Reputational Delegation*  
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Preliminary and Incomplete

Abstract

I study how a principal should delegate to an agent with career concerns. The agent and principal are assumed to have aligned intrinsic incentives. However, the market observes the chosen action, and rewards the agent more, the higher it perceives his (private) type to be. This “reputational bias” has many similarities to the classic “material bias” studied in the communication literature, for instance, both can induce the same cheap talk equilibrium sets. However, I show that it is always optimal to impose a floor on the set of available actions in reputational delegation. This is in stark contrast to delegation to an agent with a material bias, where it is never optimal to restrict the agent’s flexibility to take low actions. I further specialize to the exponential family of distributions to demonstrate that flexibility is not sub-optimal perse. I show that offering flexibility to high types (i.e. full separation) is optimal. This result uses a recursive approach novel to communication problems.

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1. Introduction

Delegation is useful when the individual with decision rights lacks relevant information possessed by another individual. For example, school administrators delegate grading to teachers who better know the true ability of each student, private equity investors cede investment decisions to the funds general partners, and law firm partners allow associates to decide how many billable hours to put into each case. The key tension is that the agent – the individual with more information – may be biased relative to the principal – the individual with decision rights. The main questions are how the principal should restrict the agent’s flexibility when delegating, and how these restrictions should depend on the agent’s biases.

A key case in the literature is when the agent is biased towards higher decisions relative to the principal, e.g. the teacher wants to give students higher grades than is warranted by their performance, a venture capitalist wants to invest more than necessary in each entrepreneur, and an associate wants to put excess hours into each case. There are many reasons that such biases may arise. For example, the teacher may like their students and want them to do well, the venture capitalist may face limited downside risk relative to the end investors, and the associate’s bonus may be tied to their hours worked. These agents have a material bias in that they intrinsically prefer higher actions. The large literature on delegation initiated by Holmstrom (1984) studies how a principal should delegate to an agent with a material bias.

Alternatively, the agents could have a reputational bias towards choosing higher actions. This derives from a positive relationship between when a higher action should be taken and when the agent has higher ability. For example, better teachers educate their students to perform better and thereby tend to give higher grades, better venture capitalists find more promising entrepreneurs who warrant greater investment, and more dedicated associates may put more work into each case. In each of these examples the agent has an incentive to choose a higher action in order to signal higher ability.

There is often a subtle distinction between when an agent’s incentives induce a reputational bias or a material bias. As a result many economic interactions that motivate material delegation studies could actually involve a reputationally biased agent. For example, bonus pay in organizations is a reason why workers may be biased towards working more hours. Law firm associates receive a bonus if they work more than some predetermined amount of hours within a year. Conversely, investment banking analysts receive a bonus that depends on their manager’s subjective view of their performance. In the law firm, bonuses induce a material bias towards working more hours, whereas in the investment
bank, bonuses induce a reputational bias towards more hours. How should a principal delegate differently when faced with a reputationally biased agent vs. one with a material bias?

My model is the same as the standard delegation model except for the agent’s preferences. It features a single principal who wants to match his chosen action to an agent’s private type, both of which are one-dimensional. The agent also wants to match the action to their type, that is, their material component is aligned with the principal. However, the market observes only the chosen action, and the agent also wants to give the market the impression that he is a higher type. The principal optimizes over incentive compatible allocations, i.e. mappings from agent types to actions.

The main takeaway from the material delegation literature is that the principal should “cap against the agent’s bias” (see Holmstrom (1984), Alonso and Matouschek (2008), Amador and Bagwell (2013)). This has two dimensions: (i) the agent is restricted from taking high actions, and (ii) the principal should never restrict the agent’s flexibility to take low actions.\(^1\) In contrast, the main result (Theorem 1) shows that in reputational delegation the principal always wants to impose a floor. That is, the principal pools some lower set of agent types by restricting them to take an inefficiently high action.

Theorem 1 leaves open whether a floor, i.e. pooling low types, is beneficial, or if it is simply optimal to discretely segment all types, as in cheap talk (Crawford and Sobel (1982)). Answering this question requires more detail about the optimal delegation set. Methods for solving the material delegation problem, e.g. those in Kleiner et al. (2021) and Amador and Bagwell (2013), do not work in the reputational delegation model. Instead, I specialize to the exponential family which facilitates solving the principal’s problem using recursive methods that are (to my knowledge) novel in communication models. Theorem 2 shows that the optimal allocation is eventually separating, i.e. above some type, the agent perfectly reveals his type with his chosen action. I also show that for exponential distributions close to the uniform limit, the optimal delegation set involves a floor and then full separation (flexibility) above some action.

The intuition for the optimality of a floor revolves around the following thought experiment. I consider an arbitrary allocation \(x\) with a threshold (a discontinuous jump in the action) at some agent type \(t\). I refer to the set of continuing allocations above \(t\) as those that would maintain incentive compatibility when juxtaposed with \(x\) below \(t\). This set depends on \(x\) only through the loss that type \(t\) experiences under \(x\). Correspondingly, I refer to

\(^1\)Even when interval delegation is not optimal in the material delegation model, it is always optimal to give the lowest type his ideal action and also make all lower actions available.
\( V(t, u) \) as the minimized continuation loss among the set of continuing allocations given that type \( t \) experiences loss \( u \) from the allocation below. For a low type \( \tilde{t} \), all allocations below \( \tilde{t} \) are very close to efficient and thereby generate similar loss for the principal. Thus, the optimal choice for the allocation below type \( \tilde{t} \) comes down to which one induces the lowest continuation loss above \( \tilde{t} \).

The key to answering this question is Proposition 1 which says that \( V(t, u) \) is increasing in \( u \). That is, increasing the agent’s loss at type \( t \) increases the principal’s minimized continuation loss above type \( t \). I show that an allocation that pools some set of types \([0, \tilde{t}]\), i.e. implements a floor, is better than allocations which distinguish types in this interval by showing that the former decreases the loss for \( \tilde{t} \). The reasons are twofold. First, it turns out that separating agent types causes distortion (distance between the chosen action and type) to accumulate very quickly for low types, where distortion is small. To see this, note that the action increases between separating types in order to deter the agent from seeking a higher reputation. When distortion is small, the action needs to increase by a lot in order to provide this deterrence, hence distortion accumulates quickly. Second, the lowest action, i.e. floor, is special in that it does not need to respect any lower types’ incentive constraints. This means that setting a non-trivial floor can bypass the first issue above.

Specializing to the exponential model in Section 4 makes the set of continuing allocations and the minimized continuation loss independent of the initial type \( t \). In addition, the continuation loss admits a recursive structure: the first pooling interval pins down both the loss for the principal on this first interval and the initial loss for the first threshold. Proposition 2 shows that a set of continuation losses that solve the associated Bellman equation must be optimal. I use this result to establish Theorem 2, i.e. that separating is eventually optimal. The intuition reverses that for the optimal floor: (i) for high types where distortion is already large, separating types increases loss very gradually relative to pooling, and (ii) unlike with the floor, the action for any other pooled set has to respect the incentive constraint for some lower type.

### 1.1. Related Literature

The delegation literature was initiated by Holmstrom (1984) which, assuming interval delegation, finds that it is optimal to cap against the agent’s bias. Alonso and Matouschek (2008) and Amador and Bagwell (2013) study generalizations of this standard model and reach interval delegation as a conclusion under palatable assumptions on the preferences and distribution of types. Kovac and Mylovanov (2009) shows that these features are robust to allowing for stochastic mechanisms. Dessein (2002) and Melumad and Shibano (1991) also study the original model and compare delegation to alternative communication
A number of papers study different delegation technologies without changing the material bias of the agent. Krishna and Morgan (2008) and Ambrus and Egorov (2017) study the delegation model with transfers or money burning. Armstrong and Vickers (2010) study an agent who has private information over which choices are available. Frankel (2014) studies a delegation model with many decisions and takes a worst case approach. Halac and Yared (2020) adds the possibility for the principal to verify the agent’s type at a cost.

My paper is also related to the large literature dealing with career concerned agents initiated by Holmström (1999), which studies a model where the outcome can be observed. My paper fits better into an alternative strand which assumes that only the agent’s choice can be observed. Examples include Morris (2001), Prendergast and Stole (1996), and Scharfstein and Stein (1990). Visser and Swank (2007), Moscarini (2007), Kartik and Van Weelden (2018) all integrate career concerns into cheap talk models. A closely motivated set of papers is Ottaviani and Sorensen (2006a) and Ottaviani and Sorensen (2006b) which attempt to compare predictions in cheap talk when the agent is motivated by career concerns vs. material concerns.

2. Model and Preliminary Results

2.1. Setup

Overview There is a principal, an agent, and an outside observer or market. The agent has private information about his type $t \in T \equiv [0, M]$ where $M \in [0, \infty)$. There is a common prior type distribution with density $f : T \rightarrow \mathbb{R}^+$ and associated measure $F : 2^T \rightarrow \mathbb{R}$. I assume the density is bounded above and below, i.e. $\exists \underline{k} \geq k > 0$ with $f(t) \in [k, \overline{k}] \ \forall t \in T$. The principal has control over an action $a \in A \equiv \mathbb{R}^+$ and delegates a set of available choices to the agent. The agent chooses an action from this delegation set. The market then observes the chosen action and updates their belief about the agent’s type.

Preferences I represent preferences in terms of losses instead of utilities as it proves more convenient throughout. The principal’s loss given action choice $a$ and type $t$ is given by $(a - t)^2$. The agent has two components to his preferences, a material component and a reputational component. The material component is the same as the principal’s. Given a belief $\mu \in \Delta T$ held by the market, an agent of type $t$ has reputational loss given by $\rho(t - \mathbb{E}[t'|t' \sim \mu])$, where $\rho > 0$ is a positive constant weight. The reputational loss is normalized (i.e. it does not affect equilibrium behavior) to be 0 if $\mu$ is degenerate on $t$. 


Given market belief \( \mu \), action \( a \), and type \( t \), the total loss of the agent is given by
\[
(a - t)^2 + \rho(t - \mathbb{E}[t'|t' \sim \mu]).
\]

**Allocations**  The principal specifies an allocation \( x : T \to A \). The market assumes the action is chosen via the allocation, so their belief over the type after observing action \( a \in x(T) \) is simply the prior conditioned on \( x^{-1}(a) \), i.e.
\[
\mu(t') \equiv \begin{cases} 
\frac{f(t')}{F(x^{-1}(a))} & t' \in x^{-1}(a) \\
0 & \text{otherwise}
\end{cases}
\]

I only specify the beliefs for on path actions, because the principal can restrict the delegation set \( x(T) \). For each \( a \in x(T) \), each allocation induces reputation \( r^a_x(a) \equiv \mathbb{E}[t'|t' \in x^{-1}(a)] \). I refer to the realized reputation as \( r^*_x(t) \equiv r_x(x(t)) \). Given allocation \( x \), the loss to the agent of type \( t \) from choosing action \( a \in x(T) \), is denoted \( L^A(a,t|x) \equiv (a - t)^2 + \rho(t - r^*_x(a)) \). I refer to the realized loss for type \( t \) as \( L^A(t|x) \equiv L^A(x(t),t|x) \). The expected loss for the principal is denoted \( L^P(x) \equiv \int_0^M (x(t') - t')^2 f(t')dt' \). Note that because the agent’s preferences over market beliefs are linear, the expected reputational loss is 0 for any allocation \( x \) and \( L^P(x) = \mathbb{E}[L^A(t|x)] \).

**Incentive Compatibility**  The principal is restricted to choose incentive compatible allocations. Given the reputations \( r_x \), the agent of each type \( t \) must be incentivized to choose their allotted action \( x(t) \). That is, \( x(t) \) is an equilibrium of the Bayesian game between the agent and the market given choice set \( x(T) \). An allocation \( x : T \to A \) is **incentive compatible** on \( T \) if
\[
L^A(t|x) = \min_{a \in x(T)} L^A(a,t|x) \quad \forall t \in T.
\]

Let the set of allocations satisfying (1) on \( T \) be \( IC(T) \). The principal seeks to minimize their loss over all incentive compatible allocations. That is, the principal solves
\[
\inf_{x \in IC(T)} L^P(x).
\]

**Discussion**  The problem in (2) can be interpreted in a few different ways. One interpretation has already been implied. Given allocation \( x \), the principal delegates a set \( x(T) \) to the agent who selects an action from this set. The market forms a belief about the agent’s
type after observing only the action. It is assumed that the principal gets his most preferred equilibrium of the bayesian game between the market and the agent.

An alternative interpretation is that the principal does nothing, i.e. delegates the full set of actions $A$ to the agent and gets his most preferred equilibrium. This interpretation would give rise to more stringent incentive compatibility conditions in order to prevent off path deviations. However, these additional constraints prove non-binding for optimal allocations. Finally, one can interpret the model from a more standard mechanism design perspective, i.e. the agent reports his type to the principal who has commitment power to choose the action. Importantly, the market does not observe the reports of the agent, and again the principal gets his most preferred equilibrium.

One can also interpret the reputational concerns of the agent in a couple different ways. The first interpretation is implied by the setup above: the reputational concern is a reduced form for the agent being compensated in the future based on the market’s belief about his type. For example, as stated in the introduction, the type could be correlated with ability and so the market’s belief could capture future hiring opportunities.

The reputational concern could also capture the expected gains in future repeated identical interactions with the same principal who lacks dynamic commitment power and can choose to replace the agent. To be succinct, I have not modeled the principal as valuing lower type agents. However, if the principal’s loss were instead given by $(a - t)^2 - t$, this would induce an incentive to replace low type agents, and clearly not change the optimal allocation. Through all possible interpretations, it is very important that only the action and not the loss or the report of the agent is observed by the market.

2.2. Preliminaries

I first characterize incentive compatible allocations. For any allocation $x$ let $J_x \subset T$ be its set of discontinuities.

**Lemma 1.** An allocation $x \in IC(T)$ if and only if

1. $x$ is increasing.
2. $J_x$ is a countable nowhere dense set. Let $t, \bar{t} \in J_x$ such that $(t, \bar{t}) \cap J_x = \emptyset$. \forall $t \in (t, \bar{t})$

\[
\begin{align*}
(a) \quad x'(t) &= 0 \quad \text{(pooling)} \\
(b) \quad x'(t) &= \frac{\rho}{2(x(t) - t)} \quad \text{(separating)}
\end{align*}
\]

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2 This would only match the delegation interpretations in the previous paragraph.
3. \( \forall t \in J_x, \)
\[
\lim_{t' \to t^+} L^A(x(t'), t|x) = \lim_{t' \to t^-} L^A(x(t'), t|x).
\]

Given the characterization, I equate each \( x \in IC(T) \) with its right continuous counterpart.\(^3\) Lemma 1 is very similar to the characterization of incentive compatibility in material delegation: it is identical to lemma 1 in Alonso and Matouschek (2008) aside from point (2b). First, all IC allocations are monotonic. The reason is that for any allocation \( x, L^A(a, t|x) \) is strictly submodular in \( (a, t) \). This monotonicity means that in each IC allocation, each action is associated with an interval of types, and so the reputations are the associated expectations on these intervals. That is \( r_x(a) = \mathbb{E}[t'|t \leq t' \leq \bar{t}] \equiv R(t, \bar{t}) \) for some \( t \leq \bar{t} \).

The second part of the result says that any \( x \in IC(T) \) segments the type space into countably many intervals on which the allocation is either constant, or a solution to the differential equation in (2b). I refer to these two types of allocations as pooling and separating respectively. While this guarantees local incentive compatibility on each interval, the last part of the lemma guarantees local incentive compatibility at the endpoints.

As mentioned above, the main difference between Lemma 1 and the analogous characterization in material delegation is the behavior of the separating allocation. A separating allocation on a given interval \( (\underline{t}, \bar{t}) \) is defined by the property that the action reveals the type, i.e. \( x^{-1}(x(t)) = t \) or \( r_x(t) = t \ \forall t \in (\underline{t}, \bar{t}) \). In this case incentive compatibility means that,
\[
t \in \arg\min_{t' \in T} (x(t') - t)^2 + \rho(t - t').
\]
Taking first order conditions gives the differential equation in (2b).

Figure 1 illustrates an example of an IC allocation with \(|J_x| = 3\). The left panel displays the action as a function of the type while the right panel displays the corresponding loss for the agent. A few observations are worth noting now. First, the action jumps whenever the allocation switches from pooling to pooling, separating to pooling, or pooling to separating, because in each of these cases there is also a jump in the corresponding reputations. This is in contrast to material delegation wherein the allocation only jumps between two pooling intervals. Second, the loss of the agent is continuous in the type and almost everywhere differentiable. Third, the separating allocation is different than that for material delegation; in particular, the separating loss to the agent is increasing in the type, while in material delegation the agent gets his ideal action on separating intervals. The separating allocation will be important for the analysis and so it will prove useful to further explore (2b).

\(^3\)This changes the allocation on a measure zero set of types and thereby does not affect the principal’s loss.
There are a continuum of increasing solutions to (2b) pinned down by the initial condition. Let $d_u(t)$ solve (2b) with initial condition $d_u(0)^2 = u$. Let $D_u(t) \equiv (d_u(t) - t)^2$ be the separating loss given that type $t = 0$ experiences loss $u$. While there is no explicit representation of $d_u(t)$, key properties are derived below. The separating loss for various initial conditions is displayed in Figure 2.

**Lemma 2.** Properties of the separating allocation:

1. If $u \leq (\geq)\rho^2/4$ then the loss of the agent $D_u(t)$, is increasing and concave (decreasing and convex) in $t$.

2. $\forall u \geq 0, \lim_{t \to \infty} D_u(t) = \rho^2/4$.

3. $\forall t, D_u(t)$ is strictly increasing in $u$.

The first two points report that the separating loss monotonically asymptotes to $\rho^2/4$. At this loss, increasing the action one to one with the type exactly deters the agent from seeking a higher reputation, so loss remains constant. The third point says that the separating allocation is increasing in the initial loss $u$. This means that the principal’s optimal separating allocation is given by $d_0(t)$ and is associated with the lowest curve in Figure 2.

I conclude this section by asserting that a minimum to (2) exists.

**Lemma 3.** There exists an allocation $x^* \in IC(T)$ that minimizes $L^P(x)$ across all $x \in IC(T)$.
3. The Optimality of a Floor

**Theorem 1** (Optimality of a Floor). \( \exists K > 0 \) such that for every solution \( x^* \) to (2), \( \exists F \geq K \) with \( x^*(t) = F \forall t \in [0, F] \).

The main result says that it is optimal to restrict the agent’s flexibility for low actions in any optimal allocation. Specifically, there is a uniform bound \( K \leq x^*(0) \) for any optimum \( x^* \). An implication of this, is that the optimal allocation also pools a non-trivial first interval of types \([0, \hat{t}]\). This is because (as will be clarified below), it is always optimal to set the minimum action lower than the highest type to which it is allocated to, i.e. \( x^*(0) < \sup\{x^{r-1}(0)\} \).

It is of course possible to construct IC allocations that violate the condition of Theorem 1. An example is the allocation in Figure 1. As the optimal action for type \( t = 0 \) is \( a = 0 \), such alternative allocations perform better than any optimal allocation for low types. Indeed, any optimal allocation sacrifices some loss on low types by setting a floor, in exchange for lowering the loss on higher types. The intuition behind this derives from the following thought experiment: given a threshold type \( t > 0 \) and that the principal is choosing the optimal allocation for types above \( t \), how should the principal select the allocation below \( t \) in order to minimize total loss?
3.1. Continuing Allocations and the Alignment Principle

Consider some type $t \in T$, an allocation $x \in IC([0, t])$ below $t$, and another allocation $y \in IC([t, M])$ above $t$. Under what conditions can one join these allocations together to form an incentive compatible allocation? That is, when is the allocation defined by

$$
z(t') \equiv \begin{cases} 
  x(t') & t < t \\
  y(t') & t \geq t
\end{cases}
$$

an element of $IC(T)$. The answer is given by point 3 of Lemma 1: the agent of type $t$ must be indifferent between $x$ and $y$. That is, the only information about $x$ needed to specify $y$ is the loss that type $t$ experiences under $x$. Motivated by this, Let $y : [t, M] \to A$ be a continuing allocation at $(t, u)$ if $y \in IC([t, M])$ and $L^A(t|y) = u \geq 0$. Denote the set of these continuing allocations as $C(t, u)$.

Define the minimized continuation loss as

$$V(t, u) \equiv \frac{1}{F([t, M])} \inf_{y \in C(t, u)} \int (y(t') - t')^2 f(t')dt'. \quad (3)$$

Note that if one solves (3) for every $(t, u)$ then this greatly simplifies finding the optimal allocation. To see this, note that the solution to (2) is the solution to

$$\inf_{a_1, t_1} \int_0^{t_1} (a_1 - t')^2 f(t')dt' + F([t, M])V(t_1, (a_1 - t_1)^2 + \rho(t - R(0, t_0))). \quad (4)$$

That is, the principal only needs to choose the first action and threshold. These choices pin down the agent’s loss at the first threshold and thereby associated minimized continuation loss. This is the method used in the next section. For now I focus on how $V(t, u)$ changes with the initial loss $u$.

An illustrative example of a continuing allocation is the separating continuing allocation at $(t, u)$ given by $d_u(t' - t) + t \forall t' > t$. As derived in Lemma 2 point 3, the principal does better with the separating continuing allocation when the initial loss $u$ is lower. The next result shows that this property extends to any optimal continuing allocation.

**Lemma 4.** $\forall u \geq 0$ and $t \in T$, there exists a solution to (3).

**Proposition 1** (The Alignment Principle). $\forall \tilde{M} < M, \exists k > 0$ such that $\forall t \geq 0, \forall u > 0, \forall \varepsilon \leq u$,

$$\frac{V(t, u) - V(t, u - \varepsilon)}{\varepsilon} > k.$$
The alignment principle says that the minimized continuation loss above type $t$ is increasing in the initial loss of type $t$. Moreover, the magnitude of this change in continuation loss is uniformly bounded away from 0. The implication is that all else being equal, the principal should seek an allocation $x$ below type $t$ that minimizes type $t$’s loss.

The intuition comes from the fact that the reputational concern induces a bias towards higher actions. Reconsider the piecewise allocation $z$ above formed by joining $x$ and $y$, and suppose that $L_A(t|x) < L_A(t|y)$. Since $r_x^*(t) \leq r_y^*(t)$, i.e. $x$ provides a worse reputation than $y$, it must be that $(x(t) - t)^2 < (y(t) - t)^2$, i.e. material loss must be less from $x$ than $y$. This means that the principal would also prefer to assign $x(t)$ rather than $y(t)$ to type $t$; in other words their preferences are aligned at type $t$.

There are many ways the principal can take advantage of the slack introduced by this alignment. The method used in the proof is illustrated in Figure 3. Consider $u_1 > u_0 \geq 0$, and a continuing allocation $x_1$ at $(t, u_1)$. The proof constructs a better continuing allocation at $(t, u_0)$ by replacing $x_1$ on $[t, \tilde{t}]$ with a separating allocation with initial loss $u_0$. Here, $\tilde{t}$ is the first type indifferent between this separating allocation and $x_1(\tilde{t})$ given the altered reputations. The left panel of Figure 3 illustrates these two continuing allocations, where the right panel shows how this process reduces the agent loss for every type. Note that the principal’s expected loss and agent’s expected loss are equal so this latter panel illustrates the improvement.

![Figure 3: The alignment principle](image)

3.2. Proof Sketch of Theorem 1

Consider that Theorem 1 does not hold. This means $\forall \varepsilon > 0$, there is an optimum $x^*$ such that $x^*(0) < \varepsilon$. Let $\tilde{t}$ be the first threshold, i.e $r_x(0) = R(0, \tilde{t})$. Because of the alignment
principle, it is dominant to choose \( R(0, \tilde{t}) \leq x^*(0) \leq \tilde{t} \). Otherwise the principal could improve both the loss on \([0, \tilde{t}]\) and the loss of the first threshold \( \tilde{t} \) by moving \( x^*(0) \) in this interval. Thus \( \tilde{t} < \varepsilon \) can be taken arbitrarily small as well.

It is useful to consider two cases based on whether the “next thresholds” under \( x^* \) can also be taken arbitrarily small as well or not.\(^4\) Consider first that the next threshold \( \tilde{s} \) is large. In this case the action for \( t \in [\tilde{t}, \tilde{s}] \) is respecting the incentive constraint of \( \tilde{t} \) at the benefit of decreasing the loss on \( t \in [0, \tilde{t}] \). However, because \( \tilde{t} \) is small, the principal finds it beneficial to get rid of this first interval and freely optimize with respect to the new first action for \( t \in [0, \tilde{s}] \).

The alternative is that there exist some next threshold \( \tilde{s} \) (not necessarily the adjacent one), that is small, i.e. \( \tilde{s} < \varepsilon \), but large relative to \( \tilde{t} \), i.e. \( \tilde{t}/\tilde{s} < \varepsilon \). In this case, one can show that the loss for type \( \tilde{s} \) is approximated by that of the separating loss for \( \tilde{s} \).

Since the loss on \([0, \tilde{s}]\) is small and second order for any (reasonable) allocation. This above approximation and the fact that \( x^* \) is optimal means that the separating allocation below \( \tilde{s} \) must minimize this type’s loss. Otherwise, the alignment principle would imply the principal could improve her loss above \( \tilde{s} \) by using some alternative allocation. I use a floor to construct such an alternative allocation; let

\[
z(t) \equiv \begin{cases} 
\tilde{s} & t < \tilde{s} \\
x^*_{\tilde{s}, \tilde{u}}(t') & t \geq \tilde{s}
\end{cases},
\]

where \( \tilde{u} \equiv \rho(t - R(0, \tilde{s}) \) and \( x^*_{\tilde{s}, \tilde{u}} \) is an optimal continuing allocation at \((\tilde{s}, \tilde{u})\). Figure 4 illustrates why \( z \) improves on \( x^* \). The left panel illustrates how, under the uniform distribution and for small \( \tilde{s} \), type \( \tilde{s} \)'s loss is lower under \( x \) than under the separating allocation. This comparison is general: for small \( \tilde{s} \), the separating loss and pooling loss for small \( \tilde{s} \) are approximated by \( \rho \tilde{s} \) and \( \rho \tilde{s}/2 \) respectively. The right panel illustrates how \( z \) improves loss above \( \tilde{s} \) relative to \( x^* \) when the \( x^*_{\tilde{s}, \tilde{u}} \) is the separating continuing allocation. While the type by type comparison in loss is specific to the separating continuing allocation, the alignment principle shows that the comparison in expected loss holds generally.

4. The Exponential Model

The previous section showed that a floor is always optimal. The proof sketch makes use of the special properties of the first interval, i.e. that the corresponding action is uncon-
strained. However, another valid interpretation (at this point) is that pooling, i.e. discretely segmenting all types, is simply necessary or preferred in the reputational delegation framework. Indeed, this is the case in cheap talk a la Crawford and Sobel (1982). Furthermore, like in cheap talk and unlike in material delegation, the principal lacks complete control over the agent’s incentives because of their reputational concerns.

Rebutting this alternative interpretation requires more details about the solution. However, standard methods used to solve the general material delegation model, e.g. those used in Amador and Bagwell (2013), and Kleiner et al. (2021), do not work in the reputational delegation framework. Instead, I specialize to the exponential model in this section, and solve the problem using recursive methods. That is, for the remainder of this section $T = [0, \infty)$ with $f(t) = \lambda e^{-\lambda t}$ $\forall t$ with $\lambda > 0$. The main takeaway is that the separating allocation is “eventually” optimal, i.e. there exists some type above which the optimal allocation is separating. This means that pooling is not good per-se in reputational delegation, rather it is specifically optimal for low types.

The memorylessness property of the exponential distribution affords key simplifications. The set of continuing allocations and thereby the minimized continuation loss at

\[ \frac{\rho \tilde{s}}{2} \]

\[ D_0(\tilde{s}) \]

\[ \sqrt{D_0(t) + t} \]

\[ z(t) \]

\[ t \]

Figure 4: Improving on $x^*$ Using a Floor.

---

5 These papers optimize over the entire allocation subject to the envelope version of the incentive constraint. They find Lagrange multipliers under which this problem is convex and use first order conditions to find the optimal allocation. In the current problem each allocation induces reputations which (i) factor into the envelope constraints and (ii) are expectations over the inverse mapping of the allocation and thereby do not change smoothly with the allocation.

6 The compactness of the type space is used in the existence results in sections 2 and 3. In this section, I will prove that the optimal continuing allocation exists and is separating above some type $M$. This reduces the problem to optimizing the allocation over the compact type space $[0, M]$ given the separating continuation loss at $M$ thereby restoring the existence results.
(t, u) both do not depend on the initial type t. Given this fact, I normalize the initial type to 0, and denote a continuing allocation as \( y : [0, \infty) \to A \), \( y \in IC(T) \) with \( L^A(0|y) = u \). Furthermore, I abuse notation and write \( C(t, u) \equiv C(u) \), \( V(t, u) \equiv V(u) \), and \( R(t_1, t_2) - t_1 \equiv R(t_2)\). The goal of this section will be to derive properties of the optimal continuing allocations at each \( u \). In specific, I will exploit the recursive structure of the minimized continuation loss.

4.1. Recursive Formulation

Let \( y \) be a continuing allocation at \( u \) with first threshold \( t \). This first threshold pins down the reputation for this interval as \( R(t) \). Since \( u = L^A(0|y) = y(0)^2 - \rho R(t) \), this first threshold also pins down the first action \( y(0) \), which in turn pins down the first thresholds loss \( L(t|y) \). Motivated by this, define

\[
\begin{align*}
    a(t, u) &\equiv \sqrt{u + \rho R(t)} \text{ and } \\
    \pi(t, u) &\equiv (a(t, u) - t)^2 + \rho(t - R(t)).
\end{align*}
\]

The minimized continuation losses must choose this first threshold optimally. That is, the following Bellman equation holds.

\[
V(u) = \inf_{t \geq 0} \int_0^t \lambda (a(t, u) - t')^2 e^{-\lambda t'} dt' + e^{-\lambda t} V(\pi(t, u)) \quad \forall u.
\]

That is the first interval must be chosen optimally given an optimal continuing allocation being used thereafter. This is a Bellman equation with both a one dimensional state variable \(- u \), and a one dimensional control variable \(- t \). The natural question is whether the converse to the above implication holds: if a set of continuation losses satisfy (5), then are these continuation losses minimized? The next result answers a version of this question affirmatively.

Proposition 2. Take \( u \leq \rho^2/4 \). Let \( \{y_u\} \) be a set of continuing allocations at each \( u \geq u \). Suppose that \( \forall u' \geq u \) \( L^P(y_u) \) is differentiable in \( u \) and \( \frac{dL^P(y_u)}{du} \leq 1 \).

\[
\begin{align*}
L^P(y_u) &= \min_{t \geq 0} \int_0^t \lambda (a(t, u) - t')^2 e^{-\lambda t'} dt' + e^{-\lambda t} L^P(\pi(t, u)) \quad \forall u \geq u \\
\Rightarrow L^P(y_u) &= V(u) \quad \forall u \geq u.
\end{align*}
\]

It is worth noting a few caveats to Proposition 2. First and most salient, one only needs to verify the recursive condition for \( u \geq u \). This is due to the following regularity property of the exponential model.
Lemma 5. In the exponential model, if \( u \leq \rho^2/4 \) then \( \forall t > 0 \ \overline{\pi}(t, u) > u \), and if \( u \geq \rho^2/4 \) then \( \forall t > 0 \ \overline{\pi}(t, u) \geq \rho^2/4 \).

That is, if initial loss \( u \leq \rho^2/4 \), then no subsequent threshold can admit a loss less than \( u \). This means that the continuing allocations at losses less than \( u \) do not have any impact on the choice of continuing allocations at losses greater than \( u \). This simplification will be especially useful in the next section.

The second caveat is that, unlike in analogous dynamic optimization results, one cannot “ignore” the continuing allocations, and simply check that an arbitrary function \( \tilde{V}(u) \) satisfies (6). For example, consider \( \tilde{V}(u) = 0 \ \forall u \geq 0 \). This satisfies (6) by simply choosing \( t = 0 \), but \( \tilde{V}(u) = 0 \) is clearly not attainable for any \( u \). For this reason, Proposition 2 requires that the continuation losses correspond to actual continuing allocations.

Finally, the continuation losses are required to have a derivative less than 1. This condition guarantees an approximation result in the proof. One may be concerned that this condition makes the result vacuous, but any set of optimal continuing allocations must also satisfy this condition. To see this, let \( y_u \) be a continuing allocation at \( u \). Now consider constructing another continuing allocation at \( u + \varepsilon \) by maintaining the same set of thresholds as \( y_u \). This means that the change in loss is only the effect of changing the action on each interval to match an increase in the constraint in the initial loss. One can show that the change in loss degrades in the type, i.e. for two types \( t'' > t' \), \( \frac{d L^A(t'|y_u)}{d u} > \frac{d L^A(t''|y_u)}{d u} \). Clearly an optimal set of continuing allocations would change the thresholds with \( u \) to further decrease the principal’s loss,\(^7\) and so this derivative assumption does not restrict the set of allocations.

Proposition 2 provides for the “guess and check” method to solve for the optimal continuation loss. One can conjecture a set of continuing allocations and check that its associated continuation losses satisfy (5). This is the approach taken in the next subsection.

4.2. Separating in the Exponential Model

A focal continuing allocation at \( u \) is the separating continuing allocation \( d_u \) introduced in the previous section.

Theorem 2. In the exponential model, \( u > \rho^2/16 \) \( \implies \) \( V(u) = L^F(d_u) \). Moreover, \( d_u \) is the unique optimal continuing allocation at \( u \) \( \forall u > \rho^2/16 \).

\(^7\)The argument is actually exact under an optimal set of continuing allocations because any change in the thresholds has a 0 effect on \( V(u) \) by an envelope theorem argument.
The proof of Theorem 2 shows that for $u > \rho^2/16$, setting $y_u = d_u$ satisfies (6). While the result says that there exists initial losses such that separating is optimal, it does not speak to whether separating will actually arise in the optimal allocation. However, using logic similar to that behind Lemma 5, one can deduce that loss increases throughout the type space until $u \geq \rho^2/4$. This means that, as long as there is an unbounded sequence of thresholds in the optimal allocation (i.e. there is no eventual pooling), then the optimal allocation will eventually be separating. This holds for small biases.

**Lemma 6.** If $\lambda \rho \leq 4$, the pooling continuing allocation at $u$ given by $x(t) = \sqrt{u + \rho/\lambda} \equiv p_u \forall t \in T$ is not optimal $\forall u$.

**Corollary 1.** An optimal allocation exists. If $\lambda \rho \leq 4$, then there exists $\bar{t}$ such that for any optimal allocation $x^*$, $\exists \bar{u}$ with $x^*(t) = d_u(t - \bar{t}) + \bar{t} \forall t \geq \bar{t}$.

**Corollary 1** first restates the existence of an optimal allocation. Because the type space is now unbounded, one cannot immediately apply the methods to prove existence from the previous sections. I first establish the fact that the allocation is separating or completely pooling after some $\bar{t}$. Then standard methods imply an optimal allocation on the compact space $[0, \bar{t}]$ given the separating or pooling continuation loss above $\bar{t}$.

While the argument for Theorem 2 is complicated, a broad intuition is as follows. There are two main reasons why a floor is optimal and in particular does better than the lowest separating allocation $d_0$. First, the first pooling allocation does not need to respect any incentive constraint to the left and can be set freely, whereas $d_0(t)$ is constructed to equalize the incentive to obtain a higher reputation with the cost of increasing material loss. Second, the separating action $d_u(t)$ increases infinitely quickly for low types.

Neither of these reasons are present when determining the continuing allocation for large initial losses. First, any continuing allocation at $u$ must respect the left incentive constraint that the initial loss is $u$. Second, the loss under the separating continuing allocation increases very slowly for large initial losses. As shown in Figure 2, at initial losses close to $\rho^2/4$, the separating loss is near constant in the type. This means that the comparison between any continuing allocation which pools some first set of types and the separating allocation is more favorable to the latter at large initial losses.

**4.3. The Uniform Limit**

The separating continuing is not be optimal for small initial losses. More generally, it is difficult to analytically solve for the optimal continuing allocation at every $u$, because there is no explicit representation of the separating continuation loss, which by Theorem 2 will
be optimal for large $u$. However, finding $V(u) \forall u$ is not necessary in solving for the optimal allocation. If one can instead show that the floor $(t_0, a_0)$ in (4), must be optimally set so that the loss at the first threshold $- (a_0 - t_0)^2 + \rho(t - R(t_0))$, is greater than $\rho^2 / 16$, then the optimal continuing allocation is separating by Theorem 2. This requires proving bounding $V'(u)$ for low initial losses. This is tractable in the uniform limit, i.e. when $\lambda \to 0$.

The result, reported below, is that the optimal allocation uses a floor and then separates thereafter. The optimal allocation in the uniform limit is illustrated in Figure 5.

**Proposition 3.** There exists $c > 0$ such that for $\lambda \leq c$, the optimal allocation is given by

$$x^*(t) \equiv \begin{cases} a_0 & t < t_0, \\ d_u(t - t_0) + t_0 & t \geq t_0, \end{cases}$$

where $u = (a_0 - t_0)^2 + \rho(t_0 - R(t_0))$, and $a_0, t_0$ solve,

$$\min_{a_0, t_0} \int_0^{t_0} (a_0 - t')^2 \lambda e^{-\lambda t'} dt' + L^P(d((a_0 - t_0)^2 + \rho(t_0 - R(t_0))) e^{-\lambda t_0}.$$  

In addition,

$$\lim_{\lambda \to 0} a_0 \to k_1 \rho$$
$$\lim_{\lambda \to 0} t_0 \to k_2 \rho$$

where $k_1 < k_2$ are constants with $(k_2 - k_1)^2 + k_2 / 2 > 1 / 16$.

5. Conclusion

This paper studies whether an agent with a reputational bias should be treated differently than an agent with a material bias. Despite the models having many superficial similarities, this paper answers this question affirmatively. Specifically, Theorem 1 showed that unlike in material delegation where it is never beneficial to restrict flexibility to low type agents, it is always optimal to impose a floor in reputational delegation. In addition, this is not due to flexibility being sub-optimal per-se. Theorem 2 shows that it is optimal to give full flexibility in the exponential model, when the initial loss is large.

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8 The important condition is a kind of convexity in the optimal continuation loss, in particular that $V'(u) \leq L^P(d, \rho / 16)$.

9 Specifically, $k_1 \equiv (31 - 7 \sqrt{17}) / (8(-11 + 3 \sqrt{17}))$, $k_2 \equiv (1/32)(-13 + 5 \sqrt{17})$, and $k_3 \equiv (k_2 - k_1)^2 + k_2 / 2$. 
To solve the exponential model, I use the recursive nature of the problem, namely that the first threshold in any continuing allocation pins down the initial loss at that threshold, and thereby the continuation loss for the principal. This method is not specific to the reputational delegation framework, and can be used in other difficult mechanism design problems. The important features are that the allocation over which the principal has commitment (the action) is monotone in the type, and the portion over which the principal does not have commitment (the reputation) is dependent only on the chosen action.

Finally, I assumed throughout the paper that the agent benefits from having a reputation of being a higher type. As discussed, this assumption fits many principal-agent interactions, however it was also partially chosen to make the model directly comparable with the standard material delegation framework in which the agent is biased towards higher actions. It would be interesting for future work to study optimal delegation to agents with different reputational biases. In particular, standard models of expertise, e.g. Ottaviani and Sorensen (2006a), would associate extreme actions with better agents rather than higher actions. Conversely, Bernheim (1994) studies agents with a "preference for conformity", and this reputational bias would push agent’s to take more middling actions.
References


A. Preliminaries

I first prove a series of lemmas that will be used in the arguments below. Lemma 7 derives the familiar envelope condition version of incentive compatibility. Lemma 8 shows that it is without loss to restrict to a compact action set. Lemma 9 shows that the minimized continuation loss is differentiable. Lemma 10 shows that the pooling action for a
By incentive compatibility, Lemma 11 uses the previous lemma to derive a lower bound on the loss of the endpoint type for any small interval. Lemma 12 provides a necessary condition for any optimal pooled interval derived by separating a small set of types near the left endpoint.

For any allocation \(x \in IC(T)\) define the endpoint functions \(\tau, \overline{\tau} : T \to T\) such that \(r_x(x(t)) = R(\tau(t), \overline{\tau}(t)) \forall t\). It will also be useful to define a sense of a “small interval”, such that if this interval is pooled, then the corresponding action is greater than the endpoint type: let \(\delta > 0\) be defined by \(\sqrt{\rho R(t, t + \delta)} > t + \delta \forall \delta < \overline{\delta}\). Since \(f\) is bounded away from zero, so is the derivative of \(R(t_1, t_2)\) with respect to \(t_1\) and \(t_2\), and so such a \(\overline{\delta}\) exists.

**Lemma 7.** For any compact \(S \subset \mathbb{R}^+\) and \(x \in IC(S)\), \(L^A(t|x) = \lim_{\epsilon \to 0} \frac{L^A(t, t+\epsilon|x) - L^A(t, t-\epsilon|x)}{2\epsilon} = \rho - 2(x(t) - t)\).

**Proof.** For \(a \in x(S)\) \(L^A(a, t|x) = \rho - 2(a - t)\) is equicontinuous in \(t\) for all \(a \in x(S)\). The result follows from theorem 3 in Milgrom and Segal (2002). Q.E.D.

**Lemma 8.** Let \(x \in IC(T)\). There exists an allocation \(\tilde{x} : T \to [0, B]\) where \(B \equiv \sqrt{R(0, M)^2 + \rho M + M^2} + M\) such that \(\tilde{x} \in IC(T)\) and \(L^F(\tilde{x}) \leq L^F(x)\).

**Proof.** If \(x(M) \leq B\) then set \(\tilde{x}(t') \equiv x(t') \forall t'\). Suppose \(x(M) > B\). Let \(L^A(0|x) \equiv u_0\). By incentive compatibility, \(x(M) = \sqrt{L^A(\tau(M)|x) + \rho(R(\tau(M), M) - \tau(M)) + \tau(M)}\). By Lemma 7,

\[
L^A(\tau(M)|x) = \int_0^{\tau(M)} \rho - 2(x(t') - t') \ dt' + u_0 < u_0 + \rho \tau(M) + \tau(M)^2.
\]

Thus \(x(M) < \sqrt{u_0 + \tau(M)^2 + \rho R(\tau(M), M) + \tau(M)} < \sqrt{u_0 + \rho M + M^2 + M}\). However \(\forall t', x(t') \geq x(0) \geq \sqrt{u_0}\). \(x(M) > B \implies u_0 > R(0, M)^2\). Thus, one can choose \(\tilde{x}(t') = R(0, M) \forall t'\) and satisfy the lemma. Q.E.D.

**Lemma 9.** \(\forall t, u V(t, u)\) has left and right derivatives in \(u\) denoted \(V_{u-}(t, u)\) and \(V_{u+}(t, u)\) respectively.

**Proof.** In order to show differentiability of \(V\), I will redefine the set of continuing allocations by their induced reputations. Let \(\mathcal{R} \equiv \{r_x^\tau(t) : x \in C(t, u)\}\), i.e. the set of reputation functions resulting from any given interval partition of \([t, M]\).
First note that for any increasing and bounded function \( r : T \to \mathbb{R}^+ \), there exists an allocation \( x : T \to A \) such that

\[
(x(t) - t)^2 + \rho(t - r(t)) = \max_{t' \in T} (x(t') - t)^2 + \rho(t - r(t')).
\]

That is, \( x \) is incentive compatible given reputations given by \( r \). This is because the envelope theorem holds and so the ordinary differential equation has a solution by theorem 1 in Persson (1975). Note that this also implies \( R \) is independent of \( u \).

I will first show that the reputation function uniquely pins down the continuing allocation. Suppose not, i.e. \( \exists x, y \in C(t, u) \) such that \( r^*_x = r^*_y \) but \( x \neq y \). Take \( s \equiv \inf\{t' : L^A(t'|x) \neq L^A(t'|y)\} \). Since the agent loss is continuous for any IC allocation and \( L^A(0|x) = L^A(0|y) = u \), it must be that \( L^A(s|x) = L^A(s|y) \). Suppose first that there exists \( t' < s + \delta \) such that \( L^A(t'|x) \neq L^A(t'|y) \) and \( \tau(t') < s \). Because the reputations are the same, it must be that \( x(t') \neq y(t') \). But since \( \tau(t') < s \), \( x(t') = x(s) \neq y(s) = y(t') \). This means that there exists \( \tilde{t} \leq s \) such that \( L^A(\tilde{t}|x) \neq L^A(\tilde{t}|y) \), which is a contradiction to the definition of \( s \).\(^{10}\)

This means that \( \forall t' \in [s, \delta) : L^A(t'|x) > L^A(t'|y) \) (a symmetric argument holds for the case in which \( L^A(t'|x) > L^A(t'|y) \)), it holds that \( \tau(t') > s \). Because agent losses are continuous in the type, there exists \( s \geq s \) such that \( L^A(t''|x) > L^A(t''|y) \) \( \forall t'' \in (s, t'] \) and \( L^A(s|x) = L^A(s|y) \). By definition of \( \delta \), this means that \( x(t'') > y(t'') > t'' \forall t'' \in (s, t'] \). But this is a contradiction, because by Lemma 7,

\[
L^A(t'|x) - L^A(t'|y) = \int_s^{t'} y(t'') - x(t'') \, dt'' + L^A(s|x) - L^A(s|y) < 0.
\]

Thus, for each \( r \in R \) define \( x_{r,u} \) as the unique continuing allocation at \( (t, u) \) with reputation function \( r^*_x(t') = r(t') \forall t' \in [t, M] \). I can rewrite the problem in (3) as,

\[
V(t, u) = \frac{1}{F([t, M])} \min_{r \in R} \int_t^M L^A(t'|x_{r,u}) \, f(t') \, dt'
\]

Because \( L^A(0|x_{r,u}) = u \forall r \in R, \frac{d}{du} L^A(t|x_{r,u}) \) is equicontinuous at \( t = 0 \). Moreover since, \( \rho - B < L_t(t'|x_{r,u}) < \rho + t' \forall t', r \in R, \frac{d}{du} L^A(t|x_{r,u}) \) is equicontinuous at every \( t \). Thus the

\(^{10}\) \( \tilde{t} \) need not be \( s \) because it could be that \( x(s) - s = s - y(s) \) in which case, despite different actions, \( L^A(s|x) = L^A(s|y) \).
Lemma 10. Let \( y \) be a continuing allocation at \((t, u)\) and \( \bar{t} \in \pi(T) \) with \( \bar{t} - t < \delta \).

\[ y(\bar{t}) \geq \sqrt{u + \rho(R(t, \bar{t}) - t) + t}. \]

Proof. Suppose the inequality does not hold. Then, since \( y \) is increasing.

\[
\int_t^{\bar{t}} 2(y(t') - (\sqrt{u + \rho(R(t, \bar{t}) - t) + t})) dt' < 0
\]

\[ \iff L^A(\bar{t}|y) > \left( \sqrt{u + \rho(R(t, \bar{t}) - t) - (\bar{t} - t)} \right)^2 + \rho(\bar{t} - R(t, \bar{t})). \]

Note that since \( \bar{t} - t < \delta \), \( y(t') > t' \) \( \forall t' < \bar{t} \). Thus \( y(\bar{t}) < \sqrt{u + \rho(R(t, \bar{t}) - t) + t} \) means that \( y \) gives \( \bar{t} \) lower material loss than the pooling allocation. However this contradicts the last line, because \( r_y^\ast(\bar{t}) \geq R(t, \bar{t}) \) since the pooling reputation is the minimum such reputation. Q.E.D.

Lemma 11. Take any continuing allocation \( x \) at \((t, u)\) such that \( \delta + t > \bar{t} \in \pi(T) \).

\[ L^A(\bar{t}|x) > \left( \sqrt{u + \rho(\bar{t} - t) - \bar{t}} \right)^2. \]

Proof. I start by proving the claim below. The claim says that if the reputations for the agent are higher for any interval, then the agent experiences lower loss. In order to write the claim I will need to notate certain dependences on the distribution. I use subscripts in the agent loss and reputation function to denote dependence on a distribution \( f - L^A_f \) and \( R_f \) respectively.

Claim 1. Let \( \bar{f} \) strictly monotone likelihood ratio dominate \( f \).\(^{11}\) Let \( x \) be a continuing allocation at \((t, u)\) under distribution \( f \) with \( \delta + t \geq \bar{t} \in \pi(T) \). Let \( \bar{x} \) be a continuing allocation at \((t, u)\) under \( \bar{f} \) such that \( \pi \) and \( x \) induce the same interval partition, i.e. \( \tau_x(t') = \tau_{\bar{x}}(t') \) \( \forall t' \). \( L^A_f(t'|x) \leq L^A_f(t'|\bar{x}) \) \( \forall t' \in [t, \bar{t}] \).

Proof of Claim: Note that \( \forall t_1 < t_2 R_f(t_1, t_2) > R_f(t_1, t_2) \) by the definition of MLR dominance. Note that at \( t' = t \), \( L_f(t'|x) = L_f(t'|\bar{x}) = u \) by the fact that these are both continuing allocations. Thus because both loss functions are continuous, it is sufficient to

\(^{11}\) That is, \( \forall t' > t \frac{\bar{f}(t')}{f(t')} \geq \frac{\bar{f}(t)}{f(t)} \).
prove that at any $t'$ such that $L^A_f(t'|x) = L^A_f(t'|\overline{x})$ and $L^A_f(t''|x) \geq L^A_f(t''|\overline{x}) \forall t \leq t'' \leq t'$, $\frac{d}{dt} \left( L^A_f(t'|x) - L^A_f(t'|\overline{x}) \right) \geq 0$. Note that,

$$L^A(t'|x) = L^A(t'|\overline{x})$$

$$\iff \left( \sqrt{L^A_f(t'|x)} + \rho(R_f(t', \tau(t')) - \tau(t')) \right)^2 + \rho(t' - R_f(t', \tau(t')))$$

Notice that the loss on either side of the above equality is decreasing in the reputation for the interval and increasing in the agent’s loss at $\tau(t')$. If $t' > \tau(t')$, then $R_f(\tau(t'), \tau(t')) > R_f(\tau(t'), \tau(t'))$. Since $L^A_f(\tau(t')|x) \geq L^A_f(\tau(t')|\overline{x})$ by assumption, this contradicts the above equality. Thus $t' = \tau(t')$.

$$\frac{d}{dt} \left( L^A_f(t'|x) - L^A_f(t'|\overline{x}) \right) \geq 0$$

$$\iff x(t') \leq \overline{x}(t')$$

$$\iff \sqrt{L^A(t'|x) + \rho(R_f(t', \tau(t') - t'))} \leq \sqrt{L^A(t'|\overline{x}) + \rho(R_f(t', \tau(t')) - t')}$$

$$\iff R_f(t', \tau(t')) \leq R_f(t', \tau(t')),$$

where the last implication is due to the assumption that $L^A_f(t'|x) = L^A_f(t'|\overline{x})$. This proves the claim.

Let $\overline{f}$ be a limiting MLRP dominating distribution such that $R_{\overline{f}}(t_1, t_2) = t_2 \forall t_1 \leq t_2$. Let $\overline{x}$ be the corresponding continuing allocation at $(t, u)$ under $\overline{f}$ defined by having the same endpoints, i.e. $\tau_x(t') = \tau_{\overline{x}}(t') \forall t'$. Because of the previous claim, $L^A_f(\overline{t}|\overline{x}) \leq L^A_f(\overline{t}|x)$. Now define an alternative continuing allocation $z$ at $(t, u)$ under distribution $\overline{f}$ that pools types below $\overline{t}$, i.e. it satisfies $z(t') \equiv \sqrt{u + \rho(\overline{t} - t) + t\overline{t}'} < \overline{t}$. Notice that under $\overline{f}$, $r^*_z(\overline{t}) = \overline{t} = r^*_z(\overline{t})$. By Lemma 10, $\overline{t} \leq z(\overline{t}) \leq x(\overline{t})$. Thus $z(t)$ delivers lower distortion and the same reputation so $(\sqrt{\rho(\overline{t} - t) + u - \overline{t}})^2 \leq L^A_f(\overline{t}|\overline{x})$.

Q.E.D.

**Lemma 12.** Consider an optimal allocation $x^*$ with $\overline{t} \equiv \tau(t) \leq \tau(t) \equiv \overline{t}$. Let $L^A(t|x^*) = u_0$, $L^A(\overline{t}|x^*) = u_1$, $x^*(\overline{t}) \equiv \overline{x}$, $R(\overline{t}, \overline{t}) \equiv \overline{R}$, and $F(|\overline{t}, \overline{t}|) \equiv \overline{F}$. It holds that,

$$\frac{2\overline{F}(\overline{x} - \overline{R})}{\overline{x} - \overline{t} + \sqrt{u_0}} - (\overline{R} - \overline{t})f(\overline{t}) + \frac{F(|\overline{t}, M|)}{\overline{F}}V_u(t, u_1) \left( \frac{2\overline{F}(\overline{x} - \overline{t})}{\overline{x} - \overline{t} + \sqrt{u_0}} - (\overline{t} - t)f(t) \right) \geq 0.$$

**Proof.** Consider the continuing allocation $x^*_{[\overline{t}, M]}$ at $(\overline{t}, u_0)$. I will show that the inequality
above results from this continuing allocation being optimal at \((t, u_0)\). Consider an alternative continuing allocation at \((t, u_0)\) parameterized by \(m\) that (i) separates between \(t\) and \(m\), (ii) pools between \(m\) and \(\bar{t}\), and (iii) chooses an optimal continuing allocation above \(\bar{t}\). Let \(a^m \equiv \sqrt{D_{u_0}(m-t) + \rho(R(m, \bar{t})-m)} + m\) and \(u^m \equiv (a^m - \bar{t})^2 + \rho(\bar{t} - R(m, \bar{t}))\). Finally, let \(x^*_{t, u}\) be an optimal continuing allocation at \((t, u)\). Specifically the alternative continuing allocation is defined as

\[
y^m = \begin{cases} 
d_{u_0}(t' - \bar{t}) + t & t \leq t' < m \\
a^m & m \leq t' < \bar{t} \\
x^*_{t, u}(t') & t' \geq \bar{t}
\end{cases}
\]

Because \(x^*\) is optimal, it must be that \(\frac{d L^P(y^m)}{dm}\bigg|_{m=t} \geq 0\). This gives the inequality in the display. \(Q.E.D.\)

**B. Proofs from Section 2**

**B.1. Proof of Lemma 1**

**Proof.** \("\implies\)\( "\) \(x(t)\) is increasing because \(\forall x : T \rightarrow A, L^A(a, t|x)\) is strictly submodular is \((a, t)\), so any minimizing selection from \(\min_{a \in x(T)} L^A(a, t|x)\) is increasing.

Since \(x(t)\) is increasing it has an at most countable set of discontinuities by Froda’s theorem, i.e. \(J_x\) is countable.

Now suppose \(J_x\) is dense on some interval \([\bar{t}, \bar{t}]\). First, consider that \(x(t_1) = x(t_2)\) for \(\bar{t} \leq t_1 < t_2 \leq \bar{t}\). Then since \(x\) is increasing \(x(t)\) is constant on \([t_1, t_2]\), i.e. \(J_x \cap (t_1, t_2) = \emptyset\) contradicting the hypothesis that \(J_x\) is dense on this interval. Thus, \(x\) is injective on \([\bar{t}, \bar{t}]\), and \(r_x(x(t)) = t \forall t \in [\bar{t}, \bar{t}]\). note that \(\forall x \in IC(T), L^A(a, t|x)\) is continuous in \(t\). Therefore, since the reputation is continuous on \([\bar{t}, \bar{t}]\) so is \(x\). This means \((\bar{t}, \bar{t}) \cap J_x = \emptyset\), which is a contradiction. This continuity of \(L^A(t|x)\) also implies the third condition in the lemma.

This means \(J_x\) is a countable nowhere dense set. Take arbitrary \(\bar{t}, \bar{t} \in J_x\) with \((\bar{t}, \bar{t}) \cap J_x = \emptyset\). First suppose that \(x\) is not strictly increasing on \((\bar{t}, \bar{t})\) Then since \(x\) is increasing there exists \(\bar{t} \leq t_1 < t_2 \leq \bar{t}\) such that \(x(t') \equiv \bar{x}\) is constant \(\forall t' \in (t_1, t_2)\), where \(t_1 \equiv \inf\{t \in T : x(t) = \bar{x}\}\) and \(t_2 \equiv \sup\{t \in T : x(t) = \bar{x}\}\). Suppose \(t_1 > \bar{t}\). Then \(\forall \varepsilon > 0, r_x(\bar{x}) - r_x(x(t_1 - \varepsilon)) > R(t_1, t_2) - t_1\). Since \(L^A(t|x)\) is continuous in \(t\) and the reputation discontinuously jumps at \(t_1\), the action must also jump. But this contradicts the fact that \((\bar{t}, \bar{t}) \cap J_x = \emptyset\), so \(t_1 = \bar{t}\). A symmetric argument shows that \(t_2 = \bar{t}\), and so \(x\) is constant on \((\bar{t}, \bar{t})\). This means that \(x\) is either constant or strictly increasing on \((\bar{t}, \bar{t})\).
Suppose $x$ is strictly increasing on $(t, \bar{t})$. Then $x$ is injective on this interval and $r_x(x(t)) = t \forall t \in (t, \bar{t})$. And so

$$L^A(x(t'), t|x) = (x(t') - t)^2 + \rho(t - t').$$

For $\varepsilon > 0$, incentive compatibility implies,

$$L^A(x(t + \varepsilon), t|x) - L^A(x(t), t|x) \geq 0$$

$$\implies x(t + \varepsilon)^2 - x(t)^2 - 2t(x(t + \varepsilon) - x(t)) \geq \rho \varepsilon$$

$$\implies \frac{x(t + \varepsilon) - x(t)}{\varepsilon} \geq \frac{\rho}{x(t) + x(t + \varepsilon) - 2t}.$$

An analogous argument for $\varepsilon < 0$ applies, so it holds that $\forall \varepsilon > 0$,

$$\frac{\rho}{x(t) + x(t - \varepsilon) - 2t} \geq \frac{x(t + \varepsilon) - x(t)}{\varepsilon} \geq \frac{\rho}{x(t) + x(t + \varepsilon) - 2t}.$$

Since $x(t + \varepsilon)$ is continuous in $\varepsilon$ for $|\varepsilon|$ small. $x'(t) = \frac{\rho}{2(x(t) - t)}$.

"$\iff$" Since $L^A(a, t|x)$ is submodular, local incentive constraints are sufficient for global constraints.\(^{12}\) First consider $t \notin J_x$. Either $x$ is constant around $t$ or solves (2b) which was shown to preserve local incentives. For $t \in J_x$ condition 3 is equivalent to local incentive compatibility.

Q.E.D.

B.2. Proof of Lemma 2

Proof. The solution to (2b) exists and is given by

$$d_u(t) \equiv t + \rho/2 \left(1 + W_0 \left(-e^{-\frac{\rho-2(\sqrt{u}+\rho)\rho-2\sqrt{u}}{\rho}}\right)\right),$$

Where $W_0(z)$ is the Lambert W-function, i.e. the principal solution to $z = W_0(z)e^{W_0(z)}$. The properties of the lemma come directly from examining (2b).

Q.E.D.

B.3. Proof of Lemma 3

Proof. Note that by Lemma 8, it is without loss to take $A \equiv [0, B]$. Take an IC allocation $x : T \to \mathbb{R}$. By the envelope theorem, $L^A_A(t|x) = \rho - 2(x(t) - t)$. Notice that this derivative is uniformly bounded because $(x(t), t)$ are from a compact set. Take the set of loss

\(^{12}\) This fact is detailed in Carroll (2012).
functions induced by IC allocations to be \( L \equiv \{ L^A(t|x) : x \in IC(T) \} \). \( L \) is a set of uniformly equicontinuous functions and is thereby compact by the Arzela – Ascoli theorem. Since \( L^p(x) = \int_0^M L^A(t'|x)dt' \), the principal’s loss is continuous in the agent’s loss and a minimum exists.

Q.E.D.

C. Proofs from Section 3

C.1. Proof of Lemma 4

**Proof.** By the same argument as in Lemma 8, all continuing allocations \( y \) at \((t, u)\) have \( y(t') < \sqrt{u + \rho(M - t) + (M - t)^2 + M} \). Thus \( C(t, u) \) is a compact set and an optimal continuing allocation exists by the same argument as in Lemma 3.

Q.E.D.

C.2. Proof of Proposition 1

**Proof.** Take \( t \geq 0 \), and \( u \geq 0 \), and let \( x^* \) be an optimal continuing allocation at \((t, u)\). Define \( \tilde{L}(t') \equiv (x^*(t') - t')^2 - R(t', \tau(t')) \). This is the agent’s loss using the \( x^* \) allocation but the higher reputation for a left censored interval. Let \( u_0 = u - \varepsilon \) and \( t_0 \equiv t \). Construct a set of sequences \( \{u_i\}, \{t_i\} \) as follows. Let \( t_{i+1} \) be the first type \( t' \geq \tau(t_i) \) to solve the equation

\[
\tilde{L}(t') = D_{u_i}(t' - \tau(t_i)).
\]

If no such solution exists, let \( t_{i+1} \equiv M \). Now set \( u_{i+1} \equiv (x^*(\tau(t_{i+1})) - \tau(t_{i+1}))^2 + \rho(\tau(t_{i+1}) - R(t_{i+1}, \tau(t_{i+1}))) \). Notice that \( L(\tau_i) = L^A(\tau_i|x^*) \geq u_i = D_{u_i}(0) \) so the LHS starts above the RHS at any iteration. Also \( \tilde{L}(\tilde{t}) \) is continuous for \( \tilde{t} \notin J_{x^*} \). Moreover \( \forall \tilde{t} \in J_{x^*} \),

\[
\lim_{t' \to t^-} \tilde{L}(t') \leq \lim_{t' \to t^-} L(t'|x^*) = \lim_{t' \to t^+} L(t'|x^*) = \lim_{t' \to t^+} \tilde{L}(t').
\]

Thus \( \tilde{L} \) is continuous except at upwards jumps, and so \( \tilde{L}(t') > D_{u_i}(t') \forall t' < t_{i+1} \).

Now define a continuing allocation \( y \) at \((t, u - \varepsilon)\) as

\[
y(t') = \begin{cases} 
  d_{u_i}(t' - \tau(t_i)) + \tau(t_i) & t' \in [\tau(t_i), t_{i+1}) \\
  x^*(t') & \text{otherwise}
\end{cases}
\]

This allocation satisfies incentive compatibility by construction. Note that \( \forall t' \in [t, M] \)
\( L^A(t|x) \geq L^A(t'|y) \). Now let \( K \equiv \inf_{0 \leq t_1 \leq t_2 \leq M} (R(t_1, t_2) - t_1) \frac{f(t_1)}{F(t_1, t_2)} \) be the minimum derivat-
tive of the reputation for an interval $R(t_1, t_2)$ with respect to its lower endpoint. Note that because $f(t) \in [k, \bar{k}]$ by assumption, $K > 0$.

$$
\frac{V(t, u) - V(t, u - \varepsilon)}{\varepsilon} \\
\geq \frac{\int_t^M ((x^*(t') - t')^2 - (y(t) - t)^2) f(t) dt'}{\varepsilon} \\
= \frac{\int_t^M (L^A(t'|x^*) - L^A(t'|y)) f(t) dt'}{\varepsilon} \\
\geq \frac{\int_t^{t_1} (L^A(t'|x^*) - L^A(t'|y)) f(t) dt'}{\varepsilon} \\
= \varepsilon - \int_t^M 2(x^*(t') - y(t')) F([t', M]) dt' \\
\geq 1 - \frac{\int_t^{t_1} 2(x^*(t') - y(t')) F([t', M]) dt'}{\int_t^{t_1} 2(x^*(t') - y(t')) + \rho(R(t', \bar{\tau}(t')) - t') \frac{f(v')}{F(v', \bar{\tau}(v'))} dt'} \\
\geq 1 - \frac{2B}{2B + \rho K}.
$$

The first line is due to the fact that $x^*$ is optimal under $(t, u)$ while $y$ is feasible under $u - \varepsilon$. The second line is due to the fact that reputations integrate out to $\rho E[t']$ for any allocation. The third line uses the fact that $\forall t' \in [t, M] L^A(t|x) \geq L^A(t'|y)$. The fourth line uses integration by parts and Lemma 7. The fifth line uses the fact that $\tilde{L}(t_1) \geq D_{u-\varepsilon}(t_1)$ and $\tilde{L}(t) = D_{u-\varepsilon}(0) + \varepsilon$, so the change in losses must be less than $\varepsilon$ over this interval, i.e. $\int_t^{t_1} 2(x^*(t') - y(t')) + \rho(R(t', \bar{\tau}(t')) - t') \frac{f(v')}{F(v', \bar{\tau}(v'))} dt' \leq \varepsilon$. The last two lines use the Cauchy mean value theorem and the bounds on the action and the derivative of the reputation function.  

Q.E.D.

C.3. Proof of Theorem 1

Proof. Suppose the theorem does not hold, i.e. there is a sequence of allocations $x_n$, each one optimal, such that $\lim_{n \to \infty} x_n(0) = 0$. Let $\underline{\tau}_n$ and $\bar{\tau}_n$ be the associated endpoint functions. Note that $\lim_{n \to \infty} \tau_n(0) = 0$ as well. This is because $x_n(0) > R(0, \tau_n(0))$. Otherwise, an alternative allocation that increases $x_n(0)$ to $R(0, \tau_n(0))$ improves the principal’s loss on the first interval $[0, \bar{\tau}_n(0)]$, and the initial loss of the agent at type $\bar{\tau}_n(0)$. By the alignment principle, this latter change also improves the principal’s loss. I begin by breaking the problem into two exhaustive cases described below.
Case 1: There exists a pair of subsequences \( t_n, s_n \in \tau_n(T) \) such that \( (t_n, s_n) \cap \tau_n(T) = \emptyset \), and as \( n \to \infty \), \( t_n \to 0 \) and \( \frac{t_n}{s_n - t_n} \to 0 \).

Case 2: There exists a pair of subsequences \( t_n, s_n \in \tau_n(T) \) such that as \( n \to \infty \), \( t_n \to 0 \), \( s_n \to 0 \), and \( \frac{t_n}{s_n - t_n} \to 0 \).

Claim 2. Either case 1 or case 2 holds.

Proof of Claim: Take a subsequence \( t_n < 1/n^3 \) \( \forall n \) which exists by the fact that \( \lim_{n \to \infty} \tau_n(0) = 0 \). Now suppose that there exists a subsequence \( s_{n_k} \) such that \( s_{n_k} \in \tau_{n_k}(T) \cap (1/n^2, 1/n) \). Then this pair of subsequences satisfies case 2. Suppose that there exists no such subsequence, i.e. \( \forall n > N, \tau_n(T) \cap (1/n^2, 1/n) = \emptyset \). Then take \( r_n \equiv \max \tau_n(T) \cap [0, 1/n^2] \) and \( s_n \equiv \min \{[1/n, M] \cap \tau_n(T) \cup \{M\} \} \). The subsequences \( r_n \) and \( s_n \) satisfy case 1 proving the claim.

Suppose case 1 holds and not case 2. This means that for the relevant subsequences \( t_n, s_n \in \tau_n(T) \), \( \exists b > 0 : s_n > b \ \forall n \). Let \( u_n \equiv L^A(s_n | x_n) \) and \( a_n \equiv x_n(t_n) \). Applying Lemma 12 to \( (t_n, s_n) \) gives

\[
\frac{2}{x_n(t_n) - t_n + \sqrt{L^A(t_n|x_n)}} \left( (a_n - R(t_n, s_n))F([t_n, s_n]) + (a_n - s_n)V_u^-(s_n, L^A(s_n | x_n))F([s_n, M]) \right)
\geq f(t_n) \left( (R(t_n, s_n) - t_n) + (s_n - t_n) \frac{F([s_n, M])}{F([t_n, s_n])} V_u^-(s_n, L^A(s_n | x_n)) \right). \tag{7}
\]

Now let \( \tilde{u}_n \equiv (a - s_n)^2 + \rho(s_n - R(0, s_n)) \) for \( a \in A \), and consider the alternative allocation

\[\tilde{x}_n^a(t) \equiv \begin{cases} a & t' < s_n \\ x_{s_n, \tilde{u}_n}(t') & t \geq s_n \end{cases}.\]

Now choose \( a^* \leq s_n \) to minimize \( L^P(x_n^{a^*}) \). The associated first order condition in \( a^* \) means that

\[(a^* - R(0, s_n)) + (a^* - s) \frac{F([s_n, M])}{F([0, s_n])} V_u^-(s_n, \tilde{u}_n) = 0.\]

Because of (7) and the fact that \( t_n \to 0 \) and \( s_n > b \), \( L^P(\tilde{x}_n^{a^*}) - L^P(x_n^{a^*}) \) is bounded away from 0. However, also since \( t_n \to 0 \), \( L^P(\tilde{x}_n^{a^*}) - L^P(x_n) \to 0 \) as \( n \to \infty \). This contradicts the fact that \( x_n \) is optimal for large \( n \).

Now consider that Case 2 holds. There exists a pair of subsequences \( t_n, s_n \in \tau_n(T) \) such
that as $n \to \infty$, $t_n \to 0$, $s_n \to 0$, and $\frac{t_n}{s_n-t_n} \to 0$. By Lemma 11,

$$L^A(s_n|x_n) \geq \left( \sqrt{L^A(t_n|x_n)} + \rho(s_n - t_n) - (s_n - t_n) \right)^2 \equiv u_n.$$  

By the alignment principle, this means that $L^P(x_n) \geq \int_0^{s_n} (x_n(t') - t')^2 f(t') dt' + F([s_n, M]) V(s_n, u_n)$. Now construct an alternate sequence of allocations $z_n$ defined by

$$z_n(t) \equiv \begin{cases} s_n & t < s_n \\ x_{s_n, \rho(s_n-R(0,s_n))}(t) & t \geq s_n \end{cases}.$$  

The loss from $z_n$ is $L^P(z_n) = \int_0^{s_n} (s_n - t')^2 f(t') dt' + F([s_n, M]) V(s_n, \rho(s_n-R(0,s_n))).$ Notice that because $\frac{t_n}{s_n-t_n} \to 0$, $u_n - \rho(s_n-R(0,s_n)) \sim \rho s_n/2$ for large $n$, i.e. it is approximately linear in $n$. Because of the alignment principle, this means that the difference in loss between $x_n$ and $z_n$ for types in $[s_n, M]$ is positive and linear in $s_n$. Moreover, because $s_n \to 0$ the difference in loss between $x_n$ and $z_n$ on $[0, s_n]$ is second order. This means that for large $n$, $z_n$ does better than $x_n$ contradicting the latter allocation’s optimality. This completes the proof for case 2 and thereby the full argument.  

Q.E.D.

D. Proofs from Section 4

D.1. Proof of Lemma 5

Proof. Note the $\pi(t, u)$ is continuously differentiable in $s \forall u, t$. In addition, $\forall u \pi(0, u) - u = 0$. This means that in order to prove the lemma I only need to establish that $\pi(t, u) - u$ is strictly single crossing from below for $u \leq \rho^2/4$, and that $\pi(t, u) - \rho^2/4$ is strictly single crossing from below for $u \geq \rho^2/4$.

Note that $\pi(t, u) = u + \rho t - 2t \sqrt{u + \rho R(t)} + t^2$. First suppose $u < l = \rho^2/4$. First, suppose $t > 0$ and $\pi(t, u) = u$, i.e.

$$\rho t - 2t \sqrt{u + \rho R(t)} + t^2 = 0.$$  

(8)
I next evaluate the derivative of $\pi(t, u)$ with respect to $t$.

\[
\frac{d(\pi(t, u))}{dt} = \rho - 2\sqrt{u + \rho R(t)} + 2t - \frac{\rho t R'(t)}{\sqrt{u + \rho R(t)}}
\]

\[
= t \left( \frac{\rho R'(t)}{\sqrt{u + \rho R(t)}} \right)
\]

\[
= \frac{1}{2\sqrt{u + \rho R(t)}} \left( 2t\sqrt{u + \rho R(t)} - 2\rho t R'(t) \right)
\]

\[
> \frac{1}{2\sqrt{u + \rho R(t)}} \left( 2t\sqrt{u + \rho R(t)} - \rho t \right)
\]

\[
= \frac{t^2}{2\sqrt{u + \rho R(t)}} > 0.
\]

The first and last equality use (8). The inequality is from the fact that $R'(t) < 1/2$ as every exponential distribution is decreasing. Note that for $t = 0$, $\frac{d(\pi(t, u))}{ds} = \rho - 2\sqrt{u}$ which is positive if $u \leq l^2/4$. This means that $\pi(t, u) - u$ is strictly single crossing from below.

Now suppose $u > \rho^2/4$ and $\pi(t, u) = \rho^2/4$ which can only occur for $t > 0$. This means that,

\[
\rho t - 2t\sqrt{u + \rho R(t)} + t^2 < 0. \tag{9}
\]

Again, I next evaluate the derivative of $\pi(t, u)$ with respect to $t$.

\[
\frac{d(\pi(t, u))}{dt} = \rho - 2\sqrt{u + \rho R(t)} + 2t - \frac{\rho t R'(t)}{\sqrt{u + \rho R(t)}}
\]

\[
> t - \frac{\rho R'(t)}{\sqrt{u + \rho R(t)}}
\]

\[
> t \left( 1 - \frac{\rho}{2\sqrt{u + \rho R(t)}} \right) > 0.
\]

The first inequality follows from (9), the second inequality follows again from $R'(t) < 1/2$, and the last inequality follows from $u > \rho^2/4$. This means that $\pi(t, u) - \rho^2/4$ is strictly single crossing from below.

Q.E.D.

D.2. Proof of Proposition 2

Proof. I begin by proving the following approximation claim on the principal’s and agent’s loss for small intervals.

Claim 3. Take any continuing allocation $x$ at $u \geq 0$ with a threshold at $\delta \in \tau(T)$ with $\delta < \tilde{\delta}$. 


Let \( z \) be an alternative pooling continuing allocation with first threshold \( \delta \), i.e. it satisfies \( z(t') = \sqrt{u + \rho R(\delta)} \forall t \in [0, \delta] \). It holds that

\[
\int_0^\delta ( (z(t') - t')^2 - (x(t') - t')^2 ) \lambda e^{-\lambda t'} dt' \\
\leq \max\{0, L^A(\delta|z) - L^A(\delta|x)\} \\
\leq 2\delta^{3/2} \sqrt{\rho} \tag{11}
\]

**Proof of Claim:** Note that,

\[
\int_0^\delta ( (z(t') - t')^2 - (x(t') - t')^2 ) \lambda e^{-\lambda t'} dt' \\
= \int_0^\delta ( L^A(t'|z) - L^A(t'|x) ) \lambda e^{-\lambda t'} dt' \\
\leq \max_{t' \leq \delta} L^A(t'|z) - L^A(t'|x),
\]

where the last inequality is due to the Cauchy mean value theorem. By Lemma 7 the derivative of \( L^A(t'|z) - L^A(t'|x) \) with respect to \( t' \) is given by \( 2(x(t') - z(t')) \). Since \( x \) is increasing and \( z \) is constant, the integrand is convex in \( t' \) and thereby maximized at its endpoints. Because both are continuing allocations at \( u, L^A(0|z) - L^A(0|x) = 0 \), which delivers the first inequality. By Lemma 11

\[
L^A(\delta|z) - L^A(\delta|x) \\
\leq - \left( \sqrt{u + \rho R(\delta)} - \delta \right)^2 + \rho(\delta - R(\delta)) - \left( \sqrt{u + \rho \delta} - \delta \right)^2 \\
= 2\delta(\sqrt{u + \rho \delta} - \sqrt{u + \rho R(\delta)}) \\
\leq 2\delta^{3/2} \sqrt{\rho}.
\]

Suppose that for some \( u \geq 0 \), \( L^P(y_u) > V(u) + 2\varepsilon \) for some small \( \varepsilon > 0 \). Let \( y^* \) be an associated optimal continuing allocation at \( u \) that achieves \( L^P(y^*) < V(u) + \varepsilon \). Suppose first that there is a highest threshold \( t = \max\{\mathcal{T}_{y^*}(T)\} \) under \( y^* \), i.e. \( y^* \) is pooling after \( t \). Then
note that
\[
\int_t^\infty \lambda(y^*(t') - t')^2 e^{-\lambda t'} dt'
= \lim_{t \to \infty} \int_0^T \lambda(a(T, u) - t')^2 e^{-\lambda t'} dt' + e^{-\lambda T} L^P(y_{\tilde{\pi}(T, u)})
> L^P(y_{L^A(t|y^*)})
\]

where the equality uses \( \frac{dL^P(y_u)}{du} \leq 1 \) and the inequality is by Lemma 5 and the assumption that \( y_u \) satisfies the Bellman equation. In this case set \( t_0 = t \).

Now suppose that \( \{\tau_{y^*}(T)\} \) is unbounded. There must exist a threshold \( s \in \tau(T) \) for which \( L^P(y_{L^A(s|y^*)})\lambda e^{-\lambda t'} dt' < \varepsilon/2 \). If not, since \( \frac{dL^P(y_u)}{du} \leq 1 \), this would imply that \( L^P(y^*) = \infty \). Take an \( s \) that satisfies \( L^P(y_{L^A(s|y^*)})\lambda e^{-\lambda t'} dt' < \varepsilon/2 \) and set \( t_0 = s \).

Take arbitrary \( \delta > \delta > 0 \). I will iteratively revise \( y^* \) at a sequence of thresholds decreasing from \( t_0 \) to 0 in finite steps and end with \( y_{\tilde{u}} \). Define \( t_{i+1} \) recursively as follows.

Case 1: there exists a first threshold \( s \in \tau_{y^*}(T) \) such that \( s \leq t_i - \delta \), i.e. \( \tau_{y^*}(t') = s \forall t' \in [s, t_i) \), in which case define \( t_{i+1} \equiv s \).

Case 2: case 1 does not hold, in which case define \( t_{i+1} = \min\{[t_i - \delta] \cap \tau_{y^*}(T)\} \).

Note that one of the two cases must hold, and that in either case \( t_i - t_{i+2} > \delta \forall i \). Now define,

\[
y_i(t') \equiv \begin{cases} y^*(t') & t' < t_i \\ y_{L^A(t_i|y^*)}(t' - t_i) & t' \geq t_i \end{cases}
\]

By construction the loss from changing from \( y^* \) to \( y_0 \) is less than \( \varepsilon/2 \). I now analyze the change in loss from \( y_i \) to \( y_{i+1} \). Consider the continuing allocation \( z \) at \( L^A(t_{i+1}|y^*) \) defined by

\[
z(t) \equiv \begin{cases} a(t_i - t_{i+1}, L^A(t_{i+1}|y^*)) & t' < t_i - t_{i+1} \\ y_{\tilde{\pi}(t_i-t_{i+1},L^A(t_{i+1}|y^*))}(t' - t_i) & t \geq t_i - t_{i+1} \end{cases}
\]

Allocation \( y_i \) uses \( y^* \) below \( t_i \) and some \( y_u \) above \( t_i \). Allocation \( z \) pools \( t_{i+1} \) to \( t_i \) and then
uses some $y_u$ above $t_i$. The losses from $z$ and $y_i$ above $t_{i+1}$ are given respectively by

$$
\int_{t_{i+1}}^{t_i} \lambda \left(a(t_i - t_{i+1}, L^A(t_{i+1}|y^*)) - (t' - t_{i+1}) \right)^2 e^{-\lambda t'} dt' + L^P \left(y_{\pi(t_i-t_{i+1}, L^A(t_{i+1}|y^*))} \right) e^{-r_{t_i}}, \tag{12}
$$

$$
\int_{t_{i+1}}^{t_i} \lambda (y^* - t')^2 e^{-\lambda t'} dt' + L^P \left(y_{L^A(t_i|y^*)} \right) e^{-r_{t_i}}. \tag{13}
$$

If $t_{i+1}$ is chosen according to case 1, then $z(t - t_{i+1}) = y_i(t) \ \forall t \geq t_{i+1}$ and the two losses are equivalent. If not, then $t_i - t_{i+1} < \delta$. Now note that the difference between (12) and (13) is

$$
\int_{0}^{t_i-t_{i+1}} \left( \left( a(t_i - t_{i+1}, L^A(t_{i+1}|y^*)) - t' \right)^2 - \left( y^*(t' + t_{i+1}) - (t' + t_{i+1}) \right)^2 \right) \lambda e^{-\lambda t'} dt' \frac{1 - e^{-\lambda(t_i - t_{i+1})}}{1 - e^{-\lambda(t_i - t_{i+1})}} e^{-\lambda t_i}
\leq 2\sqrt{p(t_i - t_{i+1})^{3/2}} (e^{-\lambda t_i} - e^{-\lambda t_i}) + 2\sqrt{p(t_i - t_{i+1})^{3/2}} e^{-\lambda t_i}
\leq 2\sqrt{\delta^{3/2}} \sqrt{\rho}. \tag{14}
$$

The first inequality applies Claim 3 twice and the fact that $\frac{d L^P(y_u)}{du} \leq 1$. The loss from $y_{i+1}$ above $t_{i+1}$ is $L^P \left(y_{L^A(t_{i+1}|y^*)} \right) e^{-\lambda t_{i+1}}$ which is less than that of $z$ by Lemma 5 and the assumption that $y_u$ satisfies the Bellman equation. Thus the expression in (14) is actually an upper bound on the difference in loss between $y_{i+1}$ and $y_i$.

Note that because $t_i - t_{i+2} > \delta$ there exists $n < 2t_0/\delta$ such that $y_n = y_u$. Thus the total change in loss at the end of the process is bounded above by the sum over (14) given by

$$
\sum_{i=1}^{n} 2\delta^{3/2} \sqrt{\rho}
\leq 4t_0 \delta^{1/2} \sqrt{\rho}
$$

which goes to 0 with $\delta$. Therefore, by choosing $\delta$ small enough, one can guarantee that this sum is less than $\varepsilon/2$, which is a contradiction to the hypothesis that $L^P(y_u) > V(u) + 2\varepsilon$.

**Q.E.D.**

### D.3. Proof of Theorem 2

**Proof.** First I compute the derivative of $V_s(u)$.  

$$
\sum_{i=1}^{n} 2\delta^{3/2} \sqrt{\rho}
\leq 4t_0 \delta^{1/2} \sqrt{\rho}
$$

which goes to 0 with $\delta$. Therefore, by choosing $\delta$ small enough, one can guarantee that this sum is less than $\varepsilon/2$, which is a contradiction to the hypothesis that $L^P(y_u) > V(u) + 2\varepsilon$.
Claim 4.

\[ V_s'(u) = \int_0^\infty \frac{\rho - 2\sqrt{D_u(t')}}{\rho - 2\sqrt{u}} \lambda e^{-\lambda t'} dt' = \lambda \frac{V_s(u) - u}{\rho - 2\sqrt{u}} \leq 1 \quad \forall u \geq 0. \]

Proof of Claim: The second equality is obtained through integration by parts, i.e.

\[ V_s(u) = \int_0^\infty D_u(t') \lambda e^{-\lambda t'} dt' = u + \int_0^\infty (\rho - 2\sqrt{D_u(t')}) e^{-\lambda t'} dt', \]

where the second inequality uses \( D_u'(t) = \rho - 2\sqrt{D_u(t')} \) which follows from (2b). The inequality in the claim follows from inspecting the second term and using the fact in Lemma 2 that \( \rho - 2\sqrt{u} \geq (\leq) 0 \implies D_u(t) \geq (\leq) u \). To see the first equality, note that because of (2b), \( \forall t_2 \geq t_1 \geq 0, D_u(t_2) = D_u(t_1)(t_2 - t_1) \). Thus define \( m_u(v) \) be the type \( t \) that solves \( D_u(t) = v \). By Lemma 2, \( m_u(v) \) is well defined for all \( u,v \) such that either \( \rho^2/4 \geq v \geq u \) or \( \rho^2/4 \leq v \leq u \). Using implicit differentiation gives \( m_u'(v) = \frac{1}{\rho - 2\sqrt{D_u(m_u(v))}} \). Thus,

\[ \frac{d}{du} \frac{D_u(t)}{d\varepsilon} \bigg|_{\varepsilon=0} = \frac{\rho - 2\sqrt{D_u(t')}}{\rho - 2\sqrt{u}}. \]

This proves the claim.

I will confirm that

\[ V_s(u) = \min_{t \geq 0} \int_0^t \lambda(a(t, u) - t')^2 e^{-\lambda t'} dt' + e^{-\lambda u} V_s(\bar{u}(t, u)) \quad (15) \]

for \( u \geq \rho^2/16 \). I will first show that this condition is tighter for higher \( \lambda \), i.e. if it holds for some fixed \( \lambda \) then it holds for all lower \( \lambda \). I will then show that there exists a large \( \lambda \) such that the condition holds for all \( \lambda' > \lambda \). Define

\[ \bar{V}(t, u) \equiv \int_0^t \lambda(a(t, u) - t')^2 e^{-\lambda t'} dt' + e^{-\lambda u} V_s(\bar{u}(t, u)) \quad (16) \]

as the principal’s loss from choosing first threshold \( t \) and separating thereafter given initial loss \( u \).

**Part 1:** There exists a large \( \bar{\lambda} \) such that \( \forall \lambda > \bar{\lambda}, \forall t > 0, \) and \( \forall u > \rho^2/16, \bar{V}(t, u) - V_s(u) > 0. \)

More specifically, for the same qualifiers, I will show that \( \frac{d\bar{V}(t,u)}{dt} \geq 0. \) Since \( \bar{V}(0, u) = V_s(u) \), this completes part 1. I begin by expanding and simplifying this derivative condi-

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\[
\begin{align*}
&d \left( \int_0^t \lambda (a(t, u) - t')^2 e^{-\lambda t'} dt' + e^{-\lambda t} V_s(\overline{u}(t, u)) \right) \\
= &\rho R'(t) \left( a(t, u) - R(t) \right) \left( 1 - e^{-\lambda t} \right) + \left( (a(t, u) - t)^2 - V_s(\overline{u}(t, u)) \right) \lambda e^{-\lambda t} \\
&+ e^{-\lambda t} V_s'(\overline{u}(t, u)) \frac{d}{dt} \overline{u}(t, u) \geq 0 \\
\iff &\rho \lambda e^{-\lambda t} (R(t) - t) \left( \frac{a(t, u) - R(t)}{a(t, u)} \right) + \left( \overline{u}(t, u) - \rho (t - R(t)) - V_s(\overline{u}(t, u)) \right) \lambda e^{-\lambda t} \\
&+ e^{-\lambda t} V_s'(\overline{u}(t, u)) \frac{d}{dt} \overline{u}(t, u) \geq 0 \\
\iff &\rho (t - R(t)) \frac{R(t)}{a(t, u)} + \left( \overline{u}(t, u) - V_s(\overline{u}(t, u)) \right) \\
&+ V_s'(\overline{u}(t, u)) \lambda \frac{d}{dt} \overline{u}(t, u) \geq 0,
\end{align*}
\]

where the second equality uses \( R'(t) = (t - R(t)) \frac{\lambda e^{-\lambda t}}{e^{-\lambda t}} \) and \( \overline{u}(t, u) = (a(t, u) - t)^2 + \rho (t - R(t)) \). Using Claim 4 gives,

\[
V_s'(\overline{u}(t, u)) \frac{d}{dt} \overline{u}(t, u) = \lambda \frac{V_s(\overline{u}(t, u)) - \overline{u}(t, u)}{l - 2 \sqrt{\overline{u}(t, u)}} \left( l - \frac{2 R'(t)}{a(t, u)} - 2(a(t, u) - t) \right).
\]

Plugging the above identity to the above inequality gives

\[
\rho (t - R(t)) \frac{R(t)}{a(t, u)} + \frac{V_s(\overline{u}(t, u)) - \overline{u}(t, u)}{\rho - 2 \sqrt{\overline{u}(t, u)}} \left( -\frac{2 a(t, u)}{\rho - 2 \sqrt{\overline{u}(t, u)}} + 2 \sqrt{\overline{u}(t, u)} + t - a(t, u) \right) \geq 0
\]

\[
\iff R(t) + \frac{V_s(\overline{u}(t, u)) - \overline{u}(t, u)}{\rho - 2 \sqrt{\overline{u}(t, u)}} \left( \frac{2 a(t, u)}{\rho - 2 \sqrt{\overline{u}(t, u)}} + \sqrt{\overline{u}(t, u)} - t \right) \geq 0. \]

The last implication uses the identity \((\sqrt{\overline{u}(t, u)} + t - a(t, u))(\sqrt{\overline{u}(t, u)} + a(t, u) - t) = \rho (t - R(t)) \). Reusing this identity and dividing both sides by \( R(t) \) reduces the above inequality to,

\[
\frac{V_s(d^2_1) - d^2_1}{l - 2d_1} \left( \frac{\frac{1}{\sqrt{d_0}}(2(d_0 + d_1) - \rho) + \rho}{(d_0 + d_1)^2} + \lambda \right) \geq 1, \quad (17)
\]

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where \( d_1 \equiv \sqrt{a(t, u)} \) and \( d_0 \equiv a(t, u) - t \).

Define \( L_{u, \lambda} \equiv \{ t : R(t) + \rho/4 < \sqrt{u + \rho R(t)} \} \).

**Claim 5.** \( t \in L_{u, \lambda} \implies d_1 + d_0 > \rho/2 \). Moreover, for \( \rho \lambda > 2 \) and \( \forall u \geq \rho^2/16 \), \( L_{u, \lambda} = [0, \infty) \).

**Proof of Claim:** Suppose instead that \( d_1 + d_0 \leq \rho/2 \). By definition \((d_1 + d_0)(d_1 - d_0) = \rho(t - R(t))\). Using the assumed inequality gives

\[
\rho/2(\rho/2 - 2d_0) \geq \rho(t - R(t)) \\
\iff \rho/2 - 2(a(t, u) - t) \geq 2(t - R(t)) \\
\iff \rho/4 + R(t) \geq \sqrt{u + \rho R(t)} \\
\implies t \notin L_{u, \lambda}
\]

The last implication follows from \( u \geq \rho^2/16 \). Next, if \( \rho \lambda > 2 \), then \( R(t) < \rho/2 \) which implies that \( \forall u \geq \rho^2/16 \), \( L_{u, \lambda} = [0, \infty) \).

**Claim 6.** \( \forall t > t > 0 \), there exists large enough \( \lambda \) such that (17) holds \( \forall t \in [\underline{t}, \overline{t}] \) and \( \lambda > \lambda \).

**Proof of Claim:** Rewrite (17) as follows.

\[
V_s'(d_1^2) \left( \frac{\frac{t}{R(t)}(2(d_0 + d_1) - l) + l/\lambda}{(d_0 + d_1)^2} + 1 \right) \geq 1
\]

Note that, by inspecting the second term in Claim 4, \( \lim_{\lambda \to \infty} V_s'(d_1^2) = 1 \).

Note that \( \lim_{\lambda \to \infty} \frac{t}{R(t)}(2(d_0 + d_1) - l) + l/\lambda \left( \frac{1}{(d_0 + d_1)^2} \right) = t \lim_{\lambda \to \infty} \frac{2(d_0 + d_1) - l}{(d_0 + d_1)^2} \). By claim 1, this latter expression is bounded away from 0 for any \( \lambda \) and fixed \( t > 0 \). Note also that for large enough \( k \), the LHS is Lipschitz continuous in \( t \) on any compact interval for some constant. The reason is that \( t/\lambda R(t), d_1, d_0 \), and \( V_s' \) are all themselves Lipschitz continuous on \([\underline{t}, \overline{t}]\) for any large \( \lambda \) with the same constant, and by the claim \( d_1 + d_0 \) is bounded away from 0. Lipschitz continuity of the first 3 follows from their explicit definitions. The Lipschitz continuity of \( V_s' \) is implied by its limit in \( \lambda \) being 1 and the fact that \( V_s'' > 0 \) which is proved below. This means that the LHS converges uniformly to its pointwise limit on any compact interval.

Since the pointwise limit is greater than 1, this completes the proof of the claim.

---

\[13\] This holds for any continuing allocation \( x \) at \( u \) as the associated change in principal loss from changing initial loss is \( \int_0^\infty d \frac{L^1(t|x) \lambda e^{-\lambda t}}{d_0} dt \), i.e. it is the expectation of changes in each type's loss from changing the loss of type 0. As \( \lambda \to \infty \) the distribution becomes a point mass on type 0, so this expectation simply becomes the change in the loss of type 0 from changing the loss on type 0.
Claim 7. $V_s''(u) \geq 0 \forall u$.

Proof of Claim: using the differential equation for $V_s$ twice gives

$$0 \leq V_s''(u)$$

$$\iff 0 \leq \lambda(V_s(u) - u) - (\rho - 2\sqrt{u}) + \frac{(V_s(u) - u)}{\sqrt{u}}$$

Note that if $u = \rho^2/4$, then $V_s(u) = \rho^2/4$ and the above holds with equality. If $u < (>\) \rho^2/4$, the above reduces to,

$$V_s'(u)(\lambda + 1/\sqrt{u}) \geq (\leq)1.$$  

But note that $V_s'(u)(\lambda + 1/\sqrt{u}) - 1$ is single crossing from above in $u$. To see this, assume $V_s'(u)(\lambda + 1/\sqrt{u}) - 1 = 0$ at some $u$ which implies $V_s''(u) = 0$. Then,

$$\frac{d (V_s'(u)(\lambda + 1/\sqrt{u}) - 1)}{du} = V_s''(u)(\lambda + 1/\sqrt{u}) - 2u^{-3/2}V_s'(u)$$

$$= -2u^{-3/2}V_s'(u) < 0$$

proving the claim.

Claim 8. (17) holds for $t = 0$ for large enough $\lambda$.

Proof of Claim: When $t = 0$, (17) reduces to

$$\frac{V_s(u) - u}{\rho - 2\sqrt{u}} + \left(\frac{8\sqrt{u} - \rho}{4u} + \lambda\right) \geq 1.$$  \hspace{1cm} (18)

Suppose first that $u \leq \rho^2/4$. I use integration by parts to rewrite $V_s(u) = u + 1/\lambda \int_0^\infty (\rho - 2\sqrt{D_u(t')}\lambda e^{-\lambda t'} dt'$. Since $\sqrt{D_u(t)}$ is concave, Jensen’s inequality implies that $V_s(u) - u \geq 1/\lambda(\rho - 2\sqrt{D_u(1/\lambda)})$. Thus (17) at $t = 0$ holds if

$$\frac{8\sqrt{u} - \rho}{4u}(\rho - 2\sqrt{D_u(1/\lambda)}) \geq 2(\sqrt{D_u(1/\lambda)} - \sqrt{u})\lambda$$
Taking the limit of both sides as $\lambda \to \infty$, this inequality reduces to,

$$\frac{8\sqrt{u} - \rho}{4u} (\rho - 2\sqrt{u}) \geq \frac{\rho - 2\sqrt{u}}{\sqrt{u}}$$

$$\iff u \geq \rho^2/16.$$ 

By taking limits one can show that $V''(\rho^2/4) = \frac{\lambda \rho}{\lambda \rho + 2}$. This means that (18) holds at $u = \rho^2/4$. Now, I will show that the LHS of (18) single crosses 1 from below in $u$ for $u \geq \rho^2/4$. To see this, note that (18) for $u \geq \rho^2/4$ is equivalent to both

$$V'_s(u) \geq \frac{4u \lambda}{8\sqrt{u} - \rho + 4u \lambda}, \text{ and}$$

$$V_s(u) \frac{8\sqrt{u} - \rho + 4u \lambda}{3u + 4u^2 \lambda} \leq 1$$

Assume a crossing at some $u \geq \rho^2/4$, i.e. that both of the above inequalities hold with equality. I will show the derivative of the LHS of the second inequality in $u$ is negative. This condition is equivalent to,

$$V'_s(u)(8\sqrt{u} - \rho + 4u \lambda) + (4\sqrt{u} + 4\lambda)V_s(u) \leq 3 + 8u \lambda$$

$$\iff 4u \lambda + (4\sqrt{u} + 4\lambda) \frac{3u + 4u^2 \lambda}{8\sqrt{u} - \rho + 4u \lambda} \leq 3 + 8u \lambda$$

$$\iff 0 \leq (3 + 4u \lambda)(4\sqrt{u} - \rho)$$

$$\iff u \geq \rho^2/16,$$

where the first equivalence uses the identities assumed by a crossing at $u$. This means that (18) holds for all $u \geq \rho^2/16$ proving the claim.

**Claim 9.** There exists small $\zeta > 0$ and $\lambda$ such that for $\lambda > \lambda$ the derivative of the LHS of (17) is positive for $t \in [0, \zeta]$.

**Proof of Claim:**

To ease notation for this claim write $d_1 + d_0 \equiv P$ and $\frac{d(d_1 + d_0)}{dt} \equiv dP$. The derivative of the
LHS of (17) is equivalent to

\[ \pi_t(t, u)V''_s(d_1^2) \left( \frac{t}{\lambda R(t)} \left( 2P - \rho \right) + \frac{\rho}{\lambda} \right) + 1 \]

\[ + V'_s(d_1^2) \frac{2P - \rho}{P^2} \frac{dt}{\lambda R(t)} \]

\[ + V'_s(d_1^2) \frac{2dP}{P^4} \frac{t}{\lambda R(t)} (P - P^2) + P/\lambda \]

(19)

The first line of (19) converges to 0 uniformly for small \( t \) as \( \lambda \to \infty \) because \( \pi_t(t, u) \) and the term in parentheses are uniformly bounded for small \( t \) and all large \( \lambda \), and \( V''_s(d_1^2) \to 0 \) uniformly as \( \lambda \to \infty \). This last point can be seen by noting that \( \forall u > 0, \)

\[ V''_s(u) = \begin{cases} 
\int_0^\infty \frac{\rho - 2\sqrt{D_u(t') - 2u}}{\rho - 2\sqrt{u}} \frac{2\sqrt{D_u(t') - 2u}}{\sqrt{u} \sqrt{D_u(t')}} \lambda e^{-\lambda t'} dt' & u \neq \rho^2/4 \\
\frac{4\lambda \rho}{(\lambda \rho + 2)(\lambda \rho + 4)} & u = \rho^2/4 .
\end{cases} \]

For any \( u > 0 \), the integrand above has a bounded derivative in \( t' \) close to 0, and so \( V''_s(u) \) converges to 0 uniformly in \( \lambda \).

Note that \( \frac{t}{\lambda R(t)} \) is convex in \( t \) for all \( \lambda \) and bounded below by \( t/3 + 2/\lambda \) with equality at \( t = 0 \). Thus, its derivative \( -\frac{dt}{\lambda R(t)} \geq 1/3 \) for all \( \lambda, t \). Since \( V'_s(d_1^2) \to 1 \) uniformly as \( \lambda \to \infty \), and \( \frac{2P - \rho}{P^2} \) is bounded away from 0 on any compact interval (both by previous claims), the second line of (19) is bounded away from 0 for small \( t \) and large \( \lambda \).

Lastly, the third line of (19) converges uniformly to 0 as \( t \to 0 \) and \( \lambda \to \infty \). This is because \( \frac{t}{\lambda R(t)} \leq t + 2/\lambda \) and the other terms are uniformly bounded for small \( t \) and large \( \lambda \). This completes the proof of the claim.

**Claim 10.** There exists \( \bar{t} \) and \( \bar{\lambda} \) such that \( \forall \lambda \geq \bar{\lambda} \) and \( s > \bar{t} \), (17) holds.

**Proof of Claim:** Note that for large enough \( t \) such that \( t > a(t, u), \)

\[ d_1 + d_0 = \sqrt{(a(t, u) - t)^2 + \rho(t - R(t))} - (t - a(t, u)) \]

\[ \leq \frac{\rho(t - R(t))}{2(t - a(t, u))} , \]

Where the inequality uses concavity of the square root function. Since this last expression
goes to $\rho/2$ uniformly for $\lambda > \bar{\lambda}$, for large enough $t$ and $\bar{\lambda}$

\[ \frac{t}{\lambda R(t)} (2(d_0 + d_1) - \rho) + \rho/\lambda \]

\[ \geq t(2(d_0 + d_1) - \rho)/\rho^2 \]

\[ \geq t/\rho^2 \left( 2 \left( \min \left\{ \sqrt{u - t} + \sqrt{u + t^2 + t(\rho - 2\sqrt{u})}, \sqrt{u + \rho/\bar{\lambda}} - t + \sqrt{u + t^2 + t(\rho - 2\sqrt{u + \rho/\bar{\lambda}})} \right\} - \rho \right) \]

To see the last inequality, note that $d_1 + d_0$ is quasiconcave in $R(t)$: a marginal change in $R(t)$ changes $d_1 + d_0$ by $1/2a(t,u) (1 - t^2 d_1)$, and $d_1$ is decreasing in $R(t)$. Thus, the two expressions in the minimum replace $R(t)$ with its minimum value of 0 and maximum value of $1/\bar{\lambda}$. Now note that,

\[ \lim_{t \to \infty} t \left( 2 \left( \sqrt{u - t} + \sqrt{u + t^2 + t(\rho - 2\sqrt{u})} \right) - \rho \right) = \rho/4 \left( 4\sqrt{u} - 1 \right) \]

\[ \lim_{t \to \infty} t \left( 2 \left( \sqrt{u + \rho/\bar{\lambda}} - t + \sqrt{u + t^2 + \left( \rho - 2\sqrt{u + \rho/\bar{\lambda}} \right)} - \rho \right) \right) = u - 1/4 \left( \rho - 2\sqrt{\rho/\bar{\lambda}} + u \right)^2 \]

Both expressions are strictly positive for $u > \rho^2/16$, completing the proof of the claim. Putting these claims together completes part 1.

**Part 2:** If for some $\tilde{\lambda}, \tilde{V}(t,u) - V_s(u) > 0 \forall u > \rho^2/16$ and $\forall t \in L_{u,\tilde{\lambda}}$, then $\forall \lambda' < \tilde{\lambda}$, $\tilde{V}(t,u) - V_s(u) > 0 \forall t \in L_{u,\lambda'}$ and $\forall u > \rho^2/16$.

Recall the definition, $L_{u,\lambda} \equiv \left\{ t : R(t) + \rho/4 < \sqrt{u + \rho R(t)} \right\}$. Note that $\forall u \geq \rho^2/16$, $L_{u,\lambda} = [0, \bar{t}_{u,\lambda})$, where $\bar{t}_{u,\lambda}$ is increasing in $\lambda, u$.

Suppose the conclusion of part 2 is false, and let $\bar{\lambda}$ be the highest witness to this contradiction less than $\tilde{\lambda}$ with associated $\bar{t}$ so that $\tilde{V}(\bar{t}, u) - V_s(u) \neq 0$. This exists because $\tilde{V}$ is differentiable in $t, \lambda$. Therefore, also by differentiability of $V$ in $t$ and $\lambda$, the following conditions are satisfied at $\bar{\lambda}$ and $\bar{t}$,

\[ \tilde{V}(\bar{t}, u) - V_s(u) = 0, \quad \text{and} \]

\[ \frac{d}{dt} \tilde{V}(\bar{t}, u) = 0. \quad \text{(21)} \]

If (20) did not hold, then a slightly higher $\lambda$ would also violate the condition. If (21) did not hold then some nearby $t$ to $\bar{t}$ would have $\tilde{V}(t, u) - V_s(u) < 0$, and then by the same logic one can find a higher $\lambda$ that also violates the condition.
It will be helpful to explicitly notate $\lambda$ in $R(t) \equiv R(t, \lambda)$. Let $k > 0$, and $u > 0$. Let $y \equiv \sqrt{\rho R(\bar{t}, k) + u}$ and $\bar{u} \equiv (y - t)^2 + \rho(s - R(\bar{t}, k))$. Define the continuing allocation

\[
x(t') \equiv \begin{cases} 
y & t' < \bar{t} \\
d_\pi(t' - \bar{t}) + \bar{t} & t' > \bar{t}.
\end{cases}
\]

Note that for $k = \bar{\lambda}$, $\bar{V}(\bar{t}, u) - V_s(u) = \int_0^\infty (L^A(t'|x) - D_u(t')) \lambda e^{-\lambda t'} dt'$. Thus,

\[
\frac{d(\bar{V}(\bar{t}, u) - V_s(u))}{d\lambda} = \frac{d(\int_0^\infty (L^A(t'|x) - D_u(t')) \lambda e^{-\lambda t'})}{d\lambda} \bigg|_{\lambda=k=\bar{\lambda}} + \frac{d(\int_0^\infty (L^A(t'|x) - D_u(t')) \bar{\lambda} e^{-\bar{\lambda} t})}{d\lambda} \bigg|_{k=\lambda=\bar{\lambda}}
\]

\[
= \frac{d(\int_0^\infty (L^A(t'|x) - D_u(t')) \lambda e^{-\lambda t'})}{d\lambda} \bigg|_{\lambda=k=\bar{\lambda}} + \int_0^\infty \frac{d(L^A(t'|x))}{dk} \bigg|_{k=\lambda=\bar{\lambda}} \bar{\lambda} e^{-\bar{\lambda} t}.
\]

The rest of the proof of part 2 shows that (22) is negative. Because of (20), this contradicts the fact that $\bar{\lambda}$ is the highest witness to a violation of $\bar{V}(\bar{t}, u) - V_s(u) > 0$ below $\bar{\lambda}$.

**Step 1:** The first term in (23) is negative.

**Claim 11.** $L^A(t|x) - D_u(t)$ is single crossing from below in $t > 0$.

**Proof of Claim:**

\[
\frac{d}{dt} \left( L^A(t|x) - D_u(t) \right) = 2(d_u(t) - x(t)).
\]

Since $d_u(t)$ is increasing, and $x(t)$ is constant for $t < \bar{t}$, $L^A(t|x) - D_u(t)$ is strictly convex in $t$ for $t < \bar{t}$. Also $2(d_u(0) - x(0)) < 0$ and $L^A(0|x) - D_u(0) = 0$, so $L^A(t|x) - D_u(t)$ is single crossing from below in $t$ for $\bar{t} > t > 0$. Since $L^A(t|x) = D_{\pi(t - \bar{t})}$ for $t \geq \bar{t}$, Lemma 2 point 3 implies $\text{sign}(L^A(t|x) - D_u(t)) = \text{sign}(\bar{u} - D_u(\bar{t}))$. Continuity of $L^A(t|x)$ in $t$ completes the proof of the claim.

This means that the first term in (23) is of the form $E[g(t)|t \sim \text{exp}(\lambda)]$, where the function $g$ is single crossing from below in $t$ and independent of $\lambda$. Note also that an increase in $\lambda$ corresponds to a downward monotone likelihood ratio shift in the exponential distribution, i.e. for $\lambda' < \lambda'', \frac{\lambda' e^{-\lambda' t}}{\lambda'' e^{-\lambda'' t}}$ is increasing in $t$. Theorem 2 from Athey (2002) delivers that the single crossing property is preserved under monotone likelihood ratio shifts. Because of (20) $\bar{\lambda}$, represents a crossing in this expectation and so its derivative in $\lambda$ must be negative in order to satisfy single crossing. This completes step 1.
Step 2: The second term in (23) is negative.

First, I rewrite condition (21)

\[
\frac{d}{dt} \left( \int_0^t (a(t, u) - t')^2 e^{-\lambda t'} dt' + e^{-\lambda t} V_s(\bar{\eta}(t, u)) \right) |_{t=\bar{t}} = \rho R_k(s, k) \left( (a(\bar{t}, u) - R(\bar{t}, k))(1 - e^{-\lambda \bar{t}}) - \bar{V}_s(\bar{\eta}(\bar{t}, u))e^{-\lambda \bar{t}} \right)
\]

Decomposing this last equality gives,

\[
((a(\bar{t}, u) - \bar{t})^2 - V_s(\bar{\eta}(\bar{t}, u))) \lambda e^{-\lambda \bar{t}} + V_s'(\bar{\eta}(\bar{t}, u))((\rho - 2(a(\bar{t}, u) - \bar{t}))e^{-\lambda \bar{t}}) \leq 0 \tag{24}
\]

The penultimate equivalence follows from \( R_k(s, k) \leq 0 \). The last expression is the second term in (23). Thus, establishing (24) is sufficient to complete the proof of step 2, and thereby part 2. I now simplify (24),

\[
((a(\bar{t}, u) - \bar{t})^2 - V_s(\bar{\eta}(\bar{t}, u))) \lambda e^{-\lambda \bar{t}} + V_s'(\bar{\eta}(\bar{t}, u))((\rho - 2(a(\bar{t}, u) - \bar{t}))e^{-\lambda \bar{t}}) \leq 0
\]

But note that the simplified version of (21) in (17) means that it suffices to prove that,

\[
\frac{2}{d_0 + d_1} \leq \frac{\lambda}{R(\bar{t})} (2(d_0 + d_1) - \rho) + \frac{\rho}{(d_0 + d_1)^2} + \lambda
\]

\[
\iff (d_0 + d_1) \geq \rho,
\]

43
which was shown to be true in part 1 since $\tilde{t} \in L_{u,\bar{\lambda}}$.

**Part 3:** $\forall \lambda > 0, \forall u \geq \rho^2/16$, and $t \notin L_{u,\lambda}$, $\tilde{V}(t, u) > V_s(u)$.

Let $u \geq \rho^2/16$, $t \notin L_{u,\lambda}$, and $0 \leq r \leq t$. Define the continuing allocation $x$ at $u$ by

$$
x(t') \equiv \begin{cases} 
  d_u(t') & t' < t \\
  a(t - r, D_u(r)) + r & r \leq t' < t \\
  d_{\Pi(t-r, D_u(r))}(t' - t) + t & t' > t 
\end{cases}
$$

That is $x_r$ separates on $[0, r)$, pools on $[r, t)$, and separates again on $(t, \infty)$. Note that $L^P(x_0) = \tilde{V}(t, u)$. Define $t - r \equiv \tilde{t}$.

**Claim 12.**

$$
\frac{d}{dr} L^P(x_r) > 0 \implies \tilde{t} \in L_{D_u(r), \lambda}.
$$

**Proof of Claim:** Let $t - r \equiv \tilde{t}$. By the same logic as the argument for Lemma 12,

$$
\frac{d}{dr} L^P(x_r) \leq 0 \\
\iff \frac{2(a(\tilde{t}, u(t)) - R(\tilde{t}))}{a(\tilde{t}, D_u(r)) + \sqrt{D_u(r)}}(1 - e^{-\lambda\tilde{t}}) - \lambda R(\tilde{t}) \\
+ \left( \frac{2(a(\tilde{t}, D_u(r)) - t)}{a(\tilde{t}, D_u(r)) + \sqrt{D_u(r)}} - \frac{\lambda t}{1 - e^{-\lambda t}} \right) V'_s(\Pi(\tilde{t}, D_u(r))) e^{-\lambda t} \leq 0 \\
\iff \frac{2(a(\tilde{t}, u(t)) - R(\tilde{t}))}{a(\tilde{t}, D_u(r)) + \sqrt{D_u(r)}}(1 - e^{-\lambda\tilde{t}}) - \lambda R(\tilde{t}) \leq 0
$$

where the last implication is because the multiplier of $V'_s(\Pi(\tilde{t}, D_u(r))) e^{-\lambda \tilde{t}}$ in the third line is greater than the last line, and $V'_s > 0$. Suppose that $\tilde{t} \notin L_{D_u(r), \lambda}$, i.e. $\sqrt{D_u(r)} + \rho R(\tilde{t}) \leq R(\tilde{t}) + \rho/4$.

$$
\frac{2(a(\tilde{t}, D_u(r)) - R(\tilde{t}))}{a(\tilde{t}, D_u(r)) + \sqrt{D_u(r)}}(1 - e^{-\lambda\tilde{t}}) - \lambda R(\tilde{t}) \leq 0 \\
\iff \frac{2(\rho/4)}{R(\tilde{t}) + \rho/4 + \sqrt{D_u(r)}}(1 - e^{-\lambda\tilde{t}}) \leq \lambda R(\tilde{t}) \\
\iff \frac{2(1 - e^{-\lambda\tilde{t}})}{\lambda} \leq 4/\rho R(\tilde{t})^2 + R(\tilde{t}) + 4/\rho R(\tilde{t}) \sqrt{D_u(r)} \\
\iff \frac{1 - e^{-\lambda\tilde{t}}}{\lambda} \leq 2/\rho R(\tilde{t})^2 + R(\tilde{t}),
$$

where the last line follows from the fact that $L_{\rho^2/16, \lambda} \subset L_{u, \lambda}$ $\forall u \geq \rho^2/16$, so $u = \rho^2/16$ is most
binding. First, I will show that if (25) holds for some \( \tilde{t} = s \), then it also holds \( \tilde{t} = s' > s \).

To see this assume (25) holds at \( s \). The derivative of the LHS of (25) is less than that of the RHS, i.e.

\[
\frac{d}{ds} \left( 1 - e^{-\lambda s} \right) \leq \frac{d}{ds} \left( \frac{2}{\rho R(s)} R(s) + R(s) \right)
\]

\( \iff \) \( e^{-\lambda s} \leq \frac{R'(s)(4/\rho R(s) + 1)}{\lambda} \)

\( \iff \) \( e^{-\lambda s} \leq \frac{\lambda e^{-\lambda s}}{1 - e^{-\lambda s}} (s - R(s))(4/\rho R(s) + 1) \)

\( \iff \) \( \frac{1 - e^{-\lambda s}}{\lambda} \leq \frac{4/\rho R(s) + 1}{\lambda} \)

\( \iff \) \( \frac{1 - e^{-\lambda s}}{\lambda} \leq 2/\rho R(s) + R(s) \),

where the penultimate line uses \( s - R(s) > s/2 \) in the exponential distribution. This means that is suffices to prove (25) for the upper boundary of \( L_{\rho^2/16, \lambda} \), i.e. \( \tilde{t} = \bar{t}_{\rho^2/16, \lambda} \). By definition \( R(\bar{t}_{\rho^2/16, \lambda}) = \rho/2 \), so (25) reduces to

\[
\frac{1 - e^{-\lambda \bar{t}_{\rho^2/16, \lambda}}}{\lambda} \leq \rho
\]

One can show that \( \bar{t}_{\rho^2/16, \lambda} \leq 2\rho/(2 - \lambda \rho) \). Thus it suffices to prove that,

\[
\frac{1 - e^{-2\rho\lambda/(2 - \rho\lambda)}}{\lambda} \leq \rho
\]

This holds because the LHS is equal to \( \rho \) at \( \lambda = 0 \) and decreasing in \( \lambda \). This completes the proof of the claim.

Note that by Lemma 5, because \( u \geq \rho^2/16 \), \( D_u(r) \geq \rho^2/16 \) \( \forall r \in [0, t] \). This means that \( L_{D_u(r), \lambda} \) remains of the form \( [0, \bar{t}_{D_u(r), \lambda}] \). Let \( r^* \in [0, t] \) be the minimum type \( r \) with \( t - r \in L_{D_u(r), \lambda} \). That is, \( \forall r < r^*, t - r \notin L_{D_u(r), \lambda} \). By the claim above, \( \frac{dL^p(x_r)}{dx} \leq 0 \) and so \( L^p(x_{r^*}) \leq \tilde{V}(t, u) \). Now note that since the continuing allocations are the same on \( [0, r^*] \), \( L^p(x_{r^*}) - V_s(u) = e^{-\lambda r^*}(\tilde{V}(t - r^*, D_u(r^*)) - V_s(D_u(r^*))) \). Because of parts 1 and 2, and because \( t - r^* \in L_u(r^*), \lambda \), it holds that \( \tilde{V}(t - r^*, D_u(r^*)) \geq V_s(D_u(r^*)) \). This means that \( \tilde{V}(t, u) \geq V_s(u) \), completing the proof of part 3 and thereby the theorem.

Q.E.D.

D.4. Proof of Lemma 6

Proof. Note that the pooling continuing allocation is not optimal at \( u \geq \rho^2/16 \) by Theorem 2. Assume the pooling allocation is optimal at some \( u \leq \rho^2/16 \). It must be better than
an alternative continuing allocation which introduces a small separating portion at the beginning of the allocation. That is, adapting the condition of Lemma 12 using $\tilde{t} = \infty$, $t = 0$, and $\tilde{x} = \sqrt{u + \rho/\lambda}$ gives,

$$1 - \frac{2(\sqrt{u + \rho/\lambda} - 1/\lambda)}{\sqrt{u + \rho/\lambda} + \sqrt{u}} \leq 0$$

$$\iff \sqrt{u + \rho/\lambda} - \sqrt{u} \geq 2/\lambda$$

$$\iff \rho/\lambda \geq 4/\lambda^2$$

$$\iff \lambda \rho \geq 4$$

Q.E.D.

D.5. Proof of Corollary 1

Proof.

Claim 13. Take an arbitrary incentive compatible allocation $x$. If $\exists \tilde{t} : \tau(\tilde{t}) > \rho/8$, then the following allocation strictly improves on $x$,

$$y(t) \equiv \begin{cases} x(t) & t < \tilde{t} \\ d_{L^A(\tilde{t}|x)}(t) + \tilde{t} & t \geq \tilde{t} \end{cases}$$

Proof of Claim: Suppose that for some $t$, $x(t) - t > \rho/4$, then clearly for the left adjacent threshold, $x(\tau(t)) - \tau(t) > \rho/4$. This means that $L^A(\tau(t)|x) > \rho^2/16$, which would imply that separating is uniquely optimal following this threshold. Now suppose that $x(t) - t \leq \rho/4 \forall t \leq \tilde{t}$. But this means that $L^A(\tau|t) = \rho - 2(x(t) - t) \geq \rho/2 \forall t < \tilde{t}$. But this means that $\tau(\tilde{t}) < \rho/8$.

Take a sequence of incentive compatible allocations $x_n$ such that $L^p(x_n) \to L$ where $L$ is the infimum loss of the principal. Take $t_n \equiv \min \{ \tau_{x_n}(t) : t \in T, L^A(\tau_{x_n}(t)|x_n) \geq \rho^2/16 \}$, where $t_n = \infty$ if the relevant set is empty. Without loss of optimality by Theorem 2, replace $\tilde{x}_n(t)$ with

$$y_n(t) \equiv \begin{cases} x_n(t) & t < t_n \\ d_{L^A(t_n|x_n)}(t) + t_n & t \geq t_n \end{cases}$$

Suppose first there exists a subsequence $t_n$ that converges to some $\tilde{t}$. Then the optimal allocation is found by optimizing the allocation on $[0, \tilde{t}]$ assuming the separating continuation loss above $\tilde{t}$, which exists by Lemma 3 and the fact that the separating continuation loss is continuous in the allocation on $[0, \tilde{t}]$. Now suppose there exists a subsequence such
that \( t_{n_k} \to \infty \), then by the claim \( \max \{ \tau_{y_{n_k}}(T) \} \leq \rho/8 \) by the claim above, and each allocation \( y_{n_k} \) is approximately pooling above \( \max \{ \tau_{y_{n_k}}(T) \} \). Take a convergent subsequence of \( \max \{ \tau_{x_{n_k}}(T) \} \) that converges to \( \bar{t} \). Thus, an optimal allocation is found by optimizing the allocation on \([0, \bar{t}]\) assuming the pooling continuation value above \( \bar{t} \). These are exhaustive cases and so an optimum exists.

Now suppose \( \rho \lambda \leq 4 \). By the claim above, if the optimal allocation has a threshold above \( \rho/8 \) then it is eventually separating. If it does not, then it is eventually pooling which is suboptimal by Lemma 6.

Q.E.D.

D.6. Proof of Proposition 3

Proof. Note that the inequality in Lemma 12 implies that for \( x^* \) optimal, it must be that \( x^*(t) > r_{x^*}(t) \). Expanding this condition gives \( \sqrt{L^A(\tau(t)|x^*) + \rho R(\tau(t) - \tau(t))} > R(\tau(t) - \tau(t)) \). Since the reputation is decreasing in \( \lambda \), this means that \( \forall \lambda \) small, any pooling interval in the continuing allocation at any \( u \) has bounded length. Moreover by inspecting this condition, this bound is uniform for small initial losses \( L^A(\tau(t)|x^*) \). Since the optimal continuing allocation is separating at initial loss \( u > \rho/16 \), this will imply that there exists some \( M \), and \( \bar{\lambda} > 0 \) such that the optimal continuing allocation \( \forall \lambda \leq \bar{\lambda} \) solves,

\[
\min_{x \in IC([0,M]: L^A(0|x)=u)} \int_0^M (x(t') - t')^2 \lambda e^{-\lambda t'} dt' + e^{-\lambda M} V_s(L^A(M|x)).
\]

I complete the proof in three steps. First I extend the result in Theorem 2 to \( u \geq \rho^2/36 \) in the uniform limit. Second I bound the derivative of the minimized continuation loss in the uniform limit. Third, I show that under this conclusion the solution to Equation 4 always chooses the loss at the first threshold to be greater than \( \rho^2/36 \).

Step 1: \( \forall \varepsilon > 0 \exists \bar{\lambda} > 0 \) such that \( \forall \lambda \leq \bar{\lambda}, V(u) = V_s(u) \forall u \geq \rho^2/36 + \varepsilon \).

Because of Corollary 1, the optimal allocation is eventually separating and \( \lim_{\lambda \to 0} V(u) = \rho^2/4 \forall u \geq 0 \). This is because as \( \lambda \to 0 \) the probability distribution puts zero relative weight on any set of lower types, and so \( \lim_{\lambda \to 0} V(u) = \lim_{t \to \infty} L^A(t|x^*) = \rho^2/4 \) where \( x^* \) is an optimal allocation and the limit is by Lemma 2. Let \( V'_0(u) \equiv \lim_{\lambda \to 0} V'(u)/\lambda \). Note that by Claim 4 and Theorem 2 \( \forall u \geq \rho^2/16 \),

\[
V'_0(u) = \lim_{\lambda \to 0} V'_s(u)/\lambda = \lim_{\lambda \to 0} \frac{V_s(u) - u}{\rho - 2\sqrt{u}} = \frac{\rho + 2\sqrt{u}}{4}
\]
Now recall (17) and take the limit as $\lambda \to 0$. Note that $\lim_{\lambda \to 0} R(s) = s/2$, and let $d_1 \equiv \lim_{\lambda \to 0} d_1$ and $d_0 \equiv \lim_{\lambda \to 0} d_0$. The limit of the LHS of (17) is given by

$$
\lim_{\lambda \to 0} \left[ V_s(d_1^2) - d_1^2 \left( \frac{s}{R(s)} \left( 2(d_0 + d_1) - l \right) + l \right) \right]
\approx \frac{\rho + 2d_1 (d_0 + d_1) - \rho}{4 (d_0 + d_1)^2}.
$$

This limit where converges uniformly for $M \geq s \geq 0$ and $u \leq \rho^2/16$. The inequality reduces to

$$
2(d_0 + d_1)d_1 + \rho(d_0 + d_1) - \rho/2d_1 - \rho^2/4 \geq (d_0 + d_1)^2
\iff d_1^2 + \rho/2d_1 + \rho d_0 - \rho^2/4 \geq d_0^2
\iff d_1^2 + \rho/2d_1 + \rho d_0 - \rho^2/4 \geq d_1^2 - \rho s/2
\iff 2s + 2d_1 + 4d_0 - \rho \geq 0.
$$

It can be checked that if $u \geq \rho^2/36$ then the inequality holds $\forall s, \rho \geq 0$. This in turn means that $\forall \varepsilon > 0 \exists \lambda > 0$ such that $\forall \lambda \leq \lambda, V(u) = V_s(u) \forall u \geq \rho^2/36 + \varepsilon$.

Step 2: I will show that $\forall \varepsilon > 0, \exists \lambda > 0$ such that $\forall u \leq \rho^2/16 V'(u)/\lambda \leq 3\rho/8 + \varepsilon$.

Consider the alternative representation of allocations by their interval partition, i.e. $x \in IC([0, M])$ is identified with its threshold function $\zeta(t)$. This is the approach taken in Lemma 9. Recall the set of all allowable threshold functions be $\mathcal{R}$. Define $x_\tau$ to be an associated continuing allocation at $u$ with threshold function $\tau \in \mathcal{R}$. Again, because $L^A(t|x)$ is uniformly bounded across all distributions and and $x \in C(u)$, $\mathcal{R}$ does not depend on $\lambda, u$, and $x_{u,\tau}$ is uniquely selected for each $u \geq 0$, $\tau \in \mathcal{R}$.

$$
\min_{\tau \in \mathcal{R}} \int_0^M (x_{u,\tau}(t') - t')^2 \lambda e^{-\lambda t'} dt' + e^{-\lambda M} V_s(L^A(M|x_{u,\tau}))
$$

Define the associated limit problem as $\lambda \to 0$ as,

$$
V_0(u) \equiv \min_{\tau \in \mathcal{R}} \int_0^M (x_{u,\tau}(t') - t')^2 dt' + \int_0^{L^A(M|x_{u,\tau})} \frac{\rho + 2\sqrt{u}}{4} du.
$$

Suppose not. This means there exists $\varepsilon > 0$, with witnesses to the contradiction $\rho^2/16 \geq u \geq 0$ and a sequence $\lambda_n \to 0$ with associated $\tau_n \in \mathcal{R}$ such that $x_{u,\tau_n}$ is an optimal continuing allocation in (26) under $\lambda_n$, and $\frac{dL^A(x_{u,\tau_n})}{du}/\lambda_n \geq \rho^2/16 + \varepsilon \forall n$. Let $u$ be the supremum initial
loss to satisfy these conditions. By step 1, \( u \leq \rho^2/36 \), because \( V'_s(u')/\lambda \) is increasing in \( u' \).

Now take the first threshold under \( x_{u,\tau} \) to be \( t_n > 0 \). Let \( t_n \to t \) passing to a subsequence if necessary, and let \( (\sqrt{u + pt/2 - t})^2 + pt/2 \equiv \tilde{u} \).

\[
\lim_{n \to \infty} \frac{dL^P(x_{u,\tau_n})}{du} \Bigg|_\lambda = \lim_{n \to \infty} \frac{(a(t_n, u) - R(t_n))(1 - e^{-\lambda t_n}) + (a(t_n, u) - t_n)\ e^{-\lambda t_n} V'(\pi(t_n, u))}{\lambda a(t_n), (a(t_n, u) - t_n)\ e^{-\lambda t_n} V'(\pi(t_n, u))}/\lambda
\]

(28)

(29)

\[
(\sqrt{u + pt/2 - t/2})t + \sqrt{u + pt/2 - t} \quad V'_0(\tilde{u}) > \rho/3
\]

\[
\iff t^2/2 - pt/3 + d_0(t - \rho/12 + d_1/2) \geq 0.
\]

By taking the limit of (17) from step 1, \( t \) must also solve \( 2t + 2d_1 + 4d_0 - \rho = 0 \) which can be shown to be incompatible with the above inequality.

Now consider that \( \tilde{u} \leq \rho^2/36 \) so that \( V'_0(\tilde{u}) \leq \rho/3 \). Consider first the case in which \( a(t, \tilde{u}) - t \geq 0 \). But then,

\[
\frac{a(t, u) - t/2}{a(t, u)} + V'(\tilde{u}) \frac{a(t, u) - t}{a(t, u)} \geq \rho/3
\]

\[
\iff \frac{a(t, u) - t/2}{a(t, u)} + \rho/3 \geq \rho/3
\]

\[
\iff a(t, u) - t \geq \rho/3 - t/2
\]

\[
\iff (a(t, u) - t)^2 \geq (\rho/3 - t/2)^2
\]

\[
\iff \tilde{u} \geq (\rho/3 - t/2)^2 + pt/2
\]

\[
\iff \rho^2/36 \geq t^2/4 + pt/6 + \rho^2/9.
\]

But the last line is a contradiction. Now suppose, \( a(t, u) - t < 0 \). \( \sqrt{u + pt/2} < t \implies t > \rho/2 \). But then \( \tilde{u} > \rho^2/4 \) which is a contradiction.

Step 3: I will show that there exists \( \lambda > 0 \), and small enough \( \varepsilon > 0 \), the optimal first
threshold agent loss in (4) has $u_1 \equiv (a_1 - t_1)^2 + \rho(t - R(t)) \geq \rho^2/36 + \varepsilon$. Step 1 then provides that the optimal allocation is separating after the first threshold. At an optimal $t_1, a_1$ in (4) the corresponding FOC in $t_1$ is given by,

$$
((a_1 - t_1)^2 - V(u_1)) \lambda e^{-\lambda t_1} + (\rho(1 - R'(t_1)) + 2(t_1 - a_1)) V'(u_1) e^{-\lambda t_1} \geq 0.
$$

Fix $u_1 = (a_1 - t_1)^2 + \rho(t - R(t)) = u < \rho^2/36$. For sufficiently low $\lambda$, it can be shown that for any such fixed $u$, if the inequality above holds for some value of $a_1$ it holds $a_1 = R(t_1)$. That is, setting $\tilde{u}_1 \equiv (t - R(t))^2 + \rho(t - R(t))$, the above inequality implies

$$
((t_1 - R(t_1))^2 - V(\tilde{u}_1)) + (\rho(1 - R'(t_1)) + 2(t_1 - R(t_1))) V'(\tilde{u}_1) / \lambda \geq 0.
$$

As $\lambda \to 0$, the above inequality converges uniformly across $t_1$ to,

$$
(t_1^2/4 - \rho^2/4 + \rho/2 + t_1) V_0'(\tilde{u}_1) \geq 0
\implies (t_1^2/4 - \rho^2/4 + \rho/2 + t_1) \rho/3 \geq 0
\iff 3t_1^2 + 4\rho t_1 - \rho^2 \geq 0
\iff t_1 \geq \frac{(\sqrt{7} - 2)\rho}{3}.
$$

But this implies $\tilde{u}_1 \geq 1 = \rho^2/36(2\sqrt{7} - 1)$ a contradiction. Q.E.D.