Impartial decision making among peers

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conflict of interest in collective decision making:

my selfish interest *corrupts* the report of my subjective opinion

non corrupted information is more valuable: it produces an *impartial evaluation*
conflict of interests pervasive in collective decisions by and about peers

eexample: evaluate the merit of a peer’s work, choose a winner among us, a ranking of us all

a necessary condition for the possibility of an impartial process:

- separate aspects of the decision related to self interest versus opinions/views

then a decision rule creates no conflict of interest if it only elicits opinions, and an agent’s report does not affect her self interest
examples where the separation is plausible

<table>
<thead>
<tr>
<th>self-interest</th>
<th>opinion</th>
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<tbody>
<tr>
<td>division of a dollar</td>
<td>division of the remainder</td>
</tr>
<tr>
<td>my share</td>
<td>who wins if not me?</td>
</tr>
<tr>
<td>award of a prize</td>
<td>do I win?</td>
</tr>
<tr>
<td>who wins if not me?</td>
<td>who is apart from me?</td>
</tr>
<tr>
<td>selecting webpages</td>
<td>am I in?</td>
</tr>
<tr>
<td>ranking by peers</td>
<td>what is my rank?</td>
</tr>
<tr>
<td>ranking of the others</td>
<td></td>
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</table>

- *Impartial award of a prize*, R. Holzman and H. Moulin, September 2010

- Sum of us: strategy-proof selection from the selectors, N. Alon, F. Fischer, A. Procaccia and M. Tennenholtz, July 2010

- strategyproof and efficient allocation of private goods: Kato and Ohseto (building on the work of Hurwicz, Zhou, Serizawa and Weymark,..)
model 1: award of a prize

\[ i \in N = \{1, 2, \ldots, n\} \]

\( i \)'s message \( m_i \in M_i \)

award rule: \( M_N \ni m \rightarrow f(m) \in N \)

\[ \rightarrow \textbf{Impartiality: } f(m|m_i) = i \iff f(m|m_i') = i, \text{ for all } i, m_i, m_i' \]
$n = 3$: impartial rule $\Leftrightarrow$ altruistic dictator

$n \geq 4$: many rules are impartial

which ones are \textit{reasonable}?
deterministic rule $\Rightarrow$ asymmetric

smoothing by lotteries:

- ensures full symmetry

- respects impartiality

*other first principles must hold* ex post
two natural message spaces:

- nomination: $M_i = N \setminus \{i\}$ or $N \setminus \{i\} \subseteq M_i \subseteq 2^{N \setminus \{i\}} \setminus \emptyset$ report acceptable candidate(s)

- full voting: $M_i = \mathcal{L}(N \setminus \{i\})$ linear ordering of other agents

Monotonicity: $\forall i, j, i \neq j \forall m \in M_N : f(m) = j \Rightarrow f(m|^{i}j) = j$

Monotonicity: lifting $j$ in $i$'s ranking does not threaten $j$'s win
plurality voting with an impartial twist

\[ M_i = N \setminus \{i\} ; \sigma_i = \text{number of nominations by } N \setminus \{i\} \]

\( i \) is a plurality* winner: \( \sigma_i(-j) > \sigma_j(-i) \) for all \( j \in N \setminus \{i\} \)

\( i \) is a q-winner: \( \sigma_i \geq q \), where \( q \geq \left\lfloor \frac{n}{2} \right\rfloor + 1 \)

pick a plurality* winner, or a q-winner if any: a nice impartial rule if No Winner outcome is feasible
first impartial nomination rule:

pick a default winner \( i^* \)

\[ \rightarrow \text{choose the plurality}^{*} \text{ winner among } N \setminus \{ i^* \}, \text{ } i^* \text{ not voting, if any} \]

\[ \rightarrow \text{else choose } i^* \]
impartial, fair within $\mathcal{N} \setminus \{i^*\}$, but

- $i^*$ is a dummy

- $i^*$ can win with no support: $\sigma_{i^*} = 0$
No Dummy: \( f(m_i, m_{-i}) \neq f(m'_i, m_{-i}) \) is possible for all \( i \in N \)

Unanimity\(^-\): \( \sigma_i = 0 \Rightarrow f(m) \neq i \)
second *impartial nomination rule*:

pick a selector $i^0$

$j = m_{i^0}$ is the default winner

choose the plurality* winner among $N \setminus \{i^0\}$, $j$ not voting, if any, else choose $j$
impartial, fair within \( N \setminus \{ i^0 \} \), and

- no one is a dummy

- can't win with no support: Unanimity~
but

- $i^0$ never wins

- even if $i^0$ is nominated by everyone else $\sigma_{i^0} = n - 1$
No Discrimination: for all $i$ there is $m$ s.t. $f(m) = i$

Unanimity$^+$: $\sigma_i = n - 1 \Rightarrow f(m) = i$
**Impossibility result** deterministic nomination rule

**Impartiality** ∩ **Unanimity** \( ^- \) ∩ **Unanimity** \( ^+ \) = \( \emptyset \)

the proof relies on

1) random smoothing

2) focusing on nomination profiles with a single loop

3) a nice combinatorial argument
random nomination rule

impartiality $\cap \text{Una}_- \Rightarrow \text{Una}_+ = \frac{n-1}{n}$

question: impartiality $\cap \text{Una}_+ \Rightarrow \text{Una}_- \geq ?$
random plurality

apply dollar division methods

fix a plurality profile

\[ \rho_{ji} = n \frac{\sigma_j(-i) - \sigma_i(-j)}{2}; \rho_{ji}^{-k} = n \frac{\sigma_j(-i,k) - \sigma_i(-j,k)}{2} \]

\[ r_i^{-k} = \frac{1}{1 + \rho_{ki} + \sum_{j \neq i,k} \rho_{ji}^{-k}} \]

probability of \( i \) winning \( x_i = \frac{1}{n}(1 + \sum_{j \neq i} (r_i^{-j} - r_j^{-i})) \)
Theorem

The random plurality rule guarantees

\[ \text{Una}^- \leq \frac{2}{n^{1/2}} ; \text{Una}^+ \geq 1 - \frac{1}{n^{2/3}} \]

also strong performance for agents with high quotas.
voting rules

\[ M_i = \mathcal{L}(N \setminus \{i\}) \] linear ordering of other agents

messages are more informative

Borda*, Condorcet* voting with an impartial twist

\[ B_i(-j) > B_j(-i) \text{ for all } j \neq i \]

\( i \) beats \( j \) for a strict majority in \( N \setminus \{i, j\} \), for any \( j \neq i \)
Unanimity$^+,−$ still make sense but very weak

*what we want*

- never choose someone too bad
- do not miss someone really good

(deterministic) lower bound on Borda score, Copeland score
it is easy to guarantee a Borda score no less than $\frac{1}{4}$ average

simple partition rules

$N = N_1 \cup N_2$ with $|N_1| - |N_2| = 0, -1$

$N_1$ uses Borda to choose the winner in $N_2$

even if we randomize the partition, ex post half of the agents are dummies, the other half can’t win
ordered partition rules

\[ N = N_1 \cup N_2 \text{ with } |N_1| - |N_2| = 0, -1 \]

*step 1:* run Borda in \( N_1 \) by \( N_1 \); stop if there is a Borda* winner, otherwise go to

*step 2:* \( N_1 \) selects the default \( j \in N_2 \) by Borda; run Borda in \( N_2 \) by \( N_2 \), default not voting; if there is no Borda* winner, choose the default
Impartial, No Discrimination, No Dummy, $\frac{1}{4}$ average Borda guaranteed

ex post critique: unequal influence of $N_1$ versus $N_2$

$i$ influences $j \overset{\text{def}}{\iff} \exists m \in M^N, m'_i \in M^i : f(m|^{i}m_i) = j \neq f(m|^{i}m'_i)$

**Full mutual Influence:** $\forall i, j \in N$: $i$ influences $j$
balanced partition rules \((n \geq 8)\)

\textit{partition} \(N = \bigcup_{k=1}^{\left\lfloor \frac{n}{d} \right\rfloor} N_k\) \textit{in} \(\left\lfloor \frac{n}{d} \right\rfloor\) \textit{districts of size} \(d\) \textit{or} \(d + 1\), \(d \geq 4\), \(\left\lfloor \frac{n}{d} \right\rfloor \geq 2\)

\textit{choose a default agent} \(i_k^*\) \textit{in each district}

\begin{itemize}
  \item \textit{step 1}: run Borda inside each district, default not voting; local winner is Borda* winner, or else the default
  \item \textit{step 2}: all non local winners use Borda to select one of the local winners
\end{itemize}
**Theorem**

Impartial, Full mutual influence, $\frac{1}{d}$ average Borda guaranteed
open questions

best Borda guarantee for an impartial rule?

best Copeland guarantee for an impartial rule?
model 2: peer ranking

assign $n$ agents to $n$ ranks

private consumption of one’s rank

$i \in N, a \in [n] = \{1, 2, \cdots, n\}$

$\Sigma(N, [n]) \ni \sigma : \text{bijection} \ N \rightarrow [n]$

$i$’s message $m_i \in M_i$

\textit{ranking mechanism:} $M_N \ni m \rightarrow \theta(m) = \sigma \in \Sigma(N, [n])$
• **Impartiality:** $\theta(m^i m_i)[i] = \theta(m'^i m'_i)[i]$, for all $i, m_i, m'_i$

• **Full Ranks:** for all $i \in N$, $a \in [n]$, for some $m \in M_N : \theta(m)[i] = a$

• **Full Range:** for all $\sigma \in \Sigma(N, [n])$ for some $m \in M_N : \sigma = \theta(m)$
Theorem

For $n = 3$, $\text{Impartiality} \cap \text{Full Ranks} = \emptyset$

For $n \geq 4$, $\text{Impartiality} \cap \text{Full Ranks} \neq \emptyset$

For $n \geq 6$, $\text{Impartiality} \cap \text{Full Range} \neq \emptyset$
For $n = 4$, Impartiality $\cap$ Full Ranks $\neq \emptyset$

$M^i = \{0, 1\}$ for all $i$, $\sigma = 3412$ means agent 3 is first, etc..

$(0, 0, 0, 0) \rightarrow 1234; \ (1, 0, 0, 0) \rightarrow 1432; \ (0, 0, 0, 1) \rightarrow 1324; \ (1, 0, 0, 1) \rightarrow 1423$

$(0, 0, 1, 0) \rightarrow 2134; \ (0, 1, 1, 0) \rightarrow 2143; \ (0, 0, 1, 1) \rightarrow 2314; \ (0, 1, 1, 1) \rightarrow 2341$

$(1, 1, 0, 0) \rightarrow 3412; \ (1, 1, 1, 0) \rightarrow 3142; \ (1, 1, 0, 1) \rightarrow 3421; \ (1, 1, 1, 1) \rightarrow 3241$

$(0, 1, 0, 0) \rightarrow 4213; \ (0, 1, 0, 1) \rightarrow 4321; \ (1, 0, 1, 0) \rightarrow 4132; \ (1, 0, 1, 1) \rightarrow 4213$

fairly symmetric treatment of the agents

range is not full (15 rankings out of 24)
we construct an impartial mechanism with full range \((n \geq 7)\)

choose ex ante three "leaders' agents 1, 2, 3, and fix \(d\) such that \(d \geq 4\)

step 1: the leaders choose impartially three ranks for themselves;

key: all assignments of \(\{1, 2, 3\}\) to \([n]\) are in the range

step 2: the leaders choose a partition of \(N \setminus \{1, 2, 3\}\) in \(\left\lfloor \frac{n}{d} \right\rfloor\) districts of size \(d\) or \(d + 1\), and a default winner in each district;

step 3: each district chooses its local winner: the Borda* winner (default not voting), or the default;

step 4: all non local winners assign the local winners to the top \(d\) ranks (remaining after step 1); the leaders assign the non local winners to the remaining ranks.
**step 1 explained:**

→**separating family** in \( A : S \subseteq 2^A \) such that

for all \( a, b \in A, a \neq b \), there exists \( S' \in S : a \in S', b \notin S' \)

→**separating family of size \( k \):** for all \( S \in S : |S| = k \)

**Fact:** for \( |A| \geq 6 \), we can find three **pairwise disjoint** separating families in \( A \), all of identical size.

(for \( |A| \leq 5 \), we can find at most two such disjoint families)
\[ A = \{a, b, c, d, e, f\} \]

<table>
<thead>
<tr>
<th>( S_1 )</th>
<th>( S_2 )</th>
<th>( S_3 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>abc</td>
<td>abd</td>
<td>abe</td>
</tr>
<tr>
<td>bcd</td>
<td>bce</td>
<td>bcf</td>
</tr>
<tr>
<td>cde</td>
<td>cdf</td>
<td>acd</td>
</tr>
<tr>
<td>def</td>
<td>ade</td>
<td>bde</td>
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<td>aef</td>
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<td>cef</td>
</tr>
<tr>
<td>abf</td>
<td>acf</td>
<td>adf</td>
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</table>

for \(|A| \geq 7, A = \{1, 2, \cdots, n\}\) fix \(1 \leq t < \frac{n}{2}\), then \(S_t = \{(a, a+t)|a \in A\}\) are separating and pairwise disjoint
step 1 continued:

choose three separating families $S_i, i = 1, 2, 3$, of identical size, pairwise disjoint

each leader chooses $S_i \in S_i$; given $(S_1, S_2, S_3) \in S_1 \times S_2 \times S_3$

assign 1 to a rank in $S_3 \cap S_2^c \neq \emptyset$

assign 2 to a rank in $S_1 \cap S_3^c \neq \emptyset$

assign 3 to a rank in $S_2 \cap S_1^c \neq \emptyset$

break ties in $S_3 \cap S_2^c$ by an onto vote of leaders 2 and 3

break ties in $S_1 \cap S_3^c$ by an onto vote of leaders 1 and 3

break ties in $S_1 \cap S_3^c$ by an onto vote of leaders 1 and 3
• many variants in step 2

• critique: the three leaders influence the rest of the agents, but not vice versa
Mutual Influence:

\[ \forall i, j \in N \exists m_i, m'_i \in M^i, m_{-i} \in M^{N\setminus i} : \theta(m^i|m_i)[j] \neq \theta(m^i|m'_i)[j] \]

we can find an impartial assignment mechanism with full range, satisfying Mutual Influence

its definition is more complex
**Open question:** Can we achieve

*Impartiality, Full Range and Monotonicity*

when the message space of all agents is $M_i = \mathcal{L}(N \setminus \{i\})$?