent and rich, and let (W_i^*, P_i^*) satisfy rational awareness for all $i \in N$. Then for any $\omega^* \in \Omega^*$, $\omega \in \Omega_i(\omega^*)$, $D \in W_j(\omega|i_{\omega^*}), D \notin W_j^*(\omega_1^*)$ for some ω_1^* such that $\mathbb{P}^{\Omega_i(\omega^*)}(\omega_1^*) = \omega \Rightarrow \tilde{D} \in \mathcal{D}^* \setminus W_i^*(\omega^*)$.*

In words, this lemma says, if in i's mind-set at ω^*, j is aware of D at ω, while j is unaware of D in some underlying full state, then it must be the case that i is unaware that j could be unaware of D at ω^*.
Proof. By definition (5.1), $D \in W_j(\omega|i_{\omega^*})$ implies there exists some full state corresponding to ω such that j is aware of D at this full state. That is, there exists ω_2^* such that $\mathbb{P}^{\Omega_i(\omega_2^*)} = \omega$ and $D \in W_j^*(\omega_2^*)$. Now since $D \notin W_j^*(\omega_1^*)$, by richness, $\tilde{D}_j \in \mathcal{D}^*$.

Suppose $\tilde{D}_j \in W_i^*(\omega^*)$. Then $\left\{A_j^D, U_j^D\right\}$ is part of the specification in i's subjective state space at ω^*, that is, $\Omega_i(\omega^*)$ can be written as the Cartesian product of sets in some $\mathcal{D} \subseteq \mathcal{D}^*$ with the set $\left\{A_j^D, U_j^D\right\}$. Now since $\mathbb{P}^{\Omega_i(\omega^*)}(\omega_1^*) = \omega$, $\mathbb{P}^{\left\{A_j^D, U_j^D\right\}}(\omega) = \mathbb{P}^{\left\{A_j^D, U_j^D\right\}}(\omega_1^*) = U_j^D$ by consistency. Let $T(\omega) = \left\{\omega_3^* \in \Omega^* : \mathbb{P}^{\Omega_i(\omega^*)}(\omega_3^*) = \omega\right\}$. It follows $\mathbb{P}^{\left\{A_j^D, U_j^D\right\}}(\omega_3^*) = U_j^D$ for all $\omega_3^* \in T(\omega)$. Use consistency again, $D \notin W_j^*(\omega_3^*)$ for all such $\omega_3^* \in T(\omega)$. Therefore $D \notin \underset{\omega_3^* \in T(\omega)}{\cup} W_j^*(\omega^*) \Rightarrow D \notin W_i^*(\omega^*) \cap [\underset{\omega_3^* \in T(\omega)}{\cup} W_j^*(\omega^*)] = W_j(\omega|i_{\omega^*})$, contradiction. □

Lemma 12 *Let* $(\Omega^*, \mathbf{W}^*, \mathbf{P}^*)$ *be consistent and rich, and let* (W_i^*, P_i^*) *satisfy rational awareness for all* $i \in N$*. Then for any* n *and any permissible* q^n*,* $(W_{i^{n+1}}(\cdot|q^n), P_{i^{n+1}}(\cdot|q^n))$ *satisfies rational awareness.*

Proof. It suffices to prove the result for the case of $n = 1$. For notational ease, I write the first-order interactive model as $(\Omega_i(\omega^*), W_j(\cdot|i_{\omega^*}), P_j(\cdot|i_{\omega^*}))$ where $\omega^* \in \Omega^*$ is arbitrary. The goal is to show: for all $\omega \in \Omega_i(\omega^*), \omega' \in P_j(\omega|i_{\omega^*}) \Rightarrow W_j(\omega|i_{\omega^*}) = W_j(\omega'|i_{\omega^*})$.

By (5.2), $\omega' \in P_j(\omega|i_{\omega^*})$ implies that there exist two full states, denoted by $t(\omega), t(\omega')$, that satisfies $\mathbb{P}^{\Omega_i(\omega^*)}(t(\omega)) = \omega$, $\mathbb{P}^{\Omega_i(\omega^*)}(t(\omega')) = \omega'$, and that $t(\omega') \in P_j^*(t(\omega))$. By hypothesis, (W_j^*, P_j^*) satisfies rational awareness, it follows $W_j^*(t(\omega)) = W_j^*(t(\omega'))$.

Let $D \in W_j(\omega|i_{\omega^*})$. By (5.1), $D \in \underset{\left\{\omega_1^* : \mathbb{P}^{\Omega_i(\omega^*)}(\omega_1^*) = \omega\right\}}{\cup} W_i^*(\omega^*) \cap W_j^*(\omega_1^*)$.

Case 1: suppose $D \in W_j^*(t(\omega))$. Then $D \in W_j^*(t(\omega'))$. By (5.1), $D \in W_j(\omega'|i_{\omega^*})$;

Case 2: suppose $D \notin W_j^*(t(\omega))$. By Lemma 11, $\tilde{D}_j \in \mathcal{D}^* \setminus W_i^*(\omega^*)$. Pick any $\omega'' \in \Omega_i(\omega^*)$. There exists some underlying full state ω_1^* such that $\mathbb{P}^{\Omega_i(\omega^*)}(\omega_1^*) = \omega''$, $\mathbb{P}^{\left\{A_j^D, U_j^D\right\}}(\omega_1^*) = A_j^D$. By consistency, $D \in W_j^*(\omega'')$, and hence $D \in W_j(\omega''|i_{\omega^*})$.

This proves $W_j(\omega|i_{\omega^*}) \subseteq W_j(\omega'|i_{\omega^*})$. Reversing the role of ω and ω' in the above argument delivers the other direction. □

Now we are ready to prove Proposition 6:

Proof. It suffices to show

$$(\Omega_{i^{k-1}}(\omega_{k-1}|q^{k-2}), W_{i^{k+1}}(\cdot|q^{k-1}), P_{i^{k+1}}(\cdot|q^{k-1})) = (\Omega_{i^k}(\omega_k|q^{k-1}), W_{i^{k+1}}(\cdot|q^k), P_{i^{k+1}}(\cdot|q^k))$$

Since $\omega_k \in P_{i^k}(s(\omega_{k-1}|q^{k-1})|q^{k-1})$, by lemma 12), $W_{i^k}(\omega_k|q^{k-1}) = W_{i^k}(s(\omega_{k-1}|q^{k-1})|q^{k-1})$.

Let $G^{(k-1):(k-2)}(\omega) = \left\{\omega' \in \Omega_{i^{k-2}}(\omega_{k-2}|q^{k-3}) : \mathbb{P}^{\Omega_{i^{k-1}}(\omega_{k-1}|q^{k-2})}(\omega') = s(\omega_{k-1}|q^{k-1})\right\}$ denote the set of subjective states in i^{k-2}'s mind-set at ω_{k-2} ascribed by i^{k-3} at ω_{k-3} ascribed by $\cdots$ by i^1 at ω_1, that correspond to the subjective state ω in i^{k-1}'s mind-set by i^{k-2} at ω_{k-2} ascribed by $\cdots$ by i^1 at ω_1. By (5.4),

$$W_{i^k}(s(\omega_{k-1}|q^{k-1})|q^{k-1}) = W_{i^{k-1}}(\omega_{k-1}|q^{k-2}) \cap [\underset{\omega' \in G^{(k-1):(k-2)}(s(\omega_{k-1}|q^{k-1}))}{\cup} W_{i^k}(\omega'|q^{k-2})]$$

Since q^n is permissible, $\omega_{k-1} \in \Omega_{i^{k-2}}(\omega_{k-2}|q^{k-3})$. By definition of $s(\omega_{k-1}|q^{k-1})$, it follows that $s(\omega_{k-1}|q^{k-1}) \in G^{(k-1):(k-2)}(s(\omega_{k-1}|q^{k-1}))$. Consequently, $W_{i^k}(\omega_{k-1}|q^{k-2}) \subseteq$

$\bigcup_{\omega' \in G^{(k-1):(k-2)}(s(\omega_{k-1}|q^{k-1}))} W_{i^k}(\omega'|q^{k-2})$. Since $i^k = i^{k-1}$, it follows $W_{i^k}(s(\omega_{k-1}|q^{k-1})|q^{k-1}) = W_{i^{k-1}}(\omega_{k-1}|q^{k-2})$, which implies $\Omega_{i^k}(\omega_k|q^{k-1}) = \Omega_{i^{k-1}}(\omega_{k-1}|q^{k-2})$.

Let $\omega \in \Omega_{i^k}(\omega_k|q^{k-1})$. Now $G^{k:(k-1)}(\omega) = \left\{\omega' \in \Omega_{i^{k-1}}(\omega_{k-1}|q^{k-2}) : \mathbb{P}^{\Omega_{i^k}(\omega_k|q^{k-1})}(\omega') = \omega\right\}$, but since $\Omega_{i^k}(\omega_k|q^{k-1}) = \Omega_{i^{k-1}}(\omega_{k-1}|q^{k-2})$, $G^{k:(k-1)}(\omega) = \omega$.

Therefore, by (5.4),

$$\begin{aligned} W_{i^{k+1}}(\omega|q^k) &= W_{i^k}(\omega_k|q^{k-1}) \cap [\bigcup_{\omega' \in G^{(k):(k-1)}(\omega)} W_{i^{k+1}}(\omega'|q^{k-1})] \\ &= W_{i^{k-1}}(\omega_{k-1}|q^{k-2}) \cap W_{i^{k+1}}(\omega|q^{k-1}) \end{aligned}$$

But again by (5.4),

$$W_{i^{k+1}}(\omega|q^{k-1}) = W_{i^{k-1}}(\omega_{k-1}|q^{k-2}) \cap [\bigcup_{\omega' \in G^{(k-1):(k-2)}(\omega)} W_{i^{k+1}}(\omega'|q^{k-2})]$$

Therefore, $W_{i^{k-1}}(\omega_{k-1}|q^{k-2}) \cap W_{i^{k+1}}(\omega|q^{k-1}) = W_{i^{k+1}}(\omega|q^{k-1})$. This proves $W_{i^{k+1}}(\omega|q^k) = W_{i^{k+1}}(\omega|q^{k-1})$.

Next consider $P_{i^{k+1}}(\cdot|q^k))$. By (5.5),

$$P_{i^{k+1}}(\omega|q^k) = \begin{cases} \mathbb{P}^{\Omega_{i^k}(\omega_k|q^{k-1})} P_{i^{k+1}}(\omega_k|q^{k-1}) & \text{for } \omega \in \mathbb{P}^{\Omega_{i^k}(\omega_k|q^{k-1})} P_{i^{k+1}}(\omega_k|q^{k-1}) \\ \mathbb{P}^{\Omega_{i^k}(\omega_k|q^{n-1})} P_{i^{k+1}}(\omega \times u_{i^k}(\omega_k|q^{k-1})|q^{k-1}) & \text{otherwise.} \end{cases}$$

Now note since $P_{i^{k+1}}(\cdot|q^{k-1})$ is defined on the interactive state space $\Omega_{i^{k-1}}(\omega_{k-1}|q^{k-2})$, but $\Omega_{i^{k-1}}(\omega_{k-1}|q^{k-2}) = \Omega_{i^k}(\omega_k|q^{k-1})$, this equation simply reduces to $P_{i^{k+1}}(\omega|q^k) = P_{i^{k+1}}(\omega|q^{k-1})$ which concludes the proof. □

7.5 Proof of Lemma 7.

The following intermediate result will be useful later.

Lemma 13 *For all $i, j \in N$, if P_j^* satisfies nice factual partition then, for any $\omega^* \in \Omega^*$ and all $\omega \in \Omega_i(\omega^*)$,*

$$P_j(\omega|i_{\omega^*}) = \mathbb{P}^{\Omega_i(\omega^*)} P_j^*(\omega \times u_i(\omega^*)$$

Proof. Let $\omega \in \mathbb{P}^{\Omega_i(\omega^*)} P_j^*(\omega^*)$. Only need to show $P_j^*(\omega \times u_i(\omega^*)) = P_j^*(\omega^*)$.

Since P_j^* induces an information partition, $\omega^* \in P_j^*(\omega^*)$. By definition of $u_i(\omega^*)$, $\omega^* = s(\omega^*|i_{\omega^*}) \times u_i(\omega^*)$. On the other hand, since P_j^* satisfies nice factual partition, it is a product set. It follows $\omega \times u_i(\omega^*) \in P_j^*(\omega^*) \Rightarrow P_j^*(\omega \times u_i(\omega^*)) = P_j^*(\omega^*)$. □

Now we are ready to prove Lemma 7:

Proof. By Lemma 12, $(W_{i^{n+1}}(\cdot|q^n), P_{i^{n+1}}(\cdot|q^n))$ satisfies rational awareness.

To see that $P_{i^{n+1}}(\cdot|q^n)$ induces an information partition over $\Omega_{i^n}(\omega_n|q^{n-1})$ for all q^n, it suffices to show $P_{i^{n+1}}(\cdot|q^n)$ satisfies nice factual partition for the case of $n = 1$. But this follows trivially from Lemma 13: the projection of an information partition is obviously an information partition of the corresponding state space; and projection preserves the product structure. □

7.6 Proof of Theorem 8.

Proof of IK1:

First notice that for any ω^* and any I^n, $s(s(\omega^*|I^{n-2}(\omega^*))|I^{n-1}(\omega^*)) = s(\omega^*|I^{n-1}(\omega^*))$. By Lemma 7, the hypotheses for Proposition 6 satisfy ($\omega_k = s(\omega^*|I^{k-1}(\omega^*))$ and $\omega_{k-1} = s(\omega^*|I^{k-2}(\omega^*))$). For notational ease, let $\Omega_{i^k}(s(\omega^*|I^{k-1}(\omega^*)|I^{k-1}(\omega^*)) = \Omega^k$. Then for all $k < m < n$,

$$(\Omega^m, W_{i^{m+1}}(\cdot|I^m(\omega^*)), P_{i^{m+1}}(\cdot|I^m(\omega^*))) =$$
$$(\Omega_{i^m}(s(\omega^*|I^{(m-1)\backslash k}(\omega^*))|I^{(m-1)\backslash k}(\omega^*)), W_{i^{m+1}}(\cdot|I^{m\backslash k}(\omega^*)), P_{i^{m+1}}(\cdot|I^{m\backslash k}(\omega^*)))$$

It follows $\tilde{K}^{i^m}(E|I^{m-1}(\omega^*)) = \tilde{K}^{i^m}(E|I^{(m-1)\backslash k}(\omega^*))$ for all m such that $k < m \leq n$. Thus, it suffices to show $\tilde{K}^{i^{k-1}}(E|I^{k-2}(\omega^*)) = \tilde{K}^{i^{k-1}}(\tilde{K}^{i^k}(E|I^{k-1}(\omega^*))|I^{k-2}(\omega^*))$.

For notational ease, let $i^k = i^{k-1} = j,\ E_{\Omega^{k-2}} = E_{k-2}$. Now by (5.6),

$$\tilde{K}^j(E|I^{k-2}(\omega^*)) = \left\{\omega \in \Omega^{k-2} : P_j(\cdot|I^{k-2}(\omega^*)) \subseteq E_{n-2}, W_j(\cdot|I^{k-2}(\omega^*)) \supseteq \mathcal{D}_E\right\}$$

This is the objective knowledge operator associated with the product model

$$(\Omega_{k-2}, W_j(\cdot|I^{k-2}(\omega^*)), P_j(\cdot|I^{k-2}(\omega^*)))$$

But by Lemma 7, $(W_j(\cdot|I^{k-2}(\omega^*)), P_j(\cdot|I^{k-2}(\omega^*)))$ is rational, and hence by Theorem 3, the iteration yields the same event:

$$\begin{aligned}
\tilde{K}^j(E|I^{k-2}(\omega^*)) &= \tilde{K}^j(\tilde{K}^j(E|I^{k-2}(\omega^*))|I^{k-2}(\omega^*)) \\
&= \tilde{K}^j(E|I^{k-2}(\omega^*) \wedge (s(s(\omega^*|I^{k-2}(\omega^*))|I^{k-1}(\omega^*)), j)) \\
&= \tilde{K}^j(E|I^{k-2}(\omega^*) \wedge (s(\omega^*|I^{k-1}(\omega^*)), j)) \\
&= \tilde{K}^j(E|I^{k-1}(\omega^*))
\end{aligned}$$

Use Theorem 3 again,

$$\begin{aligned}
\tilde{K}^{i^{k-1}}(\tilde{K}^{i^k}(E|I^{k-1}(\omega^*))|I^{k-2}(\omega^*)) &= \tilde{K}^{i^{k-1}}(\tilde{K}^{i^k}(E|I^{k-2}(\omega^*))|I^{k-2}(\omega^*)) \\
&= \tilde{K}^{i^{k-1}}(E|I^{k-2}(\omega^*))
\end{aligned}$$

□

Proof of IK2:

Let $\omega^* \in K(E|(i,j))$. Then $s(\omega^*|i_{\omega^*}) \in \tilde{K}^i_{\omega^*}\tilde{K}^j(E|i_{\omega^*})$, that is, $s(\omega^*|i_{\omega^*}) \in \tilde{K}^j(E|i_{\omega^*}) \Rightarrow$

$$P_j(s(\omega^*|i_{\omega^*})|i_{\omega^*}) \subseteq E_{\Omega_i(\omega^*)} \tag{7.1}$$

and

$$W_j(s(\omega^*|i_{\omega^*})|i_{\omega^*}) \supseteq \mathcal{D}_E \tag{7.2}$$

By definition (5.2), $P_j(s(\omega^*|i_{\omega^*})|i_{\omega^*}) = \mathbb{P}^{\Omega_i(\omega^*)}P_j^*(\omega^*)$, thus (7.1) implies $P_j^*(\omega^*) \subseteq E_{\Omega^*}$. There are two cases, depending on whether j is aware of E at ω^*.

Suppose he is aware of E. Then $\mathcal{D}_E \subseteq W_j^*(\omega^*)$, and it follows $\omega^* \in K_j(E)$;

Suppose he is unaware of E, i.e. $\mathcal{D}_E \nsubseteq W_j^*(\omega^*)$, or $\omega^* \in U_j(E)$.

To see $\omega^* \in U_iU_j(E)$, first note that $\mathcal{D}_E \nsubseteq W_j^*(\omega^*)$ implies there exists $D \in \mathcal{D}_E$ such that $D \notin W_j^*(\omega^*)$. Now consider the subjective event $\underset{D\in\mathcal{D}_E}{\cup}\{U_j^D\}$. By (7.2), $D \in W_j(s(\omega^*|i_{\omega^*})|i_{\omega^*})$. By Lemma 11, $\tilde{D}_j \notin W_i^*(\omega^*) \Rightarrow \omega^* \in U_i(\{U_j^D\}) \subseteq U_i(\underset{D\in\mathcal{D}_E}{\cup}\{U_j^D\}) = U_iU_j(E)$. This concludes the proof. □

7.7 Proof of Theorem 9.

Definition 12 *For any $\omega^* \in \Omega^*$, any n, any $i \in N$ and any I_i^n, the permissible sequence $((\omega_*, i), (\omega_2, i^2), \cdots, (\omega_n, i^n))$, denoted by $r_i^n(\omega^*)$, is* **relevant** *under I_i^n and ω^* if it satisfies: $\omega_2 \in P_i(s(\omega^*|i_{\omega^*})|i_{\omega^*})$, $\omega_k \in P_{i^{k-1}}(s(\omega_{k-1}|q_i^{k-2})|q_i^{k-2})$ for all $2 < k \leq n$, and $i, i^2, \cdots, i^k$ correspond to those in I_i^k.*

In words, $r_i^n(\omega^*)$ is a reasoning string where every player in the sequence I_i^n reasons about other's reasoning at a subjective state he himself considers possible. It is easy to see that if every player has full awareness, then the set of states permissible in $r_i^n(\omega^*)$ is just $\wedge_{j=1}^n P_j^*(\omega^*)$. With nontrivial unawareness, they are subjective states, "reachable" from ω^* *subject to awareness constraints.*

Slightly abusing notation, let $\Omega^n = \Omega_{i^n}(s(\omega_n|r_i^{n-1}(\omega^*))|r_i^{n-1}(\omega^*))$ denote the subjective state space of i^n at subjective state $s(\omega_n|r_i^{n-1}(\omega^*))$ ascribed to him under $r_i^{n-1}(\omega^*)$.[27] Let $R(\omega|r_i^{n-1}(\omega^*))$ denote the set of reachable states from ω in the higher-order product model $(\Omega^{n-1}, W_{i^n}(\cdot|r_i^{n-1}(\omega^*)), P_{i^n}(\cdot|r_i^{n-1}(\omega^*)))$, that is,

$$R(\omega|r_i^{n-1}(\omega^*)) = \wedge_{j=1}^n P_j(\omega|r_i^{n-1}(\omega^*))$$

Lemma 14 *Let $(\Omega^*, \mathbf{W}^*, \mathbf{P}^*)$ be rich, consistent and interactively rational. Let $E \in \mathcal{E}^p$ and $\omega^* \in \underline{CK}(E)$. Then, for any n, any $i \in N$, any I_i^n, the following are true for all relevant $r_i^n(\omega^*)$ under I_i^n and ω^*, all j and all $\omega' \in R(s(\omega^*|r_i^{n-1}(\omega^*))|r_i^{n-1}(\omega^*))$,*

$$R(s(\omega^*|r_i^{n-1}(\omega^*))|r_i^{n-1}(\omega^*)) \subseteq E_{\Omega^{n-1}} \tag{7.3}$$

$$W_j(\omega'|r_i^{n-1}(\omega^*)) \supseteq \mathcal{D}_E \tag{7.4}$$

[27]Note it is different from the one used in before, for instance, in the proof of Theorem 8.

Intuitively, since the full possibility sets are product sets, the corresponding subjective possibility sets are product sets as well, and are obtained by projecting the full possibility sets onto the corresponding interactive state spaces. Since projection preserves set inclusion, (7.3) follows.

Proof. For $n = 2$:

Since $W_i^*(\omega^*) \supseteq \mathcal{D}_E$, $E_{\Omega_i(\omega^*)}$ is well-defined;

For all j, $P_j(s(\omega^*|i_{\omega^*})|i_{\omega^*}) = \mathbb{P}^{\Omega_i(\omega^*)} P_j^*(\omega^*)$, thus,

$$\begin{aligned} R(s(\omega^*|i_{\omega^*})|i_\omega^*) &= \wedge_{j=1}^n \mathbb{P}^{\Omega_i(\omega^*)} P_j^*(\omega^*) \\ &= \mathbb{P}^{\Omega_i(\omega^*)} \wedge_{j=1}^n P_j^*(\omega^*) \\ &\subseteq E_{\Omega_i(\omega^*)} \end{aligned}$$

Let $\omega' \in R(s(\omega^*|i_{\omega^*})|i_\omega^*)$. Since $R(s(\omega^*|i_{\omega^*})|i_\omega^*) = \mathbb{P}^{\Omega_i(\omega^*)} \wedge_{j=1}^n P_j^*(\omega^*)$, there exists $\omega_1^* \in \wedge_{j=1}^n P_j^*(\omega^*)$ such that $\mathbb{P}^{\Omega_i(\omega^*)}(\omega_1^*) = \omega'$. By hypothesis, $\mathcal{D}_E \subseteq W_j^*(\omega_1^*)$ for all j. Now,

$$\begin{aligned} W_j(\omega'|i_{\omega^*}) &= \bigcup_{\left\{\omega_2^*:\mathbb{P}^{\Omega_i(\omega^*)}(\omega_2^*)=\omega'\right\}} W_i^*(\omega^*) \cap W_j^*(\omega_2^*) \\ &\supseteq W_i^*(\omega^*) \cap W_j^*(\omega_1^*) \\ &\supseteq \mathcal{D}_E \end{aligned}$$

Suppose (7.3) and (7.4) hold for all $1 < n \leq k$. Observe that in the case of $n = k+1$, the interactive model $(\Omega^{n-1}, \mathbf{W}(\cdot|r_i^{n-1}(\omega^*)), \mathbf{P}(\cdot|r_i^{n-1}(\omega^*)))$ plays the role of the full model $(\Omega^*, \mathbf{W}, \mathbf{P})$ in the case of $n = 2$. Now observe this model is a multi-agent product model itself, and it satisfies all the properties of the full model by previous results (Lemma 13 and Lemma 7). Therefore the conclusion follows from the induction hypothesis. □

Now we are ready to prove the main theorem.

Proof. The second set inclusion is obvious.

To prove the first set inclusion, let $\omega^* \in \underline{CK}(E)$. By Lemma 14, in every interactive model at ω^*, the subjective factual signals satisfy the classic requirement that the meet is contained in the corresponding elaboration of E as in Aumann (1976). Therefore, the complication is to ensure in all relevant subjective states, every agent is aware of E, and there is higher-order knowledge of it.

It suffices to show that for all $i \in N$, all n and all I_i^n,

$$\omega^* \in K(E|I_i^n) \tag{7.5}$$

By definition (5.7), (7.5) is equivalent to:

$$s(\omega^*|i_{\omega^*}) \in \tilde{K}_{\omega^*}^i(\tilde{K}^{i^2}(\cdots(\tilde{K}^{i^k}(E|I^{k-1}(\omega^*)))\cdots|i_{\omega^*}))$$

By definition (3.3), this is equivalent to:

$$\mathbb{P}^{\Omega_i(\omega^*)} P_i^*(\omega^*) \subseteq \tilde{K}^{i^2}(\cdots(\tilde{K}^{i^k}(E|I^{k-1}(\omega^*)))\cdots|i_{\omega^*})$$

By definition (5.6), the above amounts to: for all $\omega_2 \in P_i(s(\omega^*|i_{\omega^*})|i_{\omega^*})$,

$$P_{i^2}(\omega_2|i_{\omega^*}) \subseteq_* \tilde{K}^{i^3}(\cdots(\tilde{K}^{i^k}(E|I_i^{k-1}(\omega^*)))\cdots|I_i^2(\omega^*)) \tag{7.6}$$
$$W_{i^2}(\omega_2|i_{\omega^*}) \supseteq \mathcal{D}_E \tag{7.7}$$

Apply definition (5.6) on (7.6) recursively. It follows that (7.6) and (7.7) are equivalent to: for any relevant $r_i^{k-1}(\omega^*)$ under I_i^n and ω^*, and any $1 < h \leq k$,

$$P_{i^k}(s(\omega_k|r_i^{k-1}(\omega^*))|r_i^{k-1}(\omega^*)) \subseteq_* E_{\Omega^{k-1}} \tag{7.8}$$
$$W_{i^h}(s(\omega_h|r_i^{h-1}(\omega^*))|r_i^{h-1}(\omega^*)) \supseteq \mathcal{D}_E \tag{7.9}$$

Now notice that (7.8) and (7.9) follow easily from Lemma 14. □

7.8 Proof of Theorem 10.

Proof. Only need to show $CK(E) \subseteq \underline{CK}(E)$.

Let $\omega^* \notin CK(E)$. I show that $\omega^* \notin \underline{CK}(E)$. There are two cases to consider:
Case 1: Suppose there exists $\bar{\omega}^* \in \wedge_{i=1}^n P_i^*(\omega^*)$ such that $\bar{\omega}^* \notin E_{\Omega^*}$. Let $i \in N$. Suppose $\bar{\omega}^*$ is reachable from ω^* for i through $i^2, \cdots, i^k$, that is,

$$\begin{aligned} \bar{\omega}^* &\in P_{i^k}^*(\omega_k^*) \\ \omega_k^* &\in P_{i^{k-1}}^*(\omega_{k-1}^*) \\ &\cdots \\ \omega_2^* &\in P_i^*(\omega^*) \end{aligned}$$

Now consider $r_i^k(\omega^*) = ((\omega^*, i), (s(\omega_2^*|i_{\omega^*})), \cdots, (s(\omega_k^*|r_i^{k-1}(\omega^*)), i^k))$.

By cylinder factual signals, $s(\bar{\omega}^*|r_i^{k-1}(\omega^*)) \in P_{i^k}(s(\omega_k^*|r_i^{k-1}(\omega^*))|r_i^{k-1}(\omega^*)) \notin E_{\Omega^{k-1}}$. But then by (7.8),

$$s(\omega^*|i_{\omega^*}) \notin \tilde{K}_{\omega^*}^i(\tilde{K}^{i^2}(\cdots(\tilde{K}^{i^k}(E|r_i^{k-1}(\omega^*))\cdots)|i_{\omega^*})$$

Thus $\omega^* \notin \underline{CK}(E)$.

Case 2: suppose there exists $\bar{\omega}^* \in \wedge_{i=1}^n P_i^*(\omega^*)$ such that $\mathcal{D}_E \nsubseteq W_i^*(\bar{\omega}^*)$ for some $i \in N$.

Then there exists some $D \in \mathcal{D}_E$ such that $D \notin W_i^*(\bar{\omega}^*)$. Obviously we must have $\bar{\omega}^* \neq \omega^*$. Note $D \in W_j^*(\bar{\omega}^*)$ for all $j \in N$. Now by richness, $\tilde{D}_i \in \mathcal{D}^*$; then by rational interactive awareness, $\tilde{D}_i \in W_j^*(\omega^*)$ for all $j \neq i$. Using cylinder factual signals and rational awareness, it follows that $D \notin W_i(s(\bar{\omega}^*|j_{\omega^*})|j_{\omega^*})$.

Since $\bar{\omega}^* \in \wedge_{i=1}^n P_i^*(\omega^*)$, we can construct a relevant reasoning string $r_j^k(\omega^*) = ((\omega^*, j), (s(\omega_2^*|j_{\omega^*})), \cdots, (s(\omega_k^*|r_j^{k-1}(\omega^*)), i))$ just as above. Again using cylinder factual signals and Lemma 12) recursively, we have $D \notin W_i(s(\bar{\omega}^*|r_j^{k-1}(\omega^*))|r_j^{k-1}(\omega^*))$. By (7.9),

$$s(\omega^*|j_{\omega^*}) \notin \tilde{K}_{\omega^*}^j(\tilde{K}^{i^2}(\cdots(\tilde{K}^i(E|r_j^{k-1}(\omega^*))\cdots)|j_{\omega^*})$$

It follows that $\omega^* \notin \underline{CK}(E)$. This proves the theorem.

□

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