

Intermediation and Value Creation in an Incomplete Market: Implications for Securitization*

Preliminary Draft

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Abstract

This paper studies the impact of financial innovations on real investment decisions. We model an incomplete market economy comprised of firms, investors and an intermediary. The firms face unique investment opportunities that are not spanned by the securities traded in the financial market, and thus, cannot be priced uniquely using the no-arbitrage principle. The specific innovation we consider is securitization: the intermediary buys claims from the firms that are fully backed by cash flows from the new projects, pools these claims together, and then issues tranches of secondary securities to the investors. We first derive necessary and sufficient conditions under which pooling provides value enhancement, and the prices paid to the firms are acceptable to them compared to the no-investment option or the option of forming alternative pools. We find that there is a unique pool that is sustainable, and which may or may not consist of all projects in the intermediary's consideration set.

We then determine the optimal design of tranches, fully backed by the asset pool, to be sold to different investor classes. The new securities created by the intermediary could have up to three components, one that is a marketable claim, one that represents the arbitrage opportunities available in the market due to the special ability to design and sell securities to a subset of investors, and a third component that is the remainder of the asset pool sold to investors at a price not exceeding arbitrage-based bounds. The presence of these three components in the tranching solution has a direct bearing upon the size of the asset pool, and therefore, on value creation due to financing additional projects.

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1 Introduction

Do innovations in capital markets have an impact on real investment decisions, apart from providing arbitrage opportunities to the innovators? In other words, do such innovations permit investments in real assets that would otherwise not occur, because they are too costly to finance? The casual evidence suggests that the answers to these questions are in the affirmative, based on examples such as venture capital and private equity, project finance and securitization. In all these cases, the innovations allow financing to be provided for projects that might not be undertaken in their absence. Entrepreneurs and firm managers are able to undertake fresh investments in projects, since their “cost of capital” has been reduced as a result of the innovations, thus making the net present value of the projects positive.

The academic literature identifies three alternative mechanisms that may be responsible for the improved attractiveness of projects as a result of a financial innovation. The first is a reduction in the impact of market frictions, such as transaction costs, as a result of the innovation. The second is the effect of the innovation on the amelioration of asymmetric information effects, particularly in the context of the principal agent relationship between investors on the one hand and the entrepreneur/manager on the other. The third effect is through the improvement in the spanning across future states of world by the available securities in the market, as a result of the innovation. While the first two mechanisms have been used in a variety of models proposed in the literature to study the impact of financial innovations, such as venture capital funds, on real investment decisions, the third mechanism has not been adequately studied thus far. In this paper, we focus on the third argument based on incompleteness in markets, because we believe that it shows the effect of financial innovations on real investments, without making explicit assumptions about market frictions or information asymmetries. Instead, this line of argument explains the effect of innovation on real investment decisions through the existence of arbitrage opportunities in an incomplete market that are exploited by the innovating firm. Therefore, our paper provides the value creation by intermediaries through supporting project financing in the context of incomplete capital markets.

The phenomenon of securitization is now widespread in financial markets: mortgages, credit card receivables and various types of corporate debt instruments have been securitized using a variety of alternative structures. It began in the 1970s with the securitization of mortgage loans by Fannie Mae, Ginnie Mae and Freddie Mac, but has since expanded to other fixed income

markets, including the corporate debt market. The common feature of these structures is that an intermediary purchases claims on cash flows issued by various entities, pools these claims into a portfolio, and then tranches them into marketable securities that cater to the investment needs of particular clienteles of investors.

To take the example of collateralized debt obligations (CDOs), the basic structure is that a financial intermediary sets up a special purpose vehicle (SPV) that buys a portfolio of debt instruments - bonds and/or loans - and adds credit derivatives on individual “names.” This is referred to as pooling. The SPV then issues various claims against the pooled portfolio, which enjoy different levels of seniority; the claims issued range from a super-senior claim, which typically a high-grade AAA claim, which has a negligible probability of not meeting its promised payment, to a medium-grade claim, say rated BBB+, which has a low but not negligible probability of such default, and finally, to an equity security, which is viewed as risky. The structuring of the claims to match investor tastes is referred to as tranching and the resulting claims are referred to as tranches.

We study securitization in the context of an economy with firms, investors and financial intermediaries. In this economy, there are primary securities traded in the financial market at fair (i.e., arbitrage-free) prices. Firms have opportunities to invest in certain unique assets, and their objective is to maximize the present value of the cash flows from these assets. Investors are utility maximizers. Intermediaries purchase claims from firms that are fully backed by their cash flows, and issue two types of securities to be sold to investors. They can issue securities that are within the span of the financial market (marketable securities), or create new securities that are not spanned by the market (secondary securities). Throughout our analysis, we assume that firms are not large enough to influence the prices of the securities traded in the market, i.e., they are price-takers. We study whether, in an arbitrage-free setting, the transactions undertaken by the intermediary create value for firms and investors. The value for firms is created by permitting investments in real assets that would not be taken otherwise. The value to investors is created by satisfying demand for consumption in states that are not served by the primary securities traded in the financial market. We do not permit short sales of secondary securities by any agent and tranching of primary securities by intermediaries.

In our model, a firm is willing to invest in a new project when the project has a positive net present value, i.e., when the value of the project exceeds the value of the resources employed, characterized as the reservation price of the firm. The value of the project is usually derived from

the capital market based on the twin assumptions that the cash flows from the project can be replicated in the financial market, and that all agents are price-takers with respect to financial claims. When markets are incomplete, the cash flows from such unique investment opportunities often cannot be fully replicated in the financial market; hence, only bounds can be placed on their present values. Therefore, projects whose values are unambiguously greater than their reservation prices are financed, while those whose values unambiguously fall below their respective reservation prices are rejected. This raises the following questions: (i) What are the conditions under which projects whose reservation prices lie inside the bounds on their values can be made acceptable through securitization? (ii) What are the incentives to firms to be willing to transact with the intermediary? Our paper addresses these questions. (iii) What is the structure of the securitization transactions undertaken by the intermediary?

We carry out the analysis in two parts. First, we study the phenomenon whereby financial intermediaries pool cash flows from the assets of several firms or divisions of a firm and issue securities that are within the span of the market. We assume that firms have reservation prices for undertaking investments in opportunities that are unique to them. In order to simulate competition in the intermediation process, we assume that firms can form coalitions with some or all of the other firms. Therefore, we formulate the firms' decision problem as a cooperative game. We examine conditions under which incentives can be structured such that all firms participate in the creation of the asset pool. We show that there is a simple condition which is necessary and sufficient for all firms to participate in the game, which implies that all firms obtain at least their reservation prices and cannot do better by breaking away from the grand coalition. When this condition is not met, our analysis yields a strong result that there is a maximal pool of assets that is sustainable in the cooperative game. This pool is the one that maximizes the value enhancement provided by pooling. We characterize the composition of this pool and show that it may or may not consist of all of the firms. Finally, we show how the intermediary can allocate the value of the asset pool fairly among contributing firms.

We then study the joint pooling and tranching problem, in which a financial intermediary first pools cash flows from several firms, and then issues securities against the pool. We expect pooling and tranching to provide greater value than pooling alone. In this part of the paper, we develop a method for determining the incremental value of tranching and the structuring of incentives so that firms participate in this modified setting. Additionally, we determine the general structure of

the optimal tranches be issued to investors. The new securities created by the intermediary could have up to three components: one that is a marketable claim, one that represents the arbitrage opportunities available in the market due to special ability to design and sell securities to a subset of investors, and a third component that is the remainder of the asset pool which is sold to investors at a price not exceeding arbitrage based bounds. The presence of these three components in the tranching solution has direct bearing upon the size of the asset pool, and therefore, upon value creation due to financing additional projects.

It should be emphasized that the value creation studied in our paper is due to securities created from the cash flows of new projects. This is fundamentally different from profits that can arise in an incomplete market from arbitrage of existing securities. We exclude such arbitrage in our analysis, since we presume that it has already occurred prior to the innovation, and is reflected in the initial equilibrium in the market.

Our results can be applied to other examples of financial intermediation in the context of market incompleteness, such as the choice of investments by a venture capitalist matched by the (optimal) mix of claims issued against them to investors, or the optimal asset-liability mix of a bank. They can also be applied to traditional corporate financial problems such as mergers and acquisitions, optimal financing, and the valuation of real options. We discuss a few of these examples in §6.

This paper is organized as follows. Section 2 reviews the related literature on incomplete markets and securitization. Section 3 presents the model setup and assumptions. Section 4 analyzes the conditions under which there is value in pooling and firms willingly participate in the creation of the asset pool. Section 5 analyzes the conditions under which there is value in pooling and tranching, and determines the optimal tranching strategy, §6 presents a numerical example illustrating the results of our paper, and §7 concludes with a discussion of the implications of our analysis.

2 Literature Review

Our research is related to the fairly sparse literature on securitization, broadly defined as the issuance of securities in the capital market that are backed or collateralized by a portfolio of assets. Most of this research has focused on the rationale for the widespread use of pooling and tranching in the asset-backed securities market. This rationale is largely based on market imperfections, mainly based on transaction costs and information asymmetry. Specific examples of securitization include the academic literature on “supershares” (i.e., tranches of the portfolio of all securities in

the market), primes and scores (i.e., income and capital gains portions of a stock), and “bull” and “bear” bonds.¹ More recently, specific examples of securitization have been analyzed by researchers, e.g., the assets of insurance companies (Cummins (2004)), and those of firms in financial distress (Ayotte and Gaon (2004)).

As mentioned earlier, two alternative economic explanations have been proposed in the literature for the securitization of assets. The first relates transaction costs to the welfare improvement that can be achieved by designing, creating and selling securities to meet the preferences of particular clienteles of investors or issuers. The other type of explanation has to do with some aspect of information asymmetry and the ability of a financial intermediary to reduce the agency costs resulting from it. The first type of explanation is typified by Allen and Gale (1991), who examine the incentive of a firm to issue a new security when there are transaction costs. Allen and Gale examine the incentive of a firm to issue claims that increase the spanning of states. They study an exchange equilibrium that results in an incomplete market. In their model, firms do not behave as price takers and also incur a cost of splitting the cash flows from their asset into financial claims. If the value of the firm is unaffected by splitting the cash flows, as is the case in a complete market, the firm has no incentive to do so. In an incomplete market, it may be possible to split the return into financial claims, i.e., innovate, and benefit from selling the new claims to investors. But not every firm needs to innovate. Even if a single firm amongst many similar ones, or a financial intermediary, for that matter, does so, investors can benefit if short sales are permitted. The new claims result in readjustment of consumption by investors, which, in turn, leads to a change in asset prices that may benefit the firm. There are two implications of this: a) the *ex-post* value of similar firms may be equal, thus reducing the incentive of any one firm to innovate, and b) the firm has an incentive to innovate new claims only if the prices change, i.e., if competition is imperfect.

We draw upon the model of Allen and Gale (1991), but our approach differs from theirs in significant ways. First, our model does not use a general equilibrium approach. The reason is that we are interested in obtaining more specific results, without considering the complex feedback effects that a general equilibrium analysis would entail. Second, we use a game-theoretic setting to ensure participation by firms. Third, we do not explicitly model the cost of issuing claims, since we wish to focus on value creation in a frictionless market. Fourth, our aim is to explicitly introduce

¹See Hakansson (1978), and Jarrow and O’Hara (1989) for details. For example, Hakansson (1978) argues that options or supershares on the market portfolio improve the allocational efficiency of an existing market structure, even if the market portfolio itself is not efficient.

a third type of agent - firms - into the exchange equilibrium and study how they can benefit from intermediation. Moreover, in our framework, the problem for firms is not just whether to issue new claims against returns from existing assets; rather, the problem is also to decide whether to undertake new projects. Lastly, in order to study the effect of intermediation and whether it helps more firms to undertake investments (or firms to invest in more projects), we have to necessarily limit short sales of secondary securities by investors - otherwise, investors can also intermediate. Therefore, we confine the financial innovation activity to designated financial intermediaries.

Many researchers have studied the effect of information asymmetry between issuers and investors in the context of securitization [see, for example, DeMarzo and Duffie (1999), DeMarzo (2001) and Leland and Pyle (1977)]. Pooling assets is considered beneficial to both an uninformed issuer and an uninformed investor. The benefit to an uninformed issuer is that it reduces the issuer's incentive to gather information (Glaeser and Kallal (1997)). The benefit to uninformed investors is that pooling reduces their adverse selection problem when competing with informed investors (DeMarzo 2001). In this context, Subrahmanyam (1991) shows that security index baskets are more liquid than the underlying stocks. DeMarzo (2001) also shows that an informed issuer (or intermediary) does not prefer pure pooling, because it destroys the asset-specific information of the informed issuer. Instead, an informed intermediary prefers pooling and tranching to either pure pooling or separate asset sales because pooling and tranching enable an intermediary to design low-risk debt securities that minimize the information asymmetry between the intermediary and uninformed investors. DeMarzo calls this the "risk diversification effect" of pooling and tranching. Pooling and tranching are also beneficial to uninformed investors. For example, Gorton and Pennachi (1990) show that uninformed investors prefer to split cash flows into a risk-less debt and an equity claim. Unfortunately, even though these explanations might explain the structure of securities to an extent, they do not provide the motivation to innovate or securitize, especially in the context of originating firms, who then use the proceeds to undertake more projects.

To summarize, the differences between our paper and the prior work on securitization are as follows. Our work in this paper is based on an arbitrage-free pricing framework; however, we restrict the ability of some agents to take advantage of arbitrage opportunities. In our framework, only those who are designated as financial intermediaries are able to take full advantage of these opportunities. Also, our model assumes an incomplete securities market; but, we do not consider issues relating to transaction costs and information asymmetry, except in the indirect sense that

the financial intermediaries in our model can undertake certain transactions that other agents cannot. Another difference is that our paper analyzes the *simultaneous* enhancement of the welfare of investors and the expansion of the set of value increasing projects undertaken by firms, due to the intermediaries' intervention through the securitization of unspanned claims. We, therefore, consider multiple firms that have different assets (unlike Allen and Gale, who consider multiple firms that have the same asset). We relate securitization to the problem of financing projects and also to satisfying the needs of investor clientele.

We have chosen to use the arbitrage-based approach of Harrison and Kreps (1979),². It should not be construed that we are advocating only this approach. Instead, we believe that our methodology shows how the set of projects that can be undertaken in an incomplete market expands due to intermediation. Indeed, one can derive an alternative formulation of our framework, yielding more specific conclusions, if we impose the additional restrictions on investor preferences or the reward-to-risk ratio in the market. We illustrate these ideas with an example: Consider a firm that wishes to undertake a project requiring an immediate investment, which results in uncertain and unspanned cash flows in the future. From the preceding discussion, it is clear that the firm cannot place an exact value on its cash flows, based on arbitrage-free pricing. The ambiguity regarding the value of the project is the source of the problem considered in this paper. We do not rule out the possibility that adopting one or the other more specific approaches described above might resolve the decision problem unequivocally. However, by using an arbitrage-free framework, the question we are able to answer is whether an intermediary can enhance the “value” by pooling assets from different firms and tranching them for sale to investors.

3 Model Setup

We consider an Arrow-Debreu economy in which time is indexed as 0 and 1.³ The set of possible states of nature at time 1 is $\Omega = \{\omega_1, \omega_2, \dots, \omega_K\}$. For convenience, the state at time zero is denoted as ω_0 . All agents have the same informational structure: The true state of nature is unknown at $t = 0$ and is revealed at $t = 1$. Moreover, the K states are a complete enumeration of all possible events of interest, i.e., the subjective probability of any decision-maker is positive for each of these

²See also Ross (1976) and John (1981).

³The model described below can be extended to a multi-period setting with some added complexity in the notation. However, the basic principles and results derived would still obtain.

states and adds up to one when summed over all the states.

3.1 Securities Market

We start with a market in which N *primary securities* are traded via a financial exchange. Security n has price p_n and payoff $S_n(\omega_k)$ in state k . These securities are issued by firms and purchased by investors through the exchange. The securities market is arbitrage-free and frictionless, i.e., there are no transaction costs associated with the sale or purchase of securities. To keep the analysis uncluttered, cash flows are not discounted, i.e., the risk-free rate of interest is zero.

From standard theory, the absence of arbitrage is equivalent to postulating that there exists a set, Θ , of risk neutral pricing measures over Ω under which all traded securities are uniquely priced, i.e., $E_q[S_n] = p_n$, for all n and for all $q \in \Theta$. It is well known that the set Θ is spanned by a finite set of independent linear pricing measures.⁴ These are labelled $\{q_l, l = 1, \dots, L\}$. In particular, when the set Θ is a singleton, the market is complete, else it is incomplete.

We use the following additional notation in the sequel. Not every claim can be priced uniquely in an incomplete market. When a claim cannot be priced uniquely, the standard theory provides bounds for the price of a claim Z that pays $Z(\omega_k)$ in state k . Let $V^-(Z) = \max\{E[S] : S \leq Z, S \text{ is attainable}\}$, and let $V^+(Z) = \min\{E[S] : S \geq Z, S \text{ is attainable}\}$. $V^-(Z)$ and $V^+(Z)$ are well-defined and finite, and correspond to the lower and upper bound on the price of the claim Z . Given that the set Θ is spanned by a finite set of independent linear pricing measures labelled $\{q_l, l = 1, \dots, L\}$, this can be formalized in the following Lemma. (All proofs are in the Appendix.)

Lemma 1. (i) $V^+(Z) = \max_{l \in L} E_{q_l}[Z]$.

(ii) $V^-(Z) = \min_{l \in L} E_{q_l}[Z]$.

(iii) *If the payoffs from the claim $Z(\omega_k)$ are non-negative in all states, then these bounds are unaffected by the inability of agents to short sell securities.*

This lemma is needed for several proofs in the Appendix as well as for models in §4 and §5. Since we do not allow short sales of primary securities for the purpose of securitization, an important aspect of this lemma is that short sales restrictions do not affect the arbitrage-based price bounds if the payoffs of the contingent claim are non-negative.

⁴A linear pricing measure is a probability measure that can take a value equal to zero in some states, whereas a risk neutral probability measure is strictly positive in all states. Thus, the set Θ is the interior of the convex set spanned by the set of independent linear pricing measures. The maximum dimension of this set equals the dimension of the solution set to a feasible finite-dimensional linear program, and thus, is finite. See Pliska (1997).

3.2 Agents

We consider three types of agents in our model: investors, firms, and intermediaries. Investors are utility maximizers. Their decision problem is to construct a portfolio of *primary* securities (subject to budget constraints), so as to maximize expected utility. Investors can buy or sell primary securities, but cannot short secondary securities or issue securities. Firms own (real) assets and issue primary securities that are fully backed by the cash flows from these assets. Firms can also create new assets and sell claims against the cash-flows from these assets to intermediaries.⁵ They negotiate with intermediaries to get the highest possible value for their assets that is consistent with the prices prevailing in the financial market. Intermediaries facilitate transactions between firms and investors by repackaging the claims purchased from the firms and issuing *secondary securities* traded on the over-the-counter securities market. We stipulate that the claims sold by firms to the intermediaries must be fully backed by their asset-cash flows, and the claims issued by the intermediaries should be fully backed by the assets purchased from firms. We also do not allow for short sales of secondary securities or tranching of primary securities by intermediaries. These assumptions enable us to isolate the roles of the three types of agents, and explicitly study the phenomenon of securitization through the intermediaries. A secondary reason for these assumptions is to avoid transactions that permit default in some states, because that would lead to complex questions relating to bankruptcy and renegotiation, which are outside the purview of this paper.

Having broadly described the various agents, we set out the details of their decision making problems as below:

Investors: We model investors by classifying them into investor types. The set of investor types is finite and denoted as I . Each investor of type i has endowment $e_i(\omega_k)$ in state k . The utility derived by type i investors is given by a von Neumann-Morgenstern function $U_i : \mathfrak{R} \times \mathfrak{R} \rightarrow \mathfrak{R}^+$. U_i is assumed to be concave, strictly increasing and bounded above. Investors maximize their expected utility, subject to the constraint that consumption is non-negative in every state.

Denote the consumption of type i investors in state k as x_{ik} and let the subjective probability of

⁵The new assets created by firms may also include assets that are already in place, but not yet securitized. For example, the loans made a bank that are presently held on the asset side of its balance sheet may be candidates for securitization in a collateralized loan obligations structure. The bank would be a “firm” in the context of our model. In these cases, of course, the decision to acquire the assets in question has already been made and, to that extent, part of the analysis in this paper would not apply directly.

state k be $P_i(\omega_k)$. Then, the investor derives expected utility equal to $\sum_{k=0}^K P_i(\omega_k)U_i(x_{i0}, x_{ik})$. The portfolio of primary securities held by a type i investor is denoted as the N -tuple of real numbers $(\alpha_{i1}, \alpha_{i2}, \dots, \alpha_{iN})$, where α_{in} is the amount of security n in the portfolio. The type i investor's decision problem can be written as

$$\max \sum_{k=0}^K P_i(\omega_k)U_i(x_{i0}, x_{ik})$$

subject to

$$\begin{aligned} x_{ik} &= e_i(\omega_k) + \sum_{n=1}^N \alpha_{in}S_n(\omega_k), \quad \forall k = 1, 2, \dots, K \\ x_{i0} &= e_i(\omega_0) - \sum_{n=1}^N \alpha_{in}p(n) \\ x_{ik} &\geq 0, \quad \forall k = 0, 1, 2, \dots, K. \end{aligned}$$

The first constraint equates the consumption in each state at time 1 with the cash flow provided by the portfolio and the endowment. The second specifies the budget constraint for investment in primary securities at time 0. The third constraint specifies that the cash flow in each state at time 1 should be non-negative.

Denote the derivatives of U_i with respect to x_{i0} and x_{ik} , $k \geq 1$ as U_{i1} and U_{i2} respectively. We shall assume, as customary, that the current period consumption is strictly bounded away from zero for investor types. It follows that, at optimality,

$$\sum_{k=1}^K P_i(\omega_k) \frac{U_{i2}(x_{i0}, x_{ik})}{\sum_{k=1}^K P_i(\omega_k)U_{i1}(x_{i0}, x_{ik})} S_n(\omega_k) \leq p_n.$$

Here, we obtain an inequality because of the restriction on consumption. The inequality suggests that, in state k , type i investors are willing to buy an infinitesimal amount of consumption at a price, m_{ik} given by

$$m_{ik} = P_i(\omega_k) \frac{U_{i2}(x_{i0}, x_{ik})}{\sum_{k=1}^K P_i(\omega_k)U_{i1}(x_{i0}, x_{ik})}.$$

These values are called the state prices (also called reservation prices) of investors. We require that each security is present in the optimal portfolio of at least one investor type.

We note that the reservation price for an unspanned state may differ amongst investor types due to the incompleteness of the market. We assume that an investor of type i is willing to buy not only consumption that is specific to state k , but also secondary securities issued by the intermediary if the price of the secondary security is below that given by valuing its state dependent cash flows, using the investors' reservation prices.

Firms: Firms maximize the time 0 expected values of their investments. Firm j can create an asset X_j that is unique to it. The asset provides a positive cash flow of $X_j(\omega_k)$ in each state k , at time $t = 1$. The firm can sell claims issued against X_j to the intermediary. Claims issued against X_j should be fully backed by X_j ; in other words, the sum promised should not exceed the cash flow from X_j in any state of nature. We assume that firm j has a reservation price r_j on X_j . The reservation price could be comprised of financial, physical and transaction costs, as well as opportunity costs of the key decision-makers of the firm that are required to create the asset. The firm invests in the asset, if the net present value, given by the difference between the selling price offered by the intermediary and the reservation price, is positive. Additionally, firms cannot trade with other firms directly and also cannot issue claims that are not fully backed by their assets. Let J denote the number of firms that wish to undertake investment projects at time 0.

We assume that the total cash flow available from this set of firms in any state k , $\sum_{j=1}^J X_j(\omega_k)$, is small relative to the size of the economy. Each firm, therefore, behaves as a price-taker in the securities market. However, when the asset cannot be priced precisely, it negotiates with the intermediary for obtaining the highest possible price for securitization of the asset. In the rest of this paper, we use X_j to refer to both the j -th asset and the cash flows from the j -th asset.

Intermediaries: Intermediaries are agents who have knowledge about the firms' and investors' asset requirements. Notice that such knowledge is different from receiving a private signal regarding the future outcome. Hence, intermediaries have no superior information about future cash flows, relative to other agents in the economy. The intermediaries purchase assets from firms and repackage them to sell to investors. They seek to exploit price enhancement through securitization operations that increase the spanning of available securities. They use this superior ability to negotiate with the firms for the prices of their assets. They use the knowledge about the investors' preferences to create new claims and price them correctly. An important aspect of the model is that intermediaries act fairly by *paying* the same price for the same asset, independent of which firm is selling it to the pool, and *charging* the same price for the same product even though it is sold to different customers. The rationale for these fairness requirements is the possibility of entry and competition from other intermediaries. However, we do not explicitly model competition amongst intermediaries beyond imposing the fairness requirements and the participation constraints by firms that are discussed in the next section. Hence, in what follows, we consider the securitization problem from the viewpoint of a single intermediary.

The intermediary purchases claims from firms, *pools* them and packages them into different *tranches*, and sells them as *collateralized secondary securities*. Pooling is defined as combining the cash flows from claims issued by different firms in a proportion determined by the intermediary. The intermediary is not restricted to purchasing only all or none of a firm's cash flows. Instead, it can purchase fractions (between 0 and 1) of the available assets. Tranching is defined as splitting the pooled asset into sub-portfolios to be sold to different groups of investors, with the constraint that the sub-portfolios be fully collateralized, i.e., fully backed by the claims purchased from the firms. We assume that the intermediary can sell secondary securities to investors in a subset of the investor classes, which is denoted as $I_1 \subset I$.

4 Value of Pooling

We attribute the beneficial role played by the intermediary to two factors: the value enhancement provided by pooling alone, and the value provided by tranching. In this section, we consider the former. We analyze the problem of pooling the cash flows of some or all firms and valuing the pooled asset by replicating its cash flows in the securities market. We use the lower bound $V^-(\cdot)$ as a measure of value, and thus, compute the lowest price at which the pooled asset can be sold without presenting opportunities for arbitrage. The reason why $V^-(\cdot)$ is taken as a measure of value is that it is the price at which the claim can be sold for sure in the market.⁶

From one perspective, there is value to pooling if the lower bound on the pooled asset exceeds the sum of the lower bounds on the individual assets. This is likely to happen in an incomplete market, because we expect a larger fraction of the pooled cash flows to be marketable compared to the individual components. From an entirely different, and somewhat more subtle, perspective, which is the focus of this paper, value gets created when more projects are undertaken by firms, as a consequence of the innovation. We describe how this real effect could come about due to intermediation.

Consider any given firm j . If $r_j \leq V^-(X_j)$, then clearly, firm j can profitably invest in asset X_j , even without pooling. If $r_j \geq V^+(X_j)$, then it does not make sense for the firm to invest in the asset X_j . The interesting case is the one where $V^+(X_j) \geq r_j \geq V^-(X_j)$, because, in this case, the basis for the decision to invest in X_j is ambiguous. For example, suppose that the cash flows of the pooled

⁶Of course, a price higher than this lower bound (but lower than $V^+(\cdot)$) is possible, but by no means guaranteed, and may be the result of bargaining in the market.

asset are given by $X(\omega_k) = \sum_j X_j(\omega_k)$ for all k . Clearly, we have $V^-(X) \geq \sum_j V^-(X_j)$.⁷ This example shows that pooling improves the spanning of cash flows across states, and thus, provides value enhancement. However, we still need to consider the reservation prices of firms to determine if pooling reduces the ambiguity regarding investment in assets. We say that there is value to pooling in this latter sense if there is a linear combination of assets with weight $0 \leq \alpha_j \leq 1$ for asset j such that even though for one or more j 's $V^-(X_j) < r_j$, we obtain $V^-(\sum_j(\alpha_j X_j)) \geq \sum_j(\alpha_j r_j)$ and $\alpha_j > 0$ for at least one of the firms whose value is below its reservation price. Another way of defining this type of value creation is that set of projects fully or partially financed from payments derived from the asset pool is larger than the set of such projects prior to pooling. In our formulation, firms need not behave altruistically in creating the asset pool; therefore, as an additional condition for value creation, we require that firms should have an incentive to pool their assets only when they cannot benefit, individually or severally, from breaking away from the pool.

Theorem 1 shows the necessary condition for creating value through pooling. The rest of the section determines sufficient conditions for value creation.

Theorem 1. (i) *If there is a $q \in \Theta$ such that $r_j \geq E_q[X_j] \forall j$, then value cannot be created by pooling the X_j 's.*

(ii) *Conversely, if there is no $q \in \Theta$ such that $r_j \geq E_q[X_j] \forall j$, then value can be created by pooling the X_j 's.*

The first part of the theorem states that if the reservation price for each asset is higher than its value under a common pricing measure, then additional value cannot be created through pooling. Conversely, if the condition in part (i) of the theorem fails to hold, then part(ii) states the positive part of the result, that is, there exists a set of values $\{\alpha_j\}$ such that value is enhanced through pooling. There can be several such sets of values of $\{\alpha_j\}$. For ease of presentation, we initially assume that the condition in Theorem 1(ii) holds for $\alpha_j = 1$ for all j , i.e., there is value in pooling *all* the cash flows from all firms. We first present all the results under this assumption. Then, we generalize them to the case when the condition in Theorem 1(ii) holds, but necessarily with $0 < \alpha_j < 1$ to create value by pooling.

As mentioned before, even when the value of the pooled asset exceeds the sum of the reservation

⁷The left hand side is given by minimizing the sum of the cash flows from all assets over the set of probability measures; whereas the right hand side the sum of the minimum of each individual cash flow. The minimum of the sum is always larger than or equal to the sum of minimums.

prices, the firms may be unwilling to participate in the asset pool. For example, this could happen if one firm has a very high reservation price; therefore, the remaining firms are better off keeping it out of the pool. To see this, note that adding firms to the pool might make the combined cash flows more marketable. However, adding firms imposes incremental costs due to the additional payments necessary, which may exceed the enhancement in value. This naturally leads to the following set of questions: Can we characterize reservation prices such that there is an incentive for firms to pool their assets? Can a fair price be set for each X_j ? How many asset pools would be created and what would be the composition of these asset pools? The remainder of this section answers these questions.

We stipulate that firms will participate in the pool only if they cannot do better by forming sub-coalitions amongst themselves. Therefore, the firms' participation problem is a cooperative game, \mathcal{G} . Let J_w denote a subset of the set of all firms, J , wherein each firm j contributes a fraction $w_j \in [0, 1]$ of its cash flows with proportional reservation price $w_j r_j$. Let $J_w^c = J - J_w$ denote the complement of J_w , wherein the contribution of each firm j is $(1 - w_j)X_j$ and reservation price is $(1 - w_j)r_j$. Also let $X(J_w) = \sum_{j \in J_w} w_j X_j$. We consider the cooperative game in which the value of each coalition, $V(J_w)$, is defined as $V^-(X(J_w))$. Following standard terminology for cooperative games, we say that there is a solution to this game, i.e., its core is non-empty, if the grand coalition of all firms cannot be blocked. The theorem below provides sufficient conditions for the core of the game to be non-empty, as well as conditions that guarantee that payments can be made to the firms to cover their reservation prices.

Theorem 2. (i) *If $r_j \leq V^-(X_j)$ for all j , then the core of game \mathcal{G} is not empty.*

(ii) *There is a solution in the core to \mathcal{G} such that the payments to all firms exceeds their reservation price if and only if for every subset J_w of J , we have $V(J) \geq \max(V(J_w), \sum_{j \in J_w} w_j r_j) + \max(V(J_w^c), \sum_{j \in J_w^c} (1 - w_j) r_j)$.*

Theorem 2(i) is based on an argument from Owen (1975) and Samet and Zemel (1984). The game \mathcal{G} is a linear program (LP). These papers analyze the solution to games of this type.

Theorem 2(ii) essentially states that the necessary condition for the payments to firms to support the core is also sufficient to guarantee its existence. It is easy to see that the condition implies the condition in Theorem 1(ii). The necessary part of Theorem 2(ii) is immediate, because under every solution in the core, each coalition J_w should get at least $\max(V(J_w), \sum_{j \in J_w} w_j r_j)$. If this condition does not hold, then either some coalition does not get its value (and can do better on

its own) or the payment to the firms in some coalition cannot cover the sum of the reservation prices. Part (ii) of the theorem shows that when the condition holds for all possible J_w , all firms participate and all projects are financed in full. Notice that we do not need to verify the condition in Theorem 2(ii) for all possible partitions of J . Instead, verifying the condition for partitions of size two is sufficient.

Notice also that the inequalities in Theorem 2(ii) must be tested not only for partitions where $w_j = 0$ or 1 but also for fractional values of w_j , i.e., partitions where a firm belongs to two or more subsets and divides its cash flows between them. Thus, there is a continuum of partitions making it virtually impossible to use Theorem 2(ii) directly in practice to determine the composition of the asset pool. However, this task can be avoided. We show that there is a simple condition which is necessary and sufficient for all the inequalities in 2(ii) to be satisfied. Thus, under this condition, the cash flows from each asset X_j are included fully in the pool and the core of the cooperative game is not empty.

Theorem 3. *Let $q \in \Theta$ be a pricing measure under which $\sum_j E_q[X_j] = V^-(\sum_j X_j)$. If $E_q[X_j] \geq r_j$ for all j and some such q , then the sufficiency conditions in Theorem 2(ii) are satisfied. The converse is also true.*

We remark on the nice symmetry between this result and Theorem 1(i). The earlier result, viz., Theorem 1(i), is that if under a common pricing measure each asset's value is less than its reservation price then there is no value in pooling. The new result is that if under the extreme pricing measure that minimizes the value of the asset pool, the value of each asset equals or exceeds its reservation price then value can be created by pooling all assets. Also, value is created (in the sense of additional projects being undertaken) if some project whose value was below the reservation price gets financed through the pooling effect.

While Theorems 2 and 3 show that there exist payment schemes such that firms are willing to participate in the game \mathcal{G} , we need to address the question of actually determining the payment scheme to the firms, which we now turn to. It is possible to show that there could be many such schemes but we also require the scheme to be 'fair'. It is difficult to work with the concept of 'fairness' in full generality. However, a case can be made that if all firms are paid the same price for a unit cash flow in state k , then the scheme is surely fair. We therefore restrict ourselves to payments determined using a linear pricing measure, which does not compensate for pricing synergies across states. The following corollary complements the results so far, because it uses the

sufficient condition of the Theorem 3 to construct a linear pricing scheme.

Corollary 1. *If a pricing measure q_p exists which is either an extreme point of the set of risk neutral probability measures, or a convex combination of such extreme points, such that $\sum_j E_{q_p} X_j = V^-(\sum_j X_j)$, and the reservation prices satisfy $r_j \leq E_{q_p} X_j$, then the grand coalition of all firms can be sustained when firm j is paid $E_{q_p} X_j$.*

Corollary 1 shows that value enhancement (from the first perspective) due to pooling can be construed to be given by the change in the pricing measure that is necessary to value the assets correctly. This is readily seen by assuming that $r_j \geq V^-[X_j] = E_{q_j}[X_j]$, that is, firm j cannot decide whether to invest in the project based on the minimum valuation. Notice that the measure to determine the minimum value of each firm's asset, q_j , depends on the cash flow of the asset which is being valued and it provides the lower bound $V^-[X_j]$. The measure to determine the value when the project is considered to be part of the asset pool depends on the cash flow of the *entire* asset pool. This yields a higher value. The firm surely gains when the reservation price lies within these two bounds. Moreover, when we are restricted to compensate firms using the same pricing measure, we are assured that the gain from pooling can be used to induce all firms to participate when $r_j \leq E_{q_p} X_j$. This is the second source of value creation.

There are other interesting aspects to the corollary. The scheme is fair because it uses the same pricing formula for each firm. The measure also prices the traded securities correctly. Thus, the firms can use a market benchmark to assure themselves that the intermediary is fair. In the next section, we shall examine how far these results carry over when the intermediary can tranche the pool to create secondary securities.

The above results characterize the situations in which all firms participate and contribute all their assets. A critical condition for 'full' participation by firms is $E_{q_p}(X_j) \geq r_j$ for all j and q_p as defined in Theorem 3. Also, note that according to Theorem 1, there are situations where there is value in pooling only fractions of cash flows of the firms. Further, there may be several sets of values $\{\alpha_j\}$ that provide value in pooling. The following corollary highlights one such solution. We show that there exists a set of optimal values of $\{\alpha_j\}$, denoted $\{\alpha_j^*\}$, that maximize the value of the pool. Further, if we treat $\alpha_j^* X_j$'s as the constituent assets instead of X_j 's, then Theorems 2 and 3 still apply to this asset pool.

Corollary 2. *If the condition in Theorem 1 holds, then the value of pooling is maximized by solving the linear program: $\max V^-(\sum_j \alpha_j(X_j)) - \sum_j \alpha_j r_j$, subject to $0 \leq \alpha_j \leq 1, \forall j$. An optimal*

solution to this linear program, $\{\alpha_j^*\}$, is in the core of \mathcal{G} . Assets of firms whose value exceeded their reservation price will be included fully in this asset pool. Moreover, the leftover cash flows $\{(1 - \alpha_j^*)X_j\}$ do not provide any value in pooling.

Corollary 2 is consistent with Theorem 3 because if $E_{q_p}[X_j] \geq r_j$ for all j then it can be shown that setting $\alpha_j^* = 1$ for all j gives an optimal solution to the linear program in Corollary 2. Of course, it is difficult to construct a fair payment scheme because it will simultaneously require limiting the fraction of assets purchased at that price. Value creation from the both perspectives is possible. We do not discuss how firms that get only a fraction of their assets included in the pool will finance the balance. For this reason, we also do not discuss whether value creation is more likely due to increase in the value of the pool than due to the financing of additional projects.

In summary, this section fully characterizes the value in pooling. Theorem 1(i) and (ii) show the conditions under which there is no value in pooling and those under which there is value in pooling. In the latter case, Theorems 2 and 3 and Corollary 2 together show that there will only be one coalition formed. This coalition achieves the maximum value of pooling. It includes all the assets when the condition in Theorem 3 holds, and fractional assets otherwise. Corollary 1 guarantees the existence of a linear payment scheme for this coalition. The assets not included in this coalition cannot be constituted as a separate value enhancing pool.

5 Value of Pooling and Tranching

In this section, we assume that in addition to tranches that are replicas of primary securities already traded in the securities market, the intermediary can also sell new securities, fully backed by the pool of assets, directly to investors. We call the former as marketable tranches and the latter as non-marketable tranches or secondary securities. We say that tranching provides value enhancement beyond pure pooling if the sum of the price of the marketable tranche (which is unique) and the price of the non-marketable tranche (which is obtained by selling this tranche at investor-specific state prices m_{ik}) exceeds the value $V^-(\cdot)$ obtained by pure pooling.

In general, the cash flows from a given asset pool, say, $\sum_j w_j X_j$ can be split into several tranches, and each tranche offered to every investor type. Recall that m_{ik} denotes the state price of investor type i for a unit consumption in state k . Let

$$m_k^* = \max_{i \in I_1} m_{ik}.$$

where I_1 is the subset of investor classes to whom the intermediary can sell secondary securities. It is clear that the cash flow in state k should be sold to the investor type that values it the most. Therefore, the maximum price that the intermediary expects from a tranche sold in state k is m_k^* .

Given the asset pool, we formulate the problem of designing the optimal tranches that maximize the value of the asset pool as follows:

$$V^T(J) = \max \sum_k m_k^*(Y_k - l_k) + \sum_n p_n \beta_n \quad (1)$$

such that

$$Y_k + \sum_n \beta_n S_n(\omega_k) \leq \sum_{j \in J} w_j X_j(\omega_k) \quad \text{for all } k \quad (2)$$

$$\sum_n \beta_n S_n(\omega_k) + l_k \geq 0, \quad \text{for all } k \quad (3)$$

$$Y_k, l_k \geq 0, \quad \beta_n \text{ unrestricted for all } k, n. \quad (4)$$

Here, β_n is the weight of primary security n in the marketable tranche, l_k equals the amount of negative cash flow from the marketable tranche in state k , and $Y_k - l_k$ is the cash flow of the non-marketable tranche in state k . The objective is to maximize the combined value of the tranches. The objective function removes the cash flow, l_k , from the intermediary's profits to prevent the intermediary from exploiting any arbitrage opportunities available in the market by tranching primary securities. Constraints (2)-(3) specify that the tranches should be fully backed *only* by the asset pool. In constraint (2), we state that the sum of cash flows of the tranches must be less than the cash flow of the asset pool in each state k . In constraint (3), we preclude the possibility that the intermediary may short primary securities and use the proceeds to create a new non-marketable tranche. This formulation captures the constraint placed on SPV's that any security issued by an SPV should be backed by the asset pool and not from any market operation.⁸ Finally, the non-negativity constraints on Y_k in (4) specify that short sales of secondary securities are not allowed, i.e., the non-marketable tranche should only have positive components. This is justified by recalling that consumption should be non-negative in all states.

The optimal tranching results are based on the dual of this problem. Therefore, we formulate

⁸A less stringent constraint, allowing for partial use of the proceeds of the short sales of primary securities to augment the pool, would expand the feasible set. However, this would only introduce a somewhat different shadow price, but would be qualitatively similar to the rest of the analysis presented here.

the dual problem as below:

$$D^T(J) = \min \sum_k \lambda_k \sum_{j \in J} w_j X_j(\omega_k) \quad (5)$$

such that

$$\lambda_k \geq m_k^* \text{ for all } k \quad (6)$$

$$\delta_k \leq m_k^* \text{ for all } k \quad (7)$$

$$\sum_k (\lambda_k - \delta_k) S_n(\omega_k) = p_n \text{ for all } n \quad (8)$$

$$\lambda_k, \delta_k \geq 0, \text{ for all } k. \quad (9)$$

Here, λ_k and δ_k are the dual variables corresponding to constraints (2) and (3), respectively, of the primal problem. The dual program's objective function states that λ_k are the state prices that determine the optimal value of the asset pool realized by tranching. Constraint (8) implies that $(\lambda_1 - \delta_1, \dots, \lambda_K - \delta_K) \in \Theta$ because this vector is non-negative and prices all primary securities correctly. Thus, δ_k measure the distance of the state prices obtained by allowing tranching from the set Θ . Let S_a be the set of states in which $\delta_k > 0$ in the optimal dual solution.

The following lemma formally states that the optimal solution of the dual problem lies in a bounded region, and therefore, by implication, the primal problem does not lead to infinite arbitrage. The lemma shows that we preclude the intermediary from issuing new secondary securities by short selling primary securities, and thus, taking advantage of arbitrage in an obvious way. For the purposes of this lemma, let S_{DT} be the set of feasible solutions to the dual program, and \mathcal{B} be a bounded polyhedral convex set defined as $\prod_k [0, \max(1, \max_k m_k^*)] \times [0, \max(1, \max_k m_k^*)]$.

Lemma 2. *The optimal solution to the dual problem is obtained by evaluating the value of the asset pool at each extreme point of $\mathcal{B} \cap S_{DT}$ and taking the minimum value as the solution.*

From this lemma, the primal problem $V^T(J)$ has a finite optimal solution. Therefore, the tranching solution exploits only those arbitrage opportunities in the securities market that are available to the intermediary due to the access to the asset pool and the subset of investors I_1 . The following theorem defines such opportunities and shows that they are completely characterized by the set S_a .

Theorem 4. *(i) If there exists a non-negative contingent claim Z such that $\sum_{k'} m_{k'}^* Z(\omega'_k) > V^+(Z)$, then there is no feasible solution to the dual in Θ . In particular, S_a is not empty.*

(ii) If there exists a non-negative contingent claim Z such that $\sum_{k'} m_{k'}^* Z(\omega'_k) > V^+(Z)$, then Z is strictly positive in some state $k \in S_a$.

(iii) If $S_a \neq \emptyset$, then there exists a non-negative contingent claim Z that is strictly positive in some state(s) $k \in S_a$ and zero elsewhere, such that $\sum_{k'} m_{k'}^* Z(\omega'_k) > V^+(Z)$.

(iv) If there exists a non-negative contingent claim Z such that $\sum_{k'} m_{k'}^* Z(\omega'_k) > V^+(Z)$, then there does not exist any $q \in \Theta$ such that $q_k \geq m_k^*$ for all k .

Theorem 4(i)-(ii) show that contingent claims that present arbitrage with the given subset of investors must have positive cash flows in one or more states in the set S_a . The intermediary can short primary securities to create a contingent claim that pays off in these states and sell the tranches to the subset of investors to realize an immediate profit. This is the consequence of the value to the investors exceeding $V^+(Z)$. Note that part (ii) also implies that if a claim does not have positive cash flows in any of the states in S_a then the upper bound on the price of the claim exceeds the value to the subset of investors. Theorem 4(iii) strengthens the role of the set S_a . It states that if S_a is non-empty then there is a non-negative contingent claim with cash flows in this set of states only whose value to investors exceeds $V^+(Z)$. The last part of Theorem 4 is the dual characterization which is mathematically the most useful of the three. Using this result, we can now state the general structure of the secondary securities.

Let Y_k^*, l_k^*, β_n^* denote the optimal solution to the primal problem and λ_k^*, δ_k^* denote the optimal solution to the dual problem. We partition the optimal tranching solution into three parts that we denote as T^a, T^I and T^m . Let $T_k^a = Y_k^* - l_k^*$ if $\delta_k^* > 0$ and zero otherwise, let $T_k^I = Y_k^* - l_k^* - T_k^a$, and let $T_k^m = \sum_n \beta_n S_n(\omega_k)$. Here, T^m is the marketable tranche, T^a consists of the cash flows of the non-marketable tranche in states belonging to the set S^a , and T^I consists of the cash flows of the non-marketable tranche in the remaining states. We partition the non-marketable tranche in this manner because by the complementary slackness condition applied to δ_k^* , $\delta_k^* > 0$ implies that $l_k^* + \sum_n \beta_n^* S_n(\omega_k) = 0$, which further implies that $Y_k^* - l_k^* = \sum_j w_j X_j(\omega_k)$, i.e., all the cash flows in state k are sold as secondary securities. Thus, according to Theorem 4, T^a fully exploits the arbitrage opportunities in the securities market due to the ability to design and sell secondary securities to a subset of investors, while T^m and T^I are not based on the existence of arbitrage. T^a is zero if there are no arbitrage opportunities available to the intermediary.

Note that the complementary slackness conditions also imply that the intermediary tranches all of the cash flows in the asset pool in the states belonging to the set S^a as T^a . Indeed, we have

$T_a \cdot T_m = 0$ and $T_a \cdot T_I = 0$. Thus, the optimal solution to the primal problem $V^T(J)$ is separable into one that corresponds to the tranches T_a and another that corresponds to the rest. The value of T^a is independent of changes in the cash flows of the asset pool in states $S \setminus S^a$, and likewise, the values of T^m and T^I are independent of the cash flows in states S^a . To see this, let S^a denote the subset of states in which $\delta_k^* = 0$. Define $\widehat{X}(\omega_k) = \sum_j w_j X_j(\omega_k) - T_k^a$ as the asset pool after tranching T^a . Set $\widehat{m}_k^* = 0$ for the states where $\delta_k^* > 0$, and $\widehat{m}_k^* = m_k^*$ otherwise. Let \widehat{D}^T denote the new dual problem. Clearly, \widehat{D}^T has a feasible solution in Θ . Due to the fact that T^a is orthogonal to T^m and T^I , the optimal solution to D^T is given by T^m and T^I . Thus, the value of T^m and T^I is independent of the value of T^a . Therefore, the asset pool decomposes into an ‘arbitrage part’, a marketable part and a residual part. In securitization industry terminology, the first is often referred to as a bespoke tranche, while the last is referred to as an equity tranche.

We can now specify the complete structure of the optimal tranching solution for a given asset pool as stated in the theorem below. This theorem uses the results of Theorem 4 to show the conditions under which the different tranches come about.

Theorem 5. *The optimal solution to the tranching problem is represented by (T^a, T^m, T^I) as defined above. Further,*

- (i) *If there exists $q \in \Theta$ such that $q_k \geq m_k^*$ for all k , then $T^a \equiv 0$.*
- (ii) *If there exists $q \in \Theta$ such that $q_k \leq m_k^*$ for all k and $q_k < m_k^*$ for some k , then there exists an optimal tranching solution in which $T^m = T^I \equiv 0$.*
- (iii) *Otherwise all three types of tranches may occur in the optimal solution.*

We note from Theorem 5 that the differences among the three types of solutions to the tranching problem do not depend on the cash flows in the asset pool, but only on the set Θ and the state prices m_k^* . Thus, an intermediary can verify the results in Theorems 4-5 without knowing the cash flows in the asset pool or the willingness of individual firms to participate in the asset pool. Further, the tranches in T^m might be bought by a different set of investors than I_1 , which is the set that buys tranches T^a and T^I .

Theorem 5 also shows the incremental value realized by tranching the given asset pool $\sum_j w_j X_j$. In case (i), $\lambda^* \in \Theta$, and thus, the optimal solution to the dual problem lies inside the price bounds $V^-(\sum_j w_j X_j)$ and $V^+(\sum_j w_j X_j)$. By the constraints of the dual problem, this solution is obtained in the set $\Theta \cap \{(\lambda_1, \dots, \lambda_K) : \lambda_k \geq m_k^* \text{ for all } k\}$. Since this is a subset of Θ , pooling and tranching provides incremental value beyond $V^-(\sum_j w_j X_j)$. In case (ii), the optimal solution is given by

$E_{m^*}[T^a]$, which is greater than $V^+(T^a)$. In case (iii), the value of tranches T^m and T^I is as in case (i) and the value of tranche T^a is as in case (ii). Due to the orthogonality of T^a with T^m and T^I , the total value is equal to the sum of these two components. Thus, the value from pooling and tranching is higher than $V^-(\sum_j w_j X_j)$.

Thus far, we have provided all results in this section for a given asset pool. We now relax this assumption and examine the implications of tranching on the formation of the asset pool. In case (i), all results of §4 apply if attention is restricted to the smaller set of pricing measures $\Theta^T = \{q : q_k \geq m_k^*, q \in \Theta\}$. Thus, using the inferences in §4, the asset pool may consist of cash flows from the individual firms in fractions or in full. Further, the mix of projects that get financed may change compared to the solution in §4, however, the total value of the projects financed will always increase. In case (ii), the optimal solution is linear in the cash flows $X(\omega_k)$. Thus, the solution degenerates into a pure tranching solution and there is value from tranching, but there may not be value from pooling. The decision for each firm to participate in the pool is made separately based on whether $E_{m^*}[X_j] \geq r_j$ or not. Thus, each firm either participates in the pool in full or not at all. The mix of projects financed may again change compared to §4, however, there will be no fractional pooling in this case. In case (iii), the pooling and tranching solution lies outside the set Θ . The remaining implications in this case are the same as in case (i).

6 Numerical Example

Consider a market with four states at time 1 denoted $\Omega = \{\omega_1, \dots, \omega_4\}$ and two primary securities with payoffs $S_1 = (1, 1, 1, 1)$ and $S_2 = (1, 0, 0.5, 1.5)$ at time 1 and prices $p_1 = p_2 = 1$ at time 0. The set of risk neutral pricing measures over Ω is $\Theta = \{(x + 3y, x - y, 0.5 - 2x, 0.5 - 2y)\} \cap [0, 1]^4$ with two degrees of freedom denoted x and y . The set Θ is spanned by three linear pricing measures, $Q_1 = (0, 1/3, 0, 2/3)$, $Q_2 = (1, 0, 0, 0)$ and $Q_3 = (0, 0, 1/2, 1/2)$. Q_1 corresponds to $x = 1/4, y = -1/12$, Q_2 corresponds to $x = 1/4, y = 1/4$ and Q_3 corresponds to $x = 0, y = 0$.

Consider two investor types in this market with identical preferences given by

$$U_i(c_1, c_2) = c_0 - e^{-5c_1}, \quad \text{for } i = 1, 2,$$

where c_0 denotes consumption at time 0 and c_1 denotes consumption at time 1. The investor types differ in their endowments in different states, being given as $e_1 = (10, 0.4, 1.6, 1, 0)$ and $e_2 = (10, 0.5, 0, 0, 0)$. Both investor types have the same subjective probabilities for the four states

Table 1: Equilibrium investments and state prices prior to introduction of secondary securities

Investor type	Equilibrium demand	Consumptions		State prices	Expected Utility
	for equity (S_1, S_2)	c_0	c_1	m_{ik}	
1	(0, 0.1289)	9.8711	(0.5289, 1.6, 1.0645, 0.1934)	(0.1421, 0.0002, 0.0049, 0.5703)	9.7276
2	(0.2304, 0)	9.7696	(0.7304, 0.2304, 0.2304, 0.2304)	(0.0519, 0.1580, 0.3160, 0.4741)	9.5696

given as $P = (0.4, 0.1, 0.2, 0.3)$. Each investor solves the decision problem specified in §3.2 in order to maximize total expected utility. Table 2 shows the equilibrium investments and state prices of the investor types.

Suppose that there exist three firms willing to invest in assets $X_1 = (1, 1, 0, 0)$, $X_2 = (0, 0, 1, 0)$ and $X_3 = (0, 0, 0, 1)$. The reservation prices of the firms are denoted r_1, r_2 and r_3 , respectively, and will be specified later. X_1, X_2 and X_3 are not spanned by Θ and thus do not have unique prices in this market. The price bounds on X_1 are $[V^-(X_1), V^+(X_1)] = [0, 1]$, and on X_2 and X_3 are $[0, 1/2]$ and $[0, 2/3]$, respectively. Note that this is a simple example since $X_1 + X_2 + X_3$ gives us the risk-free bond. Even so, the example suffices to illustrate values of reservation prices that yield different pooled assets.

Conditions to determine the value of pooling. Clearly, not all values of r_1, r_2 and r_3 will lead to value creation. Theorem 1(i) tells us that there is no value in pooling X_1, X_2 , and X_3 in any proportion if there exists a pricing measure $q \in \Theta$, i.e., q is a convex combination, say (a, b, c) , of Q_1, Q_2 , and Q_3 , such that the following inequalities are satisfied:

$$\begin{aligned} \frac{1}{3}a + b - r_1 &\leq 0, \\ \frac{1}{2}c - r_2 &\leq 0, \\ \frac{2}{3}a + \frac{1}{2}c - r_3 &\leq 0 \\ a + b + c &= 1 \end{aligned}$$

Otherwise there can be value from pooling. For example, if

$$r_1 = 0.26, \quad r_2 = 0.35, \quad r_3 = 0.4 \tag{10}$$

then there is a feasible solution ($a = 0.09$, $b = 0.23$, $c = 0.68$) to the above inequalities, implying that there is no value from pooling. As another example, if

$$r_1 = 0.25, \quad r_2 = 0.5, \quad r_3 = 0.25 \tag{11}$$

then there is no feasible solution to the above inequalities, implying that there can be value from pooling.

Conditions for full pooling and fractional pooling. In the case when there is value from pooling, the pool may be a grand coalition of all three assets or may be fractional. We apply Theorem 3 to determine the values of reservation prices under which the grand coalition of all three assets is sustainable. Since the grand coalition is the riskless bond, all $q \in \Theta$ achieve $V^-(X_1 + X_2 + X_3)$. Thus, by Theorem 3, we need to find a $q \in \Theta$ such that $E_q[X_j] \geq r_j$ for all $j = 1, 2, 3$. Solving these simultaneous inequalities, as above in the application of Theorem 1, we obtain conditions on r_1, r_2 , and r_3 under which the grand coalition is sustainable. These conditions could also be obtained by reversing all the inequalities derived above.

For example, consider the values of r_2 and r_3 as given in (10) and set $r_1 = 0.25 - \delta$ for some $\delta \geq 0$. Then, it can be easily shown that there is value from pooling and the grand coalition is sustainable. On the other hand, when reservation prices are as in (11), then the grand coalition is not sustainable. To see this, all three constraints have to hold as equalities (as the pool is one unit of the riskless bond). The last two constraints yield: $a = -3/8$, which is infeasible. Recall that Theorem 1 allows us to tell that there is value from pooling in this case. To find the maximal asset pool given these reservation prices, we solve the LP given in Corollary 2:

$$\max z - 0.25\alpha_1 - 0.5\alpha_2 - 0.25\alpha_3$$

such that

$$\begin{aligned} 1/3\alpha_1 + 2/3\alpha_3 &\geq z, \\ \alpha_1 &\geq z, \\ 1/2\alpha_2 + 1/2\alpha_3 &\geq z, \\ \alpha_j &\in [0, 1] \quad \text{for all } j. \end{aligned}$$

Here, α_j is the fraction of asset j included in the pool. The optimal solution is $\alpha_1 = \frac{1}{2}$ and $\alpha_3 = 1$. The LP has an optimal value of $\frac{1}{8}$. The asset pool is given by $(\frac{1}{2}, \frac{1}{2}, 0, 0) + (0, 0, 0, 1) = (\frac{1}{2}, \frac{1}{2}, 0, 1)$.

The expected value under the extreme measures are $\frac{5}{6}, \frac{1}{2}$ and $\frac{1}{2}$. The convex combination $\frac{1}{4}$ of Q_2 and $\frac{3}{4}$ of Q_3 , yields the measure $q = (\frac{1}{4}, \frac{3}{8}, \frac{3}{8})$, under which $E_q[\frac{1}{2}X_1] = \frac{1}{4}$ and $E_q[X_3] = \frac{3}{8}$, both of which are greater than or equal to the corresponding reservation prices (Theorem 3). It can be easily seen that this fractional pool is sustainable, and that there is no value in pooling the remaining cash flows given by $0.5X_1 + X_2 = (0.5, 0.5, 1, 0)$ (the inequalities to be satisfied are $b/2 + a/6 \leq 1/8$ (corresponding to $E_q[X_1/2] \leq r_1/2$) and $c/2 \leq 1$ (corresponding to $E_q[X_2] \leq r_2$) which is trivially satisfied by $(0, 0, 1)$.

Linear payment scheme for the pooling solution. In both cases, when full pooling and fractional pooling add value, a pricing measure that satisfies the conditions in Corollary 1 gives a linear payment scheme for subdividing the value of the pool between the participating firms. This payment scheme ensures that the coalition of firms cannot be broken because none of the firms in the set J can do better by forming an alternative coalition.

For example, in the case of full pooling with $r_1 = 0.2$, $r_2 = 0.35$, $r_3 = 0.4$, there are infinitely many admissible linear payment schemes. These payment schemes are given by convex combinations of the following three state-price vectors: $(0.15, 0.05, 0.35, 0.45)$, $(0.225, 0.025, 0.35, 0.4)$, $(0.2, 0, 0.4, 0.4)$. All payment schemes provide payments to firms that are at least as large as their reservation prices. Further, they distribute the surplus $V^-(X_1 + X_2 + X_3) - \sum_j r_j$ among the firms.

In the case of fractional pooling with $r_1 = 0.25$, $r_2 = 0.5$, $r_3 = 0.25$, we showed that the pricing measure that yields the payment scheme is $q = (\frac{1}{4}, \frac{3}{8}, \frac{3}{8})$.

Optimal pooling and tranching. We now illustrate the results of §5. Table 2 shows the optimal tranching solution obtained for different asset pools. The first three rows of the table show the solutions when only X_1 and X_2 are pooled together. Case A corresponds to when the intermediary has access to investors of type 1 only, case B to investors of type 2 only, and case C to investors of both types. The remaining three rows of the table, D-F, show the solutions when all three assets, X_1, X_2 and X_3 are pooled together.

We first highlight the structure of the tranching solution. In cases A and D, the tranche T^a is zero because the intermediary is unable to exploit any arbitrage in the securities market when it has access to investors of type 1 only. In cases B and E, T^a is non-zero because the intermediary can exploit arbitrage in states ω_2 and ω_3 when it has access to investors of type 2 only. Applying Theorem 4 to these cases, the contingent claim Z that achieves a value higher than $V^+(Z)$ under m^*

Table 2: Optimal tranching solution in different cases

Case	Asset Pool	Investor classes	Cash flows in the Asset pool	Tranches			Optimal value
				T^a	T^m (β_1, β_2)	T^I	
A	X_1, X_2	1	(1, 1, 1, 0)	(0, 0, 0, 0)	(1, $-2/3$)	($2/3, 0, 1/3, 0$)	0.4297
B	X_1, X_2	2	(1, 1, 1, 0)	(0, 1, 1, 0)	(0, 0)	(1, 0, 0, 0)	0.5259
C	X_1, X_2	1,2	(1, 1, 1, 0)	(1, 1, 1, 0)	(0, 0)	(0, 0, 0, 0)	0.6161
D	X_1, X_2, X_3	1	(1, 1, 1, 1)	(0, 0, 0, 0)	(1, 0)	(0, 0, 0, 0)	1
E	X_1, X_2, X_3	2	(1, 1, 1, 1)	(0, 1, $2/3, 0$)	(0, $2/3$)	($1/3, 0, 0, 0$)	1.0526
F	X_1, X_2, X_3	1,2	(1, 1, 1, 1)	(1, 1, 1, 1)	(0, 0)	(0, 0, 0, 0)	1.1864

Note: For the non-marketable tranches, T^a and T^I , the table shows cash flows in each of the four states. For the marketable tranche, T^m , the table shows the weights, β_1 and β_2 , of the two primary securities, S_1 and S_2 .

is given by $(0, 1, 2/3, 0)$. In cases C and F, T^a includes all the cash flows in the asset pool because the claim $(1, 1, 1, 1)$ now achieves a value higher than 1 under m^* . Thus, state-wise partitioning of the asset pool is optimal when the intermediary has access to both investor types. Notice that the tranching solution corresponds to Theorem 5(i) in cases A and D, Theorem 5(ii) in cases C and F, and Theorem 5(iii) in cases B and E.

7 Discussion and Conclusion

In this paper, we have studied the effect of securitization on real investment decisions in an incomplete market. We use the fact that incompleteness causes the market to place a premium on assets or asset combinations that augment the set of traded claims. Thus, we provide a rationale for securitization, independent of transaction costs or information asymmetry. Our results show that there is a benefit from pooling if and only if under every market pricing measure, the reservation price of at least one firm is not larger than the value of its assets. Moreover, there is a maximal asset pool that is sustainable in the cooperative game between firms. This pool consists of all or fractions of the cash flows of participating firms. The value creation by pooling can be augmented by tranching the pooled asset and issuing new securities to investors. We have shown that the optimal tranching solution consists of three types of tranches, a non-marketable tranche that exploits arbitrage opportunities in the market, a marketable tranche and a non-marketable tranche that does not exploit arbitrage opportunities in the market.

The predictions of our model are broadly consistent with observations from the securitization market. We observe three types of tranches in CDO securitization structures. The “super-senior”

tranches generate cash flows that are very similar to those of our marketable tranche T^m in that they mimic the cash flows of other AAA securities traded in the market. The mezzanine tranches often have unique payoffs in relation to other securities in the market and can be customized like the T^a non-marketable tranches in our solution. The third type of tranche is the residual or equity tranche, similar to our T^I tranche, that is sold to investors such as hedge funds.

Our paper can be extended in subsequent research in several ways. First, while the results in this paper are obtained under the strict definition of arbitrage, our analysis could be combined with price bounds derived under approximate arbitrage as in the recent literature (see Bernardo and Ledoit 2000, Cochrane and Saa-Requejo 2000). Under approximate arbitrage, market incompleteness should still continue to provide a rationale for seeking value enhancement through pooling and tranching. However, the imposition of a constraint that precludes “approximate arbitrage”, instead of arbitrage, would restrict the set of feasible solutions to the optimization problems considered in this paper. Additional analysis is required to determine the optimal pooling and tranching strategies when subjected to the tighter constraints.

Second, in the analysis in this paper, we have not dealt with the problems of information asymmetry. For instance, in our model, the tranches constructed by the intermediary need to be verifiable for our results to hold. If investors can verify whether the claim has positive payoffs in a state and if investors value consumption equally in every state, then the resulting partition of claims will resemble the tranches offered in the CLO market. Of course, given the issues relating to verifiability of the states, intermediaries need to handle the associated agency problems and be more innovative in creating tranches – for example, those that pay when the economy is doing well and those when it is doing poorly. We defer these issues to subsequent research.

Finally, an interesting aspect of securitization is when the pool has to be created and managed dynamically. The problem of determining when and how much of each asset to include, remove or add is a problem faced by venture fund managers. In the dynamic case, the major differences are that firms within the pool might not have the option to leave the pool, while firms that enter later might enjoy greater bargaining power. Firms and the intermediary might have only an imperfect forecast about which assets will become available in the future.

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Appendix

Proof of Lemma 1. We prove part (i). The proof for part (ii) is similar. Consider the linear program:

$$\min z$$

subject to

$$z \geq \sum_{k=1}^K q_l(\omega_k) Z(\omega_k) \quad l = 1, \dots, L$$

z unsigned.

If $z \geq \sum_k q_l(\omega_k) Z(\omega_k)$ for all l , then $\sum_l \delta_l z \geq \sum_l \sum_k \delta_l q_l(\omega_k) Z(\omega_k)$ for all $\delta_l \geq 0, \sum_l \delta_l = 1$. Thus, $z \geq \sup_{q \in \Theta} E_q[Z(\omega_k)]$. Therefore, the optimal solution to the linear program must be greater than or equal to $V^+(Z)$. On the other hand, $z = \max_{l \in L} E_{q_l}[Z]$ is a feasible solution to the linear program. But $\max_{l \in L} E_{q_l}[Z] \leq \sup_{q \in \Theta} E_q[Z(\omega_k)]$. Thus, $V^+(Z) = \max_{l \in L} E_{q_l}[Z]$.

For the proof of part (iii), consider the problem of maximizing the minimum marketable value of a claim Z that pays $Z(\omega_k)$ in state k .

$$\max \sum_n \alpha_n p_n$$

subject to

$$\sum_n \alpha_n S_n(\omega_k) \leq Z(\omega_k) \quad k = 1, \dots, K$$

α_n unsigned $n = 1, \dots, N$.

The objective is to maximize the market value of a portfolio of primary securities that pays less cash flows than claim Z in every state k . The dual of this problem is given by:

$$\min \sum_k \lambda_k Z(\omega_k)$$

subject to

$$\sum_k \lambda_k S_n(\omega_k) = p_n \quad n = 1, \dots, N$$

$\lambda_k \geq 0 \quad k = 1, \dots, K$.

We require that $\sum_k \lambda_k^* S_n(\omega_k) = p_n$ for every security n in the optimal dual solution. But this condition, by definition, implies that the optimal dual solution is a pricing measure that belongs to the set Θ . The proof follows by applying part (i) of the lemma. The validity of the upper bound can be proven similarly. \square

Proof of Theorem 1. To prove (i) of the theorem, we show the equivalent statement that if value can be created by pooling, then there does not exist any $q \in \Theta$ such that $E_q[X_j] \leq r_j$ for all j . Consider the linear program:

$$\max v - \sum_j \alpha_j r_j$$

subject to

$$\begin{aligned} -\sum_k q_l(\omega_k) \sum_j \alpha_j X_j(\omega_k) + v &\leq 0, & l = 1, \dots, L \\ \alpha_j &\geq 0, & j = 1, \dots, J. \end{aligned}$$

Here, the vector (α_j) denotes the proportion in which the assets (X_j) are pooled together, and v denotes the value of the asset pool in the securities market. The value of the asset pool is defined as $V^-(\sum_j \alpha_j X_j)$ because this is the minimum price that the asset pool commands in the securities market. To compute the value of the asset pool, we have used Lemma 1, i.e., the expected value under each extreme pricing measure, q_l , should be greater than or equal to the value v . The first L constraints correspond to these requirements. The linear program seeks to obtain the combination of assets that will maximize the difference between its value v and the combination of reservation prices required to create the asset pool, $\sum_j \alpha_j r_j$.

If value can be created by pooling, then there exist weights α_j such that the linear program is feasible and

$$v - \sum_j \alpha_j r_j > 0. \tag{12}$$

Let $\theta_l \geq 0$ be any set of weights such that $\sum_l \theta_l = 1$. Multiply each of the L constraints with the corresponding weight θ_l and add. Because the linear program is feasible, we get

$$-\sum_k \sum_l \theta_l q_l(\omega_k) \left(\sum_j \alpha_j X_j(\omega_k) \right) + v \leq 0. \tag{13}$$

Here, $\sum_l \theta_l q_l$ is a pricing measure in Θ , which we denote by q . Hence, (13) can be rewritten as

$$-E_q \left[\sum_j \alpha_j X_j \right] + v \leq 0. \tag{14}$$

Combining (12) and (14), we get $E_q \left[\sum_j \alpha_j X_j \right] > \sum_j \alpha_j r_j$. Equivalently, there exists j such that $E_q[X_j] > r_j$.

Since θ_l are arbitrary and there is a one-to-one mapping between the sets $\{(\theta_l) : \theta_l \geq 0, \sum_l \theta_l = 1\}$ and Θ , we conclude that, if value can be created by pooling, then there does not exist any $q \in \Theta$ such that $E_q[X_j] \leq r_j$ for all j .

To prove (ii) of the theorem, consider the dual of the above linear program. The dual variables μ_l will be associated with each of the constraints related to the expected value under extreme pricing measure q_l . The dual problem is:

$$\min 0$$

subject to

$$\begin{aligned} \sum_l \mu_l &= 1 \\ \sum_k \sum_l \mu_l q_l(\omega_k) X_j(\omega_k) &\leq r_j, \quad j = 1, \dots, J \\ \mu_l &\geq 0. \end{aligned}$$

We wish to show that if no value can be created by pooling, then there exists $q \in \Theta$ such that $E_q[X_j] \leq r_j$ for all j . Notice that by choosing all $\alpha_j = 0$, the primal problem is always feasible and has a lower bound of zero. The only question is whether the primal has a bounded solution – which by strong duality theorem can only be zero from the dual program’s objective function – or an unbounded solution. The former situation is the one where pooling does not create value (and the dual is feasible), and the latter situation is the one where pooling leads to value creation (and the dual is infeasible). Thus, if no value can be created by pooling, then the primal has a bounded solution and the dual is feasible. From the dual constraints, we observe that dual feasibility implies that there exist weights μ_l such that under the pricing measure $\sum_l \mu_l q_l$, we have $E[X_j] \leq r_j$ for all j . This proves the converse. \square

Proof of Theorem 2. We first show the proof of this theorem for partitions where $w_j = 0$ or 1 and then extend it to the case of fractional w_j . Since we restrict w_j to be 0 or 1, we denote the cash flows for any subset J_w of J simply as $\sum_{j \in J_w} X_j$ and the corresponding reservation prices as $\sum_{j \in J_w} r_j$.

The proof of part (i) of the theorem follows from the work of Owen (1975) and Samet and Zemel (1984). We sketch the proof for completeness. Consider the problem of maximizing the value of the portfolio formed from the assets of coalition J_w by selling tranches of primary securities against

it. The maximum value is given by solving the linear program:

$$\max \sum_n \beta_n p_n$$

subject to

$$\begin{aligned} \sum_n \beta_n S_n(\omega_k) &\leq \sum_{j \in J_w} X_j(\omega_k), & k = 1, \dots, K \\ \beta_n &\geq 0, & i = 1, \dots, N. \end{aligned}$$

The dual to this problem is

$$\min \sum_k \lambda_k \sum_{j \in J_w} X_j(\omega_k)$$

subject to

$$\begin{aligned} \sum_k \lambda_k S_n(\omega_k) &\geq p_n, & n = 1, \dots, N \\ \lambda_k &\geq 0, & k = 1, \dots, K. \end{aligned}$$

Notice that the constraints to the dual program do not depend on the coalition formed because the X_j 's enter only the objective function. Moreover, the dual is feasible because the market is arbitrage-free, that is, any $q \in \Theta$ will satisfy the dual constraints, i.e., $\sum_k q_k S_n(\omega_k) = p_n, \forall n, q \in \Theta$. Finally, as $X_j(\omega_k) \geq 0$ for all j , the solution to the dual program is finite, as it cannot drop below zero. Solve the problem for the grand coalition of all firms and obtain the optimal dual solution λ_k^* . As $X_j(\omega_k) \geq 0$ for all j , by applying the same reasoning as in Lemma 1(iii), we can also assume that these dual values constitute a pricing measure in Θ .

Consider the following solution to the cooperative game: Let firm j receive the payment $\sum_k \lambda_k^* X_j(\omega_k)$. This is surely greater than or equal to $V^-(X_j)$ and therefore, by assumption, larger than r_j . By definition, the coalition J_w receives the sum of the payments to the firms in the coalition. This sum equals or exceeds the maximum value obtained by solving the linear program for just the coalition because: (a) the λ_k^* 's constitute a dual feasible solution to the problem for all $J_w \subseteq J$ because, as noted earlier, the constraints of the dual problem do not depend on the coalition formed; and (b) all dual feasible solutions are greater than or equal to the primal optimal solution (by weak duality). This proves part (i).

For the proof of part (ii), the problem is to demonstrate the existence of a payment scheme that works for all coalitions simultaneously. Redefine the value of a coalition without loss of generality to be $V(J_w) = \max(V^-(J_w), \sum_{j \in J_w} r_j)$. We first show that if the condition stated in part (ii) applies

to partitions comprised of two subsets, then it also applies to any arbitrary partition. That is, if for every subset J_w of J , we have $V(J) \geq \max(V(J_w), \sum_{j \in J_w} r_j) + \max(V(J_w^c), \sum_{j \in J_w^c} r_j)$, then for every partition J_1, J_2, \dots, J_k of J , the same inequalities hold. (Note that the reverse statement can also be proven, implying that the two conditions are equivalent.) The proof is by contradiction. Assume that the condition does not hold for some partition, J_1, J_2, \dots, J_k . Thus, by assumption,

$$V(J) < \sum_i \max \left(V^-(J_i), \sum_{j \in J_i} r_j \right).$$

Without loss of generality, assume that for $i = 1, 2, \dots, h$, $\max(V^-(J_i), \sum_{j \in J_i} r_j) = V^-(J_i)$, and for $i = h + 1, h + 2, \dots, k$, $\max(V^-(J_i), \sum_{j \in J_i} r_j) = \sum_{j \in J_i} r_j$. Then, by super-additivity of the value function (which follows from the definition of V^-),

$$V^- \left(\bigcup_{i=1}^h J_i \right) \geq \sum_{i=1}^h V^-(J_i).$$

Let $J_w = \bigcup_{i=1}^h J_i$. By the condition given in part (ii) of Theorem 2, the definition of $V(\cdot)$, and the discussion above, we have

$$\begin{aligned} V(J) &\geq V(J_w) + V(J_w^c) \\ &\geq \sum_{i=1}^h V^-(J_i) + \sum_{j=h+1}^k \sum_{j \in J_i} r_j \\ &= \sum_i \max(V^-(J_i), \sum_{j \in J_i} r_j). \end{aligned}$$

This provides the necessary contradiction. The proof of part (ii) now appears to be immediate because, under every solution in the core, each coalition J_w gets at least $\max(V(J_w), \sum_{j \in J_w} r_j)$. Thus, the payment is sufficient to cover the reservation price. However, it must further be shown that this can be done *simultaneously* for every coalition and not just coalition by coalition.

Consider the primal problem:

$$\min 0$$

subject to

$$\begin{aligned} \sum_{j \in J_w} \pi_j &\geq V(J_w), & \text{for all } J_w \subseteq J, \\ \sum_j \pi_j &= V(J), \\ \pi_j &\geq 0, & \text{for all } j. \end{aligned}$$

This program if feasible determines the payment schedule for the firms, i.e., firm j receives a payment π_j . The dual problem is:

$$\max \sum_{J_w \subseteq J} \lambda_{J_w} V(J_w) + \lambda V(J)$$

subject to

$$\begin{aligned} \sum_{J_w: j \in J_w} \lambda_{J_w} + \lambda &\leq 0, & j = 1, \dots, J, \\ \lambda_{J_w} &\geq 0, & \lambda \text{ unsigned.} \end{aligned}$$

The dual variables λ_{J_w} correspond to the first set of constraints in the primal problem, and the dual variable λ corresponds to the second constraint. Obviously, the dual problem is always feasible (set all variables equal to zero). The dual solution will equal zero. Moreover, λ has to be less than or equal to zero. All we need to show is that zero is the maximum possible solution to the dual. If not, then the dual will be unbounded (by scaling all variables as large as desired), and therefore, the primal will be infeasible. We proceed to show that the solution to the dual problem is bounded.

Consider the constraint to the dual corresponding to $j = 1$. This constraint along with $\lambda \leq 0$ implies that:

$$\sum_{J_w: 1 \in J_w} \lambda_{J_w} V(J_w) + \lambda \max(V(J_w) : 1 \in J_w, J_w \subseteq J) \leq 0.$$

Similarly, the constraint corresponding to $j = 2$ yields

$$\sum_{J_w: 2 \in J_w \text{ and } 1 \in J_w^c} \lambda_{J_w} V(J_w) + \lambda \max(V(J_w) : 2 \in J_w \text{ and } 1 \in J_w^c, J_w \subseteq J) \leq 0.$$

We can write analogous inequalities for larger values of j . In general, we have

$$\sum_{J_w: j \in J_w \text{ and } \{1, \dots, j-1\} \subseteq J_w^c} \lambda_{J_w} V(J_w) + \lambda \max(V(J_w) : j \in J_w \text{ and } \{1, \dots, j-1\} \subseteq J_w^c, J_w \subseteq J) \leq 0.$$

The sets where the maximum is attained over $(J_w : j \in J_w \text{ and } \{1, \dots, j-1\} \subseteq J_w^c, J_w \subseteq J)$ are disjoint and their union is less than or equal to J . Adding up these inequalities gives

$$\sum_{J_w \subseteq J} \lambda_{J_w} V(J_w) + \lambda (\max(V(J_w) : 1 \in J_w, J_w \subseteq J) + \max(V(J_w) : 2 \in J_w \text{ and } 1 \in J_w^c, J_w \subseteq J) + \dots) \leq 0.$$

Recalling that $V(J)$ is greater than equal to the sum of the $V(J_i)$'s over any partition of J we obtain

$$\sum_{J_w \subseteq J} \lambda_{J_w} V(J_w) + \lambda V(J) \leq 0.$$

Therefore, the optimal value of the dual problem is bounded above by zero. This implies that the dual problem is feasible and bounded, and therefore, has an optimal solution. Therefore, by strong duality theorem, the primal has a feasible solution.

This proves the theorem for $w_j = 0$ or 1 for all j . The same proof applies for the case of fractional w_j when the number of subdivisions of each asset is finite. Thus, if each asset is broken into finite number of parts, treating each subdivided asset as a ‘undivided’ asset we get the result (we need to check over all partitions of J into two subsets – in these partitions we need to recombine the subdivisions of each asset that are in the same partition). Now, by taking limits, we get the result for countable sub-divisions of assets. Thus, using the continuity of the function $V(J_w)$ in the w_j ’s, the verification has to be done for every partition J_w and J_w^c ; where w_j ’s can take any value between 0 and 1. \square

Proof of Theorem 3. Consider a partition of the assets, such that one coalition pools $\sum_j w_j X_j$ with a reservation price of $\sum_j w_j r_j$ and the other $\sum_j (1 - w_j) X_j$ with a reservation price of $\sum_j (1 - w_j) r_j$. By assumption, $\sum_j E_{q_l} w_j X_j \geq \sum_j w_j r_j$, and by definition $\sum_j E_{q_l} w_j X_j \geq V^-(\sum_j w_j X_j)$. Thus, $\sum_j E_{q_l} w_j X_j \geq \max(\sum_j w_j r_j, V^-(\sum_j w_j X_j))$. Similarly, $\sum_j E_{q_l} (1 - w_j) X_j \geq \max(\sum_j (1 - w_j) r_j, V^-(\sum_j (1 - w_j) X_j))$. Adding these up we get, $V^-(\sum_j X_j) \geq \max(V^-(\sum_j w_j X_j), \sum_j w_j r_j) + \max(V^-(\sum_j (1 - w_j) X_j), \sum_j (1 - w_j) r_j)$. This proves the sufficiency part of the theorem.

For the necessity, suppose the condition does not hold but the solution with $w_j = 1$ is in the core. Assume that for the unique extreme pricing measure q_l that yields the minimum value of the grand coalition, there is some j , such that $E_{q_l} X_j < r_j$. Without loss of generality, assume that the measure is q_1 and this inequality holds for $j = 1$. Let $Z = \sum_{j \neq 1} X_j$. Consider the following linear program:

$$\begin{aligned} & \max v - w_1 r_1 \\ & \text{subject to} \\ & v - w_1 E_{q_1}(X_1) \leq E_{q_1}(Z) \quad \text{for all } l, \\ & w_1 \leq 1, \\ & v \text{ unsigned, } w_1 \geq 0. \end{aligned}$$

This LP seeks the optimal fraction ($w_1 \in [0, 1]$) of X_1 to add to the contingent claim Z if we can buy the claim for its reservation price r_1 and the objective is to maximize the value of the modified

claim. The q_l are the extreme risk-neutral pricing measures. The dual program is

$$\begin{aligned} & \min \sum_l \lambda_l E_{q_l}(Z) + \gamma \\ & \text{subject to} \\ & \sum_l \lambda_l = 1 \\ & - \sum_l \lambda_l E_{q_l}(X_1) + \gamma \geq -r_1 \\ & \lambda_l, \gamma \geq 0. \end{aligned}$$

In this formulation, the λ_l 's are the dual constraints corresponding to the first set of primal constraints and γ is the dual variable corresponding to the second primal constraint. Set $\lambda_1 = 1$. Clearly, $\gamma = 0$ satisfies the second constraint. However, the constraint has slack. By complementary slackness, the primal variable w_1 should be equal to zero in all optimal primal solutions.

Thus, we see that $V^-(Z) + r_1 > V^-(X_1 + Z)$. This violates the necessary condition for the solution to be in the core, i.e., Theorem 2(ii) (consider the partition X_1 and Z).

Note that if the primal solution is not unique, the proof still goes through by a simple calculation: Consider the solution to the dual that sets λ_l such that $\sum_l \lambda_l Q_l = q$, where q is the value minimizing measure for the pool of all assets. The dual objective function value is $E_q Z$ because $\gamma = 0$ by assumption ($E_q X_1 < r_1$). By the fact that the value of any feasible solution to the dual, this is greater than the value of any feasible primal solution, therefore, also $V^-(Z + X_1) - r_1$, yielding the same conclusion. \square

Proof of Corollary 1. The choice of q_p in part (i) follows from Lemma 1. Notice that when the lower bound, $V^-(\sum_j X_j)$, is achieved at several extreme points then a linear mixture of these measures also gives the same lower bound. The second part follows from part (ii) of Theorem 2. To see this, the optimal solution to the full coalition's problem is the highest value that can be obtained by pooling all assets, which must equal $E_{q_p}(\sum_j X_j)$. Moreover, any linear pricing measure that supports the core must be an optimal dual solution to the problem of determining $V^-(\sum_j X_j)$. Also, all optimal solutions to the dual problem are obtained as convex combinations of the optimal extreme points solutions. Thus, if one such pricing measure can be found that not only supports the core but also gives a value of each X_j larger than r_j , then all firms will willingly participate in creation of the pool. \square

Proof of Corollary 2. The value is maximized because this is the highest surplus that can be generated after meeting all the reservation prices. The set of projects financed is maximal because if another project could be added to the set with an increase to the objective function then the current solution is not optimal.

Finally, let q be the measure under which the pooled assets attain their minimal value. Then, if asset j is at a positive level in the pool then, $E_q X_j > r_j$ otherwise the dual constraint of the form $-E_q X_j + \gamma > -r_j$ will have slack, which will mean that the asset j is at zero level in the primal solution. Also, the set of assets $w_j X_j$ satisfy the conditions of Theorem 3.

Finally, to show that the remaining assets cannot be pooled to create value, observe that under the extreme pricing measure that minimizes the value of the pooled fractions of assets, the expected value of the unpooled fractions of each asset is below its reservation price. Thus, applying Theorem 1(i) we get the result. \square

Proof of Lemma 2. We first note that the dual problem $D^T(J)$ has a feasible optimal solution located in the bounded convex set given by the intersection of the set of feasible region of problem $D^T(J)$ and $\mathcal{B} = \prod_k [0, \max(1, \max_k m_k^*)] \times [0, \max(1, \max_k m_k^*)]$. The proof of this assertion follows from the fact that $q_k \leq 1$, so that we may bound the region in which we search for a feasible solution by a hypercube that contains the largest values of λ_k and δ_k .

The proof of Lemma 2 now follows from the facts that the optimal dual solution is bounded above by a feasible solution in \mathcal{B} , and that the minimum is attained at an extreme point of $\mathcal{B} \cap S_{D^T}$ (cf: Lemma 1). \square

Proof of Theorem 4. (i) Assume to the contrary that there exists a solution to the dual problem, $D^T(J)$, in Θ . This solution must be obtained by setting $\delta = 0$ and $\lambda = q$ for some $q \in \Theta$. Since the solution must satisfy all the dual constraints, multiplying constraints (6) by $Z(\omega_k)$ and adding, we get $\sum_k \lambda_k Z(\omega_k) \geq \sum_k m_k^* Z(\omega_k)$. However, this gives a contradiction because $\sum_k \lambda_k Z(\omega_k) = \sum_k q_k Z(\omega_k) < V^+(Z)$. Therefore, there is no feasible solution to the dual problem in Θ . Further, since $\lambda \notin \Theta$, we must have $\delta_k > 0$ in some state k , so that $S^a \neq \emptyset$.

(ii) Assume to the contrary that there exists a non-negative contingent claim Z such that $\sum_k m_k^* Z(\omega_k) > V^+(Z)$ but which is zero in all states in S_a . Let $(\lambda_k^*, \delta_k^*)$ denote the optimal solution of the dual problem. Because $Z(\omega_k) = 0$ for all $k \in S_a$, we can set $m_k^* = 0$ for these states. Since $\delta_k^* = 0$ for all k solves the dual problem with Z as the asset pool, we have a feasible $\lambda \in \Theta$.

Thus, applying the same step as in the proof of (i) above, we find that $\sum_k \lambda_k Z(\omega_k) \geq \sum_k m_k^* Z(\omega_k)$, which contradicts the assertion that $\sum_k m_k^* Z(\omega_k) > V^+(Z)$. This proves the result.

(iii) Consider a modified primal problem where we drop the l_k variables and the corresponding constraints for $k \in S_a$. Correspondingly, in the dual problem, we drop the variables δ_k and the constraints $\delta_k \leq m_k^*$ for $k \in S_a$. Note that the modified primal problem is always feasible, but the modified dual problem is infeasible. Thus, the primal problem must be unbounded. This implies that there is a marketable portfolio with cash flows that are used only in the states S_a that can be shorted to create non-marketable tranches to be sold to investors, and yields infinite profit.

(iv) As before, let Z' be the marketable security that gives the upper bound on Z . Arguing as before, we get $E_q[Z'] < E_{m^*}[Z']$ for all $q \in \Theta$. However, because $Z' \geq 0$, if there is a $q \geq m^*$, this leads to a contradiction. \square

Proof of Theorem 5. (i): The proof of this part follows from Theorem 4(iv). If there exists $q \in \Theta$ such that $q_k \geq m_k^*$ for all k , then the set S^a is empty. Thus, by definition of T^a , we obtain that T^a is zero in all states in the optimal solution.

(ii): If there exists $q \in \Theta$ such that $q_k \leq m_k^*$ for all k and $q_k < m_k^*$ for some k , then consider the solution to the dual problem, $D^T(J)$, obtained by setting $\lambda_k = m_k^*$ and $\delta_k \geq 0$ for all k . This solution is feasible and yields an objective function value of $\sum_k m_k^* \left(\sum_j w_j X_j(\omega_k) \right)$. Also consider the solution to the primal problem, $V^T(J)$, given by $Y_k = \sum_j w_j X_j(\omega_k)$ for all k , $l_k = 0$ for all k , and $\beta_n = 0$ for all n . This also gives an objective function value of $\sum_k m_k^* \left(\sum_j w_j X_j(\omega_k) \right)$. Since the primal objective function value is equal to the dual objective function value, these solutions are optimal. Therefore, there exists an optimal tranching solution in which $T^m = T^I = 0$.

(iii): This case is the complement of the possibilities covered in (i) and (ii). Hence, the proof follows. \square