Dynamic Matching Market Design∗

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Abstract

We introduce a simple benchmark model of dynamic matching in networked markets, where agents arrive and depart stochastically and the network of acceptable transactions among agents forms a random graph. The main insight of our analysis is that waiting to thicken the market can be substantially more important than increasing the speed of transactions, and by characterizing the optimal trade frequency under discounting, we show that this insight is quite robust to the presence of waiting costs. We also show that naive local algorithms that maintain market thickness by choosing the right time to match agents, but do not exploit global network structure, can perform very close to optimal algorithms. Finally, our analysis asserts that having information about agents' departure times is highly valuable, since algorithms that employ even short-horizon information on agents' departure times perform substantially better than those that lack such information. To elicit agents' departure times when such information is private, we design an incentive-compatible continuous-time dynamic mechanism without transfers.

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1 Introduction

The theory of matching has guided the designs of many markets, from school choice, to kidney exchange, to the allocation of medical residents. In a series of classic papers, economists have extensively characterized good matching algorithms for static settings.\(^1\) In the canonical set-up, a social planner faces a set of agents who have preferences over partners, contracts, and/or combinations thereof. The planner’s goal is to find a matching algorithm with desirable properties (e.g. stability, efficiency, and/or strategy-proofness). The algorithm is run, a match is made, and the problem ends.

Of course, many real-world matching problems are dynamic. In a dynamic matching environment, agents arrive gradually over time. A social planner continually observes the agents and their preferences, and chooses how to match agents. Matched agents leave the market. Unmatched agents either persist or depart. Thus, the planner’s decision today affects the sets of agents and options tomorrow.

Some seasonal markets, such as school choice systems and the National Residency Matching Program, are well described as static matching problems without intertemporal spillovers. However, some markets are better described as dynamic matching problems. Some examples include:

- **Kidney exchange:** In paired kidney exchanges, patient-donor pairs arrive over time. They stay in the market until either they are matched to a compatible pair, their condition deteriorates, or they receive a cadaveric kidney from a waiting list.

- **Markets with brokers:** Some markets, such as real estate, aircraft, and ship charters, involve intermediary brokers who receive requests to buy or sell particular items. A broker facilitates transactions between compatible buyers and sellers, but does not hold inventory. Agents may withdraw their request if they find an alternative transaction.

- **Allocation of workers to time-sensitive tasks:** Both within firms and online labor markets, such as Uber and oDesk, planners allocate workers to tasks that are profitable to undertake. Tasks arrive continuously, but may expire. Workers are suited to different tasks, but may cease to be available.

In dynamic settings, the planner must decide not only *which* agents to match, but also *when* to match them. If the planner waits, new agents may arrive, and a more socially

\(^1\)See Gale and Shapley (1962); Crawford and Knoer (1981); Kelso Jr and Crawford (1982); Roth and Sotomayor (1992); Roth et al. (2004); Hatfield and Milgrom (2005); Roth et al. (2007); Hatfield and Kojima (2010).
Figure 1: A schematic market, where there is a line between a seller and a buyer if and only if they are mutually interested in trade. In a static setting at time $t_0$, the optimum solution matches $s_1$ to $b_1$. But in a dynamic setting a new seller and a new buyer may arrive at the market in the next period, so the planner can wait and obtain a better solution by matching $s_1$ to $b_2$ and $s_2$ to $b_1$.

A desirable match may be found. For example, as shown in Figure 1, the planner can match $s_1$ to $b_1$ at time $t_0$. But if this match is made, he cannot match $s_2$ and $b_2$ in the next period. In contrast, if he postpones matching agents for one period, he can match $s_1$ to $b_2$ and $s_2$ to $b_1$, which may be a better solution. Hence, it may not be best to conduct an ‘optimal static’ match in a dynamic environment. On the other hand, if the match is postponed, $s_1$ and $b_1$ will remain unmatched until the next period and face waiting costs in the meantime.

This paper deals with identifying features of optimal matching algorithms in dynamic environments. The previous example illustrates that static matching models do not capture important features of dynamic matching markets. Obviously, waiting may bring new agents, and thus expand the set of feasible matchings. More generally, in a static setting, the planner chooses the best algorithm for an exogenously given set of agents and their preferences. By contrast, in a dynamic setting, the set of agents and trade options at each point in time depend endogenously on the matching algorithm.

The optimal timing policy in a dynamic matching problem is not obvious a priori. In practice, many paired kidney exchanges enact static matchings algorithms (‘match-runs’) at fixed intervals.\(^2\) Even then, matching intervals differ substantially between exchanges: The Alliance for Paired Donation conducts a match-run once a weekday, the United Network for Organ Sharing conducts a match-run once a week\(^3\), the South Korean kidney exchange conducts a match-run once a month, and the Dutch kidney exchange conducts a match-run

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\(^2\)In graph theory, a matching is a set of edges that have no nodes in common.

once a quarter (Akkina et al., 2011). This shows that policymakers select different timing policies when faced with seemingly similar dynamic matching problems. It is therefore useful to identify good timing policies, and to investigate how policy should depend on the underlying features of the problem.

In this paper, we create and analyze a simple model of dynamic matching on networks. Agents arrive and depart stochastically. We use binary preferences, where a pairwise match is either acceptable or unacceptable, generated according to a known distribution. These preferences are persistent over time, and agents may discount the future. The set of agents (vertices) and the set of potential matches (edges) form a random graph. Agents do not observe the set of acceptable transactions, and are reliant upon the planner to match them to each other. We say that an agent perishes if she leaves the market unmatched.

The planner’s problem is to design a matching algorithm; that is, at any point in time, to select a subset of acceptable transactions and broker those trades. The planner observes the current set of agents and acceptable transactions, but has only probabilistic knowledge about the future. The planner may have knowledge about which agents’ needs are urgent, in the sense that he may know which agents will perish imminently if not matched. The goal of the planner is to maximize the sum of the discounted utilities of all agents. In the important special case where the cost of waiting is zero, the planner’s goal is equivalent to minimizing the proportion of agents who perish. We call this the loss of an algorithm.

In this setting, the planner faces a trade-off between matching agents quickly and waiting to thicken the market. If the planner matches agents frequently, then matched agents will not have long to wait, but it will be less likely that any remaining agent has a potential match (a thin market). On the other hand, if the planner matches agents infrequently, then there will be more agents available, making it more likely that any given agent has a potential match (a thick market).

When facing a trade-off between the frequency of matching and the thickness of the market, what is the optimal timing policy? Because we explicitly model the graph structure of the planner’s matching problem, the state space of the resulting Markov Decision Problem is combinatorially complex. Thus, it is not amenable to solution via standard dynamic programming techniques. Instead, to analyze the model, we formulate simple matching algorithms with different timing properties, and compare them to analytic bounds on optimum performance. This will enable us to investigate whether timing is an important feature of dynamic matching algorithms.

Our algorithms are as follows: The Greedy algorithm attempts to match each agent upon
arrival; it treats each instant as a static matching problem without regard for the future. The Patient algorithm attempts to match patients on the verge of leaving the market, potentially by matching them to a non-urgent patient. Both these algorithms are local, in the sense that they look only at the immediate neighbors of the agent they attempt to match rather than at the global graph structure. We also study two families of algorithms that speed up the trade frequency of the Patient algorithm. The Patient($\alpha$) algorithm attempts to match urgent cases, and additionally attempts to match each non-urgent case at some rate determined by $\alpha$. The Periodic($\alpha$) algorithm attempts to match urgent cases, and additionally implements a static maximum matching every $\alpha$ periods.

We now state our main results. First, we analyze the performance of algorithms with different timing properties, in the benchmark setting where the planner can identify urgent cases. Second, we relax our informational assumption, and thereby establish the value of short-horizon information about urgent cases. Third, we exhibit a dynamic mechanism that truthfully elicits such information from agents.

Our first family of results concerns the problem of timing in dynamic matching markets. First, we establish that the loss of the Patient algorithm is exponentially (in the average degree of agents) smaller than the loss of the Greedy algorithm. This entails that, for even moderately dense markets, the Patient algorithm substantially outperforms the Greedy algorithm. For example, suppose on average agents perish after one year. In a market where 1000 agents arrive every year and and the probability of an acceptable transaction is $\frac{1}{100}$, the loss of the Patient algorithm is no more than 7% of the loss the Greedy algorithm. Thus, varying the timing properties of simple algorithms has large effects on their performance.

Second, we find that the loss of the Patient algorithm is close to the loss of the optimum algorithm. Recall that the Patient algorithm is local; it looks only in the immediate neighborhood of the agents it seeks to match. By contrast, the optimum algorithm is global and potentially very complex; the matchings it selects depend on the entire graph structure. Thus, this result suggests that the gains from waiting to thicken the market are large compared to the total gains from considering the global network structure.

Third, we find that it is possible to accelerate the Patient algorithm and still achieve exponentially small loss. That is, we establish a bound for the tuning parameter $\alpha$ such that

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4Our analysis of the Greedy Algorithm encompasses waiting list policies where brokers make transactions as soon as they are available, giving priority to agents who arrived earlier.

5Example 3.6 shows a case in which ‘locality’ of the Patient algorithms makes it suboptimal.

6More precisely, every non-urgent agent is treated as urgent when an exogenous “exponential clock” ticks and attempted to be matched either in that instance, or when she becomes truly urgent.

7As before, the exponent is in the average degree of agents.
that the Patient(\(\alpha\)) algorithm has exponentially small loss. Given the same parameters as in our previous example, under the Patient(\(\alpha\)) algorithm, the planner can promise to match agents in less than 4 months (in expectation) while the loss is at most 37\% of the loss of the Greedy algorithm. Thus, even moderate degrees of waiting can substantially reduce the proportion of perished agents.

The Periodic(\(\alpha\)) algorithm is an alternative way to accelerate the Patient algorithm. However, our analysis suggests that it is not a desirable alternative. In particular, if the choice of \(\alpha\) is such that the Periodic(\(\alpha\)) algorithm substantially increases the speed of transactions, then the algorithm does not achieve exponentially small loss. One reason for this finding is clear: the Periodic(\(\alpha\)) algorithm is not a Markov policy, so it cannot be an optimal policy for a Markov Decision Problem. But it is worse than that: maintaining market thickness at all times (as in the Patient(\(\alpha\)) algorithm) allows the planner to react to urgent cases, while the Periodic(\(\alpha\)) algorithm makes the market very thin at some specific times (immediately after clearing the market), and a large fraction of the total loss occurs at those times.

Next, we examine welfare under discounting. We show that for a range of discount rates, the Patient algorithm delivers higher welfare than the Greedy algorithm, and for a wider range of discount rates, there exists \(\alpha\) such that the Patient(\(\alpha\)) algorithm delivers higher welfare than the Greedy algorithm. Then, in order to capture the trade-off between the trade frequency and the thickness of the market, we solve for the optimal waiting time as a function of the market parameters. Our comparative statics show that the optimal waiting time is increasing in the sparsity of the graph.

Our second family of results relaxes the informational assumptions in the benchmark model. Suppose that the planner cannot identify urgent cases; i.e. the planner has no individual-specific information about departure times. We find that the loss of the Patient algorithm, which naively exploits urgency information, is exponentially smaller than the loss of the optimum algorithm that lacks such information.\(^8\) This suggests that short-horizon information about departure times is very valuable.

On the other hand, suppose that the planner has more than short-horizon information about agent departures. The planner may be able to forecast departures long in advance, or foresee how many new agents will arrive, or know that certain agents are more likely than others to have new acceptable transactions. We prove that no expansion of the planner’s

\(^8\)This result has some of the flavor of Bulow and Klemperer’s theorem (Bulow and Klemperer, 1996) comparing simple auctions to optimal negotiations. They show that simple auctions with \(N + 1\) bidders raise more revenue than optimal mechanisms with \(N\) bidders. We show that simple matching algorithms that thicken the market by exploiting urgency information are better than optimal algorithms that do not.
information allows him to achieve a better-than-exponential loss. Taken as a pair, these results suggest that short-horizon information about departure times is especially valuable to the planner. Lacking this information leads to large losses, and having more than this information does not yield large gains.

In some settings, however, agents have short-horizon information about their departure times, but the planner does not. Our final result concerns the incentive-compatible implementation of the Patient(\(\alpha\)) algorithm.\(^9\) Under private information, agents may have incentives to mis-report their urgency so as to hasten their match or to increase their probability of getting matched. We show that if agents are not too impatient, a dynamic mechanism without transfers can elicit such information. The mechanism treats agents who report that their need is urgent, but persist, as though they had left the market. This means that agents trade off the possibility of a swifter match (by declaring that they are in urgent need now) with the option value of being matched to another agent who has an urgent need in future. We prove that it is arbitrarily close to optimal for agents to report the truth in large markets.

The rest of the paper is organized as follows. Section 2 introduces our dynamic matching market model and defines the objective. Section 3 presents our main contributions; we recommend that readers consult this section to see a formal statement of our results without getting into the details of the proofs. Section 4 analyzes two optimal policies as benchmarks and provides analytic bounds on their performance. Section 5 models our algorithms as Markov Chains and bounds the mixing times of the chains. Section 6 goes through a deep analysis of the Greedy algorithm, the Patient algorithm, and the Patient(\(\alpha\)) algorithm and bounds their performance. Section 7 takes waiting costs into account and bounds the social welfare under different algorithms. Section 8 considers the case where the urgency of an agent’s needs is private information, and exhibits a truthful direct revelation mechanism. Section 9 includes the concluding discussions.

1.1 Related Work

There have been several studies on dynamic matching in the literatures of economics, computer science, and operations research, that each fit a specific marketplace, such as the real estate market, paired kidney exchange, or online advertising. To the best of our knowledge, no previous work has offered a general framework for dynamic matching in networked markets, and no previous work has considered stochastic agent departures. This paper is also

\(^9\)Note that the Patient(\(\alpha\)) algorithm contains the Patient algorithm as a special case.
the first to produce analytic results on bilateral dynamic matching that explicitly account for discounting.\textsuperscript{10}

\textit{Kurino (2009)} and \textit{Bloch and Houy (2012)} study an overlapping generations model of the housing market. In their models, agents have deterministic arrivals and departures. In addition, the housing side of the market is infinitely durable and static, and houses do not have preferences over agents. In the same context, \textit{Leshno (2012)} studies a one-sided dynamic housing allocation problem in which houses arrive stochastically over time. His model is based on two waiting lists and does not include a network structure. In addition, agents remain in the waiting list until they are assigned to a house; i.e., they do not perish.

In the context of live-donor kidney exchanges, \textit{"Unver (2010)} studies an interesting model of the dynamic kidney exchange in which agents have multiple types. In his model, agents never perish and so, one insight of his model is that waiting to thicken the market is not helpful when only bilateral exchanges are allowed. This is very different from the insights of our paper. In the Operations Research and Computer Science literatures, dynamic kidney matching has been extensively studied, see e.g., (Zenios, 2002; Su and Zenios, 2005; Awasthi and Sandholm, 2009; Dickerson et al., 2012). Perhaps most related to our work is that of \textit{Ashlagi et al. (2013)} who construct a discrete-time finite-horizon model of dynamic kidney exchange. Unlike our model, agents who are in the pool neither perish, nor bear any waiting cost, and so they do not model agents’ incentives. Their model has two types of agents, one easy to match and one hard to match, which then creates a specific graph structure that fits well to the kidney market.

In an independent concurrent study, inspired by online labor markets such as oDesk, \textit{Arnosti et al. (2014)} model a dynamic two-sided dynamic matching market and show that reducing search and screening costs does not necessarily increase welfare. Their main goal is to analyze congestion in decentralized dynamic markets, as opposed to our goal which is to study the “when to match” question from a central planning perspective.

The problem of online matching has been extensively studied in the literature of online advertising. In this setting, advertisements are static, but queries arrive adversarially or stochastically over time. Unlike our model, queries persist in the market for exactly one period. \textit{Karp et al. (1990)} introduced the problem and designed a randomized matching

\textsuperscript{10}Other analytic results in bilateral dynamic matching concern statistics such as average waiting times, which are related to but not identical with discounted utility. See a recent study of dynamic barter exchange markets by \textit{Anderson et al. (2014)}, where agents never perish and the main objective is the average waiting time. They show that when only pairwise exchanges are allowed, the Greedy algorithm is close to the optimum, which is similar to \textbf{Theorem 3.13}.
algorithm. Subsequently, the problem has been considered under several arrival models with pre-specified budgets for the advertisers, (Mehta et al., 2007; Goel and Mehta, 2008; Feldman et al., 2009; Manshadi et al., 2012).

In contrast to dynamic matching, there are numerous investigations of dynamic auctions and dynamic mechanism design. Parkes and Singh (2003) generalize the VCG mechanism to a dynamic setting. Athey and Segal (2007) construct efficient and incentive-compatible dynamic mechanisms for private information settings. Pai and Vohra (2013) and Gallien (2006) extend Myerson’s optimal auction result (Myerson, 1981) to dynamic environments. We refer interested readers to Parkes (2007) for a review of the dynamic mechanism design literature.

2 The Model

In this section, we provide a stochastic continuous-time model for a bilateral matching market that runs in the interval \([0, T]\). Agents arrive at the market at rate \(m\) according to a Poisson process. Hence, in any interval \([t, t+1]\), \(m\) new agents enter the market in expectation. Throughout the paper we assume \(m \geq 1\). For \(t \geq 0\), let \(A_t\) be the set of the agents in our market at time \(t\), and let \(Z_t := |A_t|\). We refer to \(A_t\) as the pool of the market. We start by describing the evolution of \(A_t\) as a function of \(t \in [0, T]\). Since we are interested in the limit behavior of \(A_t\), without loss of generality, we may assume \(A_0 = \emptyset\). We use \(A^n_t\) to denote\(^{11}\) the set of agents who enter the market at time \(t\). Note that with probability 1, \(|A^n_t| \leq 1\).

Also, let \(|A^{n}_{t_0,t_0+t_1}|\) denote the set of agents who enter the market in time interval \([t_0, t_1]\).

Each agent becomes critical according to an independent Poisson process with rate \(\lambda\). This implies that, if an agent \(a\) enters the market at time \(t_0\), then she becomes critical at some time \(t_0 + X\) where \(X\) is an exponential random variable with parameter \(\lambda\). Any critical agent leaves the market immediately; so the last point in time that an agent can get matched is the time that she gets critical. We say an agent \(a\) perishes if \(a\) leaves the market unmatched.\(^{12}\)

We assume that an agent \(a \in A_t\) leaves the market at time \(t\), if any of the following three events occur at time \(t\):

- \(a\) is matched with another agent \(b \in A_t\),

\(^{11}\)As a notational guidance, we use subscripts to refer to a point in time or a time interval, while superscripts \(n, c\) refer to new agents and critical agents, respectively.

\(^{12}\)We intend this as a term of art. In the case of kidney exchange, perishing can be interpreted as a patient’s medical condition deteriorating in such a way as to make transplants infeasible.
• $a$ becomes critical and gets matched
• $a$ becomes critical and leaves the market unmatched, i.e., $a$ perishes.

Say $a$ enters the market at time $t_0$ and gets critical at time $t_0 + X$ where $X$ is an exponential random variable with parameter $\lambda$. By above discussion, for any matching algorithm, $a$ leaves the market at some time $t_1$ where $t_0 \leq t_1 \leq t_0 + X$ (note $a$ may leave sooner than $t_0 + X$ if she gets matched before getting critical). The sojourn of $a$ is the length of the interval that $a$ is in the pool, i.e., $s(a) := t_1 - t_0$.

We use $A_t^c$ to denote the set of agents that are critical at time $t$.\(^{13}\) Also, note that for any $t \geq 0$, with probability 1, $|A_t^c| \leq 1$.

For any pair of agents, the probability that a bilateral transaction between them is acceptable is $d/m$, where $0 \leq d \leq m$ and these probabilities are independent. For the sake of clarity of notation, we may use $q := 1 - d/m$. For any $t \geq 0$, let $E_t \subseteq A_t \times A_t$ be the set of acceptable bilateral transactions between the agents in the market (the set of edges) at time $t$, and let $G_t = (A_t, E_t)$ be the exchange possibilities graph at time $t$. Note that if $a, b \in A_t$ and $a, b \in A_t'$, then $(a, b) \in E_t$ if and only if $(a, b) \in E_t'$, i.e. the acceptable bilateral transactions are persistent throughout the process. For an agent $a \in A_t$ we use $N_t(a) \subseteq A_t$ to denote the set of neighbors of $a$ in $G_t$. It follows that, if the planner does not match any agents, then for any fixed $t \geq 0$, $G_t$ is distributed as an Erdös-Rényi graph with parameter $d/m$ and $d$ is the average degree\(^{14}\) of agents (Erdös and Rényi, 1960).

Let $A = \cup_{t \leq T} A^0_t$, let $E \subseteq A \times A$ be the set of acceptable transactions between agents in $A$, and let $G = (A, E)$\(^{15}\). Observe that any realization of the above stochastic process is uniquely defined given $A^0_t, A^c_t$ for all $t \geq 0$ and the set of acceptable transactions, $E$. A vector $(m, d, \lambda)$ represents a dynamic matching market. Without loss of generality, we can scale time so that $\lambda = 1$ (by normalizing $m$ and $d$). Therefore, throughout the paper, we assume $\lambda = 1$, unless otherwise specified\(^{16}\).

**Online Matching Algorithms.** A set of edges $M_t \subseteq E_t$ is a matching if no two edges share the same endpoints. An online matching algorithm, at any time $t \geq 0$, selects a

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\(^{13}\)In our proofs, we use the fact that $A^c_t \subseteq \cup_{0 \leq \tau \leq t} A^e_{t+\tau}$. In the example of the text, we have $a \in A^e_{t_0+X}$. Note that even if agent $a$ is matched before getting critical (i.e., $t_1 < t_0 + X$), we still have that $a \in A^e_{t_0+X}$. Hence, $A^c_t$ is not necessarily a subset of $A_t$ since it may have agents who are already matched and left the market. This generalized definition of $A^c_t$ is going to be helpful in our proofs.

\(^{14}\)In an undirected graph, degree of a node is equal to the total number of edges connected to that node.

\(^{15}\)Note that $E \supseteq \cup_{t \leq T} E_t$, and the two sets are not typically equal, since two agents may find it acceptable to transact, even though they are not in the pool at the same time because one of them was matched earlier.

\(^{16}\)See Proposition 6.12 for details of why this is without loss of generality.
Matching, $M_t$, in the current acceptable transactions graph $G_t$, and the endpoints of the edges in $M_t$ leave the market immediately. We assume that any online matching algorithm at any time $t_0$ only knows the current graph $G_t$ for $t \leq t_0$ and does not know anything about $G_{t'}$ for $t' > t_0$. In the benchmark case that we consider, the online algorithm can depend on the set of critical agents at time $t$; nonetheless, we will extend several of our theorems to the case where the online algorithm does not have this knowledge. As will become clear, this knowledge has a significant impact on the performance of any online algorithm.

We emphasize that the random sets $A_t$ (the set of agents in the pool at time $t$), $E_t$ (the set of acceptable transactions at time $t$), $N_t(a)$ (the set of an agent $a$’s neighbors), and the random variable $Z_t$ (pool size at time $t$) are all functions of the underlying matching algorithm. We abuse the notation and do not include the name of the algorithm when we analyze these variables.

**The Goal.** The goal of the planner is then to design an online matching algorithm that maximizes the social welfare, i.e., the sum of the utility of all agents in the market. Let $\text{ALG}(T)$ be the set of matched agents by time $T$,

$$\text{ALG}(T) := \{a \in A : a \text{ is matched by } \text{ALG} \text{ by time } T\}.$$  

We may drop the $T$ in the notation $\text{ALG}(T)$ if it is clear from context.

An agent receives zero utility if she leaves the market unmatched. If she is matched, she receives a utility of 1 discounted at rate $\delta$. More formally, if $s(a)$ is the sojourn of agent $a$, then we define the utility of agent $a$ as follows:

$$u(a) := \begin{cases} e^{-\delta s(a)} & \text{if } a \text{ is matched} \\ 0 & \text{otherwise}. \end{cases}$$

We define the social welfare of an online algorithm to be the expected sum of the utility of all agents in the interval $[0, T]$, divided by a normalization factor:

$$W(\text{ALG}) := \mathbb{E} \left[ \frac{1}{mT} \sum_{a \in \text{ALG}(T)} e^{-\delta s(a)} \right].$$

The goal of the planner is to choose an online algorithm that maximizes the welfare for
large values of $T$ (see Theorem 6.1, Theorem 6.2, and Theorem 7.1 for the dependence of our results on $T$).

It is instructive to consider the special case where $\delta = 0$, i.e., the cost of waiting is negligible compared to the cost of leaving the market unmatched. In this case, the goal of the planner is to match the maximum number of agents, or equivalently to minimize the number of perished agents. The loss of an online algorithm $\text{ALG}$ is defined as the ratio of the expected number of perished agents to the expected size of $A$,

$$L(\text{ALG}) := \frac{E[|A - \text{ALG}(T) - A_T|]}{E[|A|]} = \frac{E[|A - \text{ALG}(T) - A_T|]}{mT}.$$

When we assume $\delta = 0$, we will use the $L$ notation for the planner’s loss function. When we consider $\delta > 0$, we will use the $W$ notation for social welfare.

Each of the above optimization problems can be modeled as a Markov Decision Problem (MDP)\(^{18}\) that is defined as follows. The state space is the set of pairs $(H, B)$ where $H$ is any undirected graph of any size, and if the algorithm knows the set of critical agents, $B$ is a set of at most one vertex of $H$ representing the corresponding critical agent. The action space for a given state is the set of matchings on the graph $H$. Under this conception, an algorithm designer wants to minimize the loss or maximize the social welfare over a time period $T$.

Although this MDP has infinite number of states, with small error one can reduce the state space to graphs of size at most $O(m)$. Even in that case, this MDP has an exponential number of states in $m$, since there are at least $2^\binom{m}{2}/m!$ distinct graphs of size $m$\(^{19}\), so for even moderately large markets\(^{20}\), we cannot apply tools from Dynamic Programming literature to find the optimum online matching algorithm.

**Optimum Solutions.** In many parts of this paper we compare the performance of an online algorithm to the performance of an optimal *omniscient* algorithm. Unlike any online algorithm, the omniscient algorithm has full information about the future, i.e., it knows the

\(^{17}\)We consider the expected value as a modeling choice. One may also be interested in objective functions that depend on the variance of the performance, as well as the expected value. As will be seen later in the paper, the performance of our algorithms are highly concentrated around their expected value, which guarantees that the variance is very small in most of the cases.

\(^{18}\)We recommend Bertsekas (2000) for background on Markov Decision Processes.

\(^{19}\)This lower bound is derived as follows: When there are $m$ agents, there are $\binom{m}{2}$ possible edges, each of which may be present or absent. Some of these graphs may have the same structure but different agent indices. A conservative lower bound is to divide by all possible re-labellings of the agents ($m!$).

\(^{20}\)For instance, for $m = 30$, there are more than $10^{98}$ states in the approximated MDP.
full realization of the graph $G$. Therefore, it can return the maximum matching in this graph as its output, and thus minimize the fraction of perished agents. Let $\text{OMN}(T)$ be the set of matched agents in the maximum matching of $G$. The loss function under the omniscient algorithm at time $T$ is

$$L(\text{OMN}) := \mathbb{E} \left[ \frac{|A - \text{OMN}(T) - A_T|}{mT} \right]$$

Observe that for any online algorithm, $\text{ALG}$, and any realization of the probability space, we have $|\text{ALG}(T)| \leq |\text{OMN}(T)|$.\textsuperscript{22}

It is also instructive to study the optimum online algorithm, an online algorithm with unlimited computational power. By definition, an optimum online algorithm can solve the exponential-sized state space Markov Decision Problem and return the best policy function from states to matchings. We first consider $\text{OPT}^c$, the algorithm that knows the set of critical agents at time $t$ (with associated loss $L(\text{OPT}^c)$). We then relax this assumption and consider $\text{OPT}$, the algorithm that does not know these sets (with associated loss $L(\text{OPT})$).

Let $\text{ALG}^c$ be the loss under any online algorithm that knows the set of critical agents at time $t$. It follows that

$$L(\text{ALG}^c) \geq L(\text{OPT}^c) \geq L(\text{OMN}).$$

Similarly, let $\text{ALG}$ be the loss under any online algorithm that does not know the set of critical agents at time $t$. It follows that\textsuperscript{23}

$$L(\text{ALG}) \geq L(\text{OPT}) \geq L(\text{OPT}^c) \geq L(\text{OMN}).$$

\section{Our Contributions}

In this section, we present our main contributions and provide intuitions for them. We first introduce two simple matching algorithms, and then two classes of algorithms that vary the waiting time. The first algorithm is the Greedy algorithm, which mimics ‘match-as-you-go’ algorithms used in many real marketplaces. It delivers maximal matchings at any point in time, without regard for the future.

\textsuperscript{21}In computer science, these are equivalently called offline algorithms.

\textsuperscript{22}This follows from a straightforward revealed-preference argument: For any realization, the optimum offline policy has the information to replicate any given online policy, so it must do weakly better.

\textsuperscript{23}Note that $|\text{ALG}|$ and $|\text{OPT}|$ are generally incomparable, and depending on the realization of $G$ we may even have $|\text{ALG}| > |\text{OPT}|$. 

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**Definition 3.1** (Greedy Algorithm). *If any new agent $a$ enters the market at time $t$, then match her with an arbitrary agent in $N_t(a)$ whenever $N_t(a) \neq \emptyset$. We use $L$(Greedy) and $W$(Greedy) to denote the loss and the social welfare under this algorithm, respectively.*

Note that since $|A^c_t| \leq 1$ almost surely, we do not need to consider the case where more than one agent enters the market at any point in time. Observe that the graph $G_t$ in the Greedy algorithm is (almost) always an empty graph. Hence, the Greedy algorithm cannot use any information about the set of critical agents.

The second algorithm is a simple online algorithm that preserves two essential characteristics of $\text{OPT}^c$ when $\delta = 0$ (recall that $\text{OPT}^c$ is the optimum online algorithm with knowledge of the set of critical agents):

i) A pair of agents $a, b$ get matched in $\text{OPT}^c$ only if one of them is critical. This property is called the *rule of deferral match*: Since $\delta = 0$, if $a, b$ can be matched and neither of them is critical we can wait and match them later.

ii) If an agent $a$ is critical at time $t$ and $N_t(a) \neq \emptyset$ then $\text{OPT}^c$ matches $a$. This property is a corollary of the following simple fact: matching a critical agent does not increase the number of perished agents in any online algorithm.

Our second algorithm is designed to be the simplest possible online algorithm that satisfies both of the above properties.

**Definition 3.2** (Patient Algorithm). *If an agent $a$ becomes critical at time $t$, then match her uniformly at random with an agent in $N_t(a)$ whenever $N_t(a) \neq \emptyset$. We use $L$(Patient) and $W$(Patient) to denote the loss and the social welfare under this algorithm, respectively.*

Observe that unlike the Greedy algorithm, here we need access to the set of critical agents at time $t$. We do not intend the timing assumptions about critical agents to be interpreted literally. An agent’s point of perishing represents the point at which it ceases to be socially valuable to match that agent. Letting the planner observe the set of critical agents is a modeling convention that represents high-accuracy short-horizon information about agents’ departures. An example of such information is the Model for End-Stage Liver Disease (MELD) score, which accurately predicts 3-month mortality among patients with chronic liver disease. The US Organ Procurement and Transplantation Network gives priority to individuals with a higher MELD score, following a broad medical consensus that liver donor allocation should be based on urgency of need and not substantially on waiting
time. (Wiesner et al., 2003) Note that the Patient algorithm exploits only short-horizon information about urgent cases, as compared to the Omniscient algorithm which has full information of the future. We discuss implications of relaxing our informational assumptions in Subsection 3.3.

The third algorithm interpolates between the Greedy and the Patient algorithms. The idea of this algorithm is to assign independent exponential clocks with rates $1/\alpha$ where $\alpha \in [0, \infty)$ to each agent $a$. If agent $a$’s exponential clock ticks, the market-maker attempts to match her. If she has no neighbors, then she remains in the pool until she gets critical, where the market-maker attempts to match her again.

A technical difficulty with the above matching algorithm is that it is not memoryless; that is because when an agent gets critical and has no neighbors, she remains in the pool. Therefore, instead of the above algorithm, we study a slightly different matching algorithm (with a worse loss).

**Definition 3.3 (The Patient($\alpha$) algorithm).** Assign independent exponential clocks with rate $1/\alpha$ where $\alpha \in [0, \infty)$ to each agent $a$. If agent $a$’s exponential clock ticks or if an agent $a$ becomes critical at time $t$, match her uniformly at random with an agent in $N_t(a)$ whenever $N_t(a) \neq \emptyset$. In both cases, if $N_t(a) = \emptyset$, treat that agent as if she has perished; i.e., never match her again. We use $L(Patient(\alpha))$ and $W(Patient(\alpha))$ to denote the loss and the social welfare under this algorithm, respectively.

It is easy to see that an upper bound on the loss of the Patient($\alpha$) algorithm is an upper bound on the loss of our desired interpolating algorithm. Under this algorithm each agent’s exponential clock ticks at rate $\frac{1}{\alpha}$, so we search their neighbors for a potential match at rate $\bar{\alpha} := 1 + \frac{1}{\alpha}$. We refer to $\bar{\alpha}$ as the trade frequency. Note that the trade frequency is a decreasing function of $\alpha$.

The last algorithm that we define introduces another way to speed up the Patient algorithm by mimicking a popular algorithm seen in some real-world marketplaces.

**Definition 3.4 (The Periodic($\alpha$) algorithm).** If an agent $a$ becomes critical at time $t$ match her to an arbitrary agent in $N_t(a)$ whenever $N_t(a) \neq \emptyset$. Every $\alpha$ time steps find a maximum matching between agents that are in the pool and match the endpoints of each edge in that matching.

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24The use of exponential clocks is a modeling convention that enables us to reduce waiting times while retaining analytically tractable Markov properties.
In the rest of this section we describe our contributions. To avoid cumbersome notation, we state our results in the large-market long-horizon regime (i.e., as $m \to \infty$ and $T \to \infty$). In later sections, we explicitly study the dependency on $(m, T)$. For example, we show that the transition of the market to the steady state takes no more than $O(\log(m))$ time units. In other words, many of the large time effects that we predict in our model can be seen in poly-logarithmic time in the size of the market. In addition, we only present an overview of the proofs in this section; the rest of the paper includes detailed analysis of the model and full proofs.

### 3.1 Timing in Matching Markets

Does timing substantially affect the performance of dynamic matching algorithms? Our first result establishes that varying the timing properties of simple algorithms has large effects on their performance. In particular, we show that the number of perished agents under the Patient algorithm is exponentially (in $d$) smaller than the number of perished agents under the Greedy algorithm.

**Theorem 3.5.** For $d \geq 2$, as $T, m \to \infty$,

\[
L(\text{Greedy}) \geq \frac{1}{2d+1} \\
L(\text{Patient}) \leq \frac{1}{2} \cdot e^{-d/2}
\]

As a result,

\[
L(\text{Patient}(\alpha)) \leq (d+1) \cdot e^{-d/2} \cdot L(\text{Greedy})
\]

This theorem shows that the Patient algorithm strongly outperforms the Greedy algorithm. The intuition behind this finding is that, under the Greedy algorithm, there is no acceptable transactions among the set of agents in the pool and so all critical agents imminently perish. On the contrary, the pool is thicker since under the Patient algorithm is always an Erdős-Rényi random graph (see Proposition 5.1 for a proof of this claim), and such market thickness helps the planner to react to critical cases.

The next question is, are the gains from market thickness large compared to the total gains from optimization and choosing the right agents to match? First, in the following example, we show that the Patient algorithm is not optimal because it ignores the global graph structure.
Figure 2: If $a_2$ gets critical in the above graph, it is strictly better to match him to $a_1$ as opposed to $a_3$. The Patient algorithm, however, chooses either of $a_1$ or $a_3$ with equal probability.

**Example 3.6.** Let $G_t$ be the graph shown in Figure 2, and let $a_2 \in A^c_t$, i.e., $a_2$ is critical at time $t$. Observe that it is strictly better to match $a_2$ to $a_1$ as opposed to $a_3$. Nevertheless, since the Patient algorithm makes decisions that depend only on the immediate neighbors of the agent it is trying to match, it cannot differentiate between $a_1$ and $a_3$ and will choose either of them with equal probability.

To quantify gains of optimization, we compare the Patient algorithm to the omniscient algorithm, which obviously bounds the performance of the optimum online algorithm.

The next theorem shows that no algorithm achieves better than exponentially small loss. Furthermore, the gains from the right timing decision (moving from the Greedy algorithm to the Patient algorithm) are much larger than the remaining gains from optimization (moving from the Patient algorithm to the optimum algorithm).

**Theorem 3.7.** For $d \geq 2$, as $T, m \to \infty$,

$$\frac{e^{-d}}{d+1} \leq L(\text{OMN}) \leq L(\text{Patient}) \leq \frac{1}{2} \cdot e^{-d/2}.$$

This constitutes an answer to the “when to match versus whom to match” question. Recall that OMN has perfect foresight, and bounds the performance of any algorithm, including OPT, the globally optimal solution. Thus, Theorem 3.7 implies that the marginal effect of a globally optimal solution is small relative to the effect of making the right timing decision. In many settings, optimal solutions may be computationally demanding and difficult to implement. Thus, this result suggests that it will often be more worthwhile for policymakers to find ways to thicken the market, rather than to seek potentially complicated optimal policies.\(^{25}\)

\(^{25}\)It is worth emphasizing that this result (as well as Theorem 3.13) proves that local algorithms are close to optimal and since in our model agents are ex ante homogeneous, this shows that “who to match” is not as important as “when to match”. In settings where agents have multiple types, however, the decision of “who to match” can be an important one even when it is local. For instance, suppose a critical agent has two neighbors, one who is hard-to-match and one who is easy-to-match. Then, everything else constant, the optimal policy should match the critical agent to the hard-to-match neighbor. This result has been formally proved in a new working paper by Akbarpour, Nikzad and Roth, in which they show that breaking the ties in favor of hard-to-match agents reduces the loss.
A planner, however, may not be willing to implement the Patient algorithm for various reasons. First, the cost of waiting is usually not zero; in other words, agents prefer to be matched earlier (we discuss this cost in detail in Subsection 3.2). Second, the planner may be in competition with other exchange platforms and may be able to attract more agents by advertising reasonably short waiting times. Hence, we study the performance of the Patient($\alpha$) algorithm, which introduces a way to speed up the Patient algorithm. The next result shows that when $\alpha$ is not ‘too small’ (i.e., the exponential clocks of the agents do not tick at a very fast rate), then Patient($\alpha$) algorithm still (strongly) outperforms the Greedy algorithm. In other words, even waiting for a moderate time can substantially reduce perishings.

**Theorem 3.8.** Let $\bar{\alpha} := 1/\alpha + 1$. For $d \geq 2$, as $T, m \to \infty$,

$$L(\text{Patient}(\alpha)) \leq (d + 1) \cdot e^{-d/2\bar{\alpha}} \cdot L(\text{Greedy})$$

A numerical example clarifies the significance of this result. Consider the case of a kidney exchange market, where 1000 new patients arrive to the market every year, their average sojourn is 1 year and they can exchange kidneys with a random patient with probability $\frac{1}{100}$; that is, $d = 10$. The above result for the Patient($\alpha$) algorithm suggests that the market-maker can promise to match agents in less than 4 months (in expectation) while the fraction of perished agents is at most 37% of the Greedy algorithm. Note that if we employ the Patient algorithm, the fraction of perished agents will be at most 7% of the Greedy algorithm.

Periodic($\alpha$) algorithm is an alternative way to speed up the Patient algorithm. Unlike the Patient($\alpha$) algorithm that matches individual agents at different times, the Periodic($\alpha$) algorithm finds a maximum matching among all agents every $\alpha$ time steps. Our analysis of this algorithm shows that the loss of this algorithm is not exponentially small in $d$. This shows that the Patient($\alpha$) algorithm is a better way than Periodic($\alpha$) to reduce the average waiting time.

**Theorem 3.9.** If $d < m/2$, then for any $\alpha > 0$ independent of $d$, there exists a constant $C(\alpha)$, such that, as $m, T \to \infty$,

$$L(\text{Periodic}(\alpha)) \geq C(\alpha)/d^2$$

Intuitively, maintaining market thickness at all times allows the planner to match urgent cases. The Patient($\alpha$) algorithm increases the speed of transactions in a ‘smoothed’ way so
that the market remains relatively thick at almost all times. However, under the Periodic(\(\alpha\)) algorithm, market gets very thin for a short period of time (after implementing a maximum matching) and a large fraction of loss occurs at those intervals.

We now present a proof sketch for all of the theorems presented in this section. The details of the proofs are discuss in the rest of the paper.

Proof Overview. [Theorem 3.5, Theorem 3.7, Theorem 3.8, and Theorem 3.9] We first sketch the proof of Theorem 3.5. We show that for large enough values of \(T\) and \(m\), (i) \(L(\text{Patient}) \leq e^{-d/2}/2\) and (ii) \(L(\text{OPT}) \geq 1/(2d + 1)\). By the fact that \(L(\text{Greedy}) \geq L(\text{OPT})\) (since the Greedy algorithm does not use information about critical agents), Theorem 3.5 follows immediately. The key idea in proving both parts is to carefully study the distribution of the pool size, \(Z_t\), under any of these algorithms.

For (i), we show that pool size under the Patient algorithm is a Markov chain, it has a unique stationary distribution and it mixes rapidly to the stationary distribution (see Theorem 5.2). This implies that for \(t\) larger than mixing time, \(Z_t\) is essentially distributed as the stationary distribution of the Markov Chain. We show that under the stationary distribution, with high probability, \(Z_t \in [m/2, m]\). Therefore, any critical agent has no acceptable transactions with probability at most \((1 - d/m)^{m/2} \leq e^{-d/2}\). This proves (i) (see Subsection 6.2 for the exact analysis of the Patient Markov chain).

For (ii), note that any algorithm which lacks the information of critical agents, the expected perishing rate is equal to the pool size because any critical agent perishes with probability one, and agents get critical with rate 1. Therefore, if the expected pool size is large, the perishing rate is very high. On the other hand, if the expected pool size is very low, the perishing rate is again very high because there will be many agents with no acceptable transactions upon their arrival or during their sojourn. We analyze the trade-off between the above two extreme cases and show that even if the pool size is optimally chosen, the loss cannot be less than what we claimed in (ii) (see Theorem 4.1).

We prove Theorem 3.7 by showing that \(L(\text{OMN}) \geq e^{-d}/d + 1\). To do so, we provide a lower bound for the fraction of agents who arrive the market at some point in time and have no acceptable transactions during their sojourn (see Theorem 4.2). Note that even with full knowledge about the realization of the process, those agents cannot be matched.

We now sketch the proof of Theorem 3.8. By the additivity of the Poisson process, the loss of Patient(\(\alpha\)) algorithm in a \((m, d, l)\) matching market is equal to the loss of the Patient algorithm in a \((m, d, \bar{\alpha})\) matching market, where \(\bar{\alpha} = 1/\alpha + 1\).
The next step is to show that a matching market \((m, d, \bar{\alpha})\) is equivalent to a matching market \((m/\bar{\alpha}, d/\bar{\alpha}, 1)\) in the sense that any quantity in these two markets is the same up to a time scale (see Definition 6.11). By this fact, the loss of the Patient algorithm on a \((m, d, \bar{\alpha})\) matching market at time \(T\) is equal to the loss of Patient algorithm on a \((m/\bar{\alpha}, d/\bar{\alpha}, 1)\) market at time \(\bar{\alpha}T\). But, we have already upper bounded the latter in Theorem 3.5.

The proof of Theorem 3.9 has a different technical challenge since the Periodic(\(\alpha\)) algorithm is not a Markov algorithm and the graph of acceptable transactions is not an Erdös-Rényi random graph. To prove this theorem, we first provide a lower bound for the size of the maximum matching that the planner implements at every \(\alpha\) time intervals. Then we count the number of agents with no acceptable transactions upon their arrival or during their sojourn. This number is specifically high under this algorithm because the market gets very thin immediately after implementing a maximum matching, and a large fraction of loss corresponds to those intervals. The full proof is presented in Subsection 4.3.

\[\square\]

**Remark 3.10.** One may interpret the above result as demonstrating the value of information (i.e. knowledge of the set of critical agents) as opposed to the value of waiting. This is not our interpretation, since the Greedy algorithm cannot improve its performance even if it has knowledge of the set of critical agents. The graph \(G_t\) is almost surely an empty graph, so there is no possibility of matching a critical agent in the Greedy algorithm. The Patient algorithm strongly outperforms the Greedy algorithm because it waits long enough to have a large pool of agents with many acceptable bilateral transactions.

### 3.2 Welfare Under Discounting and Optimal Waiting Time

In this part we explicitly account for the cost of waiting and study online algorithms that optimize social welfare. It is clear that if agents are very impatient (i.e., they have very high waiting costs), it is better to implement the Greedy algorithm. On the other hand, if agents are very patient (i.e., they have very low waiting costs), it is better to implement the Patient algorithm. Therefore, a natural welfare economics question is: For which values of \(\delta\) is the Patient algorithm (or the Patient(\(\alpha\)) algorithm) socially preferred to the Greedy algorithm?

Our next result studies social welfare under the Patient, Patient(\(\alpha\)) and Greedy algorithms. We show that for small enough \(\delta\), there exists a value of \(\alpha\) such that the Patient(\(\alpha\)) algorithm is socially preferable to the Greedy algorithm.
Theorem 3.11. For any \( 0 \leq \delta \leq \frac{1}{8 \log(d)} \), there exists an \( \alpha \geq 0 \) such that as \( m, T \to \infty \),

\[
W(\text{Patient}(\alpha)) \geq W(\text{Greedy}).
\]

In particular, for \( \delta \leq 1/2d \) and \( d \geq 5 \), we have

\[
W(\text{Patient}) \geq W(\text{Greedy}).
\]

A numerical example illustrates these magnitudes. Consider a barter market, where 100 new traders arrive at the market every week, their average sojourn is one week, and there is a satisfactory trade between two random agents in the market with probability 0.05; that is, \( d = 5 \). Then our welfare analysis implies that if the cost associated with waiting for one week is less than 10% of the surplus from a typical trade, then the Patient(\( \alpha \)) algorithm, for a tuned value of \( \alpha \), is socially preferred to the Greedy algorithm.

When agents discount the future, how should the planner trade off the frequency of transactions and the thickness of the market? To answer this, we characterize the optimal trade frequency under the Patient(\( \alpha \)) algorithm. Recall that under this algorithm each agent’s exponential clock ticks at rate \( \frac{1}{\alpha} \), so we search their neighbors for a potential match at rate \( \bar{\alpha} := 1 + \frac{1}{\alpha} \). The optimal \( \bar{\alpha} \), the trade frequency, is stated in the following theorem.

Theorem 3.12. Given \( d, \delta \), as \( m, T \to \infty \), there exists \( \bar{d} \in [d/2, d] \) as a function of \( m, d \)\(^{26}\) such that the Patient\( (\frac{1}{\max\{\alpha, 1\}} - 1) \) algorithm where \( \bar{\alpha} \) is the solution of

\[
\delta - \left(\delta + \bar{d} + \frac{d\delta}{\bar{\alpha}}\right)e^{-\bar{\alpha}} = 0, \tag{3.1}
\]

attains the largest welfare among all Patient(\( \alpha \)) algorithms. In particular, if \( \delta < \bar{d}/4 \), then \( \bar{d}/\log(\bar{d}/\delta) \leq \bar{\alpha} \leq \bar{d}/\log(2\bar{d}/\delta) \).

In addition, if \( \delta < (d - 1)/2 \), then \( \bar{\alpha}^* \) is a non-decreasing function of \( \delta \) and \( d \).

Figure 3 illustrates \( \max\{\bar{\alpha}, 1\} \) as a function of \( \delta \). As one would expect, the optimal trade frequency is increasing in \( \delta \). Moreover, Theorem 3.12 indicates that the optimal trade frequency is increasing in \( d \). In Subsection 3.1, we showed that \( L(\text{Patient}) \) is exponentially smaller in \( d \) than \( L(\text{Greedy}) \). This may suggest that waiting is mostly valuable in dense graphs. By contrast, Theorem 3.12 shows that one should wait longer as the graph becomes sparser. Intuitively, an algorithm performs well if, whenever it searches neighbors of a critical

\(^{26}\)More precisely, \( \bar{d} := xd/m \) where \( x \) is the solution of equation (C.6).
agent for a potential match, it can find a match with very high probability. This probability is a function of both the pool size and $d$. When $d$ is smaller, the pool size should be larger (i.e. the trade frequency should be lower) so that the probability of finding a match remains high. For larger values of $d$, on the other hand, a smaller pool size suffices.

Finally, we note that Patient (i.e. $\alpha = \infty$) is the optimal policy for a range of parameter values. To see why, suppose agents discount their welfare, but never perish. The planner would still wish to match them at some positive rate. For a range of parameters, this positive rate is less than 1. In a world where agents perish, $\bar{\alpha}$ is bounded below by 1, and we have a corner solution for such parameter values.

**Proof Overview.** [Theorem 3.11 and Theorem 3.12] We first show that for large values of $m$ and $T$, (i) $W(\text{Greedy}) \leq 1 - \frac{1}{2d+1}$, and (ii) $W(\text{Patient}(\alpha)) \simeq \frac{2}{2-e^{-d/2\bar{\alpha}} + d/\bar{\alpha}}(1 - e^{-d/2\bar{\alpha}})$, where $\bar{\alpha} = 1 + 1/\alpha$. The proof of (i) is very simple: $1/(2d + 1)$ fraction of agents perish under the Greedy algorithm. Therefore, even if all of the matched agents receive a utility of 1, the social welfare is no more than $1 - 1/(2d + 1)$.

The proof of (ii) is more involved. The idea is to define a random variable $X_t$ representing the potential utility of the agents in the pool at time $t$, i.e., if all agents who are in the pool at time $t$ get matched immediately, then they receive a total utility of $X_t$. We show that
$X_t$ can be estimated with a small error by studying the evolution of the system through a differential equation. Given $X_t$ and the pool size at time $t$, $Z_t$, the expected utility of an agent that is matched at time $t$ is exactly $X_t/Z_t$. Using our concentration results on $Z_t$, we can then compute the expected utility of the agents that are matched in any interval $[t, t + dt]$. Integrating over all $t \in [0, T]$ proves the claim. (See Section 7 for an exact analysis of welfare under the Patient algorithm)

To prove Theorem 3.12, we characterize the unique global maximum of $W(\text{Patient}(\alpha))$. The key point is that the optimum value of $\bar{\alpha} (= 1 + 1/\alpha)$ is less than 1 for a range of parameters. However, since $\alpha \geq 0$, we must have that $\bar{\alpha} \in [1, \infty)$. Therefore, whenever the solution of Equation 3.1 is less than 1, the optimal $\bar{\alpha}$ is 1 and we have a corner solution; i.e. setting $\alpha = \infty$ (running the Patient algorithm) is optimal.

3.3 Information and Incentive-Compatibility

Up to this point, we have assumed that the planner knows the set of critical agents; i.e. he has accurate short-horizon information about agent departures. We now relax this assumption in both directions.

Suppose that the planner does not know the set of critical agents. That is, the planner’s policy may depend on the graph $G_t$, but not the set of critical agents $A^C_t$. Recall that OPT is the optimum algorithm subject to these constraints.

**Theorem 3.13.** For $d \geq 2$, as $T, m \to \infty$,

$$\frac{1}{2d + 1} \leq L(\text{OPT}) \leq L(\text{Greedy}) \leq \frac{\log(2)}{d},$$

**Theorem 3.13** shows that the loss of OPT and Greedy are relatively close. This indicates that waiting and criticality information are complements, in that waiting to thicken the market is substantially valuable only when the planner can identify urgent cases. Observe that OPT could in principle wait to thicken the market, but the gains from doing so (compared to running the Greedy algorithm) are not large. Under these new information assumptions, we once more find that local algorithms can perform close to computationally intensive global optima.

Moreover, **Theorem 3.5** and **Theorem 3.13** together show that criticality information is valuable, since the loss of Patient, which naively exploits criticality information, is exponentially smaller than the loss of OPT, the optimal algorithm without this information.
What if the planner knows *more* than just the set of critical agents? For instance, the planner may have long-horizon forecasts of agent departure times, or the planner may know that certain agents are more likely to have acceptable transactions in future than other agents\(^{27}\). However, Theorem 3.7 shows that *no* expansion of the planner’s information set yields a better-than-exponential loss. This is because \(L(\text{OMN})\) is the loss under a maximal expansion of the planner’s information; the case where the planner has perfect foreknowledge of the future.

Taken together, these results suggest that criticality information is particularly valuable. This information is necessary to achieve exponentially small loss, and no expansion of information enables an algorithm to achieve better-than-exponential loss.

However, in many settings, it is plausible that agents have privileged insight into their own departure timings. In such cases, agents may have incentives to misreport whether they are critical, in order to increase their chance of getting matched or to decrease their waiting time. We now exhibit a truthful mechanism without transfers that elicits such information from agents, and implements the Patient\((\alpha)\) algorithm.

We assume that agents are fully rational and know the underlying parameters, but they do not observe the actual realization of the stochastic process. That is, agents observe whether they are critical, but do not observe \(G_t\), while the planner observes \(G_t\) but does not observe which agents are critical. Consequently, agents’ strategies are independent of the realized sample path. Our results are sensitive to this assumption\(^{28}\); for instance, if the agent knew that she had a neighbor, or knew that the pool at that moment was very large, she would have an incentive under our mechanism to falsely report that she was critical.

The truthful mechanism, Patient-Mechanism\((\alpha)\), is described below.

**Definition 3.14** (Patient-Mechanism\((\alpha)\)). *Assign independent exponential clocks with rate \(1/\alpha\) to each agent \(a\), where \(\alpha \in [0, \infty)\). Ask agents to report when they get critical. If agent’s exponential clock ticks or if she reports becoming critical, the market-maker attempts to match her to a random neighbor. If the agent has no neighbors, the market-maker treats her as if she has perished, i.e., she will never be matched again.*

Each agent \(a\) selects a mixed strategy by choosing a function \(c_a(\cdot)\); at the interval \([t, t+dt]\)

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\(^{27}\)In our model, the number of acceptable transactions that a given agent will have with the next \(N\) agents to arrive is Bernoulli distributed. If the planner knows beforehand whether a given agent’s realization is above or below the 50th percentile of this distribution, it is as though agents have different ‘types’.

\(^{28}\)This assumption is plausible in many settings; generally, centralized brokers know more about the current state of the market than individual traders. Indeed, frequently agents approach centralized brokers because they do not know who is available to trade with them.

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after her arrival, if she is not yet critical, she reports being critical with rate $c_a(t)dt$, and when she truly becomes critical she reports that immediately. Our main result in this section asserts that if agents are not too impatient, then the Patient-Mechanism($\alpha$) is incentive-compatible in the sense that the truthful strategy profile is a strong $\epsilon$-Nash equilibrium.\footnote{Any strong $\epsilon$-Nash equilibrium is an $\epsilon$-Nash equilibrium. For a definition of strong $\epsilon$-Nash equilibrium, see Definition 8.1.}

**Theorem 3.15.** Suppose that the market is in the stationary distribution and $d = \text{polylog}(m)$. Let $\bar{\alpha} = 1/\alpha + 1$ and $\beta = \bar{\alpha}(1 - d/m)^{m/\bar{\alpha}}$. Then, for $0 \leq \delta \leq \beta$, $c_a(t) = 0$ for all $a, t$ (i.e., truthful strategy profile) is a strong $\epsilon$-Nash equilibrium for Patient-Mechanism($\alpha$), where $\epsilon \to 0$ as $m \to \infty$.

If $d \geq 2$ and $0 \leq \delta \leq e^{-d/2}$, the truthful strategy profile is a strong $\epsilon$-Nash equilibrium for Patient-Mechanism($\infty$), where $\epsilon \to 0$ as $m \to \infty$.

**Proof Overview.** There is a hidden obstacle in proving that truthful reporting is incentive-compatible: Even if one assumes that the market is in a stationary distribution at the point an agent enters, the agent’s beliefs about pool size may change as time passes. In particular, an agent makes inferences about the current distribution of pool size, conditional on not having been matched yet, and this conditional distribution is different from the stationary distribution. This makes it difficult to compute the payoffs from deviations from truthful reporting. We tackle this problem by using the concentration bounds from Proposition 6.9, and focusing on strong $\epsilon$-Nash equilibrium, which allows small deviations from full optimality.

The intuition behind this proof is that an agent can be matched in one of two ways under Patient-Mechanism($\infty$): Either she becomes critical, and has a neighbor, or one of her neighbors becomes critical, and is matched to her. By symmetry, the chance of either happening is the same, because with probability 1, every matched pair consists of one critical agent and one non-critical agent. When an agent declares that she is critical, she is taking her chance that she has a neighbor in the pool right now. By contrast, if she waits, there is some probability that another agent will become critical and be matched to her. Consequently, for small $\delta$, agents will opt to wait.

\[ \square \]

### 3.4 Technical Contributions

As alluded to above, most of our results follow from concentration results on the distribution of the pool size for each of the online algorithms that are stated in Proposition 6.5 \footnote{\text{polylog}(m) denotes any polynomial function of $\log(m)$. In particular, $d = \text{polylog}(m)$ if $d$ is a constant independent of $m.$} and...
Proposition 6.9. In this last part we describe ideas behind these crucial results.

For analyzing many classes of stochastic processes one needs to prove concentration bounds on functions defined on the underlying process by means of central limit theorems, Chernoff bounds or Azuma-Hoeffding bounds. In our case many of these tools fail. This is because we are interested in proving that for any large time $t$, a given function is concentrated in an interval whose size depend only on $d, m$ and not $t$. Since $t$ can be significantly larger than $d, m$ a direct proof fails.

In contrast we observe that $Z_t$ is a Markov Chain for a large class of online algorithms. Building on this observation, first we show that the underlying Markov Chain has a unique stationary distribution and it mixes rapidly. Then we use the stationary distribution of the Markov Chain to prove our concentration bounds.

However, that is not the end of the story. We do not have a closed form expression for the stationary distribution of the chain, because we are dealing with an infinite state space continuous time Markov Chain where the transition rates are complex functions of the states. Instead, we use the following trick. Suppose we want to prove that $Z_t$ is contained in an interval $[k^* - f(m, d), k^* + f(m, d)]$ for some $k^* \in \mathbb{N}$ with high probability, where $f(m, d)$ is a function of $m, d$ that does not depend on $t$. We consider a sequence of pairs of states $P_1 := (k^* - 1, k^* + 1), P_2 := (k^* - 2, k^* + 2), \text{ etc.}$ We show that if the Markov Chain is at any of the states of $P_i$, it is more likely (by an additive function of $m, d$) that it jumps to a state of $P_{i-1}$ as opposed to $P_{i+1}$. Using balance equations and simple algebraic manipulations, this implies that the probability of states in $P_i$ geometrically decrease as $i$ increases. In other words $Z_t$ is concentrated in a small interval around $k^*$. We believe that this technique can be used in studying other complex stochastic processes.

4 Performance of the Optimum and Periodic Algorithms

In this section we lower-bound the loss of the optimum solutions in terms of $d$. In particular, we prove the following theorems.

Theorem 4.1. If $m > 10d$, then for any $T > 0$

$$L(OPT) \geq \frac{1}{2d + 1 + d^2/m}.$$
Theorem 4.2. If $m > 10d$, then for any $T > 0$,

$$L(OMN) \geq \frac{e^{-d - d^2/m}}{d + 1 + d^2/m}$$

Before proving the above theorems, it is useful to study the evolution of the system in the case of the inactive algorithm, i.e., where the online algorithm does nothing and no agents ever get matched. We later use this analysis in this section, as well as Section 5 and Section 6.

We adopt the notation $\tilde{A}_t$ and $\tilde{Z}_t$ to denote the agents in the pool and the pool size in this case. Observe that by definition for any matching algorithm and any realization of the process,

$$Z_t \leq \tilde{Z}_t. \quad (4.1)$$

Using the above equation, in the following fact we show that for any matching algorithm $\mathbb{E}[Z_t] \leq m$.

**Proposition 4.3.** For any $t_0 \geq 0$,

$$\mathbb{P} \left[ \tilde{Z}_{t_0} = \ell \right] \leq \frac{m^\ell}{\ell!}.$$

Therefore, $\tilde{Z}_t$ is distributed as a Poisson random variable of rate $m(1 - e^{-t_0})$, so

$$\mathbb{E} \left[ \tilde{Z}_{t_0} \right] = (1 - e^{-t_0})m.$$

**Proof.** Let $K$ be a random variable indicating the number agents who enter the pool in the interval $[0, t_0]$. By Bayes rule,

$$\mathbb{P} \left[ \tilde{Z}_{t_0} = \ell \right] = \sum_{k=0}^{\infty} \mathbb{P} \left[ \tilde{Z}_{t_0} = \ell, K = k \right] = \sum_{k=0}^{\infty} \mathbb{P} \left[ \tilde{Z}_{t_0} = \ell | K = k \right] \cdot \frac{(mt_0)^k e^{-mt_0}}{k!},$$

where the last equation follows by the fact that arrival rate of the agents is a Poisson random variable of rate $m$.

Now, conditioned on the event that an agent $a$ arrives in the interval $[0, t_0]$, the probability that she is in the pool at time $t_0$ is at least,

$$\mathbb{P} \left[ X_{a_i} = 1 \right] = \int_{t_0}^{t_0} \frac{1}{t_0} \mathbb{P} \left[ s(a_i) \geq t_0 - t \right] dt = \frac{1}{t_0} \int_{t_0}^{t_0} e^{t - t_0} dt = \frac{1 - e^{-t_0}}{t_0}.$$
Therefore, conditioned on $K = k$, the distribution of the number of agents at time $t_0$ is a Binomial random variable $B(k, p)$, where $p := (1 - e^{-t_0}) / t_0$. Let $\mu = m(1 - e^{-t_0})$, we have

$$
\mathbb{P} \left[ \tilde{Z}_{t_0} = \ell \right] = \sum_{k=\ell}^{\infty} \binom{k}{\ell} \cdot p^\ell \cdot (1 - p)^{k-\ell} \frac{(mt_0)^k e^{-mt_0}}{k!}
$$

$$
= \sum_{k=\ell}^{\infty} \frac{m^\ell e^{-mt_0}}{\ell!(k-\ell)!} (1 - e^{-t_0})^\ell (t_0 - 1 + e^{-t_0})^{k-\ell}
$$

$$
= \frac{m^\ell e^{-mt_0} \mu^\ell}{\ell!} \sum_{k=\ell}^{\infty} \frac{(mt_0 - \mu)^{k-\ell}}{(k-\ell)!} = \frac{\mu^\ell e^{-\mu}}{\ell!}.
$$

### 4.1 Loss of the Optimum Online Algorithm

In this section, we prove Theorem 4.1. Let $\zeta$ be the expected pool size of the OPT,

$$
\zeta := \mathbb{E}_{t \sim \text{unif}[0,T]} [Z_t]
$$

Since OPT does not know $A_c^t$, each critical agent perishes with probability 1. Therefore,

$$
L(\text{OPT}) = \frac{1}{m \cdot T} \mathbb{E} \left[ \int_{t=0}^{T} Z_t dt \right] = \frac{\zeta T}{mT} = \zeta / m. \quad (4.2)
$$

To finish the proof we need to lower bound $\zeta$ by $m/(2d + 1 + d^2/m)$. We provide an indirect proof by showing a lower-bound on $L(\text{OPT})$ which in turn lower-bounds $\zeta$.

Our idea is to lower-bound the probability that an agent does not have any acceptable transactions throughout her sojourn, and this directly gives a lower-bound on $L(\text{OPT})$ as those agents cannot be matched under any algorithm. Fix an agent $a \in A$. Say $a$ enters the market at a time $t_0 \sim \text{unif}[0,T]$, and $s(a) = t$, we can write

$$
\mathbb{P} [N(a) = \emptyset] = \int_{t_0}^{\infty} \mathbb{P} [s(a) = t] \cdot \mathbb{E} \left[ (1 - d/m)^{|A_{t_0}|} \right] \cdot \mathbb{E} \left[ (1 - d/m)^{|A_{t_0+t+t_0}|} \right] dt \quad (4.3)
$$

To see the above, note that $a$ does not have any acceptable transactions, if she doesn’t have any neighbors upon arrival, and none of the new agents that arrive during her sojourn are
not connected to her. Using the Jensen’s inequality, we have

\[
\mathbb{P}[N(a) = \emptyset] \geq \int_{t=0}^{\infty} e^{-t} \cdot (1 - d/m)^{\mathbb{E}[Z_{t_0}]} \cdot (1 - d/m)^{\mathbb{E}[|A_{t_0,t+t_0}|]} dt
\]

\[
= \int_{t=0}^{\infty} e^{-t} \cdot (1 - d/m)\zeta \cdot (1 - d/m)^{mt} dt
\]

The last equality follows by the fact that \(\mathbb{E}[|A_{t_0,t+t_0}|] = mt\). Since \(d/m < 1/10\), \(1 - d/m \geq e^{-d/m - d^2/m^2}\),

\[
L(OPT) \geq \mathbb{P}[N(a) = \emptyset] \geq e^{-\zeta(d/m + d^2/m^2)} \int_{t=0}^{\infty} e^{-t(1 + d + d^2/m)} dt \geq \frac{1 - \zeta(1 + d/m)d/m}{1 + d + d^2/m} \quad (4.4)
\]

Putting (4.2) and (4.4) together, for \(\beta := \zeta d/m\) we get

\[
L(OPT) \geq \max\left\{ \frac{1 - \beta(1 + d/m)}{1 + d + d^2/m}, \frac{\beta}{d} \right\} \geq \frac{1}{2d + 1 + d^2/m}
\]

where the last inequality follows by letting \(\beta = \frac{d}{2d+1+d^2/m}\) be the minimizer of the middle expression.

### 4.2 Loss of the Omniscient Algorithm

In this section, we prove Theorem 4.2. This demonstrates that, in the case that the planner observes the critical agents, no policy can yield a faster-than-exponential decrease in losses, as a function of the average degree of each agent.\(^{31}\)

The proof is very similar to Theorem 4.1. Let \(\zeta\) be the expected pool size of the OMN,

\[
\zeta := \mathbb{E}_{t \sim \text{unif}[0,T]}[Z_t].
\]

By (4.1) and Proposition 4.3,

\[
\zeta \leq \mathbb{E}_{t \sim \text{unif}[0,T]}[\tilde{Z}_t] \leq m.
\]

Note that (4.2) does not hold in this case because the omniscient algorithm knows the set of critical agents at time \(t\).

Now, fix an agent \(a \in A\), and let us lower-bound the probability that \(N(a) = \emptyset\). Say \(a\)

\(^{31}\)This, in fact, proves a much more powerful claim: It shows that even if the planner knows when agents are going to be critical upon their arrival, she still cannot to faster-than-exponential decrease in losses.
enters the market at time $t_0 \sim \text{unif}[0, T]$ and $s(a) = t$, then

$$
P[N(a) = \emptyset] = \int_0^\infty \mathbb{P}[s(a) = t] \cdot \mathbb{E}\left[(1 - \frac{d}{m})^{Z_{t_0}}\right] \cdot \mathbb{E}\left[(1 - \frac{d}{m})^{A_{t_0,t+t_0}}\right] dt \geq \int_0^\infty e^{-t(1 - \frac{d}{m})^{\zeta+m}} dt \geq \frac{e^{-\zeta(1+d/m)d/m}}{1+d+d^2/m} \geq \frac{e^{-d-d^2/m}}{1+d+d^2/m},
$$

where the first inequality uses the Jensen’s inequality and the second inequality uses the fact that when $d/m < 1/10$, $1 - d/m \geq e^{-d/m} - d^2/m^2$.

### 4.3 Loss of the Periodic Matching Algorithms

In this part we lower-bound the loss of the periodic matching algorithms.

**Definition 4.4 (Periodic Matching Algorithm($\alpha$)).** If an agent $a$ becomes critical at time $t$ match her to an arbitrary agent in $N_t(a)$ whenever $N_t(a) \neq \emptyset$. Every $\alpha$ time steps find a maximum matching between agents that are in the pool and match the endpoints of each edge in that matching.

The following theorem is the main result of this section.

**Theorem 4.5.** If $2d < m$, for any $\alpha > 0$, the loss of the periodic matching algorithm is at least

$$
\mathbf{L}(\text{Periodic}(\alpha)) \geq \exp\left(-\frac{8}{3(1-e^{-\alpha})}\right) \cdot \frac{2\alpha d(2d + 1)}{2d^2/m}.
$$

It follows that if $\alpha$ is a function independent of $d$, then as $d \to \infty$, we get $\mathbf{L}(\text{Periodic}(\alpha)) \geq \Omega(1/d^2)$.

In the following lemma we prove a lower-bound on the size of the maximum matching in a graph where part of it has a distribution of an Erdős-Rényi random graph. We refer the reader to Karp and Sipser (1981) for tight bounds on the size of the maximum matching of a sparse random graph.

**Lemma 4.6.** Let $G = (S \cup T, E)$ be a graph where between each pair of vertices in $S$, there is an edge with probability $d/m$. If $2d < m$, then the maximum matching $M$ of $G$ satisfies

$$
|M| \geq \frac{|S|}{2} - \frac{1 - e^{-d|S|/m}}{2d/m - 2d^2/m^2}.
$$
Proof. We lower-bound the size of a maximal matching of $G$ and that itself lower-bounds the size of the maximum matching. Let $f(s)$ be expected size of a maximal matching in $G$ when $|S| = s$. Let $v \in S$. Then, we can write

$$f(s) \geq f(s - 2) + \mathbb{P} [v \text{ has a neighbor}] \geq f(s - 2) + 1 - (1 - d/m)^{s-1}.$$  

Therefore,

$$f(s) \geq \sum_{i=1}^{s/2} 1 - (1 - d/m)^{s+1-2i} \geq \sum_{i=1}^{s/2} 1 - e^{-d(s+1-2i)/m} = \frac{s}{2} - \frac{e^{-d/m} - e^{-(s+1)d/m}}{1 - e^{-2d/m}} \geq \frac{s}{2} - \frac{1 - e^{-sd/m}}{2d/m - 2d^2/m^2}.$$

The second inequality uses $1 - x \leq e^{-x}$ for $x < 1$ and the last inequality uses $e^{-x} \leq 1 - x + x^2/2$ for $x < 1$ and $2d < m$.

We use $A_{k\alpha}$ to denote the agents in the pool right before the execution of the $k$-th maximum matching algorithm and $A_{k\alpha+}$ to denote the agents in the pool right after that, i.e., agents in $A_{k\alpha+}$ are those who are not matched in the maximum matching that we choose.

**Lemma 4.7.** If $2d < m$, for any $\alpha > 0$, the expected number of agents in the pool after each execution of the global maximum matching algorithm is at most $\frac{4m}{3d(1 - e^{-\alpha})}$, i.e.,

$$\mathbb{E} [Z_{k\alpha+}] \leq \frac{4m}{3d(1 - e^{-\alpha})}.$$  

Proof. We prove the claim by induction. We assume that $\mathbb{E} [Z_{k\alpha+}] \leq \beta m$ for $\beta = \frac{4m}{3d(1 - e^{-\alpha})}$. Let $S_k = A_{k\alpha,(k+1)\alpha} \cap A_{(k+1)\alpha}$ be the set of the new agents that arrive to the market after time $k\alpha$ and are still in the market before the execution of $k+1$-st maximum matching.

Also let $T_k = A_{k\alpha+} \cap A_{(k+1)\alpha}$ be the set of the unmatched agents in the $k$-th maximum matching who are still in the market before the execution of $k+1$-st maximum matching.

Since any agent $a \in A_{k\alpha+}$, gets critical by time $(k+1)\alpha$ with probability $1 - e^{-\alpha}$, we have

$$\mathbb{E} [|T_k|] \leq e^{-\alpha} \mathbb{E} [|A_{k\alpha+}|] = e^{-\alpha} \mathbb{E} [Z_{k\alpha+}] \leq \beta e^{-\alpha} m.$$  

(4.5)
When we run $k + 1$-st maximum matching algorithm we can assume that there is an Erdős-Rényi random graph between agents of $S_k$ (because the periodic matching algorithm does not look into the edges between the nodes of $S_k$). By Lemma 4.6, we can upper-bound the size of $T_{k+1}$,

$$Z_{(k+1)\alpha^+} \leq |T_k| + \frac{1 - e^{-d|S_k|/m}}{d/m - d^2/m^2}$$  \hspace{1cm} (4.6)

Next, we upper-bound the second term of the RHS. Let $\mu = m(1 - e^{-\alpha})$. By (4.1), $|S_k| \leq \tilde{Z}_\alpha$, so

$$\mathbb{E}[e^{-d|S_k|/m}] \geq \mathbb{E}[e^{-d\tilde{Z}_\alpha/m}] = \sum_{i=0}^{\infty} e^{-d\tilde{Z}_\alpha/m} \cdot \frac{e^{-\mu} \mu^i}{i!} = e^{-\mu} \sum_{i=0}^{\infty} \frac{(e^{-d/m} \cdot \mu)^i}{i!} = e^{\mu(e^{-d/m} - 1)} \geq e^{-d\mu/m}.

The first equality uses Proposition 4.3 and the last inequality uses $e^{-d/m} \geq 1 - d/m$.

Taking expectations from both sides of (4.6), we get

$$\mathbb{E}[Z_{(k+1)\alpha^+}] \leq \mathbb{E}[|T_k|] + \frac{2(1 - e^{-d\mu/m})}{d/m} \leq \beta e^{-\alpha} m + \frac{4m}{3d} \leq \beta m.$$  \hspace{1cm} (4.7)

The second inequality uses (4.5) and that $2d < m$ and the last inequality uses the definition of $\beta$. \hfill \Box

**Proof of Theorem 4.5.** Without loss of generality, we assume that $T$ is a multiple of $\alpha$. Let $P_k$ be the set of perished agents that arrive after time $k\alpha$ and get critical before $(k+1)\alpha$ for all integers $0 \leq k \leq T/\alpha$. All we need to do is to lower-bound the expected size of $P_k$. Then by linearity of expectation we get

$$L(\text{Periodic}(\alpha)) \geq \frac{\sum_{k=0}^{T/\alpha-1} \mathbb{E}[|P_k|]}{mT}$$  \hspace{1cm} (4.7)

Let $\epsilon > 0$ be a parameter very close to zero (eventually we will let $\epsilon \to 0$). Fix a time $t_0 \in [0, \alpha - \epsilon]$. Say $a$ is an agent who arrives in the interval $[k\alpha + t_0, k\alpha + t_0 + \epsilon]$. Next, we
bound \( P[a \in P_k] \). Similar to (4.3) we can write,

\[
P[a \in P_k] \geq \mathbb{P}[N(a) = \emptyset] \geq \int_{t=0}^{\alpha-t_0} \mathbb{P}[s(a) = t] \cdot \mathbb{E}[(1 - d/m)^{|A_n|}] 
\cdot \mathbb{E}[(1 - d/m)^{|A_{k\alpha+t_0,k\alpha+t_0+t}|}] dt 
\geq \int_{t=0}^{\alpha-t_0} e^{-t} \cdot (1 - d/m)^{\mathbb{E}[Z_{k\alpha}]+\mathbb{E}[\mid A_{k\alpha,k\alpha+t_0}|]} dt.
\]

Now, by Lemma 4.7, \( \mathbb{E}[Z_{k\alpha}] \leq \beta m \) for \( \beta = 4/3(1 - e^{-\alpha}) \) and by Proposition 4.3, \( \mathbb{E}[|A_{k\alpha,k\alpha+t_0}|] \leq m(1 - e^{-t-t_0}) \). Therefore,

\[
P[a \in P_k] \geq \int_{t=0}^{\alpha-t_0} e^{-t} \cdot e^{-2t} \cdot m^m \cdot (1 - e^{-t-t_0})^m dt.
\]

Assuming \( 2d < m \) we have \( 1 - d/m \geq e^{-2d/m} \). Also, using \( e^{-t-t_0} \geq 1 - t - t_0 \) we can write,

\[
P[a \in P_k] \geq \int_{t=0}^{\alpha-t_0} e^{-t} \cdot e^{-2d \beta - 2d(t+t_0)} dt \geq \int_{t=0}^{\infty} e^{-(2d+1)t} \frac{e^{-2d \beta - 2d t_0}}{2d + 1} dt = \frac{e^{-2d \beta - 2d t_0}}{2d + 1}.
\]

Finally, we can use the above bound to lower-bound \( \mathbb{E}[|P_k|] \). The expected number of agents arriving in the interval \([k\alpha + t_0, k\alpha + t_0 + \epsilon]\) is \( m\epsilon \). Letting \( \epsilon \to 0 \) and integrating over \( t_0 \) we get

\[
\mathbb{E}[|P_k|] \geq \int_{t_0=0}^{\alpha} \frac{m \cdot e^{-2d \beta - 2d t_0}}{2d + 1} dt_0 = \frac{m \cdot e^{-2d \beta}}{2d + 1} \int_{t_0=0}^{\alpha} e^{-2d t_0} dt_0 \geq \frac{m \cdot e^{-2d \beta}}{2d(2d + 1)}.
\]

The theorem follows by (4.7).

5 Modeling an Online Algorithm as a Markov Chain

5.1 Background

As an important preliminary, we establish that under both of the Patient and Greedy algorithms the random processes \( Z_t \) are Markovian, have unique stationary distributions, and mix rapidly to the stationary distribution. Before getting into the details we provide a brief overview on continuous time Markov Chains. We refer interested readers to Norris (1998); Levin et al. (2006) for detailed discussions.

Let \( Z_t \) be a continuous time Markov Chain on the non-negative integers (\( \mathbb{N} \)) that starts
from state 0. For any two states $i, j \in \mathbb{N}$, we assume that the rate of going from $i$ to $j$ is $r_{i \rightarrow j} \geq 0$. The rate matrix $Q \in \mathbb{N} \times \mathbb{N}$ is defined as follows,

$$Q(i, j) := \begin{cases} r_{i \rightarrow j} & \text{if } i \neq j, \\ \sum_{k \neq i} -r_{i \rightarrow k} & \text{otherwise.} \end{cases}$$

Note that, by definition, the sum of the entries in each row of $Q$ is zero. It turns out that (see e.g., (Norris, 1998, Theorem 2.1.1)) the transition probability in $t$ units of time is,

$$e^{tQ} = \sum_{i=0}^{\infty} \frac{t^i Q^i}{i!}.$$ 

Let $P_t := e^{tQ}$ be the transition probability matrix of the Markov Chain in $t$ time units. It follows that,

$$\frac{d}{dt}P_t = P_tQ. \quad (5.1)$$

In particular, in any infinitesimal time step $dt$, the chain moves based on $Q \cdot dt$.

A Markov Chain is irreducible if for any pair of states $i, j \in \mathbb{N}$, $j$ is reachable from $i$ with a non-zero probability. Fix a state $i \geq 0$, and suppose that $Z_{t_0} = i$, and let $T_1$ be the first jump out of $i$ (note that $T_1$ is distributed as an exponential random variable). State $i$ is positive recurrent iff

$$\mathbb{E}[\inf\{t \geq T_1 : Z_t = i\}|Z_{t_0} = i] < \infty \quad (5.2)$$

The ergodic theorem (Norris, 1998, Theorem 3.8.1) entails that a continuous time Markov Chain has a unique stationary distribution if and only if it has a positive recurrent state.

Let $\pi : \mathbb{N} \rightarrow \mathbb{R}_+$ be the stationary distribution of a Markov chain. It follows by the definition that for any $t \geq 0$, $P_t = \pi P_t$. The balance equations of a Markov chain say that for any $S \subseteq \mathbb{N}$,

$$\sum_{i \in S, j \notin S} \pi(i)r_{i \rightarrow j} = \sum_{i \in S, j \notin S} \pi(j)r_{j \rightarrow i}. \quad (5.3)$$

Let $z_t(.)$ be the distribution of $Z_t$ at time $t \geq 0$, i.e., $z_t(i) := \mathbb{P}[Z_t = i]$ for any integer $i \geq 0$. For any $\epsilon > 0$, we define the mixing time (in total variation distance) of this Markov Chain as follows,

$$\tau_{\text{mix}}(\epsilon) = \inf\left\{ t : \|z_t - \pi\|_{\text{TV}} := \sum_{k=0}^{\infty} |\pi(k) - z_t(k)| \leq \epsilon \right\}. \quad (5.4)$$
5.2 Markov Chain Characterization

First, we argue that $Z_t$ is a Markov process under the Patient and Greedy algorithms. This follows from the following simple observation.

**Proposition 5.1.** Under either of Greedy or Patient algorithms, for any $t \geq 0$, conditioned on $Z_t$, the distribution of $G_t$ is uniquely defined. So, given $Z_t$, $G_t$ is conditionally independent of $Z_{t'}$ for $t' < t$.

**Proof.** Under the Greedy algorithm, at any time $t \geq 0$, $|E_t| = 0$. Therefore, conditioned on $Z_t$, $G_t$ is an empty graph with $|Z_t|$ vertices.

For the Patient algorithm, note that the algorithm never looks at the edges between non-critical agents, so the algorithm is oblivious to these edges. It follows that under the Patient algorithm, for any $t \geq 0$, conditioned on $Z_t$, $G_t$ is an Erdős-Rényi random graph with $|Z_t|$ vertices and parameter $d/m$.

The following is the main theorem of this section.

**Theorem 5.2.** For the Patient and Greedy algorithms and any $0 \leq t_0 < t_1$,

$$
\mathbb{P}[Z_{t_1} \mid Z_t \text{ for } 0 \leq t < t_1] = \mathbb{P}[Z_{t_1} \mid Z_t \text{ for } t_0 \leq t < t_1].
$$

The corresponding Markov Chains have unique stationary distributions and mix in time $O(\log(m) \log(1/\epsilon))$ in total variation distance:

$$
\tau_{\text{mix}}(\epsilon) \leq O(\log(m) \log(1/\epsilon)).
$$

The proof of the theorem can be found in the Appendix B.

This theorem is crucial in justifying our focus on long-run results in Section 3, since these Markov chains converge very rapidly (in $O(\log(m))$ time) to their stationary distributions.

6 Performance Analysis

In this section we upper bound $L(\text{Greedy})$ and $L(\text{Patient})$ as a function of $d$, and we upper bound $L(\text{Patient}(\alpha))$ as a function of $d$ and $\alpha$.

We prove the following three theorems.\footnote{We use the operators $O$ and $\tilde{O}$ in the standard way. That is, $f(m) = O(g(m))$ iff there exists a positive real number $N$ and a real number $m_0$ such that $|f(m)| \leq N|g(m)|$ for all $m \geq m_0$. $\tilde{O}$ is similar but ignores logarithmic factors, i.e. $f(m) = \tilde{O}(g(m))$ iff $f(m) = O(g(m) \log^k g(m))$ for some $k$.}
Theorem 6.1. For any $\epsilon \geq 0$ and $T > 0$,

$$L(Greedy) \leq \frac{\log(2)}{d} + \frac{\tau_{\text{mix}}(\epsilon)}{T} + 6\epsilon + O\left(\frac{\log(m/d)}{\sqrt{dm}}\right),$$

(6.1)

where $\tau_{\text{mix}}(\epsilon) \leq 2\log(m/d)\log(2/\epsilon)$.

Theorem 6.2. For any $\epsilon > 0$ and $T > 0$,

$$L(Patient) \leq \max_{z \in [1/2,1]} \left( z + \tilde{O}(1/\sqrt{m}) \right) e^{-zd} + \frac{\tau_{\text{mix}}(\epsilon)}{T} + \frac{\epsilon m}{d^2} + 2/m,$$

(6.2)

where $\tau_{\text{mix}}(\epsilon) \leq 8\log(m)\log(4/\epsilon)$.

Theorem 6.3. Let $\bar{\alpha} := 1/\alpha + 1$. For any $\epsilon > 0$ and $T > 0$,

$$L(Patient(\alpha)) \leq \max_{z \in [1/2,1]} \left( z + \tilde{O}(\sqrt{\bar{\alpha}/m}) \right) e^{-zd/\bar{\alpha}} + \frac{\tau_{\text{mix}}(\epsilon)}{\bar{\alpha}T} + \frac{\epsilon m \bar{\alpha}}{d^2} + 2\bar{\alpha}/m,$$

where $\tau_{\text{mix}}(\epsilon) \leq 8\log(m/\bar{\alpha})\log(4/\epsilon)$.

We will prove Theorem 6.1 in Subsection 6.1, Theorem 6.2 in Subsection 6.2 and Theorem 6.3 in Subsection 6.3.

6.1 Loss of the Greedy Algorithm

In this part we upper bound $L(Greedy)$. We crucially exploit the fact that $Z_t$ is a Markov Chain and has a unique stationary distribution, $\pi : \mathbb{N} \to \mathbb{R}_+$. Our proof proceeds in three steps: First, we show that $L(Greedy)$ is bounded by a function of the expected pool size. Second, we show that the stationary distribution is highly concentrated around some point $k^*$, which we characterize. Third, we show that $k^*$ is close to the expected pool size.

Let $\zeta := \mathbb{E}_{Z \sim \mu}[Z]$ be the expected size of the pool under the stationary distribution of the Markov Chain on $Z_t$. First, observe that if the Markov Chain on $Z_t$ is mixed, then the agents perish at the rate of $\zeta$, as the pool is almost always an empty graph under the Greedy algorithm. Roughly speaking, if we run the Greedy algorithm for a sufficiently long time then Markov Chain on size of the pool mixes and we get $L(Greedy) \simeq \frac{\zeta}{m}$. This observation is made rigorous in the following lemma. Note that as $T$ and $m$ grow, the first three terms become negligible.
Lemma 6.4. For any \( \epsilon > 0 \), and \( T > 0 \),

\[
L(\text{Greedy}) \leq \frac{\tau_{\text{mix}}(\epsilon)}{T} + 6\epsilon + \frac{1}{m}2^{-6m} + \frac{\mathbb{E}_{Z \sim \pi}[Z]}{m}.
\]

The theorem is proved in the Appendix C.1.

The proof of the above lemma involves lots of algebra, but the intuition is as follows: The \( \frac{\mathbb{E}_{Z \sim \pi}[Z]}{m} \) term is the loss under the stationary distribution. This is equal to \( L(\text{Greedy}) \) with two approximations: First, it takes some time for the chain to transit to the stationary distribution. Second, even when the chain mixes, the distribution of the chain is not exactly equal to the stationary distribution. The \( \frac{\tau_{\text{mix}}(\epsilon)}{T} \) term provides an upper bound for the loss associated with the first approximation, and the term \( (6\epsilon + \frac{1}{m}2^{-6m}) \) provides an upper bound for the loss associated with the second approximation.

Given Lemma 6.4, in the rest of the proof we just need to get an upper bound for \( \mathbb{E}_{Z \sim \pi}[Z] \). Unfortunately, we do not have any closed form expression of the stationary distribution, \( \pi(\cdot) \). Instead, we use the balance equations of the Markov Chain defined on \( Z_t \) to characterize \( \pi(\cdot) \) and upper bound \( \mathbb{E}_{Z \sim \pi}[Z] \).

Let us rigorously define the transition probability operator of the Markov Chain on \( Z_t \). For any pool size \( k \), the Markov Chain transits only to the states \( k + 1 \) or \( k - 1 \). It transits to state \( k + 1 \) if a new agent arrives and the market-maker cannot match her (i.e., the new agent does not have any edge to the agents currently in the pool) and the Markov Chain transits to the state \( k - 1 \) if a new agent arrives and is matched or an agent currently in the pool gets critical. Thus, the transition rates \( r_{k \rightarrow k+1} \) and \( r_{k \rightarrow k-1} \) are defined as follows,

\[
r_{k \rightarrow k+1} := m \left( 1 - \frac{d}{m} \right)^k \quad (6.3)
\]

\[
r_{k \rightarrow k-1} := k + m \left( 1 - \left( 1 - \frac{d}{m} \right)^k \right). \quad (6.4)
\]
In the above equations we used the fact that agents arrive at rate $m$, they perish at rate 1 and the probability of an acceptable transaction between two agents is $d/m$.

Let us write down the balance equation for the above Markov Chain (see equation (5.3) for the full generality). Consider the cut separating the states 0, 1, 2, \ldots, $k$ from the rest (see Figure 4 for an illustration). It follows that,

$$
\pi(k-1)r_{k-1\to k} = \pi(k)r_{k\to k-1}.
$$

(6.5)

Now, we are ready to characterize the stationary distribution $\pi(\cdot)$. In the following proposition we show that there is a number $z^* \leq \log(2)m/d$ such that under the stationary distribution, the size of the pool is highly concentrated in an interval of length $O(\sqrt{m/d})$ around $z^*$.

**Proposition 6.5.** There exists $m/(2d+1) \leq k^* < \log(2)m/d$ such that for any $\sigma > 1$,

$$
\mathbb{P}_\pi \left[ k^* - \sigma \sqrt{2m/d} \leq Z \leq k^* + \sigma \sqrt{2m/d} \right] \geq 1 - O(\sqrt{m/d})e^{-\sigma^2}.
$$

**Proof.** Let us define $f : \mathbb{R} \to \mathbb{R}$ as an interpolation of the difference of transition rates over the reals,

$$
f(x) := m(1 - d/m)^x - (x + m(1 - (1 - d/m)^x)).
$$

In particular, observe that $f(k) = r_{k\to k+1} - r_{k\to k-1}$. The above function is a decreasing convex function over non-negative reals. We define $k^*$ as the unique root of this function.

Let $k^*_{\min} := m/(2d+1)$ and $k^*_{\max} := \log(2)m/d$. We show that $f(k^*_{\min}) \geq 0$ and $f(k^*_{\max}) \leq 0$. This shows that $k^*_{\min} \leq k^* < k^*_{\max}$.

$$
\begin{align*}
f(k^*_{\min}) &\geq -k^*_{\min} - m + 2m(1 - d/m)k^*_{\min} \geq 2m \left( 1 - \frac{k^*_{\min}d}{m} \right) - k^*_{\min} - m = 0, \\
f(k^*_{\max}) &\leq -k^*_{\max} - m + 2m(1 - d/m)k^*_{\max} \leq -k^*_{\max} - m + 2me^{-(k^*_{\max})d/m} = -k^*_{\max} \leq 0.
\end{align*}
$$

In the first inequality we used equation (A.4) from Appendix A.

It remains to show that $\pi$ is highly concentrated around $k^*$. We prove this in several steps.

**Lemma 6.6.** For any integer $k \geq k^*$

$$
\frac{\pi(k+1)}{\pi(k)} \leq e^{-(k-k^*)d/m}.
$$

\[33\text{In this paper, } \log x \text{ refers to the natural log of } x.\]
And, for any \( k \leq k^\ast \), \( \pi(k - 1)/\pi(k) \leq e^{-(k^\ast-k+1)d/m} \).

**Proof.** For \( k \geq k^\ast \), by (6.3), (6.4), (6.5),

\[
\frac{\pi(k)}{\pi(k + 1)} = \frac{(k + 1) + m(1 - (1 - d/m)^{k+1})}{m(1 - d/m)^k} = \frac{k - k^\ast + 1 - m(1 - d/m)^{k+1} + 2m(1 - d/m)^{k^\ast}}{m(1 - d/m)^k}
\]

where we used the definition of \( k^\ast \). Therefore,

\[
\frac{\pi(k)}{\pi(k + 1)} \geq -(1 - d/m) + \frac{2}{(1 - d/m)^{k^\ast}} \geq \frac{1}{(1 - d/m)^{k^\ast}} \geq e^{-(k^\ast-k)d/m}
\]

where the last inequality uses \( 1 - x \leq e^{-x} \). Multiplying across the inequality yields the claim. Similarly, we can prove the second conclusion. For \( k \leq k^\ast \),

\[
\frac{\pi(k - 1)}{\pi(k)} = \frac{k - k^\ast - m(1 - d/m)^k + 2m(1 - d/m)^{k^\ast}}{m(1 - d/m)^{k^\ast-1}} \leq - (1 - d/m) + 2(1 - d/m)^{k^\ast-k+1} \leq (1 - d/m)^{k^\ast-k+1} \leq e^{-(k^\ast-k+1)d/m},
\]

where the second to last inequality uses \( k \leq k^\ast \).

By repeated application of the above lemma, for any integer \( k \geq k^\ast \), we get\(^{34}\)

\[
\pi(k) \leq \pi([k^\ast]) \leq \exp \left( -\frac{d}{m} \sum_{i=[k^\ast]}^{k^\ast-1} (i - k^\ast) \right) \leq \exp(-d(k - k^\ast - 1)^2/2m). \tag{6.6}
\]

We are almost done. For any \( \sigma > 0 \),

\[
\sum_{k=k^\ast+1+\sigma\sqrt{2m/d}}^{\infty} \pi(k) \leq \sum_{k=k^\ast+1+\sigma\sqrt{2m/d}}^{\infty} e^{-d(k - k^\ast - 1)^2/2m} = \sum_{k=0}^{\infty} e^{-d(k + \sigma\sqrt{2m/d})^2/2m} \leq \frac{e^{-\sigma^2}}{\min\{1/2, \sigma\sqrt{d/2m}\}}
\]

The last inequality uses equation (A.1) from Appendix A. We can similarly upper bound \( \sum_{k=0}^{k^\ast-\sigma\sqrt{2m/d}} \pi(k) \).

**Proposition 6.5** shows that the probability that the size of the pool falls outside an interval

\(^{34}[k^\ast]\) indicates the smallest integer larger than \( k^\ast \).
of length $O(\sqrt{m/d})$ around $k^*$ drops exponentially fast as the market size grows. We also remark that the upper bound on $k^*$ becomes tight as $d$ goes to infinity.

The following lemma exploits Proposition 6.5 to show that the expected value of the pool size under the stationary distribution is close to $k^*$.

**Lemma 6.7.** For $k^*$ as in Proposition 6.5,

$$\mathbb{E}_{Z \sim \pi}[Z] \leq k^* + O(\sqrt{m/d} \log(m/d)).$$

This lemma is proved in the Appendix C.2.

Now, Theorem 6.1. follows immediately by Lemma 6.4 and Lemma 6.7 because we have

$$\frac{\mathbb{E}_{Z \sim \pi}[Z]}{m} \leq \frac{1}{m}(k^* + O(\sqrt{m \log m})) \leq \frac{\log(2)}{d} + o(1).$$

### 6.2 Loss of the Patient Algorithm

Throughout this section we use $Z_t$ to denote the size of the pool under Patient. Let $\pi : \mathbb{N} \rightarrow \mathbb{R}_+$ be the unique stationary distribution of the Markov Chain on $Z_t$, and let $\zeta := \mathbb{E}_{Z \sim \pi}[Z]$ be the expected size of the pool under that distribution.

Once more our proof strategy proceeds in three steps. First, we show that $\text{L}(\text{Patient})$ is bounded by a function of $\mathbb{E}_{Z \sim \pi}[Z(1 - d/m)^{Z-1}]$. Second, we show that the stationary distribution of $Z_t$ is highly concentrated around some point $k^*$. Third, we use this concentration result to produce an upper bound for $\mathbb{E}_{Z \sim \pi}[Z(1 - d/m)^{Z-1}]$.

By Proposition 5.1, at any point in time $G_t$ is an Erdős-Rényi random graph. Thus, once an agent becomes critical, he has no acceptable transactions with probability $(1 - d/m)^{Z_t-1}$. Since each agent becomes critical with rate 1, if we run Patient for a sufficiently long time, then $\text{L}(\text{Patient}) \approx \frac{\zeta}{m}(1-d/m)^{\zeta-1}$. The following lemma makes the above discussion rigorous.

**Lemma 6.8.** For any $\epsilon > 0$ and $T > 0$,

$$\text{L}(\text{Patient}) \leq \frac{1}{m}\mathbb{E}_{Z \sim \pi}[Z(1 - d/m)^{Z-1}] + \frac{\tau_{\text{mix}}(\epsilon)}{T} + \frac{\epsilon m}{d^2}.$$  

*Proof.* See Appendix C.3. \qed

So in the rest of the proof we just need to lower bound $\mathbb{E}_{Z \sim \pi}[Z(1 - d/m)^{Z-1}]$. As in the Greedy case, we do not have a closed form expression for the stationary distribution, $\pi(\cdot)$.  

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Instead, we use the balance equations of the Markov Chain on $Z_t$ to show that $\pi$ is highly concentrated around a number $k^*$ where $k^* \in [m/2, m]$.

Let us start by defining the transition probability operator of the Markov Chain on $Z_t$. For any pool size $k$, the Markov Chain transits only to states $k + 1$, $k - 1$, or $k - 2$. The Markov Chain transits to state $k + 1$ if a new agent arrives, to the state $k - 1$ if an agent gets critical and the planner cannot match him, and it transits to state $k - 2$ if an agent gets critical and the planner matches him.

Remember that agents arrive with the rate $m$, they become critical with the rate of 1 and the probability of an acceptable transaction between two agents is $d/m$. Thus, the transition rates $r_{k\to k+1}$, $r_{k\to k-1}$, and $r_{k\to k-2}$ are defined as follows,

\begin{align}
 r_{k\to k+1} &:= m \quad (6.7) \\
r_{k\to k-1} &:= k \left(1 - \frac{d}{m}\right)^{k-1} \quad (6.8) \\
r_{k\to k-2} &:= k \left(1 - \left(1 - \frac{d}{m}\right)^{k-1}\right). \quad (6.9)
\end{align}

Let us write down the balance equation for the above Markov Chain (see equation (5.3) for the full generality). Consider the cut separating the states 0, 1, 2, \ldots, $k$ from the rest (see Figure 5 for an illustration). It follows that

$$
\pi(k) r_{k\to k+1} = \pi(k + 1) r_{k+1\to k} + \pi(k + 1) r_{k+1\to k-1} + \pi(k + 2) r_{k+2\to k} \quad (6.10)
$$

Now we can characterize $\pi(\cdot)$. We show that under the stationary distribution, the size of the pool is highly concentrated around a number $k^*$ where $k^* \in [m/2, m]$. Remember that under the Greedy algorithm, the concentration was around $k^* \in \left[\frac{m}{2d+1}, \frac{\log(2)m}{d}\right]$, whereas here it is at least $m/2$. 

![Figure 5: An illustration of the transition paths of the $Z_t$ Markov Chain under the Patient Algorithm](image-url)
Proposition 6.9 (Patient Concentration). There exists a number \( m/2 - 2 \leq k^* \leq m - 1 \) such that for any \( \sigma \geq 1 \),

\[
\mathbb{P}_x \left[ k^* - \sigma \sqrt{4m} \leq Z \right] \geq 1 - 2\sqrt{me^{-\sigma^2}}, \quad \mathbb{P} \left[ Z \leq k^* + \sigma \sqrt{4m} \right] \geq 1 - 8\sqrt{me^{-\frac{\sigma^2 \sqrt{m}}{2}}}. 
\]

Proof Overview. The proof idea is similar to Proposition 6.5. First, let us rewrite (6.10) by replacing transition probabilities from (6.7), (6.8), and (6.9):

\[
m\pi(k) = (k + 1)\pi(k + 1) + (k + 2)\left(1 - \left(1 - \frac{d}{m}\right)^{k+1}\right)\pi(k + 2) \quad (6.11)
\]

Let us define a continuous \( f : \mathbb{R} \rightarrow \mathbb{R} \) as follows,

\[
f(x) := m - (x + 1) - (x + 2)(1 - (1 - d/m)^{x+1}). \quad (6.12)
\]

It follows that

\[
f(m - 1) \leq 0, f(m/2 - 2) > 0,
\]

so \( f(.) \) has a root \( k^* \) such that \( m/2 - 2 < k^* < m \). In the rest of the proof we show that the states that are far from \( k^* \) have very small probability in the stationary distribution, which completes the proof of Proposition 6.9. This part of the proof involves lost of algebra and is essentially very similar to the proof of the Proposition 6.5. We refer the interested reader to the Subsection C.4 for the complete proof of this last step.

Since the stationary distribution of \( Z_t \) is highly concentrated around \( k^* \in [m/2 - 2, m - 1] \) by the above proposition, we derive the following upper bound for \( \mathbb{E}_{Z \sim \pi} [Z(1 - d/m)^Z] \), which is proved in the Appendix C.5.

Lemma 6.10. For any \( d \geq 0 \) and sufficiently large \( m \),

\[
\mathbb{E}_{Z \sim \pi} [Z(1 - d/m)^Z] \leq \max_{z \in [m/2, m]} (z + \tilde{O}(\sqrt{m}))(1 - d/m)^z + 2.
\]

Now Theorem 6.2 follows immediately by combining Lemma 6.8 and Lemma 6.10.

6.3 Loss of the Patient(\( \alpha \)) Algorithm

Our idea is to slow down the process and use Theorem 6.2 to analyze the Patient(\( \alpha \)) algorithm. More precisely, instead of analyzing Patient(\( \alpha \)) algorithm on a \((m, d, 1)\) market we
analyze the Patient algorithm on a \((m/\bar{\alpha}, d/\bar{\alpha}, 1)\) market. First we need to prove a lemma on the equivalence of markets with different criticality rates.

**Definition 6.11 (Market Equivalence).** An \(\alpha\)-scaling of a dynamic matching market \((m, d, \lambda)\) is defined as follows. Given any realization of this market, i.e., given \((A^i_t, A^n_t, E)\) for any \(0 \leq t \leq \infty\), we construct another realization \((A'^c_t, A'^n_t, E')\) with \((A'^c_t, A'^n_t) = (A^{c \cdot \alpha \cdot t}_t, A^{n \cdot \alpha \cdot t}_t)\) and the same set of acceptable transactions. We say two dynamic matching markets \((m, d, \lambda)\) and \((m', d', \lambda')\) are equivalent if one is an \(\alpha\)-scaling of the other.

It turns out that for any \(\alpha \geq 0\), and any time \(t\), any of the Greedy, Patient or Patient(\(\alpha\)) algorithms (and in general any time-scale independent online algorithm) the set of matched agents at time \(t\) of a realization of a \((m, d, \lambda)\) matching market is the same as the set of matched agents at time \(\alpha t\) of an \(\alpha\)-scaling of that realization. The following proposition makes this rigorous.

**Proposition 6.12.** For any \(m, d, \lambda\) the \((m/\lambda, d/\lambda, 1)\) matching market is equivalent to the \((m, d, \lambda)\) matching market.\(^{35}\)

Now, Theorem 6.3 follows simply by combining the above proposition and Theorem 6.2. First, by the additivity of the Poisson process, the loss of the Patient(\(\alpha\)) algorithm in a \((m, d, 1)\) matching market is equal to the loss of the Patient algorithm in a \((m, d, \bar{\alpha})\) matching market, where \(\bar{\alpha} = 1/\alpha + 1\). Second, by the above proposition, the loss of the Patient algorithm on a \((m, d, \bar{\alpha})\) matching market at time \(T\) is the same as the loss of this algorithm on a \((m/\bar{\alpha}, d/\bar{\alpha}, 1)\) market at time \(\bar{\alpha}T\). The latter is upper-bounded in Theorem 6.2.

### 7 Welfare and Optimal Waiting Time under Discounting

**Theorem 7.1.** There is a number \(m/2 - 2 \leq k^* \leq m\) (as defined in Proposition 6.9) such that for any \(T \geq 0\), \(\delta \geq 0\) and \(\epsilon < 1/2m^2\),

\[
\begin{align*}
W(\text{Patient}) & \geq \frac{T - T_0}{T} \left( \frac{1 - q^{k^* + \tilde{O}(\sqrt{m})}}{1 + \delta/2 - \frac{1}{2}q^{k^* - \tilde{O}(\sqrt{m})}} - \tilde{O}(m^{-3/2}) \right) \\
W(\text{Patient}) & \leq \frac{2T_0}{T} + \frac{T - T_0}{T} \left( \frac{1 - q^{k^* - \tilde{O}(\sqrt{m})}}{1 + \delta/2 - \frac{1}{2}q^{k^* + \tilde{O}(\sqrt{m})}} + \tilde{O}(m^{-3/2}) \right)
\end{align*}
\]

\(^{35}\)The proof is by inspection.
where $T_0 = 16 \log(m) \log(4/\epsilon)$. As a corollary, for any $\alpha \geq 0$, and $\bar{\alpha} = 1/\alpha + 1$,

$$W(\text{Patient}(\alpha)) \geq \frac{T - T_0}{T} \left( 1 - \frac{1 - q^{k^*/\bar{\alpha} + \tilde{O}(\sqrt{m})}}{1 + \delta/2\bar{\alpha} - \frac{1}{2}q^{k^*/\bar{\alpha} + \tilde{O}(\sqrt{m})}} - \tilde{O}(m^{-3/2}) \right)$$

$$W(\text{Patient}(\alpha)) \leq \frac{2T_0}{T} + \frac{T - T_0}{T} \left( 1 - \frac{1 - q^{k^*/\bar{\alpha} - \tilde{O}(\sqrt{m})}}{1 + \delta/2\bar{\alpha} - \frac{1}{2}q^{k^*/\bar{\alpha} + \tilde{O}(\sqrt{m})}} + \tilde{O}(m^{-3/2}) \right)$$

**Theorem 7.2.** If $m > 10d$, for any $T \geq 0$,

$$W(\text{Greedy}) \leq 1 - \frac{1}{2d + 1} + \frac{d^2}{m}.$$

Say an agent $a$ is arrived at time $t_a(a)$. We let $X_t$ be the sum of the potential utility of the agents in $A_t$:

$$X_t = \sum_{a \in A_t} e^{-\delta(t-t_a(a))},$$

i.e., if we match all of the agents currently in the pool immediately, the total utility that they receive is exactly $X_t$.

For $t_0, \epsilon > 0$, let $W_{t_0,t_0+\epsilon}$ be the expected total utility of the agents who are matched in the interval $[t_0, t_0 + \epsilon]$. By definition the social welfare of an online algorithm, we have:

$$W(\text{Patient}) = \mathbb{E} \left[ \frac{1}{T} \int_{t=0}^{T} W_{t,t+dt} dt \right] = \frac{1}{T} \int_{t=0}^{T} \mathbb{E} \left[ W_{t,t+dt} \right] dt$$

### 7.1 Welfare of the Patient Algorithm

In this section, we prove Theorem 7.1. All agents are equally likely to become critical at each moment. From the perspective of the planner, all agents are equally likely to be the neighbor of a critical agent. Hence, the *expected* utility of each of the agents who are matched at time $t$ under the Patient algorithm is $X_t/Z_t$. Thus,

$$W(\text{Patient}) = \frac{1}{mT} \int_{t=0}^{T} \mathbb{E} \left[ 2 \frac{X_t}{Z_t} Z_t (1 - (1 - d/m)^{Z_t}) dt \right] = \frac{2}{mT} \int_{t=0}^{T} \mathbb{E} \left[ X_t (1 - (1 - d/m)^{Z_t}) \right] dt$$

(7.1)

First, we prove the following lemma.
Lemma 7.3. For any $\epsilon < 1/2m^2$ and $t \geq \tau_{\text{mix}}(\epsilon)$,

$$\mathbb{E}[X_t] \left(1 - q^{k^* + \tilde{O}(\sqrt{m})}\right) - O(m^{-1/2}) \leq \mathbb{E}[X_t(1 - q^Z_t)] \leq \mathbb{E}[X_t] \left(1 - q^{k^* - \tilde{O}(\sqrt{m})}\right) + O(m^{-1/2}).$$

The proof of above lemma basically follows from the concentration inequalities proved in Proposition 6.9. See Appendix D for the details.

It remains to estimate $\mathbb{E}[X_t]$. This is done in the following lemma.

Lemma 7.4. For any $\epsilon < 1/2m^2$, $t_1 \geq 16\log(m)\log(4/\epsilon) \geq 2\tau_{\text{mix}}(\epsilon)$,

$$\frac{m - O(m^{-1/2})}{2 + \delta - q^{k^* - \tilde{O}(\sqrt{m})}} \leq \frac{m + O(m^{-1/2})}{2 + \delta - q^{k^* + \tilde{O}(\sqrt{m})}} \leq \mathbb{E}[X_t] \leq \frac{m + O(m^{-1/2})}{2 + \delta - q^{k^* + \tilde{O}(\sqrt{m})}}.$$

Proof. Let $\eta > 0$ be very close to zero (eventually we let $\eta \to 0$). Since we have a $(m, d, 1)$ matching market, using equation (5.1) for any $t \geq 0$ we have,

$$\mathbb{E}[X_{t+\eta}]X_t, Z_t = X_t(e^{-\eta \delta}) + m\eta - \eta Z_t \left(\frac{X_t}{Z_t} - q^Z_t\right) - 2\eta Z_t \left(\frac{X_t}{Z_t}(1 - q^Z_t)\right) \pm O(\eta^2)$$

The first term in the RHS follows from the exponential discount in the utility of the agents in the pool. The second term in the RHS stands for the new arrivals. The third term stands for the perished agents and the last term stands for the matched agents. We use the notation $A = B \pm C$ to denote $B - C \leq A \leq B + C$.

We use $e^{-x} = 1 - x + O(x^2)$ and rearrange the equation to get,

$$\mathbb{E}[X_{t+\eta}]X_t, Z_t = m\eta + X_t - \eta(1 + \delta)X_t - \eta X_t(1 - q^Z_t) \pm O(\eta^2).$$

Using Lemma 7.3 for any $t \geq \tau_{\text{mix}}(\epsilon)$ we can estimate $\mathbb{E}[X_t(1 - q^Z_t)]$. Taking expectation from both sides of the above inequality we get,

$$\frac{\mathbb{E}[X_{t+\epsilon}] - \mathbb{E}[X_t]}{\eta} = m - \mathbb{E}[X_t] (2 + \delta - q^{k^* + \tilde{O}(\sqrt{m})}) \pm O(m^{-1/2}) - O(\eta)$$

Letting $\eta \to 0$, and solving the above differential equation from $\tau_{\text{mix}}(\epsilon)$ to $t_1$ we get

$$\mathbb{E}[X_{t_1}] = \frac{m + O(m^{-1/2})}{2 + \delta - q^{k^* + \tilde{O}(\sqrt{m})}} + C_1 \exp \left(- (\delta + 2 - q^{k^* + \tilde{O}(\sqrt{m})})(t_1 - \tau_{\text{mix}}(\epsilon))\right).$$

Now, for $t_1 = \tau_{\text{mix}}(\epsilon)$ we use the initial condition $\mathbb{E}[X_{\tau_{\text{mix}}(\epsilon)}] \leq \mathbb{E}[Z_{\tau_{\text{mix}}(\epsilon)}] \leq m$, and we can
let $C_1 \leq m$. Finally, since $t_1 \geq 2\tau_{\text{mix}}(\epsilon)$ and $t_1/2 \geq 2\log(m)$ we can upper-bound the latter term with $O(m^{-1/2})$.

Let $T_0 = 16 \log(m) \log(4/\epsilon)$. Since (for any matching algorithm) the sum of the utilities of the agents that leave the market before time $T_0$ is at most $mT_0$ in expectation, by the above two lemmas, we can write

\[
W(\text{Patient}) = \frac{2}{mT} \left( mT_0 + \int_{T_0}^{T} \mathbb{E} \left[ X_t (1 - q^Z_t) \right] dt \right)
\]

\[
\leq \frac{2T_0}{T} + \frac{2}{mT} \int_{T_0}^{T} \left( \frac{m(1 - q^{k_* - \tilde{O}(\sqrt{m})})}{\delta + 2 - q^{k_* + \tilde{O}(\sqrt{m})}} + \tilde{O}(m^{-1/2}) \right) dt
\]

\[
\leq \frac{2T_0}{T} + \frac{T - T_0}{T} \left( \frac{1 - q^{k_* - \tilde{O}(\sqrt{m})}}{1 + \delta/2 - \frac{1}{2} q^{k_* + \tilde{O}(\sqrt{m})}} + \tilde{O}(m^{-3/2}) \right)
\]

Similarly, since the sum of the utilities of the agents that leave the market by time $T_0$ is non-negative, we can show that

\[
W(\text{Patient}) \geq \frac{T - T_0}{T} \left( \frac{1 - q^{k_* + \tilde{O}(\sqrt{m})}}{1 + \delta/2 - \frac{1}{2} q^{k_* - \tilde{O}(\sqrt{m})}} - \tilde{O}(m^{-3/2}) \right)
\]

### 7.2 Welfare of the Greedy Algorithm

Here, we upper-bound the welfare of the optimum online algorithm, OPT, and that immediately upper-bounds the welfare of the Greedy algorithm. Recall that by Theorem 4.1, for any $T > 1/(2d + 1 + d^2/m)$ fraction of the agents perish in OPT. On the other hand, by definition of utility, we receive a utility at most 1 from any matched agent. Therefore, even if all of the matched agents receive a utility of 1, (for any $\delta \geq 0$)

\[
W(\text{Greedy}) \leq W(\text{OPT}) \leq 1 - \frac{1}{2d + 1 + d^2/m}.
\]

### 8 Incentive-Compatible Mechanisms

In this section we design a dynamic mechanism to elicit the departure times of agents. As alluded to in Subsection 3.3, we assume that agents only have statistical knowledge about the rest of the market: That is, each agent knows the market parameters $(m, d, 1)$, her own status (present, critical, perished), and the details of the dynamic mechanism that the market-maker is executing. Agents do not observe the graph $G_t$. 

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Each agent \( a \) chooses a mixed strategy, that is she reports getting critical at an infinitesimal time \([t, t + dt]\) with rate \( c_a(t)dt\). In other words, each agent \( a \) has a clock that ticks with rate \( c_a(t) \) at time \( t \) and she reports criticality when the clock ticks. We assume each agent’s strategy function, \( c_a(\cdot) \) is well-behaved, i.e., it is non-negative, continuously differentiable and continuously integrable. Note that since the agent can only observe the parameters of the market \( c_a(\cdot) \) can depend on any parameter in our model but this function is constant in different sample paths of the stochastic process.

A strategy profile \( \mathcal{C} \) is a vector of well-behaved functions for each agent in the market, that is, \( \mathcal{C} = [c_a]_{a \in A} \). For an agent \( a \) and a strategy profile \( \mathcal{C} \), let \( \mathbb{E}[u_{\mathcal{C}}(a)] \) be the expected utility of \( a \) under the strategy profile \( \mathcal{C} \). Note that for any \( \mathcal{C}, a, 0 \leq \mathbb{E}[u_{\mathcal{C}}(a)] \leq 1 \). Given a strategy profile \( \mathcal{C} = [c_a]_{a \in A} \), let \( \mathcal{C} - c_a + \tilde{c}_a \) denote a strategy profile same as \( \mathcal{C} \) but for agent \( a \) who is playing \( \tilde{c}_a \) rather than \( c_a \). The following definition introduces our solution concept.

**Definition 8.1.** A strategy profile \( \mathcal{C} \) is a strong \( \epsilon \)-Nash equilibrium if for any agent \( a \) and any well-behaved function \( \tilde{c}_a(\cdot) \),

\[
1 - \mathbb{E}[u_{\mathcal{C}}(a)] \leq (1 + \epsilon)(1 - \mathbb{E}[u_{\mathcal{C} - c_a + \tilde{c}_a}]).
\]

Note that the solution concept we are introducing here is different from the usual definition of an \( \epsilon \)-Nash equilibrium, where the condition is either \( \mathbb{E}[u_{\mathcal{C}}(a)] \geq \mathbb{E}[u_{\mathcal{C} - c_a + \tilde{c}_a}] - \epsilon \), or \( \mathbb{E}[u_{\mathcal{C}}(a)] \geq (1 - \epsilon)\mathbb{E}[u_{\mathcal{C} - c_a + \tilde{c}_a}] \). The reason that we are using \( 1 - \mathbb{E}[u_{\mathcal{C}}(a)] \) as a measure of distance is because we know that under Patient(\( \alpha \)) algorithm, \( \mathbb{E}[u_{\mathcal{C}}(a)] \) is very close to 1, so \( 1 - \mathbb{E}[u_{\mathcal{C}}(a)] \) is a lower-order term. Thus, this definition restricts us to a stronger equilibrium concept, which requires us to show that in equilibrium agents can neither increase their utilities, nor the lower-order terms associated with their utilities by a factor of more than \( \epsilon \).

Throughout this section let \( k^* \in [m/2 - 2, m - 1] \) be the root of (C.6) as defined in Proposition 6.9, and let \( \beta := (1 - d/m)^{k^*} \). In this section we show that if \( \delta \) (the discount rate) is no more than \( \beta \), then the strategy vector \( c_a(t) = 0 \) for all agents \( a \) and \( t \) is an \( \epsilon \)-mixed strategy Nash equilibrium for \( \epsilon \) very close to zero. In other words, if all other agents are truthful, an agent’s utility from being truthful is almost as large as any other strategy.

**Theorem 8.2.** If the market is at stationary and \( \delta \leq \beta \), then \( c_a(t) = 0 \) for all \( a, t \) is a strong \( O(d^4 \log^2(m)/\sqrt{m}) \)-Nash equilibrium for Patient-Mechanism(\( \infty \)).
By our market equivalence result (Proposition 6.12), Theorem 8.2 leads to the following corollary.

**Corollary 8.3.** Let \( \bar{\alpha} = 1/\alpha + 1 \) and \( \beta(\alpha) = \bar{\alpha}(1 - d/m)^{m/\bar{\alpha}} \). If the market is at stationary and \( \delta \leq \beta(\alpha) \), then \( c_a(t) = 0 \) for all \( a,t \) is a strong \( O((d/\bar{\alpha})^4 \log^3(m/\bar{\alpha})/\sqrt{m/\bar{\alpha}}) \)-Nash equilibrium for Patient-Mechanism(\( \alpha \)).

The proof of the above theorem is involved but the basic idea is very easy. If an agent reports getting critical at the time of arrival she will receive a utility of \( 1 - \beta \). On the other hand, if she is truthful (assuming \( \delta = 0 \)) she will receive about \( 1 - \beta/2 \). In the course of the proof we show that by choosing any strategy vector \( c(\cdot) \) the expected utility of an agent interpolates between these two numbers, so it is maximized when she is truthful.

The precise proof of the theorem is based on Lemma 8.4. In this lemma, we upper-bound the utility of an agent for any arbitrary strategy, given that all other agents are truthful.

**Lemma 8.4.** Let \( Z_0 \) be in the stationary distribution. Suppose \( a \) enters the market at time 0. If \( \delta < \beta \), and \( 10d^4 \log^3(m) \leq \sqrt{m} \), then for any well-behaved function \( c(\cdot) \),

\[
\mathbb{E}[u_c(a)] \leq \frac{2(1 - \beta)}{2 - \beta + \delta} + O\left(d^4 \log^3(m)/\sqrt{m}\right) \beta,
\]

**Proof.** We present the sketch of the proof here. The full proof can be found in Appendix E.

For an agent \( a \) who arrives at the market at time \( t_0 \), let \( \mathbb{P}[a \in A_{t+t_0}] \) be the probability that agent \( a \) is in the pool at time \( t+t_0 \). Observe that an agent gets matched in one of the following two ways: First, \( a \) becomes critical in the interval \([t, t+\epsilon]\) with probability \( \epsilon \cdot \mathbb{P}[a \in A_t] (1 + c(t)) \) and if she is critical she is matched with probability \( \mathbb{E}\left[(1 - (1 - d/m)^{Z_{t-1}}) | a \in A_t\right] \). Second, \( a \) may also get matched (without being critical) in the interval \([t, t+\epsilon]\). Observe that if an agent \( b \in A_t \) where \( b \neq a \) becomes critical she will be matched with \( a \) with probability \( (1 - (1 - d/m)^{Z_{t-1}})/(Z_t - 1) \). Therefore, the probability that \( a \) is matched at \([t, t+\epsilon]\) without being critical is

\[
\mathbb{P}[a \in A_t] \cdot \mathbb{E}\left[\epsilon \cdot (Z_t - 1) \frac{1 - (1 - d/m)^{Z_{t-1}}}{Z_t - 1} | a \in A_t\right] = \epsilon \cdot \mathbb{P}[a \in A_t] \mathbb{E}\left[1 - (1 - d/m)^{Z_{t-1}} | a \in A_t\right],
\]

and the probability of getting matched at \([t, t+\epsilon]\) is:

\[
\epsilon(2 + c(t))\mathbb{E}\left[1 - (1 - d/m)^{Z_{t-1}} | a \in A_t\right] \mathbb{P}[a \in A_t].
\]
Based on this expression, for any strategy of agent \( a \) we have,

\[
\mathbb{E}[u_c(a)] \leq \frac{\beta}{m} + \int_{t=0}^{t^*} (2 + c(t)) \mathbb{E} \left[ 1 - (1 - d/m)Z_t^{-1} \right] \mathbb{P}[a \in A_t] e^{-\delta t} dt
\]

where \( t^* \) is the moment where the expected utility that \( a \) receives in the interval \( [t^*, \infty) \) is negligible, i.e., in the best case it is at most \( \beta/m \).

In order to bound the expected utility, we need to bound \( \mathbb{P}[a \in A_{t+t_0}] \). We do this by writing down the dynamical equation of \( \mathbb{P}[a \in A_{t+t_0}] \) evolution, and solving the associated differential equation. In addition, we need to study \( \mathbb{E} \left[ 1 - (1 - d/m)Z_t^{-1} \right] \mathbb{P}[a \in A_t] \) to bound the utility expression. This is not easy in general; although the distribution of \( Z_t \) remains stationary, the distribution of \( Z_t \) conditioned on \( a \in A_t \) can be a very different distribution. Therefore, we prove simple upper and lower bounds on \( \mathbb{E} \left[ 1 - (1 - d/m)Z_t^{-1} \right] \mathbb{P}[a \in A_t] \) using the concentration properties of \( Z_t \). The details of all these calculations are presented in Appendix E, in which we finally obtain the following closed form upper-bound on the expected utility of \( a \):

\[
\mathbb{E}[u_c(a)] \leq \frac{2d\sigma^5}{\sqrt{m}} \beta + \int_{t=0}^{\infty} (1 - \beta)(2 + c(t)) \exp \left( - \int_{\tau=0}^{t} (2 + c(\tau) - \beta) d\tau \right) e^{-\delta t} dt. \quad (8.1)
\]

Finally, we show that the right hand side is maximized by letting \( c(t) = 0 \) for all \( t \). Let \( U_c(a) \) be the right hand side of the above equation. Let \( c \) be a function that maximizes \( U_c(a) \) which is not equal to zero. Suppose \( c(t) \neq 0 \) for some \( t \geq 0 \). We define a function \( \tilde{c} : \mathbb{R}_+ \rightarrow \mathbb{R}_+ \) and we show that if \( \delta < \beta \), then \( U_{\tilde{c}}(a) > U_c(a) \). Let \( \tilde{c} \) be the following function,

\[
\tilde{c}(\tau) = \begin{cases} 
  c(\tau) & \text{if } \tau < t, \\
  0 & \text{if } t \leq \tau \leq t + \epsilon, \\
  c(\tau) + c(\tau - \epsilon) & \text{if } t + \epsilon \leq \tau \leq t + 2\epsilon, \\
  c(\tau) & \text{otherwise}.
\end{cases}
\]

In words, we push the mass of \( c(.) \) in the interval \( [t, t + \epsilon] \) to the right. We remark that the above function \( \tilde{c}(.) \) is not necessarily continuous so we need to smooth it out. The latter can be done without introducing any errors and we do not describe the details here. In Appendix E, we show that \( U_{\tilde{c}}(a) - U_c(a) \) is non-negative as long as \( \delta \leq \beta \), which means that the maximizer of \( U_c(a) \) is the all zero function. Plugging in \( c(t) = 0 \) into (8.1) completes the
proof of Lemma 8.4.

The proof of Theorem 8.2 follows simply from the above analysis.

Proof of Theorem 8.2. All we need to do is to lower-bound the expected utility of an agent \( a \) if she is truthful. We omit the details as they are essentially similar. So, if all agents are truthful,

\[
\mathbb{E}[u(a)] \geq \frac{2(1 - \beta)}{2 - \beta + \delta} - O\left(\frac{d^4 \log^3(m)}{\sqrt{m}}\right)\beta.
\]

This shows that the strategy vector corresponding to truthful agents is a strong \( O(d^4 \log^3(m)/\sqrt{m}) \)-Nash equilibrium.

9 Concluding Discussions

In this paper, we developed a model of dynamic matching markets to investigate the features of good policy. This paper innovates by accounting for stochastic departures and analyzing the problem under a variety of information conditions. Rather than modeling market thickness via a fixed match-function, it explicitly accounts for the network structure that affects the planner’s options. This allows market thickness to emerge as an endogenous phenomenon, responsive to the underlying constraints.

In this part of the paper, to connect our findings to real-world markets, we first review the key insights of our analysis and then discuss how relaxing our modeling assumptions would have implications for our results. At the end, we recommend some promising extensions.

9.1 Insights of the Paper

There are many real-world market design problems where the timing of transactions must be decided by a policymaker. These include paired kidney exchanges, dating agencies, and online labor markets such as oDesk. In such markets, policymakers face a trade-off between the speed of transactions and the thickness of the market. It is natural to ask, “Does it matter when transactions occur? How much does it matter?” The first insight of this paper is that waiting to thicken the market can yield substantial welfare gains. In addition, we find that naïve local algorithms that choose the right time to match can come close to optimal benchmarks that exploit the whole graph structure. This shows that the right timing decision is a first order concern in dynamic matching markets.
Another insight of the paper is that how to wait, in addition to how much to wait, is an important policy decision. In particular, a matching algorithm that waits by implementing a static maximum matching once every certain periods is not necessarily a good algorithm. That is because, as our analysis shows, maintaining market thickness at almost all times is an important feature of good matching algorithms. By implementing a static maximum matching, market gets very thin at some time intervals, and a large fraction of loss occurs in those intervals. Our results suggest that a ‘smoothed’ waiting policy can maintain market thickness and increase welfare.

A key finding of our analysis is that information and waiting time are complements: If the planner lacks information about agents’ departure times, losses are relatively large compared to simple algorithms that have access to that information. This shows that having access short-horizon information about departure times is especially valuable to the planner. When the urgency of individual cases is private information, we exhibit a mechanism without transfers that elicits such information from sufficiently patient agents.

These results suggest that the dimension of time is a first-order concern in many matching markets, with welfare implications that static models do not capture. They also suggest that policymakers would reap large gains from acquiring timing information about agent departures, such as by paying for predictive diagnostic testing or monitoring agents’ outside options.

### 9.2 Discussion of Assumptions

In order to make this setting analytically tractable, we have made several important simplifying assumptions. Here we discuss how relaxing those assumptions would have implications for our results.

First, we have assumed that agents are *ex ante* homogeneous: They have in expectation the same average degree.\textsuperscript{36} What would happen if the planner knew that certain agents currently in the pool were more likely to have edges with future agents? Clearly, algorithms that treat heterogeneous agents identically could be inefficient. However, it is an open question whether there are *local* algorithms, sensitive to individual heterogeneity, that are close to optimal.

Second, we have assumed that agents’ preferences are binary: All acceptable matches are equally good. In many settings, acceptable matches may vary in quality. We conjecture

\textsuperscript{36}Note, however, that agents are *ex post* heterogeneous as they have different positions in the trade network.
that this would reinforce our existing results, since waiting to thicken the market could allow planners to make better matches, in addition to increasing the size of the matching.\footnote{To take one natural extension, suppose that the value of an acceptable match is $v$ for both agents involved, where $v$ is a random variable drawn iid across pairs of agents from some distribution $F(\cdot)$. Suppose that the Greedy and Patient algorithms are modified to select the highest match-value among the acceptable matches. Then the value to a matched agent under Greedy is (roughly) the highest among $N$ draws from $F(\cdot)$, where $N$ is distributed Binomial($k^*_\text{Greedy}, \frac{d}{m}$). By contrast, the value to a matched agent under Patient is (roughly) the highest among $N$ draws from $F(\cdot)$, where $N$ is distributed Binomial($k^*_\text{Patient}, \frac{d}{m}$). By our previous arguments, $k^*_\text{Patient} > k^*_\text{Greedy}$, so this strengthens our result.}

Third, we have assumed that agents have the memoryless property; that is, they become critical at some constant Poisson rate. One might ask what would be different if the planner knew ahead of time which agents would be long- or short-lived. Our performance bounds on the Omniscient algorithm provide a partial answer to this question: Such information may be gainful, but a large proportion of the gains can be realized via the Patient algorithm, which uses only short-horizon information about agent’s departure times.

In addition, our assumption on agents’ departure processes can be enriched by assuming agents have a range of sequential states, while an independent process specifies transition rates from one state to the next, and agents who are at the “final” state have some exogenous criticality rate. The full analysis of optimal timing under discounting in such environment is a subject of further research. Nevertheless, our results suggest that for small waiting costs, if the planner observes critical agents, the Patient algorithm is close-to optimal, and if the planner cannot observe the critical agents, waiting until agents transit to the final state (as there is no risk of agents perishing before that time) and then greedily matching those agents who have a risk of perishing is close-to optimal.

Finally, our theoretical bounds on error terms, $O(\sqrt{m})$, are small compared to the market size only if the market is relatively large. What would happen if the market is small, e.g. if $m < 100$? To check the robustness of our results to the large market assumptions, we simulated our model for small markets. Our simulation results (see Appendix F) show that our results continue to hold for very small markets as well.

### 9.3 Further Extensions

We suggest some promising extensions. First, one could generalize the model to allow multiple types of agents, with the probability of an acceptable transaction differing across type-pairs. This could capture settings where certain agents are known \textit{ex ante} to be less likely to have acceptable trades than other agents, as is the case for patients with high Panel Re-
active Antibody (PRA) scores in paired kidney exchanges. The multiple-types framework also contains bipartite markets as a special case.

Second, one could adopt more gradual assumptions about agent departure processes; agents could have a range of individual states with state-dependent perishing rates, and an independent process specifying transition rates between states. Our model, in which agents transition to a critical state at rate $\lambda$ and then perish imminently, is a limit case of the multiple-states framework.

Third, it would be interesting to enrich the space of preferences in the model, such as by allowing matches to yield a range of payoffs. Further insights may come by making explicit the role of price in dynamic matching markets. It is not obvious how to do so, but similar extensions have been made for static models (Kelso Jr and Crawford, 1982; Hatfield and Milgrom, 2005), and the correct formulation may seem obvious in retrospect.

Much remains to be done in the theory of dynamic matching. As market design expands its reach, re-engineering markets from the ground up, economists will increasingly have to answer questions about the timing and frequency of transactions. Many dynamic matching markets have important features (outlined above) that we have not modeled explicitly. We offer this paper as a step towards systematically understanding matching problems that take place across time.

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In this section we prove several inequalities that are used throughout the paper. For any $a, b \geq 0$,

$$
\sum_{i=a}^{\infty} e^{-bi^2} = \sum_{i=0}^{\infty} e^{-b(i+a)^2} \leq \sum_{i=0}^{\infty} e^{-ba^2-2iab} = e^{-ba^2} \sum_{i=0}^{\infty} (e^{-2ab})^i
$$

$$
= \frac{e^{-ba^2}}{1 - e^{-2ab}} \leq \frac{e^{-ba^2}}{\min\{ab, 1/2\}}.
$$

The last inequality can be proved as follows: If $2ab \leq 1$, then $e^{-2ab} \leq ab$, otherwise $e^{-2ab} \leq 1/2$. 

(A.1)
For any \(a, b \geq 0\),
\[
\sum_{i=a}^{\infty} (i-1)e^{-bi^2} \leq \int_{a-1}^{\infty} xe^{-bx^2} \, dx = \frac{-1}{2b} e^{-bx^2}\bigg|_{a-1}^{\infty} = \frac{e^{-b(a-1)^2}}{2b}.
\]  \hspace{1cm} (A.2)

For any \(a \geq 0\) and \(0 \leq b \leq 1\),
\[
\sum_{i=a}^{\infty} i e^{-bi} = e^{-ba} \sum_{i=0}^{\infty} (i + a) e^{-bi} = e^{-ba} \left( \frac{a}{1 - e^{-b}} + \frac{1}{(1 - e^{-b})} \right) \leq \frac{e^{-ba}(2ba + 4)}{b^2}.
\]  \hspace{1cm} (A.3)

The Bernoulli inequality states that for any \(x \leq 1\), and any \(n \geq 1\),
\[
(1 - x)^n \geq 1 - xn.
\]  \hspace{1cm} (A.4)

Here, we prove for integer \(n\). The above equation can be proved by a simple induction on \(n\). It trivially holds for \(n = 0\). Assuming it holds for \(n\) we can write,
\[
(1 - x)^{n+1} = (1 - x)(1 - x)^n \geq (1 - x)(1 - xn) = 1 - x(n + 1) + x^2 n \geq 1 - x(n + 1).
\]

**B Proof of Theorem 5.2**

**B.1 Stationary Distributions: Existence and Uniqueness**

In this part we show that the Markov Chain on \(Z_t\) has a unique stationary distribution under each of the Greedy and Patient algorithms. By Proposition 5.1, \(Z_t\) is a Markov chain on the non-negative integers \((\mathbb{N})\) that starts from state zero.

First, we show that the Markov Chain is irreducible. First note that every state \(i > 0\) is reachable from state 0 with a non-zero probability. It is sufficient that \(i\) agents arrive at the market with no acceptable bilateral transactions. On the other hand, state 0 is reachable from any \(i > 0\) with a non-zero probability. It is sufficient that all of the \(i\) agents in the pool become critical and no new agents arrive at the market. So \(Z_t\) is an irreducible Markov Chain.

Therefore, by the ergodic theorem it has a unique stationary distribution if and only if it has a positive recurrent state (Norris, 1998, Theorem 3.8.1). In the rest of the proof we show that state 0 is positive recurrent. By (4.1) \(Z_t = 0\) if \(\tilde{Z}_t = 0\). So, it is sufficient to show
\[
\mathbb{E} \left[ \inf \{ t \geq T_1 : \tilde{Z}_t = 0 \} | \tilde{Z}_{t_0} = 0 \right] < \infty.
\]  \hspace{1cm} (B.1)
It follows that $\tilde{Z}_t$ is just a continuous time birth-death process on $\mathbb{N}$ with the following transition rates,

$$\tilde{r}_{k \to k+1} = m \text{ and } \tilde{r}_{k \to k-1} := k$$

(B.2)

It is well known (see e.g. (Grimmett and Stirzaker, 1992, p. 249-250)) that $\tilde{Z}_t$ has a stationary distribution if and only if

$$\sum_{k=1}^{\infty} \frac{\tilde{r}_{0 \to 1} \tilde{r}_{1 \to 2} \cdots \tilde{r}_{k-1 \to k}}{\tilde{r}_{1 \to 0} \cdots \tilde{r}_{k \to k-1}} < \infty.$$ 

Using (B.2) we have

$$\sum_{k=1}^{\infty} \frac{\tilde{r}_{0 \to 1} \tilde{r}_{1 \to 2} \cdots \tilde{r}_{k-1 \to k}}{\tilde{r}_{1 \to 0} \cdots \tilde{r}_{k \to k-1}} = \sum_{k=1}^{\infty} \frac{m^k}{k!} = e^m - 1 < \infty$$

Therefore, $\tilde{Z}_t$ has a stationary distribution. The ergodic theorem (Norris, 1998, Theorem 3.8.1) entails that every state in the support of the stationary distribution is positive recurrent. Thus, state 0 is positive recurrent under $\tilde{Z}_t$. This proves (B.1), so $Z_t$ is an ergodic Markov Chain.

**B.2 Upper bounding the Mixing Times**

In this part we complete the proof of Theorem 5.2 and provide an upper bound the mixing of Markov Chain $Z_t$ for the Greedy and Patient algorithms. Let $\pi(.)$ be the stationary distribution of the Markov Chain.

**B.2.1 Mixing time of the Greedy Algorithm**

We use the coupling technique (see (Levin et al., 2006, Chapter 5)) to get an upper bound for the mixing time of the Greedy algorithm. Suppose we have two Markov Chains $Y_t, Z_t$ (with different starting distributions) each running the Greedy algorithm. We define a joint Markov Chain $(Y_t, Z_t)_{t=0}^{\infty}$ with the property that projecting on either of $Y_t$ and $Z_t$ we see the stochastic process of Greedy algorithm, and that they stay together at all times after their first simultaneous visit to a single state, i.e.,

if $Y_{t_0} = Z_{t_0}$, then $Y_t = Z_t$ for $t \geq t_0$.

Next we define the joint chain. We define this chain such that for any $t \geq t_0$, $|Y_t - Z_t| \leq |Y_{t_0} - Z_{t_0}|$. Assume that $Y_{t_0} = y, Z_{t_0} = z$ at some time $t_0 \geq 0$, for $y, z \in \mathbb{N}$. Without loss
of generality assume $y < z$ (note that if $y = z$ there is nothing to define). Consider any arbitrary labeling of the agents in the first pool with $a_1, \ldots, a_y$, and in the second pool with $b_1, \ldots, b_z$. Define $z + 1$ independent exponential clocks such that the first $z$ clocks have rate 1, and the last one has rate $m$. If the $i$-th clock ticks for $1 \leq i \leq y$, then both of $a_i$ and $b_i$ become critical (recall that in the Greedy algorithm the critical agent leaves the market right away). If $y < i \leq z$, then $b_i$ becomes critical, and if $i = z + 1$ new agents $a_{y+1}, b_{z+1}$ arrive to the markets. In the latter case we need to draw edges between the new agents and those currently in the pool. We use $z$ independent coins each with parameter $d/m$. We use the first $y$ coins to decide simultaneously on the potential transactions $(a_i, a_{y+1})$ and $(b_i, b_{z+1})$ for $1 \leq i \leq y$, and the last $z - y$ coins for the rest. This implies that for any $1 \leq i \leq y$, $(a_i, a_{y+1})$ is an acceptable transaction iff $(b_i, b_{z+1})$ is acceptable. Observe that if $a_{y+1}$ has at least one acceptable transaction then so has $b_{z+1}$ but the converse does not necessarily hold.

It follows from the above construction that $|Y_t - Z_t|$ is a non-increasing function of $t$. Furthermore, this value decreases when either of the agents $b_{y+1}, \ldots, b_z$ become critical (we note that this value may also decrease when a new agent arrives but we do not exploit this situation here). Now suppose $|Y_0 - Z_0| = k$. It follows that the two chains arrive to the same state when all of the $k$ agents that are not in common become critical. This has the same distribution as the maximum of $k$ independent exponential random variables with rate 1. Let $E_k$ be a random variable that is the maximum of $k$ independent exponentials of rate 1. For any $t \geq 0$,

$$
\mathbb{P}[Z_t \neq Y_t] \leq \mathbb{P}[E_{|Y_0 - Z_0|} \geq t] = 1 - (1 - e^{-t})^{|Y_0 - Z_0|}.
$$

Now, we are ready to bound the mixing time of the Greedy algorithm. Let $z_t(.)$ be the distribution of the pool size at time $t$ when there is no agent in the pool at time 0 and let $\pi(.)$ be the stationary distribution. Fix $0 < \epsilon < 1/4$, and let $\beta \geq 0$ be a parameter that we fix later. Let $(Y_t, Z_t)$ be the joint Markov chain that we constructed above where $Y_t$ is
started at the stationary distribution and $Z_t$ is started at state zero. Then,

$$\|z_t - \pi\|_{TV} \leq \mathbb{P}[Y_t \neq Z_t] = \sum_{i=0}^{\infty} \pi(i) \mathbb{P}[Y_t \neq Z_t | Y_0 = i] \leq \sum_{i=0}^{\infty} \pi(i) \mathbb{P}[E_i \geq t] \leq \sum_{i=0}^{\beta m/d} (1 - (1 - e^{-t})^{\beta m/d}) + \sum_{i=\beta m/d}^{\infty} \pi(i) \leq \frac{\beta^2 m^2}{d^2} e^{-t} + 2e^{-m(\beta-1)^2/2d}$$

where the last inequality follows by equation (A.4) and Proposition 6.5. Letting $\beta = 1 + \sqrt{2 \log(2/\epsilon)}$ and $t = 2 \log(\beta m/d) \cdot \log(2/\epsilon)$ we get $\|z_t - \pi\|_{TV} \leq \epsilon$, which proves the theorem.

### B.2.2 Mixing time of the Patient Algorithm

It remains to bound the mixing time of the Patient algorithm. The construction of the joint Markov Chain is very similar to the above construction except some caveats. Again, suppose $Y_{t_0} = y$ and $Z_{t_0} = z$ for $y, z \in \mathbb{N}$ and $t_0 \geq 0$ and that $y < z$. Let $a_1, \ldots, a_y$ and $b_1, \ldots, b_z$ be a labeling of the agents. We consider two cases.

**Case 1)** $z > y + 1$. In this case the construction is essentially the same as the Greedy algorithm. The only difference is that we toss random coins to decide on acceptable bilateral transactions at the time that an agent becomes critical (and not at the time of arrival). It follows that when new agents arrive the size of each of the pools increase by 1 (so the difference remains unchanged). If any of the agents $b_{y+1}, \ldots, b_z$ become critical then the size of second pool decrease by 1 or 2 and so is the difference of the pool sizes.

**Case 2)** $z = y + 1$. In this case we define a slightly different coupling. This is because, for some parameters and starting values, the Markov chains may not visit the same state for a long time for the coupling defined in Case 1. If $z \gg m/d$, then with a high probability any critical agent gets matched. Therefore, the magnitude of $|Z_t - Y_t|$ does not quickly decrease (for a concrete example, consider the case where $d = m$, $y = m/2$ and $z = m/2 + 1$). Therefore, in this case we change the coupling. We use $z + 2$ independent clocks where the first $z$ are the same as before, i.e., they have rate 1 and when the $i$-th clock ticks $b_i$ (and $a_i$ if $i \leq y$) become critical. The last two clocks have rate $m$, when the $z + 1$-st clock ticks a new agent arrives to the
Figure 6: A three state Markov Chain used for analyzing the mixing time of the Patient algorithm.

Let $|Y_0 - Z_0| = k$. By the above construction $|Y_t - Z_t|$ is a decreasing function of $t$ unless $|Y_t - Z_t| = 1$. In the latter case this difference goes to zero if a new agent arrives to the smaller pool and it increases if a new agent arrives to the bigger pool. Let $\tau$ be the first time $t$ where $|Y_t - Z_t| = 1$. Similar to the Greedy algorithm, the event $|Y_t - Z_t| = 1$ occurs if the second to maximum of $k$ independent exponential random variables with rate 1 is at most $t$. Therefore,

$$P[\tau \leq t] \leq P[E_k \leq t] \leq (1 - e^{-t})^k$$

Now, suppose $t \geq \tau$; we need to bound the time it takes to make the difference zero. First, note that after time $\tau$ the difference is never more than 2. Let $X_t$ be the (continuous time) Markov Chain illustrated in Figure 6 and suppose $X_0 = 1$. Using $m \geq 1$, it is easy to see that if $X_t = 0$ for some $t \geq 0$, then $|Y_{t+\tau} - Z_{t+\tau}| = 0$ (but the converse is not necessarily true). It is a simple exercise that for $t \geq 8$,

$$P[X_t \neq 0] = \sum_{k=0}^{\infty} \frac{e^{-t}t^k}{k!} 2^{-k/2} \leq \sum_{k=0}^{t/4} \frac{e^{-t}t^k}{k!} 2^{-t/8} \leq 2^{-t/4} + 2^{-t/8}. \quad (B.3)$$

Now, we are ready to upper-bound the mixing time of the Patient algorithm. Let $z_t(.)$ be the distribution of the pool size at time $t$ where there is no agent at time 0, and let $\pi(.)$ be the stationary distribution. Fix $\epsilon > 0$, and let $\beta \geq 2$ be a parameter that we fix later. Let $(Y_t, Z_t)$ be the joint chain that we constructed above where $Y_t$ is started at the stationary
distribution and $Z_t$ is started at state zero.

$$
\|z_t - \pi\|_{TV} \leq \mathbb{P}[Z_t \neq Y_t] \leq \mathbb{P}[\tau \leq t/2] + \mathbb{P}[X_t \leq t/2] \\
\leq \sum_{i=0}^{\infty} \pi(i) \mathbb{P}[\tau \leq t/2|Y_0 = i] + 2^{-t/8+1} \\
\leq 2^{-t/8+1} + \sum_{i=0}^{\infty} \pi(i)(1 - (1 - e^{-t/2})^i) \\
\leq 2^{-t/8+1} + \sum_{i=0}^{\beta m} (it/2) + \sum_{i=\beta m}^{\infty} \pi(i) \leq 2^{-t/8+1} + \frac{\beta^2 m^2 t}{2} + 6e^{-(\beta-1)m/3}.
$$

where in the second to last equation we used equation (A.4) and in the last equation we used Proposition 6.9. Letting $\beta = 10$ and $t = 8 \log(m) \log(4/\epsilon)$ implies that $\|z_t - \pi\|_{TV} \leq \epsilon$ which proves Theorem 5.2.

C Proofs from Section 6

C.1 Proof of Lemma 6.4

Proof. By Proposition 4.3, $\mathbb{E}[Z_t] \leq m$ for all $t$, so

$$
\mathbb{L}(\text{Greedy}) = \frac{1}{m \cdot T} \mathbb{E} \left[ \int_{t=0}^{T} Z_t dt \right] = \frac{1}{mT} \int_{t=0}^{T} \mathbb{E}[Z_t] dt \\
\leq \frac{1}{mT} m \cdot \tau_{mix}(\epsilon) + \frac{1}{mT} \int_{t=\tau_{mix}(\epsilon)}^{T} \mathbb{E}[Z_t] dt \quad (C.1)
$$

where the second equality uses the linearity of expectation. Let $\tilde{Z}_t$ be the number of agents in the pool at time $t$ when we do not match any pair of agents. By (4.1),

$$
\mathbb{P}[Z_t \geq i] \leq \mathbb{P}[\tilde{Z}_t \geq i].
$$
Therefore, for \( t \geq \tau_{\text{mix}}(\epsilon) \),

\[
\mathbb{E}[Z_t] = \sum_{i=1}^{\infty} \mathbb{P}[Z_t \geq i] \leq \sum_{i=0}^{6m} \mathbb{P}[Z_t \geq i] + \sum_{i=6m+1}^{\infty} \mathbb{P}[Z_t \geq i] \\
\leq \sum_{i=0}^{6m} (\mathbb{P}[Z_{\sim \pi} \geq i] + \epsilon) + \sum_{i=6m+1}^{\infty} \sum_{\ell=1}^{m^\ell} \frac{m^\ell}{\ell!}
\]

\[
\leq \mathbb{E}_{Z_{\sim \pi}}[Z] + \epsilon 6m + \sum_{i=6m+1}^{\infty} \frac{m^i}{i!}
\]

\[
\leq \mathbb{E}_{Z_{\sim \pi}}[Z] + \epsilon 6m + \frac{4m^{6m}}{(6m)!} \leq \mathbb{E}_{Z_{\sim \pi}}[Z] + \epsilon 6m + 2^{-6m}. \quad \text{(C.2)}
\]

where the second inequality uses \( \mathbb{P}[Z_t = \ell] \leq m^\ell/\ell! \) of Proposition 4.3 and the last inequality follows by the Stirling’s approximation\(^{38}\) of \((6m)!\). Putting (C.1) and (C.2) proves the lemma.

\[\Box\]

### C.2 Proof of Lemma 6.7

**Proof.** Let \( \Delta \geq 0 \) be a parameter that we fix later. We have,

\[
\mathbb{E}_{Z_{\sim \pi}}[Z] \leq k^* + \Delta + \sum_{i=k^*+\Delta+1}^{\infty} i \pi(i). \quad \text{(C.3)}
\]

By equation (6.6),

\[
\sum_{i=k^*+\Delta+1}^{\infty} i \pi(i) = \sum_{i=\Delta+1}^{\infty} e^{-d(i-1)^2/2m} (i + k^*)
\]

\[
= \sum_{i=\Delta}^{\infty} e^{-di^2/2m} (i - 1) + \sum_{i=\Delta}^{\infty} e^{-d(i^2)/2m} (k^* + 2)
\]

\[
\leq \frac{e^{-d(\Delta-1)^2/2m}}{d/m} + (k^* + 2) \frac{e^{-d\Delta^2/2m}}{\min\{1/2, d\Delta/2m\}}, \quad \text{(C.4)}
\]

where in the last step we used equations (A.1) and (A.2). Letting \( \Delta := 1 + 2\sqrt{m/d \log(m/d)} \) in the above equation, the right hand side is at most 1. The lemma follows from (C.3) and

\(^{38}\)Stirling’s approximation states that

\[n! \geq \sqrt{2\pi n} \left(\frac{n}{e}\right)^n.\]
the above equation.

\section*{C.3 Proof of Lemma 6.8}

\textit{Proof.} By linearity of expectation,

\[ L(\text{Patient}) = \frac{1}{m \cdot T} \mathbb{E} \left[ \int_{t=0}^{T} Z_t (1 - d/m)^{Z_t - 1} dt \right] = \frac{1}{m \cdot T} \int_{t=0}^{T} \mathbb{E} \left[ Z_t (1 - d/m)^{Z_t - 1} \right] dt. \]

Since for any \( t \geq 0, \mathbb{E} \left[ Z_t (1 - d/m)^{Z_t - 1} \right] \leq \mathbb{E} [Z_t] \leq \mathbb{E} \left[ \tilde{Z}_t \right] \leq m, \) we can write

\[ L(\text{Patient}) \leq \frac{\tau_{\text{mix}}(\epsilon)}{T} + \frac{1}{m \cdot T} \int_{t=\tau_{\text{mix}}(\epsilon)}^{T} \sum_{i=0}^{\infty} (\pi(i) + \epsilon)i(1 - d/m)^{i-1} dt \]

\[ \leq \frac{\tau_{\text{mix}}(\epsilon)}{T} + \frac{\mathbb{E}_{Z \sim \pi} [Z(1 - d/m)^{Z-1}]}{m} \frac{\epsilon m}{d^2} \]

where the last inequality uses the identity \( \sum_{i=0}^{\infty} i(1 - d/m)^{i-1} = m^2/d^2. \)

\section*{C.4 Proof of Proposition 6.9}

Let us first rewrite what we derived in the proof overview of this proposition in the main text. The balance equations of the Markov chain associated with the Patient algorithm can be written as follows by replacing transition probabilities from (6.7), (6.8), and (6.9) in (6.10):

\[ m \pi(k) = (k + 1) \pi(k + 1) + (k + 2) \left( 1 - \left( 1 - \frac{d}{m} \right)^{k+1} \right) \pi(k + 2) \] (C.5)

Now define a continuous \( f : \mathbb{R} \to \mathbb{R} \) as follows,

\[ f(x) := m - (x + 1) - (x + 2)(1 - (1 - d/m)^{x+1}). \] (C.6)

It follows that

\[ f(m - 1) \leq 0, f(m/2 - 2) > 0, \]

which means that \( f(.) \) has a root \( k^* \) such that \( m/2 - 2 < k^* < m. \) In the rest of the proof we show that the states that are far from \( k^* \) have very small probability in the stationary distribution.

In order to complete the proof of Proposition 6.9, we first prove the following useful lemma.
Lemma C.1. For any integer $k \leq k^*$,

$$\frac{\pi(k)}{\max\{\pi(k+1), \pi(k+2)\}} \leq e^{-(k^*-k)/m}.$$  

Similarly, for any integer $k \geq k^*$, \( \frac{\min\{\pi(k+1), \pi(k+2)\}}{\pi(k)} \leq e^{-(k-k^*)/(m+k-k^*)} \).

Proof. For $k \leq k^*$, by equation (C.5),

$$\frac{\pi(k)}{\max\{\pi(k+1), \pi(k+2)\}} \leq \frac{(k+1) + (k+2)(1 - \frac{(1 - d/m)^{k+1}}{m})}{m} \leq \frac{(k-k^*) + (k^* + 1) + (k^* + 2)(1 - \frac{(1 - d/m)^{k^*+1}}{m})}{m} = 1 - \frac{k^* - k}{m} \leq e^{-(k-k^*)/m},$$

where the last equality follows by the definition of $k^*$ and the last inequality uses $1 - x \leq e^{-x}$.

The second conclusion can be proved similarly. For $k \geq k^*$,

$$\frac{\min\{\pi(k+1), \pi(k+2)\}}{\pi(k)} \leq \frac{m}{(k+1) + (k+2)(1 - \frac{(1 - d/m)^{k+1}}{m})} \leq \frac{m}{(k-k^*) + (k^* + 1) + (k^* + 2)(1 - \frac{(1 - d/m)^{k^*+1}}{m})} = \frac{m}{m+k-k^*} = 1 - \frac{k-k^*}{m+k-k^*} \leq e^{-(k-k^*)/(m+k-k^*)},$$

where the equality follows by the definition of $k^*$.

Now, we use the above claim to upper-bound $\pi(k)$ for values $k$ that are far from $k^*$.

First, fix $k \leq k^*$. Let $n_0, n_1, \ldots$ be sequence of integers defined as follows: $n_0 = k$, and $n_{i+1} := \arg\max\{\pi(n_i + 1), \pi(n_i + 2)\}$ for $i \geq 1$. It follows that,

$$\pi(k) \leq \prod_{i:n_i \leq k^*} \frac{\pi(n_i)}{\pi(n_{i+1})} \leq \exp\left( - \sum_{i:n_i \leq k^*} \frac{k^* - n_i}{m} \right) \leq \exp\left( - \sum_{i=0}^{(k^*-k)/2} \frac{2i}{m} \right) \leq e^{-(k^*-k)^2/4m}, \quad \text{(C.7)}$$

where the second to last inequality uses $|n_i - n_{i-1}| \leq 2$.

Now, fix $k \geq k^* + 2$. In this case we construct the following sequence of integers, $n_0 = \lceil k^* + 2 \rceil$, and $n_{i+1} := \arg\min\{\pi(n_i + 1), \pi(n_i + 2)\}$ for $i \geq 1$. Let $n_j$ be the largest number in the sequence that is at most $k$ (observe that $n_j = k - 1$ or $n_j = k$). We upper-
bound \( \pi(k) \) by upper-bounding \( \pi(n_j) \),

\[
\pi(k) \leq \frac{m \cdot \pi(n_j)}{k} \leq 2 \prod_{i=0}^{j-1} \frac{\pi(n_i)}{\pi(n_{i+1})} \leq 2 \exp \left( -\sum_{i=0}^{j-1} \frac{n_i - k^*}{m + n_i - k^*} \right) \\
\leq 2 \exp \left( -\sum_{i=0}^{(j-1)/2} \frac{2i}{m + k - k^*} \right) \leq 2 \exp \left( \frac{-(k - k^* - 1)^2}{4(m + k - k^*)} \right).
\]

(C.8)

To see the first inequality note that if \( n_j = k \), then there is nothing to show; otherwise we have \( n_j = k - 1 \). In this case by equation (C.5), \( m\pi(k - 1) \geq k\pi(k) \). The second to last inequality uses the fact that \( |n_i - n_{i+1}| \leq 2 \).

We are almost done. The proposition follows from (C.8) and (C.7). First, for \( \sigma \geq 1 \), let \( \Delta = \sigma\sqrt{4m} \), then by equation (A.1)

\[
\sum_{i=0}^{k^* - \Delta} \pi(i) \leq \sum_{i=\Delta}^{\infty} e^{-i^2/4m} \leq \frac{e^{-\Delta^2/4m}}{\min\{1/2, \Delta/4m\}} \leq 2\sqrt{me}^{-\sigma^2}.
\]

Similarly,

\[
\sum_{i=k^* + \Delta}^{\infty} \pi(i) \leq 2 \sum_{i=\Delta+1}^{\infty} e^{-(i-1)^2/4(i+m)} \leq 2 \sum_{i=\Delta}^{\infty} e^{-i/(4+\sqrt{4m}/\sigma)} \leq 2 \frac{e^{-\Delta/(4+\sqrt{4m}/\sigma)}}{1 - e^{-1/(4+\sqrt{4m})}} \leq 8\sqrt{me}^{-\frac{\sigma^2\sqrt{m}}{2\sigma + \sqrt{m}}}
\]

This completes the proof of Proposition 6.9.

C.5 Proof of Lemma 6.10

Proof. Let \( \Delta := 3\sqrt{m} \log(m) \), and let \( \beta := \max_{z \in [m/2 - \Delta, m + \Delta]} z(1 - d/m)^z \).

\[
\mathbb{E}_{Z \sim \pi} [Z(1 - d/m)^Z] \leq \beta + \sum_{i=0}^{m/2 - \Delta - 1} \frac{m}{2} \pi(i)(1 - d/m)^i + \sum_{i=m+\Delta}^{\infty} i\pi(i)(1 - d/m)^m \quad (C.9)
\]
We upper bound each of the terms in the right hand side separately. We start with upper bounding $\beta$. Let $\Delta' := 4(\log(2m) + 1)\Delta$.

\[
\beta \leq \max_{z \in [m/2,m]} z(1 - d/m)^z + m/2(1 - d/m)^{m/2}((1 - d/m)^{-\Delta} - 1) + (1 - d/m)^m\Delta \\
\leq \max_{z \in [m/2,m]} (z + \Delta' + \Delta)(1 - d/m)^z + 1. \tag{C.10}
\]

To see the last inequality we consider two cases. If $(1 - d/m)^{-\Delta} \leq 1 + \Delta'/m$ then the inequality obviously holds. Otherwise, (assuming $\Delta' \leq m$),

\[(1 - d/m)^\Delta \leq \frac{1}{1 + \Delta'/m} \leq 1 - \Delta'/2m,\]

By the definition of $\beta$,

\[\beta \leq (m + \Delta)(1 - d/m)^{m/2-\Delta} \leq 2m(1 - \Delta'/2m)^{m/2\Delta-1} \leq 2me^{\Delta'/4\Delta-1} \leq 1.\]

It remains to upper bound the second and the third term in (C.9). We start with the second term. By Proposition 6.9,

\[
\sum_{i=0}^{m/2-\Delta-1} \pi(i) \leq \frac{1}{m^{3/2}}, \tag{C.11}
\]

where we used equation (A.1). On the other hand, by equation (C.8)

\[
\sum_{i=m+\Delta}^{\infty} i\pi(i) \leq e^{-\Delta/(2+\sqrt{m})} \left( \frac{m}{1 - e^{-1/(2+\sqrt{m})}} + \frac{2\Delta + 4}{1/(2 + \sqrt{m})^2} \right) \leq \frac{1}{\sqrt{m}}. \tag{C.12}
\]

where we used equation (A.3).

The lemma follows from (C.9), (C.10), (C.11) and (C.12).
D Proofs from Section 7

D.1 Proof of Lemma 7.3

Let $\Delta := 3\sqrt{m}\log(m)$. Let $E$ be the event that $Z_t \in [k^* - \Delta, k^* + \Delta]$. First, we show the following inequality and then we upper-bound $E[X_t|E] \cdot P[\bar{E}]$.

$$E[X_t(1 - q^{k^* + \Delta})] - E[X_t|E] \cdot P[\bar{E}] \leq E[X_t(1 - q^{Z_t})] \leq E[X_t(1 - q^{k^* - \Delta})] + E[X_t|E] \cdot P[\bar{E}]$$

(\text{D.1})

We prove the right inequality and the left can be proved similarly.

By definition of expectation,

$$E[X_t(1 - q^{Z_t})] = E[X_t(1 - q^{Z_t})|E] \cdot P[E] + E[X_t(1 - q^{Z_t})|\bar{E}] \cdot P[\bar{E}]$$

Now, for any random variable $X$ and any event $E$ we have $E[X|E] \cdot P[E] = E[X] - E[X|\bar{E}] \cdot P[\bar{E}]$. Therefore,

$$E[X_t(1 - q^{Z_t})] \leq (1 - q^{k^* - \Delta})(E[X_t] - E[X_t|\bar{E}] \cdot P[\bar{E}]) + E[X_t(1 - q^{Z_t})|\bar{E}] \cdot P[\bar{E}]$$

$$\leq E[X_t] (1 - q^{k^* - \Delta}) + E[X_t|\bar{E}] \cdot P[\bar{E}]$$

where we simply used the non-negativity of $X_t$ and that $(1 - q^{k^* - \Delta}) \leq 1$. This proves the right inequality of (\text{D.1}). The left inequality can be proved similarly.

It remains to upper-bound $E[X_t|E] \cdot P[\bar{E}]$. Let $\pi(.)$ be the stationary distribution of the Markov Chain $Z_t$. Since by definition of $X_t$, $X_t \leq Z_t$ with probability 1,

$$E[X_t|E] \cdot P[\bar{E}] \leq E[Z_t|E] \cdot P[\bar{E}]$$

$$\leq \sum_{i=0}^{k^* - \Delta} i(\pi(i) + \epsilon) + \sum_{i=k^* + \Delta}^{6m} i(\pi(i) + \epsilon) + \sum_{i=6m+1}^{\infty} i \cdot P[\bar{Z}_t = i]$$

where the last term uses the fact that $Z_t$ is at most the size of the pool of the inactive policy at time $t$, i.e., $P[Z_t = i] \leq P[\bar{Z}_t = i]$ for all $i > 0$. We bound the first term of RHS using
Proposition 6.9, the second term using (C.12) and the last term using Proposition 4.3.

\[ E \left[ X_t | \mathcal{E} \right] P \left[ \mathcal{E} \right] \leq \frac{4}{\sqrt{m}} + 6m\epsilon + \sum_{i=6m}^{\infty} \frac{m^i}{i!} \leq \frac{4}{\sqrt{m}} + \frac{3}{m} + 2^{-6m}. \]

E  Proofs from Section 8

E.1 Proof of Lemma 8.4

In this section, we present the full proof of Lemma 8.4. We prove the lemma by writing a closed form expression for the utility of \( a \) and then upper-bounding that expression.

In the following claim we study the probability \( a \) is matched in the interval \([t, t+\epsilon]\) and the probability that it leaves the market in that interval.

Claim E.1. For any time \( t \geq 0 \), and \( \epsilon > 0 \),

\[ P[a \in M_{t,t+\epsilon}] = \epsilon \cdot P[a \in A_t] (2 + c(t)) E \left[ 1 - (1 - d/m)^{Z_t} | a \in A_t \right] \pm O(\epsilon^2) \]  \hspace{1cm} (E.1)

\[ P[a \notin A_{t+\epsilon}, a \in A_t] = P[a \in A_t] (1 - \epsilon(1 + c(t) + E \left[ 1 - (1 - d/m)^{Z_t-1} | a \in A_t \right]) \pm O(\epsilon^2)) \]  \hspace{1cm} (E.2)

Proof. The claim follows from two simple observations. First, \( a \) becomes critical in the interval \([t, t+\epsilon]\) with probability \( \epsilon \cdot P[a \in A_t] (1 + c(t)) \) and if he is critical he is matched with probability \( E \left[ (1 - (1 - d/m)^{Z_t-1} | a \in A_t \right] \). Second, \( a \) may also get matched (without getting critical) in the interval \([t, t+\epsilon]\). Observe that if an agent \( b \in A_t \) where \( b \neq a \) gets critical she will be matched with \( a \) with probability \((1 - (1 - d/m)^{Z_t-1})/(Z_t-1))\). Therefore, the probability that \( a \) is matched at \([t, t+\epsilon]\) without getting critical is

\[ P[a \in A_t] \cdot E \left[ \epsilon \cdot (Z_t - 1) \frac{1 - (1 - d/m)^{Z_t-1}}{Z_t - 1} | a \in A_t \right] \]

\[ = \epsilon \cdot P[a \in A_t] \cdot E \left[ 1 - (1 - d/m)^{Z_t-1} | a \in A_t \right] \]

The claim follows from simple algebraic manipulations. \( \square \)

We need to study the conditional expectation \( E \left[ 1 - (1 - d/m)^{Z_t-1} | a \in A_t \right] \) to use the above claim. This is not easy in general; although the distribution of \( Z_t \) remains stationary, the distribution of \( Z_t \) conditioned on \( a \in A_t \) can be a very different distribution. So, here we prove simple upper and lower bounds on \( E \left[ 1 - (1 - d/m)^{Z_t-1} | a \in A_t \right] \) using the
concentration properties of $Z_t$. By the assumption of the lemma $Z_t$ is at stationary at any
time $t \geq 0$. Let $k^*$ be the number defined in Proposition 6.9, and $\beta = (1 - d/m)k^*$. Also, let
$\sigma := \sqrt{6 \log(8m/\beta)}$. By Proposition 6.9, for any $t \geq 0,$
\[
E\left[1 - (1 - d/m)^{Z_t - 1} | a \in A_t \right] \leq E\left[1 - (1 - d/m)^{Z_t - 1} | Z_t < k^* + \sigma \sqrt{4m}, a \in A_t \right]
\]
\[+ P\left[Z_t \geq k^* + \sigma \sqrt{4m} | a \in A_t \right]
\]
\[\leq 1 - (1 - d/m)^{k^* + \sigma \sqrt{4m}} + \frac{P\left[Z_t \geq k^* + \sigma \sqrt{4m} \right]}{P[a \in A_t]}
\]
\[\leq 1 - \beta + \beta \left(1 - (1 - d/m)^{\sigma \sqrt{4m}} \right) + \frac{8 \sqrt{m e^{-\sigma^2/3}}}{P[a \in A_t]}
\]
\[\leq 1 - \beta + \frac{2d \sigma \beta}{\sqrt{m}} + \frac{\beta}{m^2 \cdot P[a \in A_t]} \quad (E.3)
\]
In the last inequality we used (A.4) and the definition of $\sigma$. Similarly,
\[
E\left[1 - (1 - d/m)^{Z_t - 1} | a \in A_t \right] \geq E\left[1 - (1 - d/m)^{Z_t - 1} | Z_t \geq k^* - \sigma \sqrt{4m}, a \in A_t \right]
\]
\[\cdot P\left[Z_t \geq k^* - \sigma \sqrt{4m} | a \in A_t \right]
\]
\[\geq (1 - (1 - d/m)^{k^* - \sigma \sqrt{4m}}) \frac{P[a \in A_t]}{P[a \in A_t]} - \frac{P\left[Z_t < k^* - \sigma \sqrt{4m} \right]}{P[a \in A_t]}
\]
\[\geq 1 - \beta - \beta \left(1 - (1 - d/m)^{\sigma \sqrt{4m}} \right) - \frac{2 \sqrt{m e^{-\sigma^2}}}{P[a \in A_t]}
\]
\[\geq 1 - \beta - \frac{4d \sigma \beta}{\sqrt{m}} - \frac{\beta^3}{m^3 \cdot P[a \in A_t]} \quad (E.4)
\]
where in the last inequality we used (A.4), the assumption that $2d \sigma \leq \sqrt{m}$ and the definition
of $\sigma$.

Next, we write a closed form upper-bound for $P[a \in A_t]$. Choose $t^*$ such that $\int_{t=0}^{t^*} (2 + c(t))dt = 2 \log(m/\beta)$. Observe that $t^* \leq \log(m/\beta) \leq \sigma^2/6$. Since $a$ leaves the market with
rate at least $1 + c(t)$ and at most $2 + c(t)$, we can write
\[
\frac{\beta^2}{m^2} = \exp \left( - \int_{t=0}^{t^*} (2 + c(t))dt \right) \leq P[a \in A_{t^*}] \leq \exp \left( - \int_{t=0}^{t^*} (1 + c(t))dt \right) \leq \frac{\beta}{m} \quad (E.5)
\]
Intuitively, $t^*$ is a moment where the expected utility of that $a$ receives in the interval $[t^*, \infty)$
is negligible, i.e., in the best case it is at most $\beta/m$. 

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By Claim E.1 and (E.4), for any $t \leq t^*$,

$$\frac{\mathbb{P}[a \in A_{t+\epsilon}] - \mathbb{P}[a \in A_t]}{\epsilon} \leq -\mathbb{P}[a \in A_t] \left(2 + c(t) - \beta - \frac{4d\sigma\beta}{\sqrt{m}} - \frac{\beta^3}{m^3} \cdot \mathbb{P}[a \in A_t] \pm O(\epsilon)\right)$$

$$\leq -\mathbb{P}[a \in A_t] \left(2 + c(t) - \frac{5d\sigma\beta}{\sqrt{m}} \pm O(\epsilon)\right)$$

where in the last inequality we used (E.5). Letting $\epsilon \to 0$, for $t \leq t^*$, the above differential equation yields,

$$\mathbb{P}[a \in A_t] \leq \exp\left(-\int_{\tau=0}^{t} \left(2 + c(\tau) - \frac{5d\sigma\beta}{\sqrt{m}}\right)d\tau\right) \leq \exp\left(-\int_{\tau=0}^{t} (2 + c(\tau) - \beta)d\tau\right) + \frac{2d\sigma^3\beta}{\sqrt{m}}.$$

where in the last inequality we used $t^* \leq \sigma^2/6$, $e^x \leq 1 + 2x$ for $x \leq 1$ and lemma’s assumption $5d\sigma^2 \leq \sqrt{m}$.

Now, we are ready to upper-bound the utility of $a$. By (E.5) the expected utility that $a$ gains after $t^*$ is no more than $\beta/m$. Therefore,

$$\mathbb{E}[u_c(a)] \leq \frac{\beta}{m} + \int_{t=0}^{t^*} (2 + c(t))\mathbb{E}[1 - (1 - d/m)^{Z_{t-1}}|a \in A_t] \mathbb{P}[a \in A_t] e^{-\delta t} dt$$

$$\leq \frac{\beta}{m} + \int_{t=0}^{t^*} (2 + c(t))((1 - \beta)\mathbb{P}[a \in A_t] + \beta/\sqrt{m})e^{-\delta t} dt$$

$$\leq \frac{\beta}{m} + \int_{t=0}^{t^*} (2 + c(t))\left(1 - \beta\right)e^{-\delta t} dt + \frac{3d\sigma^3\beta}{\sqrt{m}}e^{-\delta t} dt$$

$$\leq \frac{2d\sigma^5}{\sqrt{m}}\beta + \int_{t=0}^{\infty} (1 - \beta)(2 + c(t)) e^{-\delta t} dt.$$

In the first inequality we used equation (E.3), in second inequality we used equation (E.6), and in the last inequality we use the definition of $t^*$. We have finally obtained a closed form upper-bound on the expected utility of $a$.

Let $U_c(a)$ be the right hand side of the above equation. Next, we show that $U_c(a)$ is maximized by letting $c(t) = 0$ for all $t$. This will complete the proof of Lemma 8.4. Let $c$ be a function that maximizes $U_c(a)$ which is not equal to zero. Suppose $c(t) \neq 0$ for some $t \geq 0$. We define a function $\tilde{c} : \mathbb{R}_+ \to \mathbb{R}_+$ and we show that if $\delta < \beta$, then $U_{\tilde{c}}(a) > U_c(a)$.
Let \( \tilde{c} \) be the following function,

\[
\tilde{c}(\tau) = \begin{cases} 
  c(\tau) & \text{if } \tau < t, \\
  0 & \text{if } t \leq \tau \leq t + \epsilon, \\
  c(\tau) + c(\tau - \epsilon) & \text{if } t + \epsilon \leq \tau \leq t + 2\epsilon, \\
  c(\tau) & \text{otherwise}. 
\end{cases}
\]

In words, we push the mass of \( c(\cdot) \) in the interval \([t, t + \epsilon]\) to the right. We remark that the above function \( \tilde{c}(\cdot) \) is not necessarily continuous so we need to smooth it out. The latter can be done without introducing any errors and we do not describe the details here. Let

\[ S := \int_{\tau=0}^{t} (1 + c(t) + \beta) \, d\tau. \]

Assuming \( \tilde{c}'(t) \ll 1/\epsilon \), we have

\[
U_{\tilde{c}}(a) - U_{c}(a) \geq -\epsilon \cdot c(t)(1 - \beta)e^{-S}e^{-\delta t} + \epsilon \cdot c(t)(1 - \beta)e^{-S-\epsilon(2-\beta)}e^{-\delta(t+\epsilon)} + \epsilon(1 - \beta)(2 + c(t + \epsilon))(e^{-S-\epsilon(2-\beta)}e^{-\delta(t+\epsilon)} - e^{-S-\epsilon(2+c(t)-\beta)}e^{-\delta(t+\epsilon)}) \\
= -\epsilon^2 \cdot c(t)(1 - \beta)e^{-S-\delta t}(2 - \beta + \delta) + \epsilon^2(1 - \beta)(2 + c(t + \epsilon))e^{-S-\delta t}c(t) \\
\geq \epsilon^2 \cdot (1 - \beta)e^{-S-\delta t}c(t)(\beta - \delta).
\]

Since \( \delta < \beta \) by the lemma’s assumption, the maximizer of \( U_{c}(a) \) is the all zero function. Therefore, for any well-behaved function \( c(\cdot) \),

\[
\mathbb{E}[u_{c}(a)] \leq \frac{2d\sigma^5}{\sqrt{m}} + \int_{t=0}^{\infty} 2(1 - \beta) \exp \left( -\int_{\tau=0}^{t} (2 - \beta) d\tau \right) e^{-\delta t} dt \\
\leq O\left( \frac{d^4 \log^3(m)}{\sqrt{m}} \beta + \frac{2(1 - \beta)}{2 - \beta + \delta} \right).
\]

In the last inequality we used that \( \sigma = O(\sqrt{\log(m/\beta)}) \) and \( \beta \leq e^{-d} \). This completes the proof of Lemma 8.4.

## F Small Market Simulations

In Proposition 6.5 and Proposition 6.9, we prove that the Markov chains of the Greedy and Patient algorithms are highly concentrated in intervals of size \( O(\sqrt{m}/d) \) and \( O(\sqrt{m}) \), respectively. These intervals are plausible concentration bounds when \( m \) is relatively large. In fact, most of our theoretical results are interesting when markets are relatively large. Therefore, it is natural to ask: What if \( m \) is relatively small? And what if the \( d \) is not small
relative to $m$?

Figure 7 depicts the simulation results of our model for small $m$ and small $T$. We simulated the market for $m = 20$ and $T = 100$ periods, repeated this process for 500 iterations, and computed the average loss for the Greedy, Patient, and the Omniscient algorithms. As it is clear from the simulation results, the loss of the Patient algorithm is lower than the Greedy for any $d$, and in particular, when $d$ increases, the Patient algorithm’s performance gets closer and closer to the Omniscient algorithm, whereas the Greedy algorithm’s loss remains far above both of them.

Figure 7: Simulated Losses for $m = 20$. For very small market sizes and even for relatively large values of $d$, the Patient algorithm outperforms the Greedy Algorithm.