Collateral Constraints, Access to Debt Financing and Firm Growth

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Abstract

This paper studies how a firm’s growth is affected by the evolution of its external debt financing environment. Asymmetric information about quality of projects the firm can undertake makes external financing costly. Collateral can mitigate this problem, but its availability is limited by the size of the firm. As a firm grows, more collateral becomes available, broadening the firm’s access to external debt financing channels and lowering its cost of capital. The firm’s growth decision is affected by how effective additional collateral can be in lowering its cost of capital and the amount of assets it needs to accumulate to broaden its access to potential lending channels. A small firm may optimally choose to stay small when it is financially constrained and far from the size necessary to have access to formal lending. When the firm approaches the size to have access to formal lending, the strong incentive to expand makes it locally risk loving. My framework shows that high growth rates are not always associated with high value.

Keywords: Firm Financial Growth Cycle; Collateral; Firm Growth; Risk Taking

*I am grateful to my advisor Paul Pfleiderer for his guidance and support. I am indebted to Anat Admati, Shai Bernstein, Bradyn Breon-Drish, Peter DeMarzo, Han Hong, Dirk Jenter, Arvind Krishnamurthy, Tim McQuade, Andy Skrzypacz, Christopher Tonetti, Victoria Vanasco and Jeffrey Zwiebel for many fruitful discussions. All remaining errors are mine. Email: yizhoux@stanford.edu.
1 Introduction

Small and medium enterprises (SMEs) play an important role in economic growth and job creation (Neumark et al., 2011; Fort et al., 2013). However, while some small businesses grow substantially as they age, most remain small for decades (Cabrai and Mata, 2003; Angelini and Generale, 2008). Many theories have been developed to explain this observation (Levy, 2008; Bloom et al., 2013) and financial frictions are viewed as one potentially important channel. Existing literature analyzes financial frictions by simply assuming some exogenous financing costs or constraints (Cabrai and Mata, 2003; Cooley and Quadrini, 2003). However, these structures miss what may be the most striking feature for entrepreneurship finance, which is that firms of different sizes may face different external financing environments (Berger and Udell, 1998; Robb and Robinson, 2012). In particular, nascent firms mainly rely on informal lending channels such as funding from family members, friends and trade partners. As firms expand, they gain access to financial intermediaries. When they grow further and become established corporations, firms are able to borrow from the corporate bond market. The expectation of the evolution of its external financing opportunities thus may be crucial for a firm’s growth decisions.

In this paper I study the evolution of the external financing environment for SMEs and how it affects firms’ growth decisions. Various empirical studies show that the financing environment may play an important role in determining a firm’s performance and investment decisions (King and Levine, 1993; Rajan and Zingales, 1998; Beck et al., 2008; Fort et al., 2013; Adelino et al., 2014), but little is known of the mechanism behind the evolution of the firm financial growth cycle. Berger and Udell (1998) propose that the evolution of financial growth stages may be related to the firm’s information opaqueness, but they do not explain the nature of the information opaqueness and how the information asymmetries affect different potential funding sources. It is not clear why the financial growth cycle evolves in the ways we observe and what factors determine the boundaries of each stage. Furthermore, it is unclear how the evolution of the financing opportunities affects SME growth decisions.
In this paper I develop a theory that explains firms’ financial growth cycle by analyzing the availability of collateral. Collateral is a common feature of credit contracts between firms and lenders and is well known as a commitment device to mitigate asymmetric information problems (Chan and Kanatas, 1985; Bester, 1985; Besanko and Thakor, 1987). Posting collateral helps a firm signal the quality of its projects and mitigate moral hazard. Much of the existing literature on the information role of collateral implicitly assumes that firms have an unlimited supply of collateral, and the amount of collateral a firm is willing to commit is determined solely by the quality of its project.\(^1\) However, in reality, SMEs are often short of collateral and the availability of collateral plays a crucial role in small business lending. Concerns about the availability of collateral for small businesses have received particular attention during the recent financial crisis. In his speech

\(^1\)There’s another strand of literature that focuses on how collateral constraint amplifies economic shocks (Bernanke and Gertler, 1989; Kiyotaki and Moore, 1997; Krishnamurthy, 2010). In this paper I focus on how the availability of collateral affects firms funding costs and their access to external funding sources.
at the Federal Reserve’s 2010 meeting on restoring credit flow to small business, Ben S. Bernanke said ”the declining value of real estate and other collateral securing their loans poses a particularly severe challenge…it seems clear that some creditworthy businesses including some whose collateral has lost value but whose cash flows remain strong have had difficulty obtaining the credit that they need to expand…”. Mann (2014) documents that firms post patents, which are traditionally viewed as intangible assets, as collateral to increase the available amount of collateral. This paper incorporates the possibility that entrepreneurs and their firms face collateral shortages by introducing the size of the firm as a natural constraint on the amount of collateral the entrepreneur can post. As the size of a firm increases, the firm is able to commit more assets as collateral, reducing the possible information asymmetries. The size of a firm thus determines the availability of the firm’s funding sources. Large firms have enough assets to be posted as collateral and don’t need any information technology to mitigate the asymmetric information problem, enabling them to borrow from the corporate bond market. Middle size firms don’t have enough assets and thus face a collateral shortage problem. They need the costly monitoring technology from financial intermediaries (e.g., banks) and rely on bank lending to finance their projects. Small firms are denied access to bank lending because the profits a bank can earn from lending to an ”infant” entrepreneur is not sufficient to compensate for the monitoring costs. These nascent firms are financially constrained and must rely on informal lending channels.

The collateral shortage also plays an important role in SMEs growth decisions. Given limited collateral, expansion not only increases the firm’s daily operating cash flows, but also enables the entrepreneur to post more assets as collateral, lowering the firm’s future funding costs and potentially broadening its access to external funding sources. The benefit of a lower cost of capital potentially gives firms a strong incentive to expand, but can only be realized when firms are able to find a lender to finance their projects. When firms gain access to formal lending channels (bank lending, corporate bond market etc.), every good projects will be financed and the benefit of a lower cost of capital is fully
realized. SMEs are willing to expand until they reach a mature stage where additional collateral has no value in lowering funding cost. However, when firms are so small that they must rely on informal lending channels, funding for good investment opportunities may not be available because informal lenders, the sole source of funding, can themselves be financially constrained and unable to offer funding. When only informal lending is available, the benefit of a lower cost of capital is limited since a good project may not be financed, making expansion less desirable. When firms find themselves are small relative to the size they must be to gain access to formal lending sources, they optimally choose not to expand and pay all cash flows as dividend.

This result that collateral considerations may create a “growth trap” in which firms lack the incentives to take the steps needed to grow provides an alternative explanation for several empirical observations. In development economics, empirical studies (Banerjee and Duflo, 2005; Mel et al., 2008; Banerjee and Duflo, 2014) find that even though the marginal gross return rates for small enterprises are fairly high, those SMEs often stay small and are characterized by low growth rates. This “growth trap” puzzle has received considerable attentions and a popular perception is that this is because those SMEs are severely financially constrained and don’t have enough fund to support expansion. This hypothesis fails to explain why those small businesses don’t reinvest their own profits to expand. My model provides an alternative explanation that these owners actually don’t want to expand even when they are able to do so by retaining earnings.

Firm size distribution is closely related to firm growth decision. Cross section evidence (Cooley and Quadrini, 1992; Cabrai and Mata, 2003; Angelini and Generale, 2008) shows that firm size distribution is heavily skewed toward small firms. Most existing explanations are based on specific assumptions about production technologies shocks, exogenous financial constraints and market structure. The model I present below provides a potentially more general explanation that is based on the endogenous evolution of financial constraints firms face in early stages due to limited collateral.

Small businesses are important because they create most new jobs in the economy
(Neumark et al., 2011; Fort et al., 2013), and job creation is associated with firms’ growth decisions. The “stay small” result is consistent with an emerging literature on the implications of firm age on job creation. Based on detailed datasets, Haltiwanger et al. (2013) and Adelino et al. (2014) find that within the small firm category it is the young firms that contribute almost all the net creation of jobs. The growth patterns implied by the model presented in this paper are consistent with these findings. In particular, because for many small firms the optimal decision is to follow policies that keep the firm small and this means that in any cross section of small firms of different ages, the highest growth rates will tend to be observed in younger firms.

Besides corporate finance predictions, this paper also has implications for asset pricing and specifically for the risk premium associated with entrepreneurial activities. Access to formal lending channels completely changes a firm’s external funding environment and its expectation of availability of future funding. When a firm’s size reaches the level needed to obtain funding from financial intermediaries. The desire to reach this threshold introduces a nonconcavity in the firm’s value function. The benefit of having access to formal lending channels makes entrepreneurs that are risk-neutral in terms of dividend flows locally risk loving. This local risk shifting behavior can potentially explain the low private equity risk premium (Hamilton, 2000; Heaton and Lucas, 2000; Moskowitz and Vissing-Jorgensen, 2002).

The paper proceeds as follows. Section 2 reviews related literature. Section 3 introduces the model and in section 4 the model is solved and the firms optimal growth strategy is identified. Section 5 analyzes how the endogenous firm financial growth cycle affects a firms growth strategy. Section 6 studies how the desire to reach next financing stage affect firm’s risk taking behavior. Section 7 evaluates firm growth rates as a proxy for firm value in empirical research and section 8 discusses extensions. Section 9 concludes.
2 Literature Review

This paper contributes to the literature on firm financial growth cycle. Berger and Udell (1998) is the first paper to document the evolution of the financing environment for firms at different stages. They argue that information opaqueness may be the key factor to generate patterns in funding. Avery et al. (1998) and Robb and Robinson (2012) confirm the patterns examined in Berger and Udell (1998) with new and more detailed datasets. A large body of literature tries to explain entrepreneur’s choice of financing method at certain stages (Boot and Thakor, 1994; Peterson and Rajan, 1994). A notable exception which focuses on the life cycle of firm financing is Rajan (2012). In that paper it is argued that entrepreneur’s choice between internal and external funding sources is determined by the trade-off between differentiating her enterprise in ways that potentially generate higher net present value and standardizing her enterprise to lower external funding costs. I believe that my paper is the first to analyze the evolution of external financing environment in the firm’s life cycle based on collateral constraints.

Collateral as a committing device to mitigate asymmetric information problem has been widely studied both theoretically (Chan and Kanatas, 1985; Bester, 1985; Besanko and Thakor, 1987) and empirically (Avery et al., 1998; Peterson and Rajan, 1994; Voordecker and Steijvers, 2006). Different from most models that assume unlimited supply of available collateral, this paper introduces firm size as a natural constraint on the availability of collateral. Collateral shortages endogenously determine the evolution of external financing environments and firms’ growth decisions.

There is a large strand of literature that focuses on economic shocks and collateral constraint (Bernanke and Gertler, 1989; Kiyotaki and Moore, 1997; Krishnamurthy, 2010). In this paper I abstract from economic shocks and focus on how firm growth relates to the availability of collateral and its funding cost and access to external funding sources.

This paper is related to both the development economics and the entrepreneurial finance literature that focuses on growth-traps for small enterprises. Several empirical studies (Banerjee and Duflo, 2005; Mel et al., 2008; Banerjee and Duflo, 2014) show that
entrepreneurs may keep their firms small even with high marginal returns. They view it as evidence that firms are extremely financially constrained and thus are unable to expand. My paper proposes an alternative explanation that entrepreneurs may be able to expand but find that this is not justified by the long term rewards.

Another closely related research topic is firm size distribution. Cabrai and Mata (2003); Angelini and Generale (2008) find that a large fraction of firms is centered at the low end of the firm size distribution. Existing models (Jovanovic, 1982; Hopenhayn, 1992; Cooley and Quadrini, 1992) rely on specific production technology shocks, market structure assumptions or exogenous financial constraints to explain this generic pattern. Based on collateral shortage, my paper provides a potentially more generic explanation for firm dynamics with endogenous financial frictions.

There’s an emerging literature on job creation and firm age. Haltiwanger et al. (2013) and Adelino et al. (2014) find that job creation by small firms is preponderantly due to young small firms. My paper offers an explanation that those small and mature firms are generally those who have chosen to stay small and accordingly have low growth rates.

Empirical evidence finds that the returns on entrepreneurial activities are surprisingly low (Hamilton, 2000; Heaton and Lucas, 2000; Moskowitz and Vissing-Jorgensen, 2002). Most hypotheses that have been offered are based on the idea that entrepreneurs have a different set of preferences or beliefs (e.g., risk tolerance or overoptimism), while some explain it by introducing switching back to worker as the outside option for the entrepreneur (Manso, 2014). An interesting explanation is given by Vereshchagina and Hopenhayn (2009) who assume the entrepreneur has a new production technology that is only available when the entrepreneur’s wealth reaches certain threshold. This discrete choice of production technology introduces nonconcavity in the value function and creates a local risk taking incentive. Similar to Vereshchagina and Hopenhayn (2009), this paper also introduces risk shifting behavior as a result of the nonconcavity in the firm value function. As opposed to their analysis, my model doesn’t rely on the specific production technology assumption. Instead the nonconcavity in the firm value function arises in my
model form the evolution of external financing environment.

My paper also contributes to empirical assessments on SMEs (King and Levine, 1993; Rajan and Zingales, 1998; Beck et al., 2008; Fisman and Love, 2003). Because it is difficult to observe SMEs value in the data, firm growth rate often serves as a popular proxy to assess firm values in empirical researches. However, my paper shows that firm growth rates may not be a good measure for firm values.

3 The Model

Consider a continuous time economy with an entrepreneur who runs a firm over an infinite horizon. At any time $t$, the firm operates its current assets and generates cash flows. The market value of its current assets is the presented value of all future cash flows the asset would generate discounted by a constant size-adjusted market asset return rate $r_a$. The size of the firm is denoted by the market value of its asset $A_t$ and the operating profit it generates per unit of time can be written as $r_a A_t$. Besides those cash flow, the entrepreneur also invests her own human capital to generate growth opportunities for the firm. Those growth opportunities are modeled as projects that the firm may take at any time. Each project requires an investment of $I(A_{t-})$, where $A_{t-} \equiv \lim_{s \uparrow t} A_s$ since the size of the firm may jump whenever it successfully implements a project. $I(A_{t-}) = \gamma A_{t-}^\alpha$, where $\gamma > 0$ is a constant and $\alpha \in (0, 1)$. So $I(A_{t-})$ is strictly increasing and strictly concave in $A_{t-}$. The firm needs external funding to finance its projects. This may because its current assets are illiquid or the project is built on its current operation. The firm and potential external financial sources are all risk-neutral. While potential lenders discount cash flow at the market asset return rate $r_a$, the entrepreneur value future dividend flows at a higher discount rate $r > r_a$ to compensate her human capital investment.
3.1 The Project

The standard project would generate a permanent cash flow \( I(A_t^-)\pi \) when it succeeds and 0 when it fails. The firm can also deviate to a risky project which is high risk \( P_R < P_B \) and high return \( \pi_R > \pi \) such that \( P_R \pi_R < r_L \). Besides the choice of types of projects, the firm also decides when to implement a project. Over each infinitesimal period of time \( [t; t+dt] \), the firm privately observes a good signal with probability \( \lambda dt \), and the good signal generating process is described as \( \{N_t\} \). With a good signal the project would be successful with probability \( P_G \), which is normalized to 1 while without the signal the probability of success is \( P_B \). One can interpret the good signal as some private information on good market conditions. I assume that \( P_B \pi < r_L < P_G \pi \), where \( r_L \) is the funding cost for lenders. Thus it is socially optimal to only invest in the standard project when the firm observes a good signal. For simplicity I denote a standard project with good signal as a type \( G \) project, standard project without good signal as a type \( B \) project, and risky project as a type \( R \) project.

3.2 The External Financing Environment

All projects need external funding to be implemented. This assumption is natural in the sense that the firm’s current assets may be illiquid or the project is built on its current operation. Since the firm needs the entrepreneur’s human capital to generate growth opportunities, similar to Rajan (1992), external equity financing may introduce principal-agent problem and affects the entrepreneur’s human capital investment decisions. To focus on the evolution of the firm’s external financing environment, here I abstract from this potential moral hazard problem by simply assuming that external equity financing is not available. A lot of empirical studies (Berger and Udell, 1998; Robb and Robinson, 2012) have shown that for SMEs debt is the dominating external funding source. For example, Robb and Robinson (2012) reported that only 5% of firms in their data had external equity. It seems that for most SMEs equity funding is not a feasible option and it is reasonable to focus on debt financing while keeping in mind that there
exists some important friction that shuts down the external equity financing channel. By focusing on debt financing this paper does not cover some important problems, such as innovative firms funded by venture capital where the external equity financing channel is non-negligible.

When a firm decides to start a project, it approaches all potential lending parties and each of them proposes a loan contract. A loan contract can be described as \( \{A_p, r_p\} \), where \( A_p \) is the collateral requirement and \( r_p \) is the interest payment. To be more specific, in this paper I only consider perpetual debt contracts and the firm promises to pay a permanent cash flow rate \( r_p \). Introducing maturity of debts would introduce all remaining debt maturities as additional state variables, making the model difficult to solve. The firm compares all proposed contracts and decides whether to abandon the project or to accept one of the contracts to make the investment. When the project is invested and succeeds, the firm receives the project cash flow net off its funding cost \( I(A_t- r_p) \). When it fails, the firm gets nothing and has to pay the lender the amount of collateral \( 0 \leq A_p \leq A_{t-} \). Here \( A_p \leq A_{t-} \) represents a natural constraint on the maximum amount of collateral available to the firm. To focus on the role of collateral in mitigating the asymmetric information problem, I simply assume that the collateral has no value to lenders.

There are two types of lenders in the economy: informal lenders and formal lenders. Lenders all face the same funding cost \( r_L \), and I assume \( r_L \geq r_a \) so lenders would have no incentive to raise funds and purchase assets themselves, which also rules out the possibility that the entrepreneur raises fund without taking on any project. Formal and informal lenders differ in terms of their funding capacities and information technologies. Informal lenders can be interpreted as the entrepreneur’s family members, friends or trade partners. Being common individuals or small firms, informal lenders are not specialized in lending to businesses and are often themselves financially constrained, so they may not

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2 One can also rewrite the model into a mathematically equivalent version in which all projects instantaneously generate a lump sum payment, and the interest rate \( r_p \) becomes a lump sum charge \( R_p = I(A_{t-}r_p) \).

3 For simplicity, in this paper I abstract from the difference between tangible and intangible assets.

4 Alternatively, one can assume that the liquidation value of collateral is sufficiently low.
be able to fund the entrepreneur’s project whenever she needs. In the model I assumed that at any time $t$ informal lending channel is only available to the entrepreneur with a probability $P_I$. The model is tractable even when the informal financing probability is a function of the project size, but here for simplicity I just assume $P_I$ to be a constant. Motivated by empirical findings that firms are severely financial constrained even with informal funding, I assume that:

**Assumption 1.** $r > r_a + P_I \lambda (\frac{1}{P_B} - 1)$.

The term $(\frac{1}{P_B} - 1)$, as shown later, is the benefit of a lower cost of capital when the firm expands and has more assets that can be posted as collateral. This parameter condition suggests that when the firm only relies on informal lending, the benefit of a lower cost of capital is limited because only some of the future good projects would be able to be financed, making the expansion benefits too low for the entrepreneur. This assumption highlights the negative effect of firm financial constraint on firm value.

Another important feature of the informal lending channel is its information technology to facilitate lending. Here the information technology is modeled as the ability to tell whether the firm is doing a risky project or not. This specification is rooted in the fact that soft information may help lenders to monitor firm’s operation, but is not a perfect solution to the asymmetric information problem. In this model, since taking a risky project fundamentally changes the project’s payoff structure, it has to be associated with taking a different project, or operating the project in a different way. Those are behaviors that are somehow detectable. On the other hand, the private signal can be interpreted as a good market condition or good timing, which may rely on specialized professional expertise or the entrepreneur’s private observations and are often difficult verify. This soft information is generated from daily personal or business interactions, so the information cost can be treated as a sunk cost for informal lenders. At time $t$, conditional on his proposed loan contract being accepted by the firm, an informal lender’s expected lending profit is:
\[ \text{Profit}_{\text{informal}}(A_{t^-}, r_p) = \sum_{i \in \{B, G, R\}} P_{\text{project}\_i}^{\text{informal}} I(A_{t^-}) (P_i \frac{r_p - r_L}{r_a} - 1); \] (1)

Where \( P_{\text{project}\_i}^{\text{informal}} \) is the informal lender’s belief about type \( i \) project given his loan contract is chosen.

Formal lenders are deep-pocketed, so they are able to finance every project the entrepreneur proposes. Similar to informal lenders, they can monitor the firm and tell whether it is a risky project but don’t know whether the firm observes a good signal or not. However, monitoring technology is costly for formal lenders. One can interpret formal lenders with information technology as banks and formal lenders without private information as investors in the corporate bonds market. When formal lenders monitor, they spend a one time effort cost \( c \). Since informal lending is the most convenient and possibly the fastest way to raise funds, the firm would rely on it whenever it is not dominated by other financing channels in terms of funding costs.\(^5\) When the informal lending channel is unavailable, conditional on all banks offering the same attractive loan contract, each bank would win the loan contract with probability \( P_{\text{bank}} \). The probability \( P_{\text{bank}} \) can be interpreted as a measure of banking sector competitiveness. At time \( t \), conditional on his loan contract being accepted, a formal lender’s expected lending profit is:

\[ \text{Profit}_{\text{formal}}(A_{t^-}, r_p, c) = (1 - P_1) P_{\text{bank}} \sum_{i \in \{B, G, R\}} P_{\text{project}\_i}^{\text{formal}} I(A_{t^-}) (P_i \frac{r_p - r_L}{r_a} - 1) - c1_c; \] (2)

Where \( P_{\text{project}\_i}^{\text{formal}} \) is the formal lender’s belief about type \( i \) project given his loan contract is chosen, and \( 1_c \) is the indicator of usage of the bank’s monitoring technology.

\(^5\)Another way to model this is to assume an arbitrarily small filing cost for formal funding sources.
3.3 The Firm

The firm’s strategy includes both cash flow decisions and project decisions. At any time $t$ the firm decides how much cash flow to reinvest and the remaining cash flow would be paid out as dividends. The cash flow reinvestment decisions overtime are described as $I_{t}^{\text{Cash}}$, subject to $I_{t}^{\text{Cash}} \in [0, r_{a}A_{t-}]$. At any time $t$, the project implementation decision is described as $I_{t}^{p} = \{G, B, R\}$, where $I_{t}^{p} = G$ means only investing in a standard project when a good signal is available. Associated is the process of type $i$ projects implementation, denoted as $\{N_{i}\}$. For type $i$ projects, at time $t$ the set of available lending contracts is $L_{i}^{t}(A) \equiv \{L(A_{ip}(A_{t-})), r_{ip}(A_{t-})\} \cup \{L(0, \pi_{i})\}$, where the element $L(0, \pi_{i})$ means lenders decide not to finance this project. I denote the firm’s project investment process and its type $i$ available loan contract path as $I^{p} = \{I_{t}^{p}\}$ and $L_{i}(A) = \{L_{i}^{t}(A)\}$, respectively. The firm will go bankrupt when $A_{t}$ drops to 0 and the entrepreneur gets her outside option value, which is normalized to 0. Given the set of lending contracts for different types of projects $L$, the firm chooses its optimal project investment decision $I_{p}$, loan contract choice $(A_{F}, r_{F}) \in L_{I_{p}}$ and cash flow reinvestment policy $I_{t}^{\text{Cash}}$ to maximize its expected discounted value of dividend:

$$V_{F}(A) = \max_{S} E^{S}\{\int_{0}^{\tau} e^{-rt}\left[r_{a}A_{t-} - I_{t}^{\text{Cash}}\right]dt\mid A_{0} = A\}$$

(3)

Where $S = (\{I_{F}, L_{F}(A_{F}, r_{F}), I_{t}^{\text{Cash}}, I_{p}\})$ is the firm’s optimal policy, and the stopping time $\tau \equiv \inf\{s : A_{s} = 0\}$ describes the firm’s bankruptcy time. The cost of project failure is implicitly included in the stopping time $\tau$ since the firm has to pay the collateral $A_{F}$ once its project fails.

The firm’s discounted rate, $r$, is assumed to satisfy the following assumption:

**Assumption 2.** $r < r_{a} + \lambda(\frac{1}{\mu_{a}} - 1)$.

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6In section 8 I consider the case where the outside option is nonzero and may evolve as the size of the firm changes.
Intuitively, assumption 2 suggests that if the firm can finance its projects whenever it observes a good signal, the value of the firm would be higher than the size of its assets (that is to say, the market value of the firm’s assets). Otherwise the firm would not have incentive to grow.

To make monitoring technology a valuable tool for formal lenders, the information cost $c$ should be sufficiently low, which is stated in the following technical assumption:

**Assumption 3.**

$$
\frac{c}{(1 - P_1)P_{\text{bank}}} \leq \min\{\gamma^{1-\alpha}\left[\frac{\pi_R - \pi}{r_a(P_R - P_B)}\right]^{1-\alpha} - \left(P_B^{\frac{1}{K_1}} - 1\right)\frac{\gamma(\pi_R - \pi)}{r_a(P_R - P_B)}\}^{1-\alpha},
$$

$$
\gamma^{1-\alpha}\left[\frac{\pi_R - \pi}{r_a(P_R - P_B)}\right]^{1-\alpha} - \left(P_B^{\frac{1}{K_1}} - 1\right)\frac{\gamma(\pi_R - \pi)}{r_a(P_R - P_B)}\}^{1-\alpha}.
$$

Where $K_1$ is a constant defined in the following section. This assumption, as shown later, guarantees that formal lenders would not finance any projects without gathering information when the firm is small.

As in most continuous time model, here I focus on Markov Equilibrium, which is defined as:

**Definition 1.** A Markov Equilibrium of the game is the entrepreneur’s strategy $S$, lenders’ loan contracts $L$ and formal lenders’ monitoring decision $1_c$ such that:

1. Given $L$, $S$ maximizes the entrepreneur’s expected discounted value of dividend flows;
2. Given $S$ and other lenders’ strategies, each lender picks his loan contract $L$ and possibly monitoring decision $1_c$ to maximize his expected lending profit; 3. $S$, $L$ and $1_c$ only depends on $\{A_t\}$.

This is a partial equilibrium in the sense that there’s no market clear condition for credit. Lenders may get a non zero lending profit because the asymmetric information problem makes interest rate $r_P \geq r_L$. However, one can construct a general equilibrium.
in which every lender has to pay some operation cost and free entry suggests that the lending benefit equals his operation cost.

At any time \( t \), the sequence of events during the infinitesimal time interval \([t, t + dt]\) can heuristically be described as follows:

Step 1: The firm generates cash flows and pays out dividends.

Step 2: The firm may or may not observe a good signal, and may propose a project to all lending parties.

Step 3: Informal lenders and formal lenders (if they gather information), learn whether the project is risky or not, and every lender offers a loan contract \( L_i = \{r_p, A_p\} \).

Step 4: The firm makes the project investment decision, and chooses one loan contract to finance its project.

Step 5: The project payoff is realized and both the firm and the lender get paid according to the loan contract.

4 Solution of the Model

Before the analysis, the following lemma shows that without loss of generality one can focus on the separating markov equilibrium.

**Lemma 1.** For each pooling markov equilibrium there exists a corresponding separating markov equilibrium that is Pareto Superior to the pooling equilibrium.

**Proof.** See Appendix.

To find a separating markov equilibrium, it is useful to consider IC constraints for different types of projects. Since no information technology can tell when the firm observes the good signal, in the separating markov equilibrium there always exists the following IC constraint:
\[ V_F(A_{t-}) \geq P_B V_F(A_{t-} + I(A_{t-}) \frac{\pi - r_P}{r_a}) \]
\[ + (1 - P_B) V(A_{t-} - A_p) \]
\[ \text{Firm value when project succeeds} \]
\[ + (1 - P_B) V(A_{t-} - A_p) ; \]
\[ \text{Firm value when project fails} \]

Equation (5) is referred as the signal IC constraint. When formal lenders do not use monitoring technology, they may also consider the IC constraint for type \( R \) projects, which is referred as the risk shifting IC constraint.

\[ V_F(A_{t-}) \geq P_R V_F(A_{t-} + I(A_{t-}) \frac{\pi_R - r_P}{r_a}) \]
\[ + (1 - P_R) V(A_{t-} - A_p) ; \]
\[ \text{Firm value when project succeeds} \]
\[ + (1 - P_R) V(A_{t-} - A_p) ; \]
\[ \text{Firm value when project fails} \]

In the separating Markov equilibrium, the signal IC constraint would always hold. For the entrepreneur the feasible loan terms can be no better than terms that make the signal IC constraint binding. Thus it is natural to start the analysis by stating the following conjectures about the Markov equilibrium.

**Conjecture 1.** *In the separating Markov equilibrium, without observing the good signal the firm will be indifferent between taking or not taking the standard project.*

Latter I will show that with assumption 3 this is the unique separating equilibrium. There are many loan contracts that satisfy this signal IC constraint, but the following lemma shows that the loan contracts in the Markov equilibrium must be unique, and the signal IC constraint is always binding.

**Lemma 2.** *In the equilibrium, for a firm with asset \( A_{t-} \), \( A_p = A_{t-} \).*

*Proof. See Appendix.*
This lemma basically states that lenders would require as large an amount of collateral as possible unless the firm has big enough assets. This is fairly intuitive since posting more collateral is costly when the firm does not observe a good signal.

The model is difficult to solve because of the endogenous evolution of the external debt financing environment and the interaction between lenders’ strategies and the firm’s policy $S$. To get around those problem, I solve this model in four steps.

Step 1: Use IC constraint to partially solve the firm’s value function;

Step 2: Given the partial characterization of the firm’s value function, solve loan contracts in different scenarios;

Step 3: Determine the evolution of the firm’s financial growth cycle;

Step 4: Pin down the firm value function, growth decision and loan contracts.

4.1 Partially Solving the Firm’s Value Function

In the separating markov equilibrium, lenders only want to finance type $G$ projects. Taking that into account, a firm’s Hamilton-Jacobi-Bellman equation is:

$$ rV_F(A_t^-) = \max_S \left\{ V_F'(A_t^-)ICash(A_t^-) + r_a A_t^- - ICash(A_t^-) \right\} $$

\begin{equation}
\begin{aligned}
&\text{Asset expansion from asset cash flow} \\
&\text{Dividend from asset cash flow} \\
&+ P_P \lambda (V_F(A_t^- + \gamma(A_t^-) \frac{\pi}{r_a} - r_P) - V_F(A_t^-))
\end{aligned}
\end{equation}

(7)

where $P_P$ is the probability that a standard project with a good signal will be financed. It would be $P_I$ when only informal lenders are available and 1 when formal lenders are willing to lend.

Let $r_p(A)$ be the lowest interest rate to satisfy the signal IC constraint. Now consider the case $A_t^- \in \{A'|r_p(A') \geq r_L\}$. Given the conjecture and lemma 2, signal IC constraint must be binding and $A_p = A_t^-$, so the signal IC constraint can be written as:
\[ V_F(At-) = P_B[V_F(At^- + \gamma(At^-) \frac{\pi - r_P}{r_a})] + (1 - P_B)V(0); \] (8)

where \( V(0) \) is the entrepreneur’s outside option when the firm bankrupts. Substituting the signal IC constraint into the firm’s HJB equation, one finds:

\[ rV_F(At-) = \max_S \{V'_F(At^-)I^{Cash}(At^-) + r_aAt^- - I^{Cash}(At^-) + P_P \lambda(\frac{1}{P_B} - 1)V_F(At^-) \}; \] (9)

Notice that equation (9) is independent of lenders’ lending strategies and the slope of the firm’s value function. This comes from the fact that the good signal only changes the probability of success, but not the exact payoff when the project succeeds or fails. No matter how complex the lending policies and the slope of the firm’s value function might be, with the help of the signal IC constraint, one can always express the firm’s gain from project success as a function of its current firm value and its outside option value.\(^7\) Now one can analyze the firm’s value function without considering loan contracts. Since the entrepreneur is risk-neutral, the optimal cash flow reinvestment policy would be corner solutions: \( I^{Cash}(At^-) = r_aAt^- \) whenever \( V'_F(At^-) \geq 1 \) and \( I^{Cash}(At^-) = 0 \) when \( V'_F(At^-) < 1 \). So when \( At^- \leq \inf \{A'|r_p(A') = r_L\} \), there are four possible scenarios: Reinvesting cash flow and only informal lending; paying out cash flow and only informal lending; reinvesting cash flow and with access to formal lending; paying out cash flow and with access to formal lending. Taking the case “reinvesting cash flow and with access to formal lending” as an example, the firm’s HJB function would be

\[ rV_F(At-) = V'_F(At^-)r_aAt^- + \lambda(\frac{1}{P_B} - 1)V_F(At^-); \] (10)

\(^7\) In the basic model I only consider the case when the outside option value is always 0. In section 7 I study a case in which the entrepreneur’s outside option value is a function of \( A_t^- \).
Solve this ODE one finds:

$$V_F(A_t) = Z_1 A_t^{K_1}$$  \hspace{1cm} (11)$$

Where $Z_1$ is a strictly positive constant and $K_1 = \frac{r - \lambda (\frac{1}{PB} - 1)}{r_a} < 1$. Since it is still unclear for what size of asset $A_t$ formal lenders would be willing to lend and $V_F'(A_t) \geq 1$, one cannot determine the boundary condition and thus cannot pin down the constant $Z_1$. Similarly, one can also partially solve firm’s value function in other scenarios:

Reinvesting cash flow and only informal lending:

$$V_F(A_t) = Z_0 A_t^{K_0};$$ \hspace{1cm} (12)$$

Where $Z_0$ is a strictly positive constant and $K_0 = \frac{r - P_I \lambda (\frac{1}{PB} - 1)}{r_a} > 1$.

Paying out cash flow and with access to formal lending:

$$V_F(A_t) = \frac{r_a}{r - \lambda (\frac{1}{PB} - 1)} A_t = \frac{1}{K_1} A_t;$$ \hspace{1cm} (13)$$

By assumption 2, in this scenario $V_F'(A_t) > 1$. So with access to formal financing channels, firms would reinvest their daily operation cash flows whenever the interest rate is higher than $r_L$.

Paying out cash flow and only informal lending:

$$V_F(A_t) = \frac{r_a}{r - P_I \lambda (\frac{1}{PB} - 1)} A_t = \frac{1}{K_0} A_t;$$ \hspace{1cm} (14)$$

So far I partially solve the firm’s value functions in different scenarios whenever $A_t \leq \inf \{A' | r_p(A') = r_L\}$. To determine the evolution of different scenarios and value functions when the firm’s assets are large enough to take the lowest possible interest rate $r_L$, one needs to solve the optimal lending policies for lending parties.
4.2 Evolution of the Debt Financing Environment

I start the analysis by looking at the case where the firm has access to formal lending channels. Given the partial solution for the firm’s value function and the binding signal IC constraint, one finds:

\[ Z_{1}A_{t}^{K_{1}} = P_{B}Z_{1}[A_{t-} + I(A_{t-})\pi - rP^{*}A_{t-}]^{K_{1}}; \quad (15) \]

So the implied interest rate is \( r_{PS}(A_{t-}) = \pi - r^{1-\alpha}(P_{B}^{1-\frac{1}{\gamma}} - 1) \). It is straightforward to see that the interest rate is decreasing in the size of the firm.

Other than the signal IC constraint, formal lenders may need to take the firm’s risk shifting behavior into account. If they do not exert effort to gather information, then their loan contracts need to satisfy another IC constraint:

\[ Z_{1}A_{t}^{K_{1}} = P_{R}[Z_{1}(A_{t-} + \gamma(A_{t-})\pi - rP^{*}A_{t-})^{K_{1}}]; \quad (16) \]

This IC constraint is referred as the risk shifting IC constraint. The implied interest rate is \( r_{PR}(A_{t-}) = \pi - r^{1-\alpha}P_{R}^{1-\frac{1}{\gamma}}(P_{R}^{1-\frac{1}{\gamma}} - 1) \). Notice that \( r_{PR}(A_{t-}) < r_{PB}(A_{t-}) < 0, \) and \( r_{PR}(A_{t-}) \geq r_{PS}(A_{t-}) \) whenever \( A_{t-} \leq \left[ \frac{\pi_{R} - rP^{*}}{r_{R}^{1-\gamma} - P_{R}^{1-\gamma}} \right]^{\frac{1}{1-\gamma}} \); which can be shown in figure 2.

The collateral requirement is more costly for firms taking risky projects. When firms become large, they find taking risky projects so costly that the risk shifting IC constraint is implied by the signal IC constraint. Since information technology is costly, formal lenders would prefer not to gather information and only rely on collateral requirements. When \( A_{t-} \geq \left[ \frac{\pi_{R} - rP^{*}}{r_{R}^{1-\gamma} - P_{R}^{1-\gamma}} \right]^{\frac{1}{1-\gamma}} \equiv A_{C} \), the firm would get access to formal lending channels without information technology. In other words, firm can issue corporate bonds in the financial market.

When \( A_{t-} < A_{C} \), collecting information enables formal lenders to relax the risk shifting IC constraint, lowering the interest rate they offer. Collecting information would be profitable whenever \( I(A_{t-})\frac{r_{P} - r_{L}}{r_{a}} \geq \frac{c}{(1-P_{L})P_{c}} \). It is easy to verify that \( I(A_{t-})\frac{r_{P} - r_{L}}{r_{a}} \geq
The figure shows the interest rates implied by the signal and risk shifting IC constraints. The annual based parameter choice is reported in table 1. See Appendix.

is a strictly concave function starting from 0 and from assumption 3 one can conclude $I(A_C) \frac{r_p - r_l}{r_a} > \frac{e}{(1 - P_l)p_{bank}}$. Thus formal lenders find it profitable to lend with information technology whenever $A_B \leq A_t^\leq A_C$, where $A_B = \inf\{A : I(A) \frac{r_p - r_l}{r_a} = \frac{e}{(1 - P_l)p_{bank}}\}$. So for formal lenders, they would start to gather information whenever $A_t \geq A_B$ and stop doing that whenever $A_t \geq A_C$.

When $A_t^\leq A_B$, formal lenders may gather information and charge a interest rate higher than $r_{PS}(A_t^\leq)$, or do not exert effort and just charge $r_{PR}(A_t^\leq)$. The following lemma ensures that those strategies are undesirable on the equilibrium path.

**Lemma 3.** Given assumption 3, $I(A_B) \frac{r_p(A_B) - r_l}{r_a} \leq \frac{e}{(1 - P_l)p_{bank}}$ and $r_{PR}(A_B) \geq \pi$, so formal lenders won’t lend to the firm whenever $A_t^\leq < A_B$.

**Proof.** See Appendix.

To summarize, formal lenders would not lend to the firm when its size is below $A_B$ because lending profit cannot compensate for the monitoring cost. When the size of the
firm is sufficiently large ($A > A_C$), collateral can fully show the quality of projects and formal lenders would rely on public information to lend, suggesting that the firm gets access to the corporate bonds market.

4.3 Firm Dynamic

In this model, the size of a firm can increase by both type $G$ project success and cash flow reinvestment. Since the arrival of a type $G$ project is exogenous, firm growth by project is out of the entrepreneur’s control. On the other hand, the cash flow reinvestment is controlled by the entrepreneur. So here in the basic model I refer to the firm’s cash flow reinvestment policy as its growth decision. In section 8 I study an extension that allows the entrepreneur to liquidate some of its assets, and in that case the entrepreneur can also control how much project profit to reinvest, and the main result is similar to the basic model.

Given the evolution of the firm’s financial growth cycle, one can back out firm value function and associated cash flow reinvestment policies. Similar to the analysis above, I start with the final stage when the firm gains access to formal lending channels.

To pin down the constant $Z_1$, one only needs to find the proper upper boundary condition. Since whenever $r_{RS} > r_L$, equation (14) shows that the firm would always want to expand until it faces the lowest interest rate $r_L$. A natural candidate for boundary condition is when the firm reaches the lowest possible interest rate $r_{PS}(A) = r_L$, denote the size of the firm when it reaches that point as $A$. Since the firm would then pay out all cash flows as dividends, the value of the firm is:

$$P_B(V_F(A) + I(A)\frac{\pi - r_L}{r_a}) = V_F(A);$$  \hspace{1cm} (17)

However, the value of ae firm with size $A$ is difficult to solve explicitly and thus there is no reduced form solution for $Z_1$. Since it just functions as a scale factor, the exact value of $Z_1$ would not change the firm dynamic. In section 8 I study an extension in which the firm can liquidate some of its assets, and in that case $Z_1$ can be solved explicitly. So the
firm’s value function is $V_F(A) = Z_1 A^{K_1}$ when $A \in [A_B, \overline{A}]$. The firm would reinvest all its cash flows to expand when $A \in [A_B, \overline{A}]$. When $A > \overline{A}$, the firm would pay out all cash flows as dividends.\footnote{There is another case in which the firm may still want to expand since expansion increase the scale of projects in the future. In that case, the upper bound would be slightly different. For simplicity I only focus on the basic case.} Figure 5 illustrates the firm’s value function and the evolution of its external financing environment.

When $A \in [0, A_B]$, the firm would rely solely on informal lending. The following lemma plays an important role to pin down the firm’s value function:

**Lemma 4.** The firm’s value function $V_F(A)$ is strictly increasing and continuous.

**Proof.** For any $A_1 > A_2$, the firm with size $A_1$ can always copy project choice and cash flow reinvestment strategy from the firm with size $A_2$, and commit all its assets as collateral, and accept the same interest rate as the firm with size $A_2$. It would receive strictly more cash flows and have higher evaluation. For any $A > 0, \forall \epsilon \in (0, A)$, consider $A' = A \left( \frac{V_F(A) - \epsilon}{V_F(A)} \right)^{1/\alpha}$. The firm can always choose not to do any project and only reinvest its cash flow until it reaches $A$, and its value would be $V(A) - \epsilon$. Then for all $A_t \in [A', A])$, $V_F(A_t) \geq V_F(A) - \epsilon$.

Given lemma 4, the boundary condition is given by $V_F(A_B) = Z_1 A^{K_1}$, where $A_B$ is determined by:

$$\frac{c}{(1 - P_I)P_{bank}} = \gamma A_B^{a} \pi - \frac{r_P}{r_a} - \left( (P_B)^{- \frac{1}{K_1}} - 1 \right) A_B; \quad (18)$$

Which stated that at $A_B$ the expected lending profit just compensates for the information cost. Suppose a firm without access to bank loans experiences both cash flows reinvestment and cash flows dividend stages, there should exists at least one point $A_I$ such that $V_F(A_I) = Z_0 A_I^{K_0} = \frac{1}{K_0} A_I$, then it would be the case that $V_F'(A_I) = Z_0 K_0 A_I^{K_0 - 1} = \frac{1}{K_0} A_I K_0 A_I^{1-K_0} = 1$. Since $Z_0 A_I^{K_0}$ is a strictly convex function, and it only holds when the firm is willing to reinvest its cash flows, it must be the case that $A_I$ must be the starting
point of a stage of cash flows reinvestment.\textsuperscript{9} Since $A_I$ cannot be the ending point of a cash flow reinvestment stage, it must be the case that there exists a unique $A_I < A_B$ such that the firm would reinvest its cash flows when $A_I \leq A_t^- \leq A_B$ and would pay out all cash flows as dividends when $A < A_I$. Then the firm’s value function can be described as

$$V_F(A_B) = Z_1(A_B)^{K_1} = Z_0(A_B)^{K_0};$$

and

$$Z_0 A_I^{K_0} = \frac{1}{K_0} A_I;$$

Given $A_B$, one can solve $Z_0 = Z_1(A_B)^{K_1-K_0}$ and $A_I = (Z_0 K_0)^{-\frac{1}{K_0-K_1}}$. Figure 6 illustrate the firm’s value function and the dynamic of its cash flow reinvestment policy.

The firm’s value function is $V_F(A_t) = \frac{1}{K_0} A_t$ when $A_t < A_I$, and all daily operating cash flows would be paid as dividends. When $A_I \leq A_t \leq A_B$, firm’s value is $V_F(A_t) = K_0 (A_t)^{K_0}$, and it would reinvest all daily operating cash flows to expand.

4.4 Lending Policies

With the firm’s value function and reinvestment strategy, one can now solve the optimal lending policies for different lending parties.

As discussed above, formal lenders’ monitoring decisions can be described as \{${A_B, A_C}$\}, where $A_B$ is the threshold size of the firm for formal lenders to start gathering information, and $A_C$ is the end point. When $A_t^- \geq A_C$, the firm can get funding through the corporate bonds market. When $A_t^- < A_B$, only informal lending channel is available and the firm is financially constrained in the sense that not every type $G$ project can be financed.

All lending parties share the same optimal lending contracts \{$(A_P, r_P)$\}. As lemma 2 states, when $A_t^- \leq \overline{A}$ they would ask $A_P = A_t^-$ and charge the interest rate that satisfies

\textsuperscript{9}Similarly, one can verify that $Z_0 K_0 A_R^{K_1-1} > Z_1 K_1 A_R^{K_1-1} > 1$, so $A_I < A_B$.\textsuperscript{9}
the signal IC constraint:

\[ V_F(A_t^-) = P_B V_F(A_t^- + I(A_t^-) \frac{\pi - r_P}{r_a}); \] (21)

when \( A_t^->\bar{A} \), they would charge \( r_P = r_L \) and the collateral requirement satisfies:

\[ V_F(A_t^-) = P_B [I(A_t^-) \frac{\pi - r_L}{r_a} + V_F(A_t^-)] + (1 - P_B)V_F(A_t^- - A_P); \] (22)

A more detailed description of the firm’s financial growth cycle is reported in Table 3.

Now one can verify that the conjecture is true with assumption 3. And the following proposition shows that this is the unique separating markov equilibrium.

**Proposition 1.** *Under assumption 3, the signal IC constraint is always binding in the unique separating markov equilibrium.*

*Proof.* See Appendix. \( \Box \)

## 5 Collateral Constraints and Firm Growth Decisions

In the equilibrium, the evolution of the firm’s external financial environment stems from the fact that the firm only has limited assets that can be posted as collateral. This collateral constraint limits the firm’s ability to fully signal the quality of its projects, resulting in high cost of capital and limited access to potential funding sources. Expansion would loose the collateral constraint, lowering its cost of capital in the future and broadening its access to funding sources. To evaluate those effects, I first shut down the evolution of access to potential lending channels\(^{10}\) and ask the following question: At what discount rate \( r_{benchmark} \) would the firm be indifferent between reinvesting its cash flow or paying dividends, that is to say, \( V_F(A) = A? \)

\(^{10}\)That is to say, assuming that the firm would only have access to informal leading forever, or it would have access to formal lending channels and more collateral can always lower the associated interest rate.
When the firm has access to formal lending channels, the HJB equation, conditional on \( V_F(A) = A \), becomes

\[
rA = r_a A + \lambda \left( \frac{1}{P_B} - 1 \right) A; \tag{23}
\]

Hence:

\[
r_{\text{benchmark}} = r_a + \lambda \left( \frac{P_G}{P_B} - 1 \right) A; \tag{24}
\]

Equation 24 implies that after shutting down the evolution of the firm's external financing environment, the benefit of cash flow reinvestment can be decomposed into two parts. More assets not only generate more cash flow over time, but also enable the firm to post more collateral, lowering its future cost of capital. By assumption 2, \( r < r_{\text{benchmark}} \). When all type \( G \) projects would be taken, the benefit of the lower cost of capital is fully realized. The firm finds accumulating assets attractive as long as expansion lowers the cost of capital. Thus when the firm gains access to formal lending channels, it would reinvest all cash flows to expand until its funding cost reaches the lowest possible rate \( r_L \). Notice that \( r_{\text{benchmark}} \) is a constant, so without considering the evolution of the firm’s external financing environment, firm value function \( V_F(A) \) should be linear. However, the firm expects that the interest rate \( r_P \) reaches the lowest possible rate \( r_a \) when \( A_{t-} \geq \overline{A} \), and there would be no more benefit of lower cost of capital. So when \( A \in [A_B, \overline{A}] \), as \( A \) increases, the firm is reaching the end of the benefit of the lower cost of capital, and thus the marginal return of expansion is decreasing and the firm value function should be concave. At \( A = \overline{A} \), the interest rate reaches the lower bound and there would be no more benefit of future lower cost of capital. Taking that into account, the firm optimally chooses to stop cash flow reinvestment.

Similarly, when \( A \in (0, A_B] \), assuming that the firm can only have access to the
informal lending channel, the HJB equation implies:

\[ r_{\text{benchmark}} = \underbrace{r_a}_{\text{Cash flows}} + \underbrace{P_I \lambda \left( \frac{P_G}{P_B} - 1 \right)}_{\text{Lower cost of capital}} \]  \hspace{1cm} (25)

Similar to equation 24, shutting down the evolution of the firm’s external financing environment, the benefit of expansion comes from both more cash flow and lower cost of capital. In this case because the firm can only rely on the informal lending channel, type G projects can only be financed with probability \( P_I \). Since the benefit of the lower cost of capital is realized when the firm’s project is actually financed, it is limited by the probability \( P_I \). By assumption 1, \( r > r_{\text{benchmark}} \). The firm finds reinvesting asset marginally costly because only a fraction of future projects would be financed. Again, considering the evolution of the firm’s external financing environment, firm value function \( V_F(A) \) should be linear. Expecting to gain access to formal lending at \( A_B \), when \( A \in [A_I, A_B] \), as \( A \) increases, the firm is reaching the threshold of the bank lending stage, and thus the marginal return of expansion is increasing and the firm value function becomes convex. When the firm is too small, that is to say, \( A \in (0, A_I] \), it finds the threshold \( A_B \) too far to reach and optimally chooses not to reinvest its daily operating cash flows.

The endogenous collateral constraint and its effect on firm growth decisions suggest that when firms are small and severely financially constrained, it may find expansion too costly and optimally chooses to stay small. This ”stay small” result fits several empirical findings.

5.1 Firm’s Financial Growth Cycle and the Growth Trap

In development economics, empirical studies (Banerjee and Duflo, 2005; Mel et al., 2008; Banerjee and Duflo, 2014) find that even though the marginal gross return rates for small enterprises are fairly high, those SMEs often stay small and maintain low growth rates. For example, based on a directed lending program in India, Banerjee and Duflo (2014) estimate that the marginal return rates for average small firms in India is 105%.
This “growth trap” puzzle has received considerable attentions and the existing literature explains this by arguing that those SMEs are highly financially constrained and thus do not have enough funds to support firm expansion. However this theory cannot explain why those small enterprises cannot expand with their own profits.

My paper contributes to the “growth trap” puzzle by offering an alternative explanation. Similar to existing literature, my model also attributes this “stay small” result to the external financial constraint faced by SMEs, but the underlying mechanism is different. Unlike the traditional view that financial constraints resulting in insufficient funds to support firm growth, in this paper the firm can always reinvest its cash flow to expand. This is realistic because extremely high rates of return imply that firms may generate a significant amount of net cash flow over time, which is a missing part in the traditional explanations. My model introduces an explanation that is consistent with the cash flow part. As discussed earlier, with the financial constraints, the firm is unable to finance all of its future projects, lowering the benefit of a lower cost of capital. Thus the firm may find expansion too costly when it is highly financially constrained, and would optimally pay out all cash flows as dividends.

This explanation has a fairly different policy implication. Here ”stay small” is not a suboptimal result due to the lack of funds to invest, but an optimal decision by the firm because the benefit of expansion is limited by the financial constraints. While the traditional view suggests that small firms may need some funding to support their expansion, my model emphasizes loosing firms financial constraints to increase the benefit of the lower cost of capital, giving firms incentives to grow.

5.2 Firm’s Financial Growth Cycle and Firm Size Distribution

The ”stay small” result is also related to the academic discussion on firm size distribution. It is well documented that the size distribution of firms is heavily right skewed Cabrai and Mata (2003); Angelini and Generale (2008). Figure 6(a) illustrates the empirical observation of firm size distribution. The existing literature(Hopenhayn, 1992; Cooley
and Quadrini, 1992) tries to explain this with some productivity technology shocks and market structure assumptions. While few firms that are lucky to receive some positive productivity technology shocks expand, most firms cannot expand due to some exogenous limitations on market entry and firm imitation behaviors. Some theoretical models (Cabrai and Mata, 2003; Cooley and Quadrini, 2003) incorporates financial constraints by introducing some exogenous fixed cost of financing, making financing too costly for small firms.

This paper highlights the importance of the evolution of a firm’s external debt financing environment on firm size distribution. My model introduces endogenous collateral constraint as the main factor to determine the firm’s growth decisions. Figure ?? shows the simulated firm size distribution and its evolution. Consistent with empirical observations, the size distribution is heavily skewed towards small firms. While most existing theories often rely on specific production technology shock or market structure assumptions to generate the firm size distribution, this financial environment channel may potentially provide a more general explanation to firm size distribution.

5.3 Firm’s Financial Growth Cycle and Job Creation

One reason for the importance of SMEs in academic research and policy debates is that small businesses create a large fraction of jobs (Neumark et al., 2011; Fort et al., 2013). Job creation is also a measure of firm growth. Recently there is a growing literature (Haltiwanger et al., 2013; Adelino et al., 2014) exploring the implication of firm age on job creation. As shown by figure 8(a), based on detailed datasets, they find that within the small businesses, the young firms contribute almost all the net creation of jobs, suggesting that many older small firms do not have high growth rates. This cross sectional evidence matches my model prediction that for many small firms the optimal decision is to follow policies that keep the firm small, so in any cross section of small firms of different ages, the highest growth rates will tend to be observed in younger firms. Figure 8(b) presents the simulated small firm growth rates based on ages. This implies
that if a firm has survived for years but still remains small, its optimal decision is to continue to stay small and the growth rate would be fairly low.

Furthermore, Haltiwanger et al. (2013) document an "up or not" dynamic among young firms. That is, conditional on survival, young firms either grow rapidly or exit. In later session I consider an extension in which the entrepreneur has the option to liquidate the firm’s assets. In that case, the entrepreneur will optimally choose to liquidate her firm when its size is too small and thus $V_F(A) \leq A$.

6 Risk Taking by Entrepreneurs

Earlier sections have demonstrated that the firm value function performs nonconcavity. The convex firm value function in the informal lending stage suggests that entrepreneurs may be locally risk-loving even though they are risk-neutral in terms of evaluating dividend flows.

In the basic model, the daily operation would generate a risk free cash flow. To analyze the locally risk loving behavior, here I consider an extension in which the entrepreneur can also choose to operate her firm in a risky way. Besides the cash flow $r_a A_t$, this risky strategy also gives the firm some growth opportunities that would increase the size of the firm from $A$ to $\eta A$, where $\eta > 1$. However it may also lead the firm to default. The probability of growth opportunities to arrive and the firm to default during the infinitesimal time interval $(t, t + dt]$ is $\lambda_G$ and $\lambda_B$, respectively. I assume that:

Assumption 4. $\lambda_G(\eta - 1) = \lambda_B$;

Assumption 4 suggests that the risky operation strategy and the original operation strategy generate the same expected return. If the firm implements the risky operation strategy in informal lending stage and is willing to reinvest, similar to the basic model, conjecture that the new firm value function is $V_F(A_t) = Z_0 A_t^{K'}$, then the HJB equation
becomes:

\[ rV_F(A_{t^-}) = r_a A_{t^-} V'_F(A_{t^-}) + \lambda_G(\eta K' - 1)V_F(A_{t^-}) - \lambda_B V_F(A_{t^-}) + P_I \lambda \left( \frac{1}{P_B} - 1 \right) V_F(A_{t^-}); \]  

(26)

Solving this ODE one finds:

\[ V_F(A_t) = Z_0 A_t^K'; \]  

(27)

Where \( K' = r - P_I \lambda (\frac{1}{P_B} - 1) - \lambda_G(\eta K' - 1) \). The conjecture is true if and only if \( K' = K_1 \), then \( K' \) is determined by \( r - P_I \lambda (\frac{1}{P_B} - 1) + \lambda_B > K' + \frac{\lambda G}{r_a} (\eta K' - 1) \). Given assumption 4, it is straightforward to see that \( 1 \leq K' \leq K_0 \). That is to say, with the risky operating strategy, the firm still finds expansion costly, but it is less costly than the original operating strategy. Since \( \lambda_G(\eta K' - 1) > \lambda_B \), the firm would take the risky operation strategy whenever it wants to reinvest the cash flow and only has access to the informal lending channel.

I have shown that the entrepreneur would be locally risk loving even though she might be risk-neutral in terms of evaluating dividend flows. Even though the risky operation strategy has the same expected return in term of market value of the firm assets, the firm value function convexity makes it attractive. This risk taking behavior delivers an interesting asset pricing implication. Entrepreneurial activity is risky and poorly diversified, and standard asset pricing models would suggest that entrepreneurial risk should be compensated by a significant premium in returns (in terms of market value of the firm assets). This model, however, implies a low entrepreneurial risk premium.

Empirical evidence finds that the premium to entrepreneurial activity is surprisingly low (Hamilton, 2000; Heaton and Lucas, 2000; Moskowitz and Vissing-Jorgensen, 2002). Most hypotheses that have been offered are based on the idea that entrepreneurs have a different set of preferences or beliefs (e.g., risk tolerance or overoptimism), and Manso (2014) explains this by introducing switching back to worker as a put option for the entrepreneur. Another interesting explanation is Vereshchagina and Hopenhayn (2009),
they assume the entrepreneur has a new production technology which is only available when the entrepreneur’s wealth reaches a certain threshold. This discrete choice of production technology introduces the nonconcavity in the value function, creating a locally risk taking incentive. Similar to Vereshchagina and Hopenhayn (2009), this paper also introduces risk shifting behavior as a result of nonconcavity in the firm value function, but through the evolution of the firm’s external debt financing environment, which is arguably a more general and natural channel.

This risk taking behavior also has some implications on firm size distribution across different industries. Industries may perform heterogeneously in terms of availability and capacity for risk taking behaviors. While in some industries like high technology it is easier to take more risk, it is relatively more difficult to take risks in some other industries like traditional manufacturing and service. Small firms in high technology industry may take more risk and are likely to be either successful or defaults. Figure 8 shows the firm size distributions for different risk taking industries. Taking risk would help very few firms become more successful, but at the cost that a large fraction of small firms fail and default.

7 Application: Firm Growth Rate as a Proxy for Firm Value

Given the importance of SMEs, there are a lot of policy debates and academic research focusing on how certain shock or policy reform would affect SMEs’ welfare. However, it is difficult to directly measure firm value of those non-public firms. Alternatively, empirical studies often employ some proxies such as firm growth rates to measure firm value. It is intuitive to claim that firms are better off if they have higher growth rates after some shock, and this proxy is popular in a large body of empirical assessments. Thus it is important to know how effectively firm growth rates can reflect firm value. The analytic framework in my paper enables one to investigate whether those popular proxies serve
as good measures for firm value. My model shows that growth rates and other related measures (return rates etc.) may not measure firm welfare properly.

The basic intuition is best illustrated by figure 3. While firm value solely depends on the level of the firm’s value function, firm growth rate is largely determined by its slope. The most important factor in firm growth rate is its funding cost. The funding cost, however, depends on the slope of the firm’s value function. To be more specific, when the slope of the firm value function is steep, the potential value added from reinvesting one dollar of project profit is high, making type B and type R projects more attractive. Taking that into account, lenders would ask for a high interest rate, resulting in a low project profit. Similarly, when the firm value function has a flat curve, the potential benefit from project profit reinvestment is low and the firm is less likely to take type B and type R projects, and lenders would ask a low interest rate, resulting in a high project profit. Thus firm value and firm growth rates reflect two different characteristics of the firm value function.

To better illustrate this point, I take Fisman and Love (2003) as an example. In the study the authors argue that since trade credit provides an alternative source of funds, industries with higher dependence on trade credit financing would be relatively better in
countries with weaker financial institutions. More specifically, they tested the hypothesis with the following regression:

\[ \text{Growth}_{ci} = \alpha_i + \zeta_c + \beta \text{Priv}_c(Apay/TA)_i + \epsilon_{ci}; \quad (28) \]

where \( c \) denotes country and \( i \) describes industry. \( Apay/TA \) is the proxy for trade credit dependence measured by ratio of accounts payable over total assets, and \( \text{Priv} \) is the proxy for financial market development measured by ratio of private domestic credit held by monetary authorities and depositary institutions scaled by GDP. They find that \( \beta \) is significant negative, suggesting that industries that are more dependent on trade credit will grow relatively faster in countries with less developed institutional finance. Based on this empirical test, they conclude that more available trade credit would make firms better off.

This story is fairly intuitive because more trade credit alleviates a firm’s financial constraints and makes it possible for more projects to be financed. Firms exhibit higher rates of growth for two reasons. The first is that more projects would be implemented and generate more cash flows. The second is that since firms know they have more growth opportunities in the future, taking risks to implement projects other than type \( G \) projects becomes less attractive and the lender would thus charge a lower interest rate. Both factors contribute to a flatter value function, implying a higher growth rate.

However, this flat shape does not necessarily imply a higher level of firm value. More supply of trade credit makes firms less likely to borrow from financial institutions, lowering expected lending profit. Taking that into account, banks would optimally delay their entry time. In other words, though firms are better off in the short run by having more projects financed by the informal lending channel, they may be worse off in the long run because the bank lending stage is delayed.

Following the original paper, I use the model to simulate a data set consisting of 6 countries and 5 industries. For each industry in each country, I randomly simulate 5000 different firms. Countries are different in terms of bank monitoring cost, while industries
have different probability for informal lending. Based on the simulated firm level data, I then run the following two regressions:

\[ \text{Growth}_{ci,f} = C + \alpha P_I + \zeta r_L + \beta (1 - P_I)c + \epsilon_{ci,f}; \]  
\[ \text{Value}_{ci,f} = C + \alpha P_I + \zeta r_L + \beta (1 - P_I)c + \epsilon_{ci,f}; \]

Where \( P_I \) is the probability for a firm to get informal lending and \( c \) is the information technology cost for banks. I use \( P_I \) to proxy the availability of trade credit and \( c \) to measure the development of financial institutions. The main results are reported in Table 1.

### Table 1: Firm Growth Rate and Firm Value

<table>
<thead>
<tr>
<th>Sample</th>
<th>Fisman and Love (2003)</th>
<th>Simulated Data</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Firm Growth Rates</td>
<td>Firm Growth Rates</td>
</tr>
<tr>
<td>( \beta )</td>
<td>-2.01</td>
<td>-0.0852</td>
</tr>
<tr>
<td></td>
<td>(0.597)</td>
<td>(0.0131)</td>
</tr>
<tr>
<td>( N )</td>
<td>1217</td>
<td>150000</td>
</tr>
<tr>
<td>Adjusted ( R^2 )</td>
<td>0.28</td>
<td>0.04</td>
</tr>
</tbody>
</table>

This table reports the key coefficient \( \beta \) in three different regressions. The first column represents the original result in Fisman and Love (2003), while the other two show the results based on simulated data. The regressions on simulated data have lower \( R^2 \) because the data is firm level data and the firm value function is a nonlinear function of firm size. In the simulation, countries is characterized by different information technology cost ranging from \([3, 6]\) and industries are different in terms of the informal lending probability \( P_I \in [0.1, 0.4] \).

As Table 1 shows, firms with more access to trade credit do exhibit higher growth rate, while its value may be lower. This example suggests that firm growth rate is not necessarily a good proxy for firm value. Empirical assessments with firm growth rates may produce misleading empirical results. Thus one should be cautious when interpreting those empirical findings.
8 Extensions and Discussions

8.1 General Project Payoff Structure

In the basic model I assume a simple lottery payoff structure. The model is also robust to more general payoff structure assumptions. To be more specific, let $F(x), x \in [0, \pi]$ be the CDF of standard project outcome with a good signal. Similar to the basic model, the CDF of standard project outcome without good signal $F_B(x)$ satisfies $F_B(x) = P_B F(x) + (1 - P_B)$. That is to say, without a good signal the project would fail with probability $1 - P_B$, and would generate exactly the same outcome as the project with a good signal with probability $P_B$. Taking the case when the firm gains access to formal lending channels and wants to reinvest its cash flows as an example, now one can rewrite firm’s HJB equation as

$$rV_F(A_{t-}) = r_a A_{t-} V_F'(A_{t-}) + \lambda E \int_{r_p}^\pi V_F(A_{t-} + I(A_{t-}) \frac{x - r_p}{r_a}) dF(x) - V_F(A_{t-}); \quad (31)$$

And the signal IC constraint becomes:

$$V_F(A_{t-}) = P_B E \int_{r_p}^\pi V_F(A_{t-} + I(A_{t-}) \frac{x - r_p}{r_a}) dF(x); \quad (32)$$

The expectation terms in both the HJB equation and the signal IC constraint are complicated to solve directly. However, based on the payoff structure assumption, one can still substitute the signal IC constraint into the HJB equation and get:

$$rV_F(A_{t-}) = r_a A_{t-} V_F'(A_{t-}) + \lambda \left( \frac{1}{P_B} - 1 \right) V_F(A_{t-}); \quad (33)$$

The new firm’s HJB equation is exactly the corresponding HJB equation in the basic model, so one can still solve firm value function, and the implied interest rate is determined by the new signal IC constraint (32).
8.2 Non Constant Outside Option

So far I have assumed that the entrepreneur’s outside option is a constant which is normalized to 0. Anecdotal evidence suggests that operating a larger firm may give the entrepreneur more social recognition and access to a broader network, increasing the value of her outside option. Results in this paper are robust to the non-constant outside option assumption. Denote $V_0(A)$ as the entrepreneur’s outside option value when her firm size is $A$. For simplicity, let $V_0(A) = A^{\alpha_0}$. Similar to the general project payoff structure extension, taking the case when the firm gains access to bank loans and wants to reinvest its cash flows as an example, then the firm HJB equation (10) becomes:

$$rV_F(A_{t-}) = V'_F(A_{t-})r_A + \lambda\left(\frac{1}{PB} - 1\right)[V_F(A_{t-}) - V_0(A_{t-})]; \quad (34)$$

One can still substitute the signal IC constraint into the firm HJB equation and the new ODE can be solved as:

$$V_F(A_t) = ZA^K_1 + \lambda\left(\frac{1}{PB} - 1\right)\frac{r_A}{\alpha_0 - K_1}A^{\alpha_0}; \quad (35)$$

Now the new firm value function has one more term for the outside option. Similarly, when only the informal lending channel is available, the value function becomes $V_F(A_t) = ZA^K_0 + \frac{P_I\lambda(PB - 1)}{r_\alpha(\alpha_0 - K_0)}A^{\alpha_0}$. As long as $\alpha_0 \in (K_1, 1]$, the value function would still be convex in the informal lending stage and would be concave when the firm can borrow from formal lenders. Thus the non-constant outside option assumption doesn’t qualitatively change the firm value function.

8.3 Retaining Cash

Firms that are financially constrained often try to save some cash to finance their future projects. In the basic model the firm can either reinvest its cash flow or pay it out as dividends. In this extension I allow the firm to save some cash within the firm and use
it to finance the next project. However, the following analysis would show that this does not change the equilibrium and firm cash flow policy.

The main argument is that saving cash is strictly dominated by cash flow reinvestment strategy. Now denote $C_t$ as the total amount of cash saved in the firm, and $dC_t$ is the cash flow that saved within the firm at time $t$. Now without loss of generality, suppose the firm decides to start to save cash at time $t_0$ following some cash saving policy $C^*$. Now consider the following alternative strategy: reinvest all cash flows that would be saved according to $C^*$, and besides that also follow the firm’s cash flow reinvest policy. That is to say, the new cash flow reinvestment $I^{Cash}$ satisfies $I^{Cash}_t = I^{*c}_t + dC^*_t$, where $I^{*c}$ is the firm’s cash flow reinvestment policy when it implements $C^*$. Now consider at some time $t_1$, the firm wants to finance a new project. Similar to the basic model, the binding signal IC constraint implies that for each project the value of the firm would have a jump of size $\frac{1}{P_B} - 1$, independent of the firm’s cash saving strategy. Also, the alternative strategy always pays out $r_a C^*_t$ additional dividend flow and thus dominates the cash saving strategy.

8.4 Asset Liquidation

In the basic model the firm always keeps its assets and the size of the firm only decreases when a project fails and the collateral is claimed by the lender. However, firms in the real world may liquidate some of their assets so it is useful to understand how the liquidation option may change the results. In this extension I assume that there exist some outside buyers that can also operate the firm’s asset, generating $r_a A$ cash flows overtime. Those buyers have a discount rate $r_a$, so the firm can liquidate its asset at its face value $A$. Now when the firm gains access to formal lending and is willing to reinvest its cash flow, the signal IC constraint becomes:

$$V_F(A_t) = P_B[V_F(A_t- + \gamma(A_t-) \frac{\pi - r_F}{r_a} - D_t) + D_t];$$

(36)
Where $D_t$ is the amount of asset the firm wants to liquidate when the project succeeds. Substituting the signal IC constraint into the firm HJB equation, one gets:

$$rV_F(A_{t-}) = V_F'(A_{t-})r_a A_{t-} + \lambda \left( \frac{1}{P_B} - 1 \right) V_F(A_{t-});$$

(37)

Which is the same as in the basic model. Similarly, one can solve the firm value function in different scenarios. The firm would only liquidate its asset whenever $V_F'(A) \leq 1$. So the firm would liquidate some of its asset when $A \geq \overline{A}$ and the amount of asset liquidation is $A - \overline{A}$. Also it would liquidate all of its asset when $V_F(A) \leq A$, that is to say, when the firm is sufficiently small.

When the firm can liquidate asset, it would liquidate all project profits when $A \geq \overline{A}$, the one gets the boundary condition:

$$V_F(\overline{A}) = \frac{r_a \overline{A} + \lambda I(\overline{A}) \frac{\pi - r_L}{r}}{r};$$

(38)

Equations (38) and (17) thus imply that

$$\overline{A} = \left[ \frac{P_B}{1 - P_B} \frac{r - \lambda \left( \frac{1}{P_B} - 1 \right)}{r_a} \frac{\pi - r_L}{r} \right]^{1/\gamma}$$

(39)

and $Z_1$ is:

$$Z_1 = \gamma \frac{\pi - r_L}{r_a} \frac{P_B}{1 - P_B} A^{a - k_1}.$$  

(40)

### 8.5 Risk-Averse Entrepreneur

Since entrepreneurial activity is risky and poorly diversified, it is reasonable to consider the case where the entrepreneur is risk-averse. In the risk aversion case, the entrepreneur would always pay out a non-zero cash flow as dividends overtime, and the optimal cash flow reinvestment decision is no longer a corner solution. This feature suggests that
Figure 4: Firm value with risk-averse entrepreneur. The risk aversion parameter is $\beta = 1$.

one cannot categories the firm into several scenarios, making it difficult to solve the model explicitly. However, one can solve the model numerically. Figure 4 illustrates the numerical solution for the firm value function when the entrepreneur has CARA utility function.

As shown in figure 4, there is some subtle difference between the case of the risk-neutral entrepreneur and the case of the risk-averse entrepreneur. In the risky-neutral case, since marginal utility for additional consumption flow is a constant, the main factor is how additional assets would affect the firm’s external funding environments. Since expansion is costly before the firm gains access to bank loans, the firm value function at the informal lending stage performs convexity, and the firm would optimally choose to stay small whenever its size is below certain threshold $V_{Ic}$. When the entrepreneur is risk-averse, the marginal return for additional consumption flow is high and this gives the entrepreneur incentive to expands its business, however when the firm is not too small, the marginal return is diminishing and the cost of capital effect becomes more important, making the firm optimally choose to stay small. So the firm value function is concave in
the informal lending stage. However, similarly to the risk-neutral case, the threshold of formal lending introduces a different external financing environment, generating a kink in the firm value function. The firm value function thus exhibits nonconcavity around that threshold and the firm is risk-loving even when it is risk-averse in terms of evaluating its consumption flow.

9 Conclusion

This paper studies the evolution of the external financial environment for SMEs at different growth stages. Asymmetric information on the quality of projects makes external financing costly for firms. Collateral can mitigate this problem, but its availability is determined by the size of the firm. This natural constraint endogenously determines growth patterns and influences firms’ growth decisions. As a firm grows, more collateral becomes available and this extends its access to external financing channels and lowers its cost of capital. Growth decisions are affected by how effective more collateral is in lowering the cost of capital. The firm may optimally choose to stay small when it is financially constrained and the firm is unlikely or it would take too much time to grow to a size that would allow it to borrow from formal lenders. When the firm’s size is close to the level needed for formal lending, it has a strong incentive to grow and may become locally risk loving. Insights about the role of collateral and firm size play in the firm’s access for funding helps in understanding and evaluating some popular empirical proxies for firm value. I show that high growth rates or profit rates are not necessarily associated with high firm value.
References


Appendix A

Proof of Lemma 1

Proof. Because only a type $G$ project generates positive NPV, lenders only wants to finance type $G$ projects. Since the type $G$ projects only arrive with posision rate $\lambda$ while both type $R$ and type $B$ projects can be taken at anytime, in any pooling markov equilibrium, the entrepreneur would find type $B$ or type $R$ projects (when formal lenders choose not to monitor) not attractive (that is to say, at best be indifferent between taking or not taking the project). Otherwise the probability of type $G$ projects would be almost 0 and lenders would have no incentive to finance any project. Now consider an alternative firm projects implement strategy that the entrepreneur would never take type $B$ or type $R$ projects. Since lenders cannot offer more favorable loan terms (or the entrepreneur would find it attractive to take some negative NPV project), $\mathcal{L}$ would be the same. In the separating equilibrium lenders will get higher expected lending profit, which implies that the separating markov equilibrium is associated with a lower bank lending threshold $V_B$. For the firm, since it is always indifferent between taking or not taking some negative NPV projects, it is better off in the separating markov equilibrium because it is more likely to have all good projects financed in early stage. \hfill $\blacksquare$

Proof of Lemma 2

Proof. Since type $G$ projects would be successful for sure, collateral is not costly for firms with type $G$ projects, so from equation (5) they are willing to commit higher collateral requirement to lower the interest rate. Lenders thus would charge the highest possible collateral requirement $A_p = A_t^\ast$. \hfill $\blacksquare$
**Proof of Lemma 3**

Proof. At the ending boundary of banking stage $A_C$, each bank’s profit from lending is

$$\gamma \frac{P_3 \pi - r_L}{\alpha - \alpha} \left[ \frac{I(\pi_R - P)}{P - P_B} \right] \frac{1}{\alpha - \alpha}.$$  

So when $c \leq \frac{\lambda \gamma \frac{P_3 \pi - r_L}{\alpha - \alpha} \left[ \frac{I(\pi_R - P)}{P - P_B} \right] \frac{1}{\alpha - \alpha}}{P - P_B}$

banks find informational lending profitable and would do so.

Another possibility is that before the bank lending stage, banks may just charge $r_{PR}$. However, when $c \leq \lambda \gamma \frac{P_3 \pi - r_L}{\alpha - \alpha} \left[ \frac{I(\pi_R - P)}{P - P_B} \right] \frac{1}{\alpha - \alpha}$, it is easy to verify that $r_{PR} \geq \pi$ whenever $A \leq A_{RP}$. Since $A_B < A_{RP}$, interest rates offered by banks are always too high for firms before the banking stage.

**Proof of Proposition 1**

Proof. Since all lenders compete with each other, and the signal IC constraint should always hold. The only possible case is that when $A_t < A_B$, the banks would charge interest rate such that the risk shifting IC constraint is binding. Under assumption 3, lemma 3 shows that the implied interest rate is bigger than the standard project success cash flow $\pi$, so the firm would have no incentive to take type $G$ projects with those loan contracts.
### Appendix B Tables

#### Table 2: Parameter Choice

<table>
<thead>
<tr>
<th>Variables</th>
<th>Value</th>
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<tbody>
<tr>
<td>$r$</td>
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<td>$r_L$</td>
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</tr>
<tr>
<td>$r_a$</td>
<td>0.04</td>
</tr>
<tr>
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<td>$P_r$</td>
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<td>$P_t$</td>
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<tr>
<td>$P_c$</td>
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</tr>
<tr>
<td>$c$</td>
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</tr>
<tr>
<td>$\alpha$</td>
<td>0.35</td>
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</tbody>
</table>

This table reports the value choice of variables used in the quantitative simulations.

#### Table 3: Firm Financial Growth Cycle

<table>
<thead>
<tr>
<th>Stage</th>
<th>Size</th>
<th>Formal Lending</th>
<th>Cash Flow</th>
<th>$V_F$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Growth by Project</td>
<td>$[0, A_{Ic}]$</td>
<td>No</td>
<td>Dividend</td>
<td>$\frac{1}{K_0} A$</td>
</tr>
<tr>
<td>Growth(Informal)</td>
<td>$[A_{Ic}, A_B]$</td>
<td>No</td>
<td>Reinvestment</td>
<td>$Z_0 A^{K_0}$</td>
</tr>
<tr>
<td>Growth(Banks)</td>
<td>$[A_B, A_C]$</td>
<td>Bank Loans</td>
<td>Reinvestment</td>
<td>$Z_1 A^{K_1}$</td>
</tr>
<tr>
<td>Growth(Financial Market)</td>
<td>$[A_C, A]$</td>
<td>Financial Market</td>
<td>Reinvestment</td>
<td>$Z_1 A^{K_1}$</td>
</tr>
<tr>
<td>Mature</td>
<td>$[A, \infty)$</td>
<td>Financial Market</td>
<td>Dividend</td>
<td>$\frac{r_a A + \lambda I(A)(\pi - r_L)/r_a}{r}$</td>
</tr>
</tbody>
</table>
Appendix C Figures
Figure 5: Firm Value and Financial Growth Stages

The figure reports the evolution of the firm’s external financing environment. The informal lending stage refers to the scenario when the firm has to rely on informal lending channels, while the firm is able to borrow from banks when it is at the formal lending stage. The financial market stage is characterized as the stage when the firm can borrow when banks don’t need to gather information.
Figure 6: Firm Value and Growth Decision

The figure illustrates the firm’s growth decision dynamic. When additional asset can reduce its cost of capital, the firm would reinvest all its cash flows in the formal lending stage. In the informal lending stage, when the firm is not far away from the threshold to get access to formal lending, the firm would still reinvest. When the firm is too small, it would choose to pay out its cash flow as dividends.
Figure 6(a) reports the empirical firm size distribution in Angelini and Generale (2008). Based on firm level data from an Italy bank, Angelini and Generale (2008) document the firm size distribution in different firm age categories. Figure 6(b) shows the corresponding simulated firm size distribution. The parameter in the simulation is reported in table 2. Following empirical findings, in the simulation the initial firm size distribution is a truncated normal distribution with mean $A_i$ and variance 2, and the domain is $[0.5A_i, A_{max}]$.  

Figure 7: Firm Size Distribution and Firm ages
Figure 8 reports the simulated firm size distribution for firms with different operation strategies at age 25. The risky operation strategy is characterized by parameters $\eta = 1.5$, $\lambda_G = 0.1$ and $\lambda_B = 0.1$. 
Figure 9: Firm Size Distribution and Firm ages

Figure 8(a) reports the empirical job creation by firm ages in Haltiwanger et al. (2013). Figure 8(b) shows the corresponding simulated firm growth rates. The parameter choice is reported in table 2. In the simulation the initial firm growth is treated as a project success.