Accounting for Compensation: Dynamic Moral Hazard and Optimal Accruals*  

Jonathan Bonham†  Seung Y. Lee‡  

October 19, 2020

Abstract

We investigate the impact of accrual accounting policies on contracting and productive efficiency in a continuous-time moral hazard framework. An agent who controls unobservable fundamental performance is compensated via a contract written on cash flows, which contain timing errors, and accounting earnings, which correct timing errors at the expense of introducing estimation errors. Deferred compensation and accruals act as substitutes in solving the incentive problems created by timing errors in cash flows. Accruals are the primary control mechanism late in the agency relationship, when the most efficient accrual policy is time-invariant and standardized—it depends only on the controllability and measurability of fundamental performance as opposed to agency-specific parameters such as risk aversion and discount rates. By contrast, deferred compensation is the primary control mechanism early in the agency relationship, when the most efficient accrual policy changes over time, is non-standardized, and corrects fewer timing errors than the standardized policy.

*Preliminary draft. Please do not quote or circulate without permission. We thank participants at the 17th BYU Accounting Research Symposium for valuable comments and discussions.
†University of Chicago, Booth School of Business
‡University of Southern Denmark, Department of Business and Economics
1 Introduction

A hallmark of accrual accounting is its ability to recognize revenues and expenses in different periods than their associated cash inflows and outflows. Because cash flows often occur before or after the value-creating activities that generate them, accounting practice has traditionally aimed the accrual technology at recognizing earnings in the periods in which the associated cash flows are earned rather than the periods in which they occur in an effort to generate a superior measure of periodic fundamental performance.\(^1\) This aim is almost always tempered by a requirement that the cash flows in question be amenable to reliable estimation before any accounting-based income shifting is permitted.\(^2\) In this paper we provide formal theoretical support for this generally-accepted accounting practice in a setting where the primary use of performance measures is in management compensation contracts. We also rationalize non-GAAP performance measure customization by the firm near the beginning of management’s employment horizon, consistent with recent empirical findings.\(^3\)

We draw our conclusions from a novel dynamic framework that explicitly models the role of accounting accruals in correcting timing errors in cash flows. Our framework utilizes the institutional property that cash flows and earnings represent noisy measures of the same underlying economic construct, fundamental performance, which represents the present value of cash flows generated by the agent’s endogenous effort choice at each instant in time. Cash flows contain a timing error that represents the mismatch between the timing of cash realizations and the underlying performance that generates them. Accruals aim to correct (a portion of) these timing errors, but can do so only at the expense of introducing estimation errors since fundamentals can only be imperfectly measured. Because fundamentals are inherently unobservable, the principal and the agent can only use cash flows and earnings when recursively updating their beliefs about past fundamental performance, and the contract can only reference these measures when determining the agent’s compensation.

\(^1\)For example, the Financial Accounting Standards Board (FASB) defines revenues as “inflows or other enhancements of assets of an entity or settlements of its liabilities (or a combination of both) from delivering or producing goods, rendering services, or other activities...” (FASB 1985 par. 78). In particular, the definition is focused is on the activities that contribute to net assets rather than the cash flows that arise from them.

\(^2\)The FASB states that “[t]o be included in a particular set of financial statements, an item must not only qualify under the definition of an element but also must meet criteria for recognition and have a relevant attribute (or surrogate for it) that is capable of reasonably reliable measurement or estimate” (FASB 1985 par. 23). This principle is reflected, for example, in the immediate expensing of R&D due to the difficulty in estimating its future benefits.

\(^3\)Curtis, Li, and Patrick (2018) and Bloomfield, Gipper, Kepler, and Tsui (2020) respectively show that non-GAAP adjustments and certain accrual omissions are more prevalent in management compensation contracts early in the manager’s tenure.
The foundational insight in our paper is that deferred compensation and accruals act as substitutes in resolving the incentive problems created by cash flow timing errors. Specifically, deferring compensation allows any timing errors in cash flows to reverse before the agent is paid, whereas accruals offset these timing errors explicitly in short-term contracts written on current earnings. Each approach has its own deficiencies: deferring compensation exposes the agent to fundamental risk as the vesting date is pushed into the future, whereas accruals introduce estimation error into current earnings. As such, the relative cost of these control mechanisms depends on the severity of the incentive problems created by estimation errors. We show that estimation error is especially harmful early in the manager’s employment, when the manager’s short performance history makes it easy to shirk unnoticed. As a result, the principal prefers to solve the incentive problem by deferring compensation early in the agency relationship and by relying on accruals later on.

Furthermore, we show that the optimal accrual policy to use for incentive purposes late in the agency relationship is stationary and standardized: it corrects a constant amount of timing error at each point in time and depends only on the volatility of fundamentals, cash flow timing error, and accounting estimation error, as opposed to agency-specific parameters such as risk aversion and discount rates. This policy prescribes that more timing error be corrected when fundamentals are less volatile (for most parameter values) and when accruals negate more error than they create. This resembles the use of a standardized accrual policy (i.e., GAAP) that conditions income recognition only on the controllability and measurability of fundamentals. By contrast, the optimal accrual policy early in the agency relationship is non-stationary and non-standardized: it is characterized by dynamic adjustments to the standardized policy that depend heavily on agency-specific parameters. Notably, this non-standardized accrual policy always corrects less timing error than does the standardized policy, analogous to the exclusion of non-cash revenue and/or expense items from non-GAAP earnings.

Our model is closely related to the accrual accounting model of Nikolaev (2018), which also takes the perspective that the primary role of accrual accounting is to facilitate performance measurement by correcting the timing errors in cash flows. Nikolaev (2018) formalizes the performance measurement role of accruals by separating them into performance and estimation error components, which are co-mingled in conventional proxies for accounting quality. Nikolaev (2018) then uses this representation to construct a novel empirical measure of accounting quality. While fundamentally similar in its structure, our model is distinct from that of Nikolaev (2018) in several notable ways. First, we generalize our accrual repre-

---

4See also Choi (2018), which takes a similar learning-based perspective of accounting as we do.
5For instance, Dechow, Kothari, and Watts (1998) and Dechow and Dichev (2002).
sentation to incorporate any deterministic accrual reversal horizon, rather than restricting attention to working capital accruals that reverse in the short run. Second, our focus is on the optimal degree of timing error correction when the accrual policy can be chosen strategically in a principal-agent context. Along this dimension, our model can incorporate the Nikolaev (2018) accruals model as a special case in which all timing errors are fully corrected. Finally, we abstract away from the discrete nature of accounting-based performance measurement that is present in Nikolaev (2018), and instead couch our model in continuous time to take advantage of the flexibility and analytical tractability that this specification affords. This abstraction allows us to consider fully-repeated strategic interactions rather than one-shot or exogenously-imposed short-run relationships, which is crucial for the study of inherently dynamic accrual policies.

Our paper contributes to a more general body of literature that studies the effects of accrual policies on managerial production decisions and contracting efficiency. Several related papers following Rogerson (1997) and Reichelstein (1997, 2000) study goal-congruent performance measures that induce a privately-informed agent to undertake all positive and forego all negative NPV projects when faced with contracts that are increasing in reported performance in every period. These papers broadly support the efficiency of residual income paired with an accrual policy that annuitizes each project’s NPV over the investment horizon, which is loosely consistent with correcting all timing errors in cash flows. Wagenhofer (2003) also studies optimal accrual policies in a dynamic moral hazard setting, and shows how accruing the expected return from investment to the period in which the investment is made can motivate efficient investment while shielding the agent from risk in the future. Whereas the contracts in Wagenhofer (2003) are constrained to be linear and short-term, Drymiotes and Hemmer (2013) show that risk-sharing benefits can also arise from stochastic accrual reversals in long-term nonlinear contracts. Finally, Glover and Lin (2018) examine an asymmetric accrual policy in which losses are accrued and gains are deferred in a limited liability setting, showing that this conservative policy reduces the agent’s rents. Apart from many technical differences between our model and those just mentioned, a primary qualitative distinction is our focus on the inherent trade-off between timing and estimation errors when deciding whether to make an accrual, and as a result our conclusions differ significantly from those obtained in this literature.

Our paper is also related to the dynamic contracting literature that incorporates uncertainty over fundamentals in moral hazard problems. Our framework represents a departure from these studies, as it endogenizes fundamental performance and formalizes the mapping

---

between fundamentals, cash flows, and accounting earnings. Cash flow timing errors and accounting estimation errors in our model reverse over time; the nature of these reversals utilizes the technology of long-termism in Sannikov (2014), in which current effort affects current and future output levels.\footnote{Marinovic and Varas (2019) and Gryglewicz, Mayer, and Morellec (2020) also address the provision of long-term incentives in continuous-time contracting settings.} A related study to ours is Lee (2020), where the marginal impact of accounting information is to increase the efficiency of the learning process over exogenous firm fundamentals, which is analogous to the role of accounting information in our model. However, to our knowledge, we are the first to formalize the intertemporal relationship between cash flows and earnings by incorporating accrual accounting and the correction of cash flow timing errors in a fully dynamic setting with moral hazard and learning.

The remainder of the paper is organized as follows. Section 2 sets up our dynamic model of accounting accruals. Section 3 applies our model to a principal-agent setting under uncertainty and formally examines the choice of accrual accounting policies in a setting with dynamic moral hazard. Finally, Section 4 concludes.

2 A Dynamic Framework of Accounting Accruals

Consider an infinite-horizon model where the firm’s economic performance is subject to measurement over time. Time is continuous and indexed by $t \in [0, \infty)$, where the infinite time horizon establishes the firm as a going concern. Fix a complete probability space under which $P$ is a standard probability measure.

While underlying fundamental performance is inherently unobservable, it is imperfectly reflected by two observable noisy signals: cash flows and accounting earnings. Because fundamental performance is defined as the present value of cash flows generated at any given instant in time, the two constructs are closely related. However, cash flows tend to occur in different periods than do the activities that generate them, thereby creating temporary disagreements that we refer to as timing errors. By contrast, accounting earnings are generated by an accrual policy that reverses a portion of the timing errors in cash flows at the expense of introducing estimation errors that arise from the unobservability of fundamental performance. Because cash flow timing errors eventually reverse, estimation errors are also ex post observable and therefore reverse as well. As a result of these error reversals, both aggregate cash flows and aggregate accounting earnings converge to aggregate fundamental performance in the long run.
2.1 Cash Flows and Timing Errors

Let \( \pi = \{\pi_t\}_{t \geq 0} \) be a stochastic process representing cumulative fundamental performance, i.e., the present value of the firm’s cash flows over time. Relatedly, let the differential \( d\pi_t \) represent time-\( t \) fundamental performance, i.e., the change in the present value of the firm’s cash flows at time \( t \). For now, assume \( \pi \) follows an arbitrary Itô diffusion process (we will model its specific law of motion in the next section). The firm’s cash balance \( C = \{C_t\}_{t \geq 0} \) provides a noisy signal of aggregate fundamental performance subject to timing errors \( \theta = \{\theta_t\}_{t \geq 0} \). Formally, the cash balance satisfies

\[
C_t = \pi_t + \theta_t, \tag{1}
\]

where \( C_0 = \pi_0 + \theta_0 \in \mathbb{R} \), the differential \( dC_t \) is the time-\( t \) cash flow, and the differential \( d\theta_t \) is the time-\( t \) incremental timing error. Let \( \mathcal{F}^C = \{\mathcal{F}^C_t\}_{t \geq 0} \) be the augmented filtration of the \( \sigma \)-fields generated by the sample paths of \( C \) satisfying the usual conditions.\(^8\)

Analogous to Nikolaev (2018), the timing errors represent the misallocation of performance over time that manifests in cash flows. For example, a positive timing error at time \( t \), \( d\theta_t > 0 \), could manifest as the result of unearned revenue that must be serviced by way of partially-offsetting cash outflows in the future, which would cause the present value of all cash flows generated by this transaction to be less than the unearned revenue collected at time \( t \). By contrast, a negative timing error at time \( t \), \( d\theta_t < 0 \), could manifest as the result of a cash investment that generates future cash inflows, thereby causing the NPV of cash flows generated at time \( t \) to be greater than the cash outflow at time \( t \).

To capture timing error reversals, we constrain timing errors to evolve over time according to the following stochastic differential equation (SDE):

\[
d\theta_t = (\mu_\theta - \kappa \Theta_t) \, dt + \sigma_\theta dB^\theta_t, \tag{2}
\]

where \( \theta_0 \in \mathbb{R} \), \( \mu_\theta \) is the expected timing error (e.g., \( \mu_\theta > 0 \) for unearned revenues and \( \mu_\theta < 0 \) for cash investments),

\[
\Theta_t = \int_0^t e^{-\kappa(t-s)} \theta_s ds
\]

is the depreciated stock of past timing errors, \( \kappa \in [0, 1] \) is a constant representing the reversal rate of these timing errors, \( \sigma_\theta > 0 \) is the volatility of cash flows, and \( B^\theta = \{B^\theta_t\}_{t \geq 0} \) is a Brownian motion under \( P \). In other words, the reversal of the timing error follows a process that mean-reverts to zero at the rate \( \kappa \). For instance, with unearned revenue, \( \kappa \) can

\(^8\)The usual conditions are outlined in Karatzas and Shreve (1998), Definition 1.2.25.
represent the rate at which performance is delivered.

Note that \( \kappa \int_0^\infty e^{-\kappa t} dt = 1 \) which, as we will formally show in Lemma 1, implies that all of the timing errors eventually reverse; i.e., all earned performance will eventually manifest via realized cash flows. Importantly, the stochastic portion of \( \Theta_t \), which shows up through \( \theta \), ensures that the variability of the timing errors also reverses over time. As such, unlike the accrual policies modeled in Drymiotes and Hemmer (2013) that exhibit stochastic reversals over two periods, accrual reversals here are deterministic in the long run.

For intuition, suppose the timing error takes the form of a one-time misallocation \( \theta_s = \epsilon \) at time \( s \). Then, in expectation, the time-\( s \) timing error \( \epsilon \) has an instantaneous effect on the rate of cash flows of \( (1 - \kappa)\epsilon \). That is, \( \kappa \epsilon \) is instantly reversed, while \( (1 - \kappa)\epsilon \) will reverse over time. Specifically, the portion of the timing error that is not immediately reversed adds \( -\kappa \epsilon e^{-\kappa(t-s)} \) to the cash flow rate at any future time \( t > s \). As such, \( \kappa \epsilon \) is the timing error that is immediately reversed and so does not have an intertemporal effect, whereas the rest of the timing error reverses over time at rate \( \kappa \).

In (2), the diffusion component of cash flows represents shocks to cash flows that are independent of fundamental performance. As such, the volatility term can be interpreted as liquidity shocks that are not indicative of, nor are they affected by, underlying performance. Alternatively, \( \sigma_\theta \) can be thought of as parameterizing the uncertainty over cash flow collection that is not affected by fundamentals.

### 2.2 Accounting Accruals

To facilitate performance measurement, retained earnings \( E = \{E_t\}_{t \geq 0} \) complement the cash account in providing noisy information about fundamental performance over time. Specifically, accounting earnings evolve according to the following SDE:

\[
dE_t = dC_t - \psi (d\theta_t - d\lambda_t)
= d\pi_t + (1 - \psi) d\theta_t + \psi d\lambda_t = dC_t + dA_t,
\]

(3)

where \( \lambda_t \) is the accounting estimation or measurement error about cumulative fundamental performance at time \( t \), \( A = \{A_t\}_{t \geq 0} \) represents the net balance of all accrual and deferral accounts on the firm’s balance sheet at time \( t \), and the differential \( dA_t \) represents net accruals at time \( t \). Let \( \mathcal{F}^E = \{\mathcal{F}^E_t\}_{t \geq 0} \) be the augmented filtration of the \( \sigma \)-fields generated by the sample paths of \( E \). The (potentially dependent) variable \( \psi \in [0, 1] \) denotes the accrual policy that governs the portion of the timing error that earnings aims to correct and, in turn, the amount of estimation error injected into earnings.

Similar to Nikolaev (2018), the estimation errors \( d\lambda_t \) here represent overstatements or
understatements in estimated fundamental performance that reverse in the long run. For
example, a positive estimation error at time $t$, $d\lambda_t > 0$, could manifest as the result of un-
derestimating the portion of uncollectible accounts receivable associated with credit sales at
time $t$. Supposing for the moment that the accrual policy aims to correct all timing errors
in cash flows ($\psi = 1$), this estimation error would cause an inadvertent overstatement in
net accounts receivable, an understatement in bad debt expense, and an overstatement in
accounting earnings relative to fundamental performance. More generally, positive estima-
tion errors would lead an accountant to inadvertently (i) accrue revenues that will never be
collected, (ii) fail to defer revenue that has not yet been earned, (iii) defer expenses that will
never generate revenues, or (iv) fail to accrue expenses when an economic obligation arises.\(^9\)

The accounting measurement error process $\lambda = \{\lambda_t\}_{t \geq 0}$ has a similar mean-reverting
nature to the timing error process: it evolves over time as

$$d\lambda_t = (\mu_\lambda - \eta \Lambda_t) dt + \sigma_\lambda dB^\lambda_t,$$

where $\mu_\lambda$ is the expected estimation error,

$$\Lambda_t = \int_0^t e^{-\eta(t-s)\lambda_s} ds$$

is the depreciated stock of measurement errors, $\eta \in [0, 1]$ is the reversal rate, $\sigma_\lambda > 0$ is
the measurement error volatility, and $B^\lambda = \{B^\lambda_t\}_{t \geq 0}$ is a Brownian motion under $P$ that is
independent of $B^\psi$.

Consistent with Nikolaev (2018), $\mu_\lambda$ captures drift in the estimation error that represents
systematic bias in the accountant’s judgment when estimating fundamental performance.
For example, a pessimistic or risk-averse accountant may inject a negative drift, $\mu_\lambda < 0$, into
the estimation process by inadvertently low-balling his estimate of the collectibility of out-
standing accounts receivable, whereas an accountant who caters to management opportunism
may inject a positive drift by high-balling this estimate.

While judgment-induced drift in estimation errors could lead to bias in accounting earn-
ings, this is conceptually distinct from the accrual policy $\psi$, which determines the portion
of timing errors the accountant or accounting regulator aims to correct given the noisy fun-

\(^9\)Symmetrically, a negative estimation error at time $t$, $d\lambda_t < 0$, could manifest when $\psi = 1$ as the
result of overestimating uncollectible accounts, which would lead to an inadvertent understatement in net
accounts receivable, an overstatement in bad debt expense, and an understatement in accounting earnings
relative to fundamental performance. More generally, negative estimation errors would cause an accountant
to inadvertently (i) fail to accrue revenue that has been earned and that will be collected in the future, (ii)
defer revenue that has already been earned and collected, (iii) fail to defer expenses that will generate future
cash inflows, or (iv) accrue expenses that will never be paid.
damental performance estimate. For example, consider a firm that invests $1,000 in each of $N$ R&D projects, and suppose that the firm’s accountant estimates that only $n$ of these projects will successfully generate revenues in the future. Given the accrual policy $\psi$, then the accountant would capitalize $\psi \cdot n \cdot \$1,000 and expense $(N - \psi \cdot n) \cdot \$1,000$. If the accountant is pessimistic ($\mu_\lambda < 0$), then $n$ is less than the true number of successful projects in expectation; however, if $\sigma_\lambda$ is very large, then there is a high probability that $n$ is greater than the true number of successful projects despite the accountant’s pessimism. Turning to the accrual policy, if $\psi = 1$ then the accountant capitalizes $n \cdot \$1,000 and expenses $(N - n) \cdot \$1,000$, implying that accounting earnings are fully susceptible to this estimation error and potential bias in the estimate $n$. By contrast, if $\psi = 0$ then the entire $N \cdot \$1,000 is immediately expensed and earnings are impervious to the error and bias associated with the accountant’s R&D estimates.

This distinction between the correction of timing errors and estimation error allows us to cleanly separate the components of the accounting system that affect (1) when performance is recognized through the correction of cash flow timing errors and (2) the noisiness in earnings caused by imperfect estimates when timing errors are corrected.

**Definition 1.** The accounting system $(\psi, \mu_\lambda, \sigma_\lambda, \eta)$ consists of an accrual policy $\psi \in [0, 1]$ that corrects a portion of the timing error in cash flows, a drift parameter $\mu_\lambda$ that induces bias in the estimation process, a volatility parameter $\sigma_\lambda$ that induces dispersion in the estimation process, and a correction rate $\eta$ that reverses the realization of these errors over time.

The accruals process $A = \{A_t\}_{t \geq 0}$ is the difference between the accounting earnings and cash flow processes. It evolves over time according to

$$dA_t = dE_t - dC_t = \psi (d\lambda_t - d\theta_t).$$

(5)

Accruals correct a portion of the un-reversed timing error in cash flows but are subject to estimation error due to imperfections in the accounting measurement system and the inherent unobservability of fundamental performance. That is, correcting more of the timing error in cash flows ensures that earnings more closely reflect estimated fundamental performance, which may be a better or worse measure than cash flows depending on the quality of the accounting estimate. This is the primary trade-off in our model: correcting more timing error requires injecting more estimation error into earnings.

The following result verifies an important characteristic of the accruals process (5) and ensures that our representation of accruals satisfies a clean surplus condition.

**Lemma 1.** For any $(\mu_\theta, \mu_\lambda)$, both $\theta_t$ and $\lambda_t$ approach zero as $t$ increases. That is, the following clean surplus condition is satisfied: $\lim_{t \to \infty} E_t = \lim_{t \to \infty} C_t = \lim_{t \to \infty} \pi_t$. 

8
While cash flows and earnings both represent the same underlying economic construct, the nature of their imperfections differs. While cash flows are subject to a timing error, earnings are impacted by the accruals process (5), which eliminates a portion of the cash flow timing error but adds volatility in the form of estimation error. By Lemma 1, accounting errors also fully reverse over time, and thus cash equals retained earnings in the long run.

The nature of the (potentially conditional) accrual policy $\psi$ is a key component of our model; accruals here play a corrective role by “fixing” a portion of the timing errors in cash flows. The accrual process in Nikolaev (2018) is the specific case in which $\psi = 1$ regardless of the nature of the timing and estimation errors. This represents the “full correction” benchmark in our model whereby the accrual policy fully corrects the timing error in cash flows at every instant in time. While this benchmark setting is reasonably descriptive of working capital accruals, many accrual policies in practice fall short of full correction; the R&D illustration above is a case in point. As we will demonstrate in the next section, an accrual policy that fully corrects timing errors in cash flows is not always preferable to one that does not.

Thus far, we have been agnostic on how the fundamental performance process $\pi$ evolves over time. To study our accruals model in a strategic setting, we now apply the model to a dynamic contracting setting where fundamental performance is a controlled process with uncertainty.

### 3 Principal-Agent Problem under Uncertainty

Consider a dynamic agency setting where a risk-neutral principal contracts with a risk-averse agent to control an asset owned by the principal through costly productive effort. The agent’s action choice is a progressively measurable process $a = \{a_t\}_{t \geq 0}$, where $a_t \geq 0$. Then, fundamental performance follows the mean-reverting SDE

$$d\pi_t = (a_t - k\pi_t)\,dt + \sigma_\pi dB^\pi_t,$$

where $\sigma_\pi > 0$ is the volatility of fundamental performance, $k \in [0, 1]$ is the depreciation rate of cumulative fundamentals, and $B^\pi = \{B^\pi_t\}_{t \geq 0}$ is a standard Brownian motion independent of $B^\theta$ and $B^\Lambda$. Normalize $\pi_0 = 0$ without loss of generality. Equation (6) states that without any productive action taken ($a_t = 0$), fundamentals will exhibit negative autocorrelation. A low value of $k$ means that firm fundamentals exhibit a high degree of persistence. This structure seems fairly representative of productive activity in practice: machines rust, ideas grow stale, etc.
The fundamental volatility parameter $\sigma_\pi$ is assumed to be exogenous and can represent underlying market risk that only impacts cash flows and earnings to the extent that it affects fundamental performance. As such, a low $\sigma_\pi$ implies that the agent has a large degree of control over the underlying asset, and vice-versa.

3.1 Bayesian Learning

Since fundamental performance, $\pi$, is inherently unobservable and (assumed to be) payoff-relevant, the two observable signals, $C$ and $E$, are used to make inferences over $\pi$ by both the principal and the agent. Let $\mathcal{F} = \{\mathcal{F}_t\}_{t\geq 0} = \mathcal{F}_C \otimes \mathcal{F}_E$ be the augmented filtration of the joint $\sigma$-fields generated by the sample paths of $C$ and $E$. That is, $\mathcal{F}_t$ represents the public information available at time $t$.

Suppose the principal and agent share the same common priors over fundamental performance at the outset of the contracting relationship: $\pi_0 \sim \mathcal{N}(\hat{\pi}_0, \hat{\sigma}_0)$, where $\hat{\pi}_0 = 0$ and $\hat{\sigma}_0 = 0$ without loss of generality. Furthermore, suppose the optimal contract prescribes the effort policy $a^\ast$ while the agent actually follows the policy $a$. The principal recursively updates his beliefs over fundamental performance as if $\pi_t \sim \mathcal{N}(\hat{\pi}_t^a, \hat{\sigma}_t^a)$ at every time $t$, where $\hat{\pi}_t^a = \mathbb{E}[\pi_t|\mathcal{F}_t, a^\ast]$ and $\hat{\sigma}_t^a = \mathbb{E}[(\pi_t - \hat{\pi}_t^a)^2|\mathcal{F}_t, a^\ast]$ are the principal’s posterior mean and variance given the information up to $t$. On the other hand, the agent revises his beliefs as if $\pi_t \sim \mathcal{N}(\tilde{\pi}_t^a, \tilde{\sigma}_t^a)$, where $\tilde{\pi}_t^a = \mathbb{E}[\pi_t|\mathcal{F}_t, a, a]$, and $\tilde{\sigma}_t^a = \mathbb{E}[(\pi_t - \tilde{\pi}_t^a)^2|\mathcal{F}_t, a, a]$ are the agent’s posterior mean and variance over fundamental performance, respectively.

Thus, although neither the principal nor the agent observes fundamental performance directly, the agent knows strictly more than the principal. While the principal only knows the action that he has instructed the agent to take via the optimal contract, the agent knows his actual action process. By deviating, the agent can influence the principal’s beliefs over fundamental performance. The following assumption is made for simplicity.

We now examine the role of accounting accruals in the learning process. The following result shows how beliefs about fundamental performance are updated over time.

**Lemma 2.** Given an accrual policy $\psi$, the principal’s beliefs are updated recursively according to

$$
    d\hat{\pi}_t^a = (a_t^\ast - k\hat{\pi}_t^a)\ dt + \Sigma_t(\psi)\ d\tilde{Z}_t,
$$

where

$$
    \Sigma_t(\psi) = \frac{1}{\Sigma_C^4} \sqrt{\Sigma_C^4 \beta_t^C(\psi)^2 + \beta_t^E(\psi)^2 (\sigma_\pi^2 + (1 - \psi) \sigma_\theta^2 + \psi \sigma)^2},
$$

$$
    \sigma = \sqrt{\sigma_\pi^2 (\sigma_\theta^2 + \sigma_\lambda^2) + \sigma_\theta^2 \sigma_\lambda^2}, \quad \beta_t^C(\psi) = \beta_t(\psi)(\sigma_\lambda^2 \psi - \sigma_\theta^2 (1 - \psi)) \quad \text{and} \quad \beta_t^E(\psi) = \beta_t(\psi) \sigma_\theta^2
$$

are
the sensitivity of beliefs to cash flow and earnings shocks, \( \beta_t(\psi) \) is the aggregate conditional covariance defined as

\[
\beta_t(\psi) = \frac{\sigma^2_\pi - k\gamma_t}{\sigma^2_\psi},
\]

and \( \hat{Z} = \{\hat{Z}_t\}_{t \geq 0} \) is a standard Brownian motion from the principal’s perspective.

The aggregate responsiveness of beliefs to news, as measured by \( \beta(\psi) \), is decreasing in the rate of mean reversion \( k \) for a given posterior variance level \( \gamma \) and accrual policy \( \psi \). This is because a higher \( k \) will induce a tighter long-run distribution over fundamentals, so beliefs will be less responsive in general.

Also, the accrual policy \( \psi \) has two countervailing effects on the impulse responsiveness of beliefs to news: while the time-\( t \) responsiveness of beliefs to cash flow news \( \beta^C_t(\psi) \) is increasing in \( \psi \), the responsiveness of beliefs to earnings news \( \beta^E_t(\psi) \) is decreasing in \( \psi \). Both relations are monotonic. The intuition is as follows. A higher \( \psi \) means that more of the timing error in cash flows is corrected. Ceteris paribus, this increases the likelihood that performance shocks are due to measurement errors associated with accruals rather than cash flow timing errors. As such, beliefs will be relatively less responsive to earnings news than cash flow news.

Lemma 2 formalizes the role of accruals in mitigating uncertainty in the learning process over fundamentals: accruals allow for more efficient belief updating. Suppose the principal naively relies solely on cash flows to form posterior beliefs over \( \pi_t \). Then, any beliefs will be under- or over-weighted depending on the direction of the timing error. Thus, the accrual policy eliminates a portion of this “weighting error.”

### 3.1.1 Benchmark Cases

To see this, consider a benchmark setting without accounting accruals: \( dA_t = 0 \) or equivalently, \( \psi = 0 \). Then, by (3), earnings contain no additional information beyond that in cash flows. Since earnings then are equivalent to cash flows, only cash flows are used to form and update beliefs over fundamental performance. Under this benchmark case, let \( \pi^a_t \) and \( \pi^a_t \) be the principal’s and agent’s respective time-\( t \) posterior beliefs over fundamental performance and let \( \gamma_t \) be the posterior conditional variance. The following corollary compares this benchmark case to the setting with accruals.

**Corollary 1.** The benchmark case without accounting accruals has a higher stationary posterior variance than the setting with accruals in Lemma 2: \( \gamma_t > \gamma_t \) at all \( t \geq 0 \). This means that the learning process without accounting accruals is rendered more inefficient for
the principal:

\[ |\hat{\pi}_t^a - \pi_t| < |\bar{\pi}_t^a - \pi_t|, \]

relative to the setting with accruals, and similarly for the agent.

Corollary 1 shows the marginal benefit of accruals in this setting: they help in forming posterior beliefs over unobservable fundamental performance. Specifically, they bring posterior beliefs closer to the actual fundamentals by reducing the level of posterior variance. That is, while the mean of fundamental performance is controlled by the optimal contract, accruals reduce the dispersion in the belief updating process.

Consider another benchmark case in which the news in cash flows is orthogonal to the news in earnings. The relative conditional covariance terms on cash flow and earnings news in this case are \( \beta_i^{C,O} = \psi \beta_i(\psi) \sigma^2_{\theta} \) and \( \beta_i^{E,O} = \psi \beta_i(\psi) \sigma^2_{\lambda} \), respectively. Thus, by correcting timing errors in cash flows, accruals induce the filter to follow the cash flow and earnings measurements more closely, relative to the orthogonal case.

### 3.2 Intertemporal Information Asymmetry

As noted previously, the agent has an informational advantage over the principal because he could secretly deviate from the action prescribed in the optimal contract and thereby manipulate the principal’s beliefs over fundamental performance. Of course, the contract is written so that any such deviations never occur in equilibrium, and the principal and the agent therefore have the same information sets on the equilibrium path.

The difference between the agent’s and the principal’s beliefs over fundamental performance is called the belief distortion: \( \Delta_t^\pi = \bar{\pi}_t^a - \hat{\pi}_t^a \). By Lemma 2, the belief distortion evolves as

\[ d\Delta_t^\pi = L_t (a_t - a_t^* - k\Delta_t^\pi) \, dt, \]

where the learning parameter \( L_t \geq 0 \) satisfies

\[ L_t = 1 - \beta_i^C(\psi) - \beta_i^E(\psi) = \frac{k\gamma_t (\sigma^2_{\theta} + \sigma^2_{\lambda}) + \sigma^2_{\theta} \sigma^2_{\lambda}}{\sigma_t^2}, \tag{7} \]

where \( \sigma \) is defined in Lemma 2. Then, by the assumption of common initial priors,

\[ \Delta_t^\pi = \int_0^t e^{-\int_u^t kL_u \, du} L_s (a_s - a_s^*) \, ds. \tag{8} \]

Along the equilibrium path, the agent takes the action prescribed by the optimal contract,
which is known to the principal; thus $\Delta_t^e = 0$ in equilibrium. However, the principal has to provide the agent with appropriate incentives to prevent him from going down off-equilibrium paths and generating nonzero $\Delta_t^e$, which implies that incentive compatibility is an intertemporal issue.

Due to his informational advantage over the principal, the agent can take advantage of these belief distortions by manipulating the principal’s beliefs over fundamentals. We will show in the next section that this gives rise to an endogenous demand for long-term incentives. However, by (8), the agent’s informational advantage decreases exponentially over time at rate $L$ as the principal observes new information. By (7), the rate of decay of the agent’s informational advantage is driven by the variance of the timing and accounting errors $\sigma_0^2$ and $\sigma_1^2$.

Note that by Lemma 2, $\beta_t^C(\psi) (\beta_t^E(\psi))$ is the proportion of the variation in the time-$t$ cash flow (earnings) signal that is caused by variation in fundamental performance, and represents the weight the principal and agent place on cash flow (earnings) news when updating their beliefs. As such, $L_t$ is the proportion of the total variation of the time-$t$ cash flow and earnings signals that is due to timing error and measurement volatility. A higher $L_t$ signals a greater degree of informational uncertainty in the agency at time $t$, whereas $1 - L_t$ represents the degree to which beliefs respond to time-$t$ cash flow and earnings signal observations. Put differently, when updating their beliefs recursively, the principal and agent change their beliefs over fundamentals due to two factors: the trend in fundamental performance itself and unexpected cash flows/earnings news. Specifically, $L_t$ is the weight placed on the expected trend at $t$, while $1 - L_t = \beta_t^C(\psi) + \beta_t^E(\psi)$ is the aggregate weight given to the observed information shocks from cash flows and earnings realizations.

The following corollary formalizes the impact of accrual accounting on belief distortions in this setting by comparing the above filtering model to this benchmark setting without accounting accruals.

**Corollary 2.** Define $\Delta_t^e = \pi_t^a - \pi_t^{a^*}$ as the belief distortions in the benchmark setting without accounting accruals. Then, $|\Delta_t^e| < |\Delta_t^a|$ at all times $t \geq 0$.

Corollary 2 extends the implications of Lemma 2: the presence of accruals reduces the absolute value of the distortions themselves. Figure 1 shows this when the agent shirks for an instant of time. While the belief distortions with and without accruals approach zero due to the learning process, accounting accruals reduce the instantaneous impact of the agent’s deviation. Thus, the value of accrual accounting here is in its marginal impact on information asymmetry relative to cash flows. While both earnings and cash flows measure the same underlying fundamental performance process, because accruals decrease belief distortions
through their performance measurement role, they reduce the scope for the agent to take advantage of his informational advantage through $L$.

### 3.3 The Contracting Setting

The risk-averse agent is compensated via a progressively measurable process $S = \{S_t\}_{t \geq 0}$ by the principal for undertaking the personally costly productive action $a_t \in \mathbb{R}_+$. The agent’s utility from $(S_t, a_t)$ is represented by a twice continuously differentiable function $u : \mathbb{R} \times \mathbb{R}_+ \to \mathbb{R}$. As noted by equation (6), the agent controls the drift of the fundamental performance process via this action. Suppose the agent has CARA utility with quadratic effort costs of the form

$$u(S_t, a_t) = -\frac{1}{c} \exp \left\{ -c \left( S_t - \frac{1}{2} a_t^2 \right) \right\},$$

where $c > 0$ is the agent’s coefficient of absolute risk aversion. Then, the long-term contract is formally defined as follows.

**Definition 2.** A contract $\{(S_t, a_t, \psi)\}_{t \geq 0}$ specifies, at each point in time, the agent’s compensation flow and productive action, as well as the accrual policy.

Thus, the principal designs not only the contract but also the accrual policy in an effort to efficiently motivate productive activity from the agent. The principal’s lifetime expected profit is the discounted value of future cash flows, less the total compensation paid to the agent. Integrating by parts, the principal’s profit over the course of the entire agency can be written as

$$\mathbb{E} \left[ \int_0^\infty e^{-rt} (dC_t - S_t dt) \right] = \mathbb{E} \left[ \int_0^\infty e^{-rt} (a_t - k \tilde{\pi}_t - S_t) dt \right],$$

where $r > 0$ is the common discount rate for both the principal and the agent.

Thus, the principal’s problem is to find a contract $(S_t, a_t, \psi)$ at each instance of time $t \geq 0$ that solves

$$\max_{S, a, \psi \in [0, 1]} \mathbb{E}^a \left[ \int_0^\infty e^{-rt} (a_t - k \tilde{\pi}_t - S_t) dt \right],$$

s.t. $a \in \arg\max_{a^*} \mathbb{E}^{a^*} \left[ \int_0^\infty e^{-rt} u(S_t, a_t^*) dt \right],$

$$\mathbb{E}^a \left[ \int_0^\infty e^{-rt} u(S_t, a_t) dt \right] \geq W_0,$$

$$d\tilde{\pi}_t^a = (a_t - k \tilde{\pi}_t^a) dt + \sum_t(\psi) d\tilde{Z}_t,$$
where (12) follows from Lemma 2 and $W_0$ is the agent’s reservation utility at the outset of the contractual relationship. The expectation operators $E^a$ and $E^{a^*}$ signify that the agent’s effort choice controls the probability distribution over cash flow and earnings signals.

We now turn to the agent’s problem, which involves the choice of the productive action process $a$. In particular, we aim to characterize the relation between the accrual policy and the manager’s productive activity.

### 3.3.1 The Agent’s Problem

The agent’s time-$t$ continuation value $W_t$ is his expected value from the contract (in utility space) from time $t$ onward:

$$W_t = E_{t}^{a} \left[ \int_{t}^{\infty} e^{-r(s-t)} u(S_s, a_s) ds \right],$$

where $E_{t}^{a}[\cdot]$ is the conditional expectation operator under the probability measure induced by the agent’s action $a$ taken with respect to information at time $t$. By a standard martingale representation argument, there exists an $\mathcal{F}$-progressively measurable process $\delta = \{\delta_t\}_{t \geq 0}$ such that $W_t$ follows the following SDE:

$$dW_t = (rW_t - u(S_t, a_t)) dt - rcW_t \Sigma_t(\psi) \delta_t d\hat{Z}_t. \quad (13)$$

Note that the volatility of the continuation value $\delta_t$ represents how the agent’s value changes with conjectured performance: $\delta_t = dW_t/d\hat{\sigma}_t$. As such, $\delta$ represents the agent’s pay-performance sensitivity (PPS). PPS $\delta$ is in monetary units, while $W_t$ is clearly in units of utility. Multiplying PPS by the agent’s marginal utility of compensation which, as we will show, is $-rcW_t$, yields utility-performance sensitivity.

The agent’s continuation value will be used as a state variable in the principal’s problem. However, due to uncertainty and the agent’s ability to manipulate the principal’s beliefs over fundamental performance, $W_t$ is not a sufficient statistic for the continuation contract after $t$. This is because of the intertemporal belief distortions between the principal and the agent, which persist over time due to the serial correlation in the belief updating process. By Lemma 2, current beliefs over fundamental performance feed into how the agent will be evaluated in the future. Thus, the marginal benefit the agent receives from this belief distortion also needs to be a state variable. The following theorem formally shows this and solves the agent’s problem.

---

10 See Proposition 1 in Sannikov (2008).
Theorem 1. The agent’s continuation value follows a Brownian martingale:

\[
    dW_t = -rcW_t\delta_t \Sigma_t(\psi)d\hat{Z}_t = -rcW_t\delta_t \left( \beta^C_t(\psi)d\hat{Z}_t^C + \beta^E_t(\psi)d\hat{Z}_t^E \right).
\]

The agent’s long-term incentives are represented by a process \( \rho = \{\rho_t\}_{t \geq 0} \) that reflects the marginal benefit he receives from the belief distortions \( \Delta_t^x \) in monetary units:

\[
    \rho_t = \frac{1}{-rcW_t} \mathbb{E}^q_t \left[ \int_t^\infty e^{-\int_t^s (r+kL_u)du} rcW_s k (1 - L_s) \delta_s ds \right]
    = \frac{1}{-rcW_t} \mathbb{E}^q_t \left[ \int_t^\infty e^{-r(s-t)} ku_a(S_s, a_s) ds \right],
\]

which has the recursive representation as an infinite-horizon backward stochastic differential equation (BSDE):

\[
\begin{align*}
    d\rho_t &= (r + kL_t) \rho_t dt + \delta_t \left[ k (1 - L_t) + rc\Sigma_t(\psi)\nu_t \right] dt + \nu_t d\hat{Z}_t, \\
    \lim_{t \to \infty} \rho_t &= 0,
\end{align*}
\]

for some progressively measurable process \( \nu = \{\nu_t\}_{t \geq 0} \). A necessary condition for the contract to be incentive-compatible is that the agent’s action satisfies

\[
a_t = (1 - L_t) \delta_t + L_t \rho_t.
\]

If the volatility of long-term incentives satisfies a regularity condition, then condition (16) is also sufficient for optimality.

The agent’s continuation value \( W_t \) is adjusted for output surprises through the PPS control process \( \delta_t \). However, the relative adjustment weights placed on cash flows and earnings are determined by the filtering process outlined in Lemma 2. That is, at time \( t \), the agent gains in proportion to \( \delta_t \beta^C_t(\psi) \) from positive cash flow news and \( \delta_t \beta^E_t(\psi) \) from positive earnings news. The agent is punished for negative news analogously.

Theorem 1 shows that the accrual policy \( \psi \) does not affect the agent’s productive activity directly. Rather, it has an indirect effect through the incentive components in the contract. This is because accruals do not alter the underlying informational characteristics of aggregate earnings: earnings still convey information about fundamentals, but the accrual policy alters when performance is reported. It is because the principal and agent care about the time-path of performance that accruals have an impact on fundamentals. The agent’s long-term incentives \( \rho_t \) represent his expected intertemporal marginal impact from having a larger
information set than the principal. In (15), \(-rcW_t > 0\) is a scaling factor that converts incentives in cash space to utility space,\(^{11}\) which is a necessary transformation due to CARA utility.

The intuition for the incentive-compatibility (IC) constraint (16) is as follows. Suppose the agent, at \(t\), shirks for an instant of time \(dt\) by taking \(a_t < a_t^\ast\), but will follow the prescribed effort policy \(a_t^\ast\) thereafter, for \(s > t\). His marginal benefit from taking a lower action than the one prescribed by the contract is represented by a decrease in effort cost: 
\[
u_a(S_t, a_t)dt = cu(S_t, a_t)a_tdt.
\]
Since shirking harms fundamental performance, the agent’s time-\(t\) continuation value falls by \((1 - L_t)\delta_tdt\), as the principal weights cash flow and earnings shocks in proportion to \((1 - L_t) = \beta_t^C(\psi) + \beta_t^E(\psi)\).\(^{12}\) However, the presence of uncertainty here also causes deviations of this form to have a long-term effect.

By (8), shirking causes the agent’s time-\(t\) beliefs over fundamentals to be depressed relative to the principal’s by an amount proportional to \(L_t\). Shirking induces lower contemporaneous cash flow and earnings realizations. While the agent knows that these lower signals are due to his shirking negatively impacting fundamental performance itself, the principal attributes this to noise due to the timing error or measurement error beyond the agent’s control. As such, the principal will not revise his expectation of performance downwards.

Given these potential effort deviations, in order to match the principal’s beliefs over fundamentals at any future time \(s > t\), the agent’s effort level \(a_s\) must satisfy

\[
\tilde{\pi}_s^a = \tilde{\pi}_t^a + \int_t^s e^{-\int_t^{\tau} kL_u du} \left\{ L_u a_u^* dt' + \beta_u^C(\psi)dC_u + \beta_u^E(\psi)dE_u - \psi \beta_u(\psi) \left( \sigma_\psi^2 \mathbb{E}[d\theta_u] + \sigma_\lambda^2 \mathbb{E}[d\lambda_u] \right) \right\}
\]

\[
= \tilde{\pi}_t^a - \Delta_t^\pi + \int_t^s e^{-\int_t^{\tau} kL_u du} \left\{ L_u a_u^* dt' + \beta_u^C(\psi)dC_u + \beta_u^E(\psi)dE_u - \psi \beta_u(\psi) \left( \sigma_\psi^2 \mathbb{E}[d\theta_u] + \sigma_\lambda^2 \mathbb{E}[d\lambda_u] \right) \right\}.
\]

Thus, the agent will have to work an additional \(-\Delta_t^\pi > 0\) units of effort in expectation to match the principal’s expectations of performance in the future. The present value at \(t\) of this additional required effort is \(\rho_t\). This is precisely why long-term incentives are valuable in this setting. Shirking on the agent’s part therefore allows him to take lower effort while leaving the principal’s beliefs over fundamentals unaffected but is costly due to a higher burden to meet the performance target \(\beta_t^C(\psi) \mathbb{E}_t^a [dC_t] + \beta_t^E(\psi) \mathbb{E}_t^a [dE_t]\) in the future. As such, the principal can more finely control the agent’s effort level by utilizing long-term incentives.

\(^{11}\)Specifically, it is the agent’s marginal benefit from the compensation rule \(S_t\).

\(^{12}\)See the discussion following equation (8).
The concern that the agent may benefit from his informational advantage over the principal is highest when the level of posterior uncertainty over fundamentals is at its highest, which tends to be the case towards the beginning of the agency relationship. The principal will therefore tend to defer more incentives at the start of the agency relationship and allow incentives to vest as learning reduces the potential belief distortions between the principal and the agent.\footnote{This is analogous to the “belief matching” effect documented in Lee (2020).} We will show this formally when solving the principal’s problem. Because $L_t$ is essentially a measure of the persistence of these belief distortions, the higher $L_t$ is, the more the principal will utilize long-term incentives to motivate the agent.

Theorem A1 shows that as long as the volatility of long-term incentives is reasonably bounded, the effort policy (16) is both necessary and sufficient for optimality. This condition, while not without total loss of generality, makes sense in our setting. Because the agent is risk-averse, instituting path-dependence in the agent’s long-term incentives that exceeds these bounds is prohibitively costly to the principal. To clarify the demand for long-term incentives in our model, we now consider a benchmark case without measurement error in earnings.

\section*{No Measurement Error} Consider the dynamic incentive problem when there is no measurement error. The following corollary to Theorem 1 shows that there is no need for long-term incentives in this case.

\begin{corollary}
Suppose there is no measurement error: $\sigma_\lambda = 0$. Then, $\psi = 1$ and the agent’s optimal effort policy in this particular case, $a^{\sigma_\lambda = 0} = \{a_{t}^{\sigma_\lambda = 0}\}_{t \geq 0}$, satisfies

$$a_{t}^{\sigma_\lambda = 0} = \delta_{t}^{\sigma_\lambda = 0},$$

for all $t \geq 0$, where $\delta^{\sigma_\lambda = 0} = \{\delta_{t}^{\sigma_\lambda = 0}\}_{t \geq 0}$ is a progressively measurable process.

Thus, without measurement error, the need to use long-term incentives to motivate the agent’s effort disappears. With $\sigma_\lambda = 0$ and $\psi = 1$, the accounting system is booking a “perfect” accrual that completely eliminates the timing error in cash flows without injecting any additional measurement error. In this case, earnings satisfy

$$dE_{t} = d\pi_{t} + (\mu_{\lambda} - \eta\Lambda_{t})\,dt.$$ 

Because the drift of $\lambda_{t}$ is observable, this effectively makes fundamental performance observable and thus contractible. As such, the belief distortions between the principal and the
agent are eliminated, negating the need to provide the agent with long-term incentives to elicit productive activity.

The lack of long-term incentives here is not a result of measurement error per se; rather, it is due to the ability of the accrual accounting system to costlessly correct cash flow timing errors. Thus, without measurement error, long-term incentives are replaced with a “perfect” accrual policy for incentive purposes. This foreshadows the substitution effect between accruals and long-term incentives that we formalize after solving the principal’s problem in the next section.

3.4 The Principal’s Problem

The solution to the agent’s problem shows that both $W_t$ and $\rho_t$ have to serve as incentive state variables in the principal’s problem. To solve the principal’s problem, we first eliminate $W_t$ as a state variable and rewrite the principal’s profit function accordingly.

Lemma 3. The agent’s optimal compensation satisfies

$$ S_t = \frac{1}{2}a_t^2 - \frac{1}{c} \ln (-rcW_t). $$

Then, the principal’s objective function (9) can be written as

$$ \max_{\delta, \psi \in [0,1]} \mathbb{E}^a \left[ \int_0^\infty e^{-rt} \left( a_t(\delta_t) - k\hat{\pi}_t^{a(\delta)} - \frac{1}{2}a_t^2(\delta_t) - \frac{1}{2}rc\Sigma_t^2(\psi)\delta_t^2 \right) dt \right], $$

where $a_t(\delta_t)$ is the incentive-compatible effort level that satisfies equation (16).

Define the principal’s value function from problem (9) as $J(\hat{\pi}_t^a, \rho_t)$. In order to ensure that the IC condition (16) always binds, the following technical assumption on the shape of the principal’s value function is required.

Assumption 1. The principal’s value function $J(\hat{\pi}_t^a, \rho)$ is jointly concave in $(\hat{\pi}_t^a, \rho)$.

Since the IC condition (16) binds, the principal can control the agent’s action and evolution of long-run incentives through the choice of sensitivities $\delta$ and $\nu$, as well as the accrual policy $\psi$. Specifically, because the action process $a(\delta) = \{a_t(\delta_t)\}_{t \geq 0}$ solves the agent’s problem, in equilibrium, the agent will have no incentive to deviate from the recommended effort policy and manipulate the principal’s beliefs. This implies that $\hat{\pi}_t^{a(\delta)} = \hat{\pi}_t^{a(\delta)}$ at all times $t \geq 0$. Denote $\hat{\pi} = \{\hat{\pi}_t\}_{t \geq 0}$ as this common belief process. Then, the principal’s value
function $J(\hat{\pi}, \rho)$ must satisfy the Hamilton-Jacobi-Bellman (HJB) equation

$$r J(\hat{\pi}, \rho) = \max_{\delta, \nu, \psi \in [0,1]} \left[ a(\delta) - k\hat{\pi} - \frac{1}{2} a^2(\delta) - \frac{1}{2} r c \Sigma^2(\psi) \delta^2 + (a(\delta) - k\hat{\pi}) J_\hat{\pi}(\hat{\pi}, \rho) + \frac{1}{2} \left( (r + kL) \rho + \delta (k (1 - L) + r c \Sigma(\psi) \nu) \right) J_\rho(\hat{\pi}, \rho) + \frac{1}{2} \Sigma^2(\psi) J_\psi(\hat{\pi}, \rho) + 2 \Sigma(\psi) \nu J_{\hat{\pi}\rho}(\hat{\pi}, \rho) + \nu^2 J_{\rho\rho}(\hat{\pi}, \rho) \right], \quad (19)$$

where the subscripts denote partial derivatives. The HJB equation (19) is a parabolic partial differential equation (PDE) that fully characterizes the optimal contract in this continuous-time setting. Figure 3 depicts a simulation of the principal’s value function over the state space $(\hat{\pi}, \rho)$. It increases with $\hat{\pi}_t$ and is concave in $\rho_t$, which corresponds with the characteristics in Assumption 1.

The following theorem solves the principal’s problem and derives the relation between the dynamic incentive components $(\delta, \nu)$ and the accrual policy $\psi$.

**Theorem 2.** Let $a = \{a_t\}_{t \geq 0}$ be the action policy that solves the agent’s problem satisfying (16). Then, the optimal incentive components $(\delta, \nu)$ satisfy

$$\delta = \frac{(1 - L) \left( 1 + J_\hat{\pi} - L \rho + k J_\rho \right) J_{\rho\rho} - r c \Sigma^2(\psi) J_{\rho\rho} J_{\hat{\pi}\rho} \left( r c \Sigma^2(\psi) + (1 - L)^2 \right) J_{\rho\rho} + r^2 c^2 \Sigma^2(\psi) J^2_{\rho}}{J_{\rho\rho}},$$

$$\nu = -\frac{(r c \Sigma(\psi) J_\rho \delta + J_{\hat{\pi}\rho})}{J_{\rho\rho}}.$$

The contract that maximizes the principal’s profit initializes the agent’s level of long-term incentives at

$$\rho_0 = \frac{1}{L_0} \left( 1 + \frac{k}{r + k} \right).$$

Since $\rho_t$ represents the present value at time $t$ of future pay-performance sensitivities by (14), it can be thought of as a measure of unvested incentives, which are path-dependent.\(^\text{14}\) As incentives vest, they become earned, and $\rho$ decreases while $\delta$ increases as a result. The learning process is the endogenous mechanism through which incentives vest over time: as the agent’s informational advantage over the principal decreases, the principal will allow incentives to vest since the benefit of having path-dependent incentives in the first place is in mitigating the agent’s gains from the belief discrepancy.

The interaction between the learning and accrual processes has significant implications for the evolution of these path-dependent incentives in the optimal long-term contract. Figure 2 presents Monte Carlo simulations of the incentive variables: long-term incentives $\rho$ and

\(^{14}\)See Marinovic and Varas (2019) and Lee (2020) for similar interpretations.
the PPS $\delta$ over time. By the left panel of Figure 2, accruals depress the need to rely on long-term incentives and allows the principal to implement an incentive policy with a higher level of short-term incentives. The more of the timing error in cash flows that is fixed by the accruals process, the more the contract shifts from long-term to short-term incentives, which is shown in the right panel of Figure 2. This implies that fixing timing errors through accruals is analogous to “implicit incentives” that allow the principal to marginally reduce the duration of the agent’s compensation without distorting the agent’s incentives themselves. As such, the duration of compensation is inherently linked with the choice of accrual policy.

We now examine the accrual policy choice in greater detail.

### 3.4.1 Choice of Accrual Policy

In conjunction with Theorem 2, Theorem 3 completes the analysis of the principal’s problem by examining the optimal choice between accrual policies.

**Theorem 3.** Let $J(\hat{\pi}, \rho, \psi)$ be the principal’s value function over the state space $(\hat{\pi}, \rho)$ using the accrual policy $\psi$. Then, there exists a threshold level of long-term incentives $\rho^* \in (0, \rho_0]$ such that the value function solving the principal’s HJB equation (19) satisfies

$$J(\hat{\pi}, \rho) = \begin{cases} J(\hat{\pi}, \rho, \psi^*) & \text{for } \rho > \rho^*, \\ J(\hat{\pi}, \rho, \overline{\psi}) & \text{for } \rho \leq \rho^*. \end{cases}$$

(20)

$\overline{\psi}$ is the stationary accrual policy defined by

$$\overline{\psi} = \min \left\{ \frac{2\sigma_\theta^2 (\sigma_\pi^2 + \sigma_\lambda^2)}{\sigma^2 + \sigma_\theta^2 (2\sigma_\theta^2 - \sigma)}, 1 \right\},$$

(21)

where $\sigma = \sqrt{\sigma_\pi^2 (\sigma_\theta^2 + \sigma_\lambda^2) + \sigma_\theta^2 \sigma_\lambda^2}$. $\psi^* = \{\psi^*_t\}_{t \geq 0}$ is the non-stationary accrual policy satisfying

$$\psi^*_t = \min \left\{ \frac{2\sigma_\theta^2 (\sigma_\pi^2 + \sigma_\lambda^2)}{\sigma^2 + \sigma_\theta^2 (2\sigma_\theta^2 - \sigma) + \sqrt{\frac{2(\sigma_\pi^2 + \sigma_\lambda^2) \xi_t}{(\sigma^2 + \sigma_\theta^2 \sigma_\lambda^2)^2} - (\sigma^2 + \sigma_\theta^2 \sigma_\lambda^2)^2}}, 1 \right\},$$

(22)

where

$$\xi = \frac{(1 - L) J_{\rho\rho}}{rcJ^2_\rho + J_{\rho\rho}} \left( -\frac{1 - L}{rc} + \frac{(1 - L) J_\rho J_{\rho\rho} + (rcJ^2_\rho + J_{\rho\rho}) (1 + J_{\hat{\pi}} + kJ_\rho - L \rho)}{\sqrt{rcJ_{\rho\rho} (J_{\hat{\pi}} (rcJ^2_\rho + J_{\rho\rho}) - J_{\hat{\pi}}^2)} \right) > 0.$$ 

At $\rho^*$, $J_{\rho}(\hat{\pi}, \rho^*) = \phi'(\rho^*)$ and $J_{\rho\rho}(\hat{\pi}, \rho^*) = \phi''(\rho^*)$ for all $\hat{\pi}$, where $\phi : \mathbb{R}_+ \to \mathbb{R}$ is the unique
Theorem 3 shows that the principal will choose to utilize two accrual policies with distinctly different characteristics. The first policy \( \bar{\psi} \) in (21) is independent of time and only depends on the exogenous volatility parameters \( \sigma_\theta \), \( \alpha \), and \( \sigma_\pi \). On the other hand, the policy \( \psi_t^* \) in (22) is time-inhomogeneous and depends on the agency-specific parameters \( r \) (discounting) and \( c \) (risk aversion), as well as the depreciation rate of fundamentals \( k \). Therefore, the former policy is reminiscent of a standardized policy that does not depend on the characteristics of the agency relationship itself. Note that \( \psi_t^* < \bar{\psi} \) for all \( t \). As such, introducing agency-specific parameters and time dynamics into the accruals policy results in less timing error correction than the stationary case.

The following corollary to Theorem 3 outlines when it is not optimal to correct all of the timing errors in cash flows.

**Corollary 4.** The stationary accrual policy \( \bar{\psi} \) does not fully correct the timing error \( (\bar{\psi} < 1) \) if and only if

\[
\sigma_\lambda^2 > \frac{\sigma_\theta^2 \left( 2\sigma_\pi^2 + \sigma_\theta \left( \sigma_\theta + \sqrt{8\sigma_\pi^2 + \sigma_\theta^2} \right) \right)}{2 (\sigma_\pi^2 + \sigma_\theta^2)} \equiv \sigma_\lambda^*,
\]

for given \( (\sigma_\pi, \sigma_\theta) \).

The measurement error threshold (23) follows immediately from (21). Thus, the optimal stationary accrual policy calls for less than full correction of cash flow timing errors when the estimation error volatility exceeds some threshold that depends on \( \sigma_\pi \) and \( \sigma_\theta \). This is a result of the tradeoff embedded in the learning process: while correcting cash flow timing error helps eliminate timing error volatility, it introduces estimation error volatility that may negatively affect the learning process. If measurement error is sufficiently low, it will always be efficient to induce full correction of cash flow timing errors.

Note that the agent is indifferent between accrual policies because his IC condition (16) is independent of \( \psi \). Therefore, the pertinent preference over accrual policies in equilibrium is the principal’s. Which accrual policy will the principal choose to implement in equilibrium? Interestingly, Theorem 3 and the left panel of Figure 4 shows that neither accrual policy derived in Theorem 3 dominates always. The preference of accrual policy depends on long-term incentives \( \rho_t \), and is independent of conjectured fundamentals \( \hat{\pi}_t \). When long-term incentives are relatively unvested, \( (\rho_t \) is relatively high), the principal attains a higher value under the non-stationary accrual policy. On the other hand, as vesting occurs and \( \rho_t \) declines towards zero, the stationary accrual policy begins to dominate. By Theorem 2, this means

---

15See Definition II.4.1 in Fleming and Soner (2006). The use of viscosity solutions here is necessitated by the non-differentiability of \( J \) at \( \rho^* \).
that towards the beginning of the agency relationship, when incentives tend to be more
delayed, the principal will choose to implement the non-stationary accrual policy and will
find it desirable to switch to the stationary accrual policy when a certain amount of incentives
have vested.

The switching strategy between accrual policies here is rooted in the unobservability of
fundamental performance and the associated learning process. Recall that incentives are
most deferred when potential belief distortions between the principal and the agent are at
their highest, as the principal prefers to provide (short-term) incentives during times when
uncertainty over fundamentals is comparatively low. While long-term incentives are stochas-
tic and therefore costly to provide, measurement error has a relatively high marginal impact
when potential belief distortions are large. Put differently, when the agent’s scope for taking
advantage of his informational advantage over the principal is relatively high, measurement
error injects even more uncertainty into the agency problem that makes the principal’s
control problem more difficult. Therefore, relative to the stationary accrual policy, the non-
stationary policy is more effective at mitigating the harmful impact of measurement error
while allowing the principal to marginally decrease the duration of the agent’s incentives.
This is supported by Corollary 4. When \( \psi^*_t < \bar{\psi} \) (that is, both policies are not equal to one),
the magnitude of measurement error is high enough to make less correction of timing error
desirable during these times.

As the learning process progresses, the agent’s scope for taking advantage of his informa-
tional advantage declines and the principal will allow long-term incentives to vest. This
implies that the marginal impact of measurement error is comparatively low. However, the
lowered duration of incentives means that the proper allocation of fundamental performance
becomes increasingly significant. Thus, when long-term incentives have declined past the
threshold \( \rho^* \), the principal will shift to the stationary accrual policy, which corrects more
cash flow timing error than the non-stationary accrual policy. This enables the principal to
marginally increase the vesting of long-term incentives without distorting the agent’s effort
itself. Thus, the duration of pay and the implementation of accrual policies are inherently
linked. Theorem 3 states that the firm will make more non-GAAP adjustments to executive
compensation contracts when the duration of executive pay is relatively high.

Consider then the response of the principal’s value function to the two accrual policies
across the agency-specific parameters \( r, c, \) and \( k \). As shown in Figure 5, the choice of accrual
policy implementation is unaffected by discounting \( r \) or the agent’s risk aversion \( c \). On the
other hand, the principal prefers the stationary accrual policy for relatively high \( k \) and the
non-stationary accrual policy for lower \( k \). Put differently, ceteris paribus, for agencies in

---

\[16\] This follows from the concavity of the principal’s value function.
which the evolution of underlying firm fundamentals is highly persistent, the principal will
tend to choose to implement the stationary accrual policy, and vice-versa. This is because
a higher $k$ implies that shocks to fundamentals will tend to be less persistent. Thus, the
posterior variance decays quicker (see the proof of Lemma 2), and the reliance on the non-
stationary accrual policy to provide incentives diminishes as a result.

### 3.4.2 Accrual Policy Comparative Statics

Comparative statics for the stationary accrual policy $\overline{\psi}$ with respect to $\sigma_{\theta}$, $\sigma_{\lambda}$, and $\sigma_{\pi}$
are shown in Figure 6. While $\overline{\psi}$ is increasing in the timing error volatility $\sigma_{\theta}$, it is decreasing
in measurement error volatility $\sigma_{\lambda}$. For the parameter values used in Figure 6, it is also
decreasing in fundamental volatility $\sigma_{\pi}$. Interestingly, the optimal stationary accrual policy
is not one everywhere. The second panel shows that full correction of timing errors is not
necessary optimal for relatively high levels of measurement error. This is due to the trade-off
embedded in the SDE for earnings (3): more accruals fixes more of the timing error but also
introduces more estimation error into the earnings process.

However, the first and third panels of Figure 6 show that this intuition is somewhat
incomplete, as the relation between $\overline{\psi}$ and $\sigma_{\lambda}$ depends on $\sigma_{\pi}$ and $\sigma_{\theta}$ as well. This is stated
formally in the following corollary.

**Corollary 5.** Over the set $\{\sigma_{\lambda} : \sigma_{\lambda}^2 > \sigma_{\pi}^2\}$, $\overline{\psi}$ is strictly increasing in $\sigma_{\theta}$ and strictly decreasing
in $\sigma_{\lambda}$. If $\sigma_{\theta}^2 \leq \frac{\sigma_{\pi}^2 (3 + 2\sqrt{2})}{\sigma_{\pi}^2 + 2\sigma_{\theta}^2}$, then $\overline{\psi}$ is always decreasing in $\sigma_{\pi}$ over this set. Otherwise, $\overline{\psi}$ is
decreasing in $\sigma_{\pi}$ if $\sigma_{\lambda}^2$ satisfies

$$
\sigma_{\lambda}^2 > \frac{\sigma_{\theta}^2 (3 + 2\sqrt{2}) - \sigma_{\pi}^2}{\sigma_{\pi}^2 + \sigma_{\theta}^2} \equiv \overline{\sigma}_{\lambda}^2 > \overline{\sigma}_{\lambda}^2.
$$

Finally, the non-stationary accrual policy $\{\psi_{t}^{*}\}_{t \geq 0}$ is increasing in $(r, c)$, and is decreasing
in $k$.

For the set of values such that $\overline{\psi}$ is less than one, the accrual policy is increasing in the
volatility of timing error $\sigma_{\theta}$ and decreasing in the volatility of measurement error $\sigma_{\lambda}$. This is
due to the fundamental tradeoff in our model: correcting timing error improves efficiency by
enhancing the learning process but comes at the cost of introducing estimation error. The
optimal accrual policy will balance these considerations when correcting cash flow timing
error.

The second part of Corollary 5 states that the amount of timing error correction is
decreasing in the volatility of firm fundamentals $\sigma_{\pi}$ for most values of $\sigma_{\pi}$, $\sigma_{\theta}$, and $\sigma_{\lambda}$. Recall
that $\sigma_{\pi}$ represents the agent’s lack of control over the underlying asset. Thus, the fact
that the accrual policy is decreasing in $\sigma_{\pi}$ means that the accounting system will optimally correct less timing error the less control the manager has over fundamentals. However, this is not always the case. If $\sigma_{\theta}$ is high enough relative to $\sigma_{\pi}$ and $\sigma_{\lambda}$ is relatively low, $\sigma_{\lambda}^2 \in (\bar{\sigma}_{\lambda}^2, \bar{\sigma}_{\lambda}^2)$, then the amount of timing error correction is actually increasing in $\sigma_{\pi}$. This is due to the co-mingling of fundamental and cash flow timing error volatilities when forming beliefs. Specifically, when forming beliefs over fundamentals, for a given cash flow shock, the principal (and the agent) can never be certain if this shock was due to fundamental volatility or a cash flow timing error surprise. For a high $\sigma_{\theta}$ relative to $\sigma_{\pi}$, the shock was more likely than not due to cash flow timing errors. In this case, the efficiency gain from increasing the correction of timing errors swamps the induced decline from a lack of controllability. Furthermore, when $\sigma_{\lambda}^2 < \bar{\sigma}_{\lambda}^2$, the countervailing effect of introducing measurement error is comparatively weak. Thus, it is only in this case where a marginal increase in $\sigma_{\pi}$ induces a higher correction of cash flow timing errors. The implication here is that the “controllability” role of accrual accounting depends crucially on the characteristics of cash flow timing error volatility and measurement error volatility.

4 Conclusion

We develop a fully dynamic framework of accounting accruals to study the impact of performance measurement on learning and productive activity in the face of uncertainty over fundamental performance and moral hazard. Our model explicitly incorporates the role of intertemporal accruals in correcting timing errors in cash flows for the purposes of more efficient performance measurement. Our framework is flexible enough to capture properties of many accounting standards seen in practice that correct the timing errors in cash flows over time caused by the misallocation of fundamental performance.

We apply this framework to examine the role of accounting accruals in improving productive and contracting efficiency in a long-run principal-agent setting with uncertainty over fundamental performance. If measurement error is small relative to the timing error, then the optimal accrual policy corrects more timing errors in equilibrium, which in turn allows the principal to implement an incentive scheme that is less stochastic and more short-term in nature. We show that the optimal stationary accrual policy does not depend on agency-specific parameters and is therefore well-suited for GAAP. On the other hand, the optimal non-stationary accrual policy changes over time and depends on risk-aversion, time discounting, and the depreciation of underlying firm fundamentals.

We show that the implementation of accrual policies for contracting purposes depends on the dynamics of incentive provision, which is driven by the learning process over fun-
damental performance. We do so by establishing a novel link between incentive dynamics and the implementation of accounting accrual policies. The principal will choose to implement the non-stationary accrual policy earlier on in the agency relationship when the agent’s long-term incentives are relatively more deferred. As the learning process over fundamental performance progresses and incentives vest as a result, the principal will switch to the stationary accrual policy when deferred incentives fall below a certain threshold. Thus, we show that a standardized accrual policy is particularly well-suited to fulfill a stewardship role for agents that are later in their tenures. We are therefore able to provide a novel theoretical justification for non-GAAP adjustments and accrual omissions in compensation contracts that depend on managerial tenure.
References


Marinovic, Iván and Felipe Varas (2019). “CEO Horizon, Optimal Pay Duration, and the
Seminar on Stochastic Analysis. Ed. by David Nualart and Marta Sanz Solé. Basel:
Pham, Huyên (2009). Continuous-time Stochastic Control and Optimization with Financial
Applications. Berlin: Springer-Verlag.
2nd ed. Boca Raton, FL: Chapman and Hall/CRC.
Prat, Julien and Boyan Jovanovic (2014). “Dynamic Contracts when the Agent’s Quality is
Wagenhofer, Alfred (2003). “Accrual-Based Compensation, Depreciation and Investment
Yong, Jiongmin and Xun Yu Zhou (1999). Stochastic Controls: Hamiltonian Systems and
Appendices

A Proofs

Proof of Lemma 1

Proof. First consider the timing error component of accruals. Changing the order of integration in equation (2) using Fubini’s theorem,

\[ \theta_t = \mu_\theta t - \int_0^t (1 - e^{-\kappa(t-s)}) \theta_s ds + \sigma_\theta B^\theta_t, \]

which is a stochastic Volterra integral equation of the second kind.\(^\text{17}\) Its solution is

\[ \begin{aligned}
\theta_t &= \mu_\theta t + \sigma_\theta B^\theta_t - \frac{2\kappa}{\sqrt{\kappa(4-\kappa)}} \int_0^t e^{-\frac{\kappa}{2}(t-s)} \sin \left( \frac{1}{2} \sqrt{\kappa(4-\kappa)}(t-s) \right) \left( \mu_\theta s + \sigma_\theta B^\theta_s \right) ds \\
&= \mu_\theta \left( 1 - e^{-\frac{\kappa}{2}t} \right) \left( \cosh \left( \frac{\sqrt{\kappa(4-\kappa)}}{2} t \right) - \frac{(2 - \kappa) \sinh \left( \frac{\sqrt{\kappa(4-\kappa)}}{2} t \right)}{\sqrt{\kappa(4-\kappa)}} \right) \\
&\quad + \sigma_\theta B^\theta_t - \frac{2\kappa}{\sqrt{\kappa(4-\kappa)}} \int_0^t e^{-\frac{\kappa}{2}(t-s)} \sin \left( \frac{\sqrt{\kappa(4-\kappa)}}{2} (t-s) \right) \sigma_\theta B^\theta_s ds.
\end{aligned} \]

Simplifying and using the limiting properties of Brownian motion yields \( \lim_{t \to \infty} \theta_t = 0. \) Integrating by parts,

\[ \mathbb{E} [d\theta_t] = e^{-\frac{\kappa}{2}t} \mu_\theta \left( \cos \left( \frac{\sqrt{\kappa(4-\kappa)}}{2} t \right) + \kappa \sin \left( \frac{\sqrt{\kappa(4-\kappa)}}{2} t \right) \right) dt. \]

It then follows that \( \lim_{t \to \infty} \mathbb{E} [d\theta_t] = 0. \)

\(^{17}\)The solution concept used here applies the solution formulation for deterministic Volterra integral equations (Polyanin and Manzhirov 2008) to stochastic Volterra equations. See Øksendal and Zhang (1993) for a more thorough treatment of the stochastic form.
Proof of Lemma 2

Proof. To apply the multidimensional version of the Kalman-Bucy filter, write the observation processes representing cash flows and accounting earnings as

\[
\begin{bmatrix}
\frac{dC_t}{dt}
\
\frac{dE_t}{dt}
\end{bmatrix}
= \begin{bmatrix}
a_t^* + (\mu_\theta - \kappa \Theta_t)
\end{bmatrix}
- \begin{bmatrix}
k
\end{bmatrix}
\pi_t
+ \begin{bmatrix}
\sigma_\pi
\
\sigma_\pi
\end{bmatrix}
\begin{bmatrix}
\sigma_\theta
0
\end{bmatrix}
\begin{bmatrix}
\sigma_\theta
(1 - \psi)
\sigma_\psi
\end{bmatrix}
\begin{bmatrix}
\sigma_\theta
\sigma_\psi
\end{bmatrix}
\begin{bmatrix}
\sigma_\theta
\sigma_\psi
\end{bmatrix}
\begin{bmatrix}
dB_t^\pi

\end{bmatrix}
.
\]

Then, by stochastic filtering theory, the principal’s posterior mean over fundamental performance evolves according to

\[
d\hat{\pi}^a_t = (a_t^* - k\hat{\pi}^a_t) dt
+ \begin{bmatrix}
\sigma_\pi
\
\sigma_\pi
\end{bmatrix}
T
\begin{bmatrix}
\sigma_\pi
\sigma_\pi
\end{bmatrix}
+ \begin{bmatrix}
\sigma_\theta
0
\end{bmatrix}
\begin{bmatrix}
\sigma_\theta
(1 - \psi)
\sigma_\psi
\end{bmatrix}
\begin{bmatrix}
\sigma_\theta
\sigma_\psi
\end{bmatrix}
\begin{bmatrix}
dC_t
\end{bmatrix}
+ \begin{bmatrix}
\sigma_\theta
\sigma_\psi
\end{bmatrix}
\begin{bmatrix}
dE_t
\end{bmatrix}

= (a_t^* - k\hat{\pi}^a_t) dt
+ \frac{(\sigma_\pi^2 - k^2 \hat{\pi}_t^a)}{\sigma_\theta^2 \sigma_\pi^2 + \sigma_\pi^2 (\sigma_\theta^2 + \sigma_\psi^2) \psi} d\hat{Z}_t^C
+ \frac{\sigma_\theta^2 (\sigma_\pi^2 - k^2 \hat{\pi}_t^a)}{\sigma_\theta^2 \sigma_\pi^2 + \sigma_\pi^2 (\sigma_\theta^2 + \sigma_\psi^2) \psi} d\hat{Z}_t^E,
\]

where \( \hat{Z}_t^C \) and \( \hat{Z}_t^E \) are the innovation processes with respect to cash flows and earnings defined by

\[
d\hat{Z}_t^C = dC_t - (a_t^* - k\hat{\pi}^a_t + (\mu_\theta - \kappa \Theta_t)) dt,
\]

\[
d\hat{Z}_t^E = dE_t - (a_t^* - k\hat{\pi}^a_t + (1 - \psi)(\mu_\theta - \kappa \Theta_t) + \psi (\mu_\lambda - \eta \Lambda)) dt.
\]

By Lévy’s theorem, the following scaled innovation processes are standard Brownian motions:

\[
\frac{1}{\sqrt{\sigma_\pi^2 + \sigma_\theta^2}} d\hat{Z}_t^C,
\]

\[
\frac{1}{\sqrt{\sigma_\pi^2 + (1 - \psi)^2 \sigma_\theta^2 + \psi^2 \sigma_\psi^2}} d\hat{Z}_t^E.
\]

The conditional variance satisfies a Riccati ordinary differential equation (ODE):

\[
\frac{d\hat{\pi}_t^{a*}}{dt} = -2k\hat{\pi}_t^{a*} + \sigma_\pi^2 - \frac{(\sigma_\pi^2 + \sigma_\lambda^2) (\sigma_\pi^2 - k\hat{\pi}_t^{a*})^2}{\sigma_\theta^2 \sigma_\pi^2 + \sigma_\pi^2 (\sigma_\theta^2 + \sigma_\lambda^2)}.
\]

\[\text{See Theorem 12.7 in Liptser and Shiryaev (2001).}\]
The solution to this ODE is
\[
\phi^- = \frac{2a_0 - \phi^-}{\gamma_0 - \phi^-} \exp \left\{ \frac{2k_0}{\sqrt{\sigma_0^2 + \sigma_0^2 (\sigma_0^2 + \sigma_0^2)}} t \right\}, \\
\gamma_t = \frac{1}{1 - \frac{2a_0 - \phi^-}{\gamma_0 - \phi^-}} \exp \left\{ \frac{2k_0}{\sqrt{\sigma_0^2 + \sigma_0^2 (\sigma_0^2 + \sigma_0^2)}} t \right\},
\]
where
\[
\phi^+ = \frac{\sigma_0 \lambda \left( \sqrt{\sigma_0^2 + \sigma_0^2 (\sigma_0^2 + \sigma_0^2)} - \sigma_0 \lambda \right)}{k (\sigma_0^2 + \sigma_0^2)}, \quad \phi^- = -\frac{\sigma_0 \lambda \left( \sqrt{\sigma_0^2 + \sigma_0^2 (\sigma_0^2 + \sigma_0^2)} + \sigma_0 \lambda \right)}{k (\sigma_0^2 + \sigma_0^2)}.
\]

An identical procedure yields the agent’s posterior mean and variance processes. The innovation processes \( \hat{Z}^C = \{ \hat{Z}^C_t \}_{t \geq 0} \) and \( \hat{Z}^E = \{ \hat{Z}^E_t \}_{t \geq 0} \) are defined analogously for the agent, where \( \frac{1}{\Sigma_C} \hat{Z}^C_t \) and \( \frac{1}{\Sigma_E(\psi)} \hat{Z}^E_t \) are Brownian motions from the agent’s perspective. Thus, the principal’s and agent’s beliefs are updated recursively according to
\[
d\hat{\pi}^C_t = (a_t - k \hat{\pi}^C_t) \, dt + \beta_t^C(\psi) \, d\hat{Z}^C_t + \beta_t^E(\psi) \, d\hat{Z}^E_t, \\
d\hat{\pi}^E_t = (a_t - k \hat{\pi}^E_t) \, dt + \beta_t^C(\psi) \, d\hat{Z}^C_t + \beta_t^E(\psi) \, d\hat{Z}^E_t,
\]
respectively. The cash flow and earnings sensitivity parameters are \( \beta_t^C(\psi) = \beta_t(\psi)(\sigma_0^2 \psi - \sigma_0^2 (1 - \psi)) \) and \( \beta_t^E(\psi) = \beta_t(\psi) \sigma_0^2 \), respectively, where \( \beta_t(\psi) \) is the aggregate conditional covariance defined as
\[
\beta_t(\psi) = \frac{\sigma_0^2 - k \gamma_t}{(\sigma_0^2 + \sigma_0^2) + \sigma_0^2 \lambda \psi}.
\]
Lévy’s theorem implies that the scaled innovation processes \( \hat{B}^C_t \) and \( \hat{B}^E_t \) evolving as
\[
d\hat{B}^C_t = \frac{1}{\Sigma_C} d\hat{Z}^C_t, \quad d\hat{B}^E_t = \frac{1}{\Sigma_E(\psi)} d\hat{Z}^E_t,
\]
are standard Brownian motions under the principal’s information set where the volatility parameters are \( \Sigma_C = \sqrt{\sigma_0^2 + \sigma_0^2} \) and \( \Sigma_E(\psi) = \sqrt{\sigma_0^2 + (1 - \psi)^2 \sigma_0^2 + \psi^2 \sigma_0^2} \). Thus, the principal’s posterior belief updating process can be written as
\[
d\hat{\pi}^C_t = (a_t - k \hat{\pi}^C_t) \, dt + S_t \, d\hat{B}_t,
\]
where
\[
S_t = \begin{bmatrix} \beta_t^C(\psi) \Sigma_C & 0 \\ 0 & \beta_t^E(\psi) \Sigma_E(\psi) \end{bmatrix}, \quad \hat{B}_t = \begin{bmatrix} \hat{B}^C_t \\ \hat{B}^E_t \end{bmatrix}.
\]
the weights on cash flow and earnings surprises are

\[ \beta^C_t(\psi) = \frac{(\sigma^2_\lambda - \sigma^2_\theta (1 - \psi)) (\sigma^2_\theta - k \gamma^*_a)}{\sigma^2_\theta \sigma^2_\lambda + \sigma^2_\pi (\sigma^2_\theta + \sigma^2_\lambda)} \psi, \quad \beta^E_t(\psi) = \frac{\sigma^2_\theta (\sigma^2_\theta - k \gamma^*_a)}{\sigma^2_\theta \sigma^2_\lambda + \sigma^2_\pi (\sigma^2_\theta + \sigma^2_\lambda)} \psi, \]

respectively, and \( \hat{\mathbf{B}}_t \) is a bidimensional vector of correlated Brownian motions. The covariance process of \( \hat{\mathbf{B}}_t \) is

\[ \mathbf{C} = \begin{bmatrix} 1 & \frac{\sigma^2_\theta (1 - \psi) \sigma^2_\theta}{\Sigma \Sigma E(\psi)} \\ \frac{2 \Sigma + (1 - \psi) \sigma^2_\theta}{\Sigma \Sigma E(\psi)} & 1 \end{bmatrix}. \]

Let \( \mathbf{D}_t \) be the Cholesky decomposition of \( \mathbf{S}_t \mathbf{C} \mathbf{S}_t^T \); it is the upper triangular matrix

\[ \mathbf{D}_t = \begin{bmatrix} \Sigma \beta^C_t(\psi) & \frac{\beta^E_t(\psi) (\sigma^2_\theta (1 - \psi) \sigma^2_\theta)}{\Sigma} \\ 0 & \frac{\beta^E_t(\psi) \Sigma \Sigma E(\psi)}{\Sigma} \end{bmatrix}. \]

Define the bidimensional process \( \hat{\mathbf{Z}}_t \) evolving according to

\[ d\hat{\mathbf{Z}}_t = \mathbf{D}_t^{-1} \mathbf{S}_t d\hat{\mathbf{B}}_t, \]

such that the principal’s belief updating process becomes

\[ d\hat{\pi}^*_a = (a^*_t - k \hat{\pi}^*_a) \, dt + \mathbf{D}_t d\hat{\mathbf{Z}}_t. \]

Then, since

\[ (d\mathbf{Z}_t) (d\mathbf{Z}_t)^T = (\mathbf{D}_t^T)^{-1} \mathbf{S}_t \mathbf{C} \mathbf{S}_t^T \mathbf{D}_t^{-1} = \mathbf{I} dt, \]

where \( \mathbf{I} \) is the bidimensional identity matrix, by Lévy’s theorem, \( \mathbf{Z} \) is a vector of independent Brownian motions. Using the Cholesky decomposition \( \mathbf{D}_t \) and another Lévy transformation gives the result.

**Proof of Corollary 1**

**Proof.** By Theorem 12.1 in Liptser and Shiryaev (2001), the principal’s posterior mean without accruals evolves according to the SDE

\[ d\hat{\pi}^*_a = (a^*_t - k \hat{\pi}^*_a) \, dt + \frac{\sigma^2_\pi - k \gamma^*_a}{\sigma^2_\theta + \sigma^2_\theta} \left[ d\mathbf{C}_t - (a^*_t - k \pi^*_a + (\theta_t - \kappa \Theta_t)) \, dt \right] \]

\[ = (a^*_t - k \hat{\pi}^*_a) \, dt + \frac{\sigma^2_\pi - k \gamma^*_a}{\sigma^2_\theta + \sigma^2_\theta} d\hat{\mathbf{Z}}_t, \]
and the conditional variance satisfies the ODE

\[
\frac{d\gamma^a_t}{dt} = -2k\gamma^a_t + \sigma^2_\pi - \frac{(\sigma^2_\pi - k\gamma^a_t)^2}{\sigma^2_\pi + \sigma^2_\theta}.
\]

As before, the solution to this ODE is

\[
\gamma^a_t = \frac{\phi^- - \frac{\gamma^a_0 - \phi^-}{\gamma^a_0 - \phi^+} \phi^+ \exp \left\{ \frac{-2k\sigma_\theta}{\sqrt{\sigma^2_\pi + \sigma^2_\theta}} t \right\}}{1 - \frac{\gamma^a_0 - \phi^-}{\gamma^a_0 - \phi^+} \exp \left\{ \frac{-2k\sigma_\theta}{\sqrt{\sigma^2_\pi + \sigma^2_\theta}} t \right\}},
\]

where

\[
\phi^+ = \frac{\sigma_\theta \left( \sqrt{\sigma^2_\pi + \sigma^2_\theta} - \sigma_\theta \right)}{k}, \quad \phi^- = -\frac{\sigma_\theta \left( \sqrt{\sigma^2_\pi + \sigma^2_\theta} + \sigma_\theta \right)}{k}.
\]

The conditional variance relation \( \gamma < \gamma \) follows since

\[
\gamma = \frac{(\sigma^2_\theta \sigma^2_\lambda + \sigma^2_\pi (\psi^2 \sigma^2_\theta + \sigma^2_\lambda)) \sqrt{\frac{\sigma^2_\theta \sigma^2_\lambda}{\sigma^2_\pi \sigma^2_\theta + \sigma^2_\lambda \psi^2 \sigma^2_\theta + \sigma^2_\lambda}} - \sigma^2_\theta \sigma^2_\lambda}{k \left( \psi^2 \sigma^2_\theta + \sigma^2_\lambda \right)}
\]

\[
< \frac{(\sigma^2_\theta \sigma^2_\lambda + \sigma^2_\pi (\sigma^2_\theta + \sigma^2_\lambda)) \sqrt{\frac{\sigma^2_\theta}{\sigma^2_\pi + \sigma^2_\theta}} - \sigma^2_\theta \sigma^2_\lambda}{k \left( \sigma^2_\theta + \sigma^2_\lambda \right)} < \frac{(\sigma^2_\pi + \sigma^2_\theta) \sqrt{\frac{\sigma^2_\theta}{\sigma^2_\pi + \sigma^2_\theta}} - \sigma^2_\theta}{k} = \gamma,
\]

where the last inequality follows because \( \sqrt{\frac{\sigma^2_\theta}{\sigma^2_\pi + \sigma^2_\theta}} < 1. \)

\[\square\]

**Proof of Corollary 2**

**Proof.** Using the proof of Corollary 1, the belief distortions in the benchmark setting without accounting accruals evolve according to

\[
d\Delta^\pi_t = ((a_t - a^*_t) - k\Delta^\pi_t) dt + \frac{\sigma^2_\pi - k\gamma^a_t}{\sigma^2_\pi + \sigma^2_\theta} \left( k\Delta^\pi_t - (a_t - a^*_t) \right) dt
\]

\[
= L_t (a_t - a^*_t - k\Delta^\pi_t) dt,
\]

where

\[
L_t = \frac{\sigma^2_\theta + k\gamma^a_t}{\sigma^2_\pi + \sigma^2_\theta}
\]
is the learning parameter in the benchmark case. By Corollary 1, $L_t > L_t$. Then, again by Corollary 1,

$$L_t e^{-\int_s^t k \gamma_a \, du} \geq L_t \left(1 - k \gamma_a(t - s)\right) = L_t \frac{k \sigma_0^2(t - s)}{\sigma_0^2 + \sigma_0^2} > \frac{L_t}{1 + k L_t(t - s)} \geq L_t e^{-\int_s^t k L_u \, du},$$

for all $0 \leq s \leq t$, which implies that $|\Delta^\pi_t| > |\Delta^\pi_t|$.

\[\square\]

**Proof of Theorem 1**

Proof. We solve the agent’s problem by the change of measure approach applied to multiple output signals.\(^{19}\) Suppose the optimal contract prescribes the action process $a^\star = \{a^\star_t\}_{t \geq 0}$ but that the agent deviates to $a = \{a_t\}_{t \geq 0}$. Let $\Delta^a_t = a_t - a^\star_t$ be the deviation process over the agent’s action, $\Delta^\pi_t = S_t - S^\star_t$ the deviation process over the agent’s consumption, and $\Delta^\pi_t = \tilde{\pi}_t^a - \tilde{\pi}_t^a$ the belief distortion process. Define the Brownian motions

$$d\hat{Z}_t = \frac{1}{\Sigma_t(\psi)} \left( \beta^C_t(\psi) d\hat{Z}^C_t + \beta^E_t(\psi) d\hat{Z}^E_t \right),$$

$$d\tilde{Z}_t = \frac{1}{\Sigma_t(\psi)} \left( \beta^C_t(\psi) d\tilde{Z}^C_t + \beta^E_t(\psi) d\tilde{Z}^E_t \right).$$

By Lemma 2, $\hat{Z}_t$ and $\tilde{Z}_t$ are indeed Brownian motions. By the multidimensional change-of-measure approach, such a deviation induces a relative change of measure by

$$d\hat{Z}_t - d\tilde{Z}_t = \frac{1}{\Sigma_t(\psi)} \left( \beta^C_t(\psi) \left( d\hat{Z}^C_t - d\tilde{Z}^C_t \right) + \beta^E_t(\psi) \left( d\hat{Z}^E_t - d\tilde{Z}^E_t \right) \right)$$

$$= \frac{1}{\Sigma_t(\psi)} (1 - L_t) (\Delta^a_t - k \Delta^\pi_t) \, dt,$$

where $L_t = 1 - \beta^C_t(\psi) - \beta^E_t(\psi)$ is the learning parameter defined in (7). Define the density process

$$\mathcal{E}_t = \exp \left\{ \int_0^t \frac{1}{\Sigma_s(\psi)} (1 - L_s) (\Delta^a_s - k \Delta^\pi_s) \, d\tilde{Z}_s - \frac{1}{2} \int_0^t \frac{1}{\Sigma^2_s(\psi)} ((1 - L_s) (\Delta^a_s - k \Delta^\pi_s)^2) \, ds \right\}.$$

By Itô’s lemma, $\mathcal{E}_t$ follows the forward SDE

$$\begin{cases}
    d\mathcal{E}_t = \frac{1}{\Sigma_t(\psi)} (1 - L_t) \mathcal{E}_t (\Delta^a_t - k \Delta^\pi_t) \, d\hat{Z}_t,
    \\
    \mathcal{E}_0 = 1.
\end{cases}$$

\(^{19}\)The change of measure approach is outlined in Williams (2015), with the multidimensional case outlined in Lee (2020).
Since $\Delta^\pi$ is a square-integrable process, Novikov’s condition is satisfied, making $\mathcal{E}_t$ an exponential martingale. Then, the agent’s problem can be written in terms of the aforementioned deviation processes as

$$
\max_{\Delta^a, \Delta^S} \mathbb{E}^{\pi^* + \Delta^a} \left[ \int_0^\infty e^{-rt} \mathcal{E}_t u(S_t, a^*_t + \Delta^a_t) dt \right],
$$

s.t. $d\Delta^\pi_t = L_t (\Delta^a_t - k\Delta^\pi_t) dt$,
$$
d\mathcal{E}_t = \frac{1}{\Sigma_t(\psi)} (1 - L_t) \mathcal{E}_t (\Delta^a_t - k\Delta^\pi_t) d\tilde{Z}_t,
$$
$$
dS_t = (rS_t - \Delta^S_t) dt.
$$

This is an optimal stochastic control problem that can be solved via the stochastic maximum principle. Define $p^\pi, p^\mathcal{E}$ as the multipliers on $\Delta^\pi$ and $\mathcal{E}$, respectively. They have volatility parameters $\nu^\pi$ and $\nu^\mathcal{E}$. Let $\Delta^S$ be the deviation process over consumption with multiplier $p^S$. Then, the current-value Hamiltonian for the agent’s problem is

$$
\mathcal{H}(\Delta^a, \Delta^\pi, \Delta^S, \mathcal{E}, p^\pi, p^S, \nu^\mathcal{E}) = \mathcal{E} u(S + \Delta^S, a + \Delta^a) + p^\pi L (\Delta^a - k\Delta^\pi) + p^S (rS - \Delta^S) + \nu^\mathcal{E} \frac{1}{\Sigma(\psi)} (1 - L) \mathcal{E} (\Delta^a - k\Delta^\pi).
$$

Differentiating the Hamiltonian with respect to $\Delta^a$ and $\Delta^S$ at $\Delta^a = \Delta^S = 0$ yields the necessary first-order conditions

$$
-u_a(S_t, a_t) = L_t p_t^\pi + (1 - L_t) \frac{1}{\Sigma_t(\psi)} \nu_t^\mathcal{E},
$$
$$
-cu(S_t, a_t) = p_t^S,
$$

since $u_a(S_t, a_t) = ca_t u(S_t, a_t)$, $u_S(S_t, a_t) = -cu(S_t, a_t)$, and $\mathcal{E}_t = 1$ at $\Delta^a_t = \Delta^S_t = 0$. The multipliers on the state variables ($\Delta^\pi, \mathcal{E}$) follow the laws of motion

$$
dp_t^\pi = (r + kL_t) p_t^\pi dt + \frac{1}{\Sigma_t(\psi)} k (1 - L_t) \nu_t^\mathcal{E} dt + \nu_t^\pi d\tilde{Z}_t,
$$
$$
dp_t^\mathcal{E} = (rp_t^\mathcal{E} - u(S_t, a_t)) dt + \nu_t^\mathcal{E} d\tilde{Z}_t.
$$
Set $p^\pi_t = W_t$ and $\nu^\pi_t = -r\Sigma_t(\psi)W_t\delta_t$. Substituting these values into the SDE for $p^\pi_t$ gives

$$
dp^\pi_t = (r + k\delta_t) p^\pi_t dt + \frac{1}{\Sigma_t(\psi)} k (1 - L_t) \nu^\pi_t dt + \nu^\pi_t d\tilde{Z}_t
$$

$$
= (r + k\delta_t) p^\pi_t dt - rcW_t k (1 - L_t) \delta_t dt + \nu^\pi_t d\tilde{Z}_t
$$

$$
= (rp^\pi_t - ku_a(S_t,a_t)) dt + \nu^\pi_t d\tilde{Z}_t.
$$

Set $\nu^\pi_t = -r\Sigma_t(\psi)W_t\nu^\rho_t$ for some progressively measurable process $\nu^\rho = \{\nu^\rho_t\}_{t\geq 0}$, which implies that

$$
p^\pi_t = \mathbb{E}^a_t \left[ \int_t^\infty e^{-\int_s^t (r + k\delta_u) du} rcW_s k (1 - L_s) \delta_s ds \right] = \mathbb{E}^a_t \left[ \int_t^\infty e^{-r(s-t)} ku_a(S_s,a_s) ds \right].
$$

The following lemma is a standard result with CARA preferences (He 2011).

**Lemma A1.** With CARA preferences, the optimal no-savings contract satisfies

$$
rW_t = u(S_t,a_t).
$$

(27)

It is without loss of generality to restrict attention to no-savings contracts.

By Lemma A1 and using the derivations above, the first-order condition with respect to effort (25) can be written as

$$
rcW_t a_t = (1 - L_t) rcW_t \delta_t - L_t p^\pi_t.
$$

Furthermore, the agent’s continuation value follows the Brownian martingale

$$
dW_t = -rcW_t \delta_t \Sigma_t(\psi) d\tilde{Z}_t.
$$

(28)

Thus, by the scale-invariance property of negative exponential utility, the state space can be condensed from $(W_t,p^\pi_t)$ to $\rho_t$, where $\rho_t = p_t/(-rcW_t)$. Integrating by parts using (28),

$$
d\rho_t = (r + k\delta_t) \rho_t dt + \delta_t \left[ k (1 - L_t) + r\Sigma_t(\psi) (\nu^\rho_t + rc\delta_t \rho_t) \right] dt + \Sigma_t(\psi) (\nu^\rho_t + rc\delta_t \rho_t) d\tilde{Z}_t,
$$

which yields the desired result by setting $\nu_t = \Sigma_t(\psi)(\nu^\rho_t + rc\delta_t \rho_t)$ and noting that $\tilde{Z}_t = \hat{Z}_t$ in equilibrium. Rearranging the first-order condition gives the expression (16) for the agent’s action. The following theorem gives the conditions for sufficiency in our setting.

**Theorem A1.** The following condition is required for the agent’s effort policy (16) to be sufficient for global incentive-compatibility. For a given accrual policy $\psi$ imposed by the
principal,

\[ |\nu_t| \leq \begin{cases} 
L_t y_t(\psi) + \sqrt{x_t(\psi)(r + L_t)}(r + 2kL_t) + L_t^2 y_t^2(\psi) \\
2x_t(\psi)L_t 
\end{cases} \] 

if \( rc\Sigma_t^2(\psi) > 3(1 - L_t)^2 \),

otherwise,

where

\[ x_t(\psi) = \frac{1}{2} \left( rc - 3 \frac{(1 - L_t)^2}{\Sigma_t^2(\psi)} \right), \quad y_t(\psi) = \frac{1}{2\Sigma_t(\psi)} \left( 1 + \frac{r + 2kL_t}{L_t} \right), \]

for all times \( t \geq 0 \).

Proof. The proof of sufficiency is by construction and is analogous to those in Sannikov (2014) and He et al. (2017). Specifically, we fix an optimal compensation/effort policy \( \{(S_t, a_t)\}_{t \geq 0} \) and construct an upper bound on the agent’s continuation payoff following any effort deviation from the optimal policy \( a_t \) of the form

\[ W^\Delta_i = W_i \exp \left\{ -rc \left( S_t + \rho_t \Delta_t^\pi + \frac{1}{2} K(\Delta_t^\pi)^2 \right) \right\}, \quad (29) \]

where \( \Delta_t^a = a_t^\Delta - a_t, \ a^\Delta = \{a_t^\Delta\}_{t \geq 0} \) is an arbitrary deviation policy, and \( \Delta_t^\pi = \hat{\pi}_t^a - \hat{\pi}_t^a \) is the resulting difference in beliefs. \( K \) is a positive constant chosen appropriately such that \( W^\Delta_i \) is the upper bound on the agent’s deviation payoff. Let \( P^\Delta \) be the probability measure induced by this deviation. For the arbitrary deviation strategies \( \{(S_t^\Delta, a_t^\Delta)\}_{t \geq 0} \), define the agent’s total gain function

\[ G^\Delta_t = \int_0^t e^{-rs} u(S_s^\Delta, a_s^\Delta) ds + e^{-rt} W^\Delta_t. \]

The goal is to show that \( G^\Delta = \{G^\Delta_t\}_{t \geq 0} \) is a \( P^\Delta \)-supermartingale. Under \( P^\Delta \), the process \( Z^\Delta = \{Z^\Delta_t\}_{t \geq 0} \) defined by

\[ Z^\Delta_t = Z_t - \int_0^t \frac{1}{\Sigma_s(\psi)} (1 - L_s) (\Delta_s^a - k \Delta_s^\pi) ds, \]

is a standard Brownian motion. Under the probability measure \( P^\Delta \), the relevant state
variables evolve as

\[ dW_t = -rcW_t \delta_t (1 - L_t) (\Delta_t^a - k\Delta_t^\pi) dt - rcW_t \delta_t \Sigma_t(\psi) dZ_t^\Delta, \]

\[ d\rho_t = \left( (r + kL_t) \rho_t + \delta_t [k (1 - L_t) + rc\Sigma_t(\psi)\nu_t] + \frac{\nu_t}{\Sigma_t(\psi)} (1 - L_t) (\Delta_t^a - k\Delta_t^\pi) \right) dt + \nu_t dZ_t^\Delta, \]

\[ dS_t = (rS_t - \Delta_t^S) dt, \]

\[ d\Delta_t^\pi = L_t (\Delta_t^a - k\Delta_t^\pi) dt, \]

where \( \Delta_t^S = S_t^\Delta = S_t \). Then, by the construction of the upper bound (29),

\[ dW_t^\Delta = \frac{W_t^\Delta}{W_t} dW_t + W_t d \left( \frac{W_t^\Delta}{W_t} \right) + r^2 c^2 \frac{W_t^\Delta}{W_t} \delta_t \Sigma_t(\psi) \Delta_t^\pi \nu_t dt \]

\[ = -rcW_t^\Delta \left\{ \delta_t ((1 - L_t) (\Delta_t^a - k\Delta_t^\pi) dt + \Sigma_t(\psi) dZ_t^\Delta) + dS_t + \Delta_t^\pi d\rho_t + (K \Delta_t^\pi + \rho_t) d\Delta_t^\pi - \frac{rc}{2} (\Delta_t^\pi)^2 \nu_t^2 dt - rc\delta_t \Sigma_t(\psi) \Delta_t^\pi \nu_t dt \right\}. \]

Algebraic manipulation then shows that the drift of \( G_t^\Delta \) satisfies

\[ e^{rt} \frac{dG_t^\Delta}{dt} = rW_t^\Delta \left[ \exp \left\{ -c \left( \Delta_t^S - \frac{1}{2} (a_t^\Delta)^2 - a_t^2 \right) + rc \left( S_t + \rho_t \Delta_t^\pi + \frac{1}{2} K (\Delta_t^\pi)^2 \right) - c \left( \delta_t (1 - L_t) (\Delta_t^a - k\Delta_t^\pi) + (rS_t - \Delta_t^S) + L_t (K \Delta_t^\pi + \rho_t) (\Delta_t^a - k\Delta_t^\pi) + \Delta_t^\pi \left( (r + kL_t) \rho_t + \delta_t [k (1 - L_t) + rc\Sigma_t(\psi)\nu_t] + \frac{\nu_t}{\Sigma_t(\psi)} (1 - L_t) (\Delta_t^a - k\Delta_t^\pi) \right) - \frac{rc}{2} (\Delta_t^\pi)^2 \nu_t^2 - rc\delta_t \Sigma_t(\psi) \Delta_t^\pi \nu_t \right] \right] \]

\[ \leq rW_t^\Delta \left[ -c \left( \Delta_t^S - \frac{1}{2} (a_t^\Delta)^2 - a_t^2 \right) + rc \left( S_t + \rho_t \Delta_t^\pi + \frac{1}{2} K (\Delta_t^\pi)^2 \right) - c \left( \delta_t (1 - L_t) (\Delta_t^a - k\Delta_t^\pi) + (rS_t - \Delta_t^S) + L_t (K \Delta_t^\pi + \rho_t) (\Delta_t^a - k\Delta_t^\pi) + \Delta_t^\pi \left( (r + kL_t) \rho_t + \delta_t [k (1 - L_t) + rc\Sigma_t(\psi)\nu_t] + \frac{\nu_t}{\Sigma_t(\psi)} (1 - L_t) (\Delta_t^a - k\Delta_t^\pi) \right) - \frac{rc}{2} (\Delta_t^\pi)^2 \nu_t^2 - rc\delta_t \Sigma_t(\psi) \Delta_t^\pi \nu_t \right] \right] \]

\[ = rcW_t^\Delta \left[ \frac{1}{2} \left( \Delta_t^a - \left( \frac{\nu_t}{\Sigma_t(\psi)} (1 - L_t) + KL_t \right) \Delta_t^\pi \right)^2 + \left( \frac{r}{2} K + \frac{\nu_t}{\Sigma_t(\psi)} (1 - L_t) k + kKL_t + \frac{rc}{2} \nu_t^2 - \frac{1}{2} \left( \frac{\nu_t}{\Sigma_t(\psi)} (1 - L_t) + KL_t \right)^2 \right) (\Delta_t^\pi)^2 \right], \]
for all \( t \geq 0 \), where the first equality comes from Theorem 1 and the last equality follows by completing the square. Since \( W_t^\Delta \leq 0 \), the above expression is negative if
\[
\frac{r}{2}K + \frac{\nu_t}{\Sigma_t(\psi)} (1 - L_t) k + kKL_t + \frac{rc}{2} \nu_t^2 - \frac{1}{2} \left( \frac{\nu_t}{\Sigma_t(\psi)} (1 - L_t) + KL_t \right)^2 \geq 0,
\]
which can be rewritten in terms of \( K \) as
\[
\frac{1}{2} L_t^2 K^2 + \left( \frac{r}{2} + kL_t - \frac{L_t (1 - L_t) \nu_t}{\Sigma_t(\psi)} \right) K + \frac{rc}{2} \nu_t^2 + \frac{(1 - L_t) \nu_t}{\Sigma_t(\psi)} \left( k - \frac{(1 - L_t) \nu_t}{2\Sigma_t(\psi)} \right) \geq 0. \tag{30}
\]
To ensure that inequality (30) holds, choose \( K \) to solve
\[
\max_K \left\{ \frac{1}{2} L_t^2 K^2 + \left( \frac{r}{2} + kL_t - \frac{L_t (1 - L_t) \nu_t}{\Sigma_t(\psi)} \right) K \right\}.
\]
The solution to this problem is
\[
K = \frac{1}{L_t} \left( \frac{(1 - L_t) \nu_t}{\Sigma_t(\psi)} - k - \frac{r}{2L_t} \right).
\]
Substituting this value of \( K \) into (30) and rewriting the resulting expression in terms of \( \nu_t \),
\[
\frac{1}{2} \left( rc - \frac{3(1 - L_t)^2}{\Sigma_t^2(\psi)} \right) \nu_t^2 + \left( 1 - L_t \right) \left( \frac{1}{2\Sigma_t(\psi)} + \frac{r + 2kL_t}{2L_t \Sigma_t(\psi)} \right) \nu_t \leq \frac{(r + L_t)(r + 2kL_t)}{4L_t^2}.
\]
Let \( p^\pm \) denote the solutions to the quadratic
\[
\frac{1}{2} \left( rc - \frac{3(1 - L_t)^2}{\Sigma_t^2(\psi)} \right) p^2 + \left( 1 - L_t \right) \left( \frac{1}{2 \Sigma_t(\psi)} + \frac{r + 2kL_t}{2L_t} \right) p - \frac{(r + L_t)(r + 2kL_t)}{4L_t^2} = 0.
\]
That is,
\[
p^\pm = \frac{-L_t(1- L_t)}{2\Sigma_t(\psi)} \left( \frac{1}{2} + \frac{r + 2kL_t}{2L_t} \right) \pm \sqrt{\frac{1}{2} \left( r + L_t \right) \left( r + 2kL_t \right) \left( rc - \frac{3(1-L_t)^2}{\Sigma_t^2(\psi)} \right) + \frac{L_t^2(1-L_t)^2}{\Sigma_t^2(\psi)} \left( \frac{1}{2} + \frac{r + 2kL_t}{2L_t} \right)^2}.
\]
If \( rc \Sigma_t^2(\psi) > 3(1-L_t)^2 \), then (30) holds if \( \nu_t \geq p^- \). On the other hand, if \( rc \Sigma_t^2(\psi) > 3(1-L_t)^2 \), then (30) holds if \( \nu_t \leq p^+ \). This yields the conditions in the statement of the theorem. Thus, as long as these conditions hold, \( G_t^\Delta \) is a supermartingale under \( P^\Delta \). By the supermartingale
property,

\[ G_t^\Delta \geq E_t^\Delta [G_\infty^\Delta] = E_0^\Delta \left[ \int_0^\infty u(S_t^\Delta, a_t^\Delta)dt \right], \]

where \( E^\Delta \) is the expectation operator under \( P^\Delta \). This verifies that \( \bar{W}_t^\Delta \) in (29) is indeed the upper bound on the agent’s continuation payoff following any deviation. However, since \( G_t^\Delta \) is a supermartingale, \( \bar{W}_t^\Delta \) never exceeds \( W_t \), the continuation value attained if the agent follows the prescribed policy. Since the deviation policies were arbitrary, sufficiency of the optimal effort policy follows.

Thus, the IC condition (16) is necessary for optimality and if the conditions in Theorem A1 are satisfied, then it is also sufficient.

**Proof of Corollary 3**

*Proof.* The optimality of \( \psi = 1 \) when \( \sigma_\lambda \) follows from the threshold for the general case derived in Corollary 4. Then, by Lemma 2, \( \beta_t^C(\psi = 1) = 0 \) and \( \beta_t^E(\psi = 1) = 1 \). This implies that \( L_t = 1 - \beta_t^C(\psi = 1) - \beta_t^E(\psi = 1) = 0 \). Therefore, there are no belief distortions between the principal and the agent, and \( \Delta^\pi \) ceases to be a state variable in solving the agent’s problem. The density process (24) then becomes

\[ E_t = \exp \left\{ \int_0^t \frac{1}{\sigma_\pi} \Delta_s^\omega d\tilde{Z}_s - \frac{1}{2} \int_0^t \frac{1}{\sigma_\pi^2} (\Delta_s^\omega)^2 ds \right\}. \]

Solving the agent’s problem in the exact same manner as in the proof of Theorem 1 then yields the desired result.

**Proof of Lemma 3**

*Proof.* The agent’s compensation (17) follows from the first-order condition (26) and Lemma A1. Furthermore, Lemma A1 and (13) imply that the agent’s continuation value follows a Brownian martingale:

\[ dW_t = -rc\Sigma_t(\psi)W_t\delta_t d\tilde{Z}_t. \]

By (17), the principal’s profit function can be written as

\[ \mathbb{E} \left[ \int_0^\infty e^{-rt} \left( a_t - k\tilde{\pi}_t^\omega - \frac{1}{2} a_t^2 - \frac{\ln (rc) + \ln (-W_t)}{c} \right) dt \right] = \mathbb{E} \left[ \int_0^\infty e^{-rt} \left( a_t - k\tilde{\pi}_t^\omega - \frac{1}{2} a_t^2 - \frac{\ln (rc) + \ln (-W_0)}{c} - \frac{1}{2} r c\Sigma_t^2(\psi)\delta_t^2 \right) dt \right], \]
where the equality follows by Itô’s lemma and changing the order of integration. Since $rc$ and $W_0$ are constants, (18) follows.

Proof of Theorem 2

Proof. Since the IC constraint (16) binds, the HJB equation (19) for the principal’s problem can be written as

$$rJ(\hat{\pi}, \rho) = \max_{\delta, \nu, \psi \in [0,1]} (1 - L)\delta + L\rho - k\hat{\pi} - \frac{1}{2}((1 - L)\delta + L\rho)^2 - \frac{1}{2}rc\Sigma^2(\psi)\delta^2$$

$$+ ((1 - L)\delta + L\rho - k\hat{\pi})J_\hat{\pi} + [(r + kL)\rho + \delta(k(1 - L) + rc\Sigma(\psi)\nu)]J_\rho$$

$$+ \frac{1}{2}(\Sigma^2(\psi)J_{\hat{\pi}\pi} + \nu^2J_{\rho\rho}) + \Sigma(\psi)\nu J_{\rho\rho}.$$ 

We first determine the optimal incentive components $\delta$ and $\nu$. Solving for $(\delta, \nu)$ using their respective first-order conditions gives

$$\delta = \frac{(1 - L)(1 + J_\hat{\pi} - L\rho + kJ_\rho) + rrc\Sigma(\psi)J_\rho\nu}{(1 - L)^2 + rrc\Sigma^2(\psi)}$$

$$= \frac{(1 - L)(1 + J_\hat{\pi} - L\rho + kJ_\rho)J_{\rho\rho} - rrc\Sigma^2(\psi)J_\rho J_{\rho\pi}}{(rrc\Sigma^2(\psi) + (1 - L)^2)J_{\rho\rho} + r^2c^2\Sigma^2(\psi)J^2_\rho},$$

$$\nu = -\frac{\Sigma(\psi)(rcJ_\rho\delta + J_{\rho\pi})}{J_{\rho\rho}}$$

$$= -\frac{\Sigma(\psi)J_{\rho\pi}(1 - L)^2 + rrc\Sigma^2(\psi)) + rrc\Sigma(\psi)(1 - L)J_\rho(1 + J_\hat{\pi} - L\rho + kJ_\rho)}{(rrc\Sigma^2(\psi) + (1 - L)^2)J_{\rho\rho} + r^2c^2\Sigma^2(\psi)J^2_\rho}. $$

By Itô’s lemma, the dual variables $J_{\hat{\pi}}$ and $J_\rho$ evolve according to

$$dJ_{\hat{\pi}} = \left(((1 - L)\delta + L\rho - k\hat{\pi})J_{\hat{\pi}\rho} + [(r + kL)\rho + \delta(k(1 - L) + rc\Sigma(\psi)\nu)]J_{\rho\rho}$$

$$+ \frac{1}{2}(\Sigma^2(\psi)J_{\hat{\pi}\hat{\pi}} + 2\Sigma(\psi)\nu J_{\hat{\pi}\rho} + \nu^2J_{\rho\rho})\right)dt + \Sigma(\psi)J_{\hat{\pi}\rho} + \nu J_{\rho\rho})dZ,$$

$$dJ_\rho = \left(((1 - L)\delta + L\rho - k\hat{\pi})J_\hat{\pi}\rho + [(r + kL)\rho + \delta(k(1 - L) + rc\Sigma(\psi)\nu)]J_{\rho\rho}$$

$$+ \frac{1}{2}(\nu^2J_{\rho\rho} + 2\Sigma(\psi)\nu J_{\rho\rho} + \Sigma^2(\psi)J_{\rho\rho})\right)dt + (\nu J_{\rho\rho} + \Sigma(\psi)J_{\rho\rho})dZ.$$
Furthermore, by the envelope theorem,

\[ rJ_\pi = -k - kJ_\pi + [(1 - L) \delta + L\rho - k\hat{\pi}] J_{\hat{\pi}} + [(r + kL) \rho + \delta (k (1 - L) + r\Sigma(\psi)\nu)] J_{\pi,\rho} + \frac{1}{2} \left( \Sigma^2(\psi) J_{\pi,\pi} + 2\Sigma(\psi)\nu J_{\pi,\rho} + \nu^2 J_{\rho,\rho}\right), \]

where \( J_{\pi,\rho} = L - L ((1 - L) \delta + L\rho) + LJ_\pi + [(1 - L) \delta + L\rho - k\hat{\pi}] J_{\rho,\pi} + [(r + kL) \rho + \delta (k (1 - L) + r\Sigma(\psi)\nu)] J_{\rho,\rho} + \frac{1}{2} \left( \nu^2 J_{\rho,\rho} + 2\Sigma(\psi)\nu J_{\rho,\pi} + \Sigma^2(\psi) J_{\pi,\pi}\right). \]

Substituting gives

\[
dJ_\pi = [(r + k) J_\pi + k] dt + (\Sigma(\psi) J_{\pi,\pi} + \nu J_{\pi,\rho}) d\hat{Z},
\]

\[
dJ_\rho = -L [1 + J_\pi - L\rho + kJ_\rho - (1 - L) \delta] dt + (\nu J_{\rho,\rho} + \Sigma(\psi) J_{\rho,\pi}) d\hat{Z}
= -L (1 + J_\pi + kJ_\rho - a^*) dt - r\Sigma(\psi)\delta J_\rho d\hat{Z},
\]

where \( a^* = \{a_t^*\}_{t \geq 0} \) is the effort policy that satisfies the IC condition (16). Solving for \( J_\rho(\hat{\pi}_t, \rho_t) \),

\[
J_\rho(\hat{\pi}_t, \rho_t) = -\int^t_0 e^{-\int^s_0 r L a du} - \int^s_0 r\Sigma(\psi)\delta u d\hat{Z} - \frac{1}{2} \int^s_0 r^2 \Sigma^2(\psi)\delta^2 u du \left( L (1 + J_\pi(\hat{\pi}_s, \rho_s) - a^*_s) ds \leq 0, \right.
\]

where the inequality follows since

\[
a^* = \frac{(1 - L) (1 + J_\pi - L\rho + kJ_\rho) J_{\rho,\rho} - r\Sigma^2 J_\rho J_{\pi,\rho} + L \left( (r\Sigma^2 + (1 - L)^2) J_{\rho,\rho} + r^2 \Sigma^2 J_{\rho,\rho}^2 \right)}{(r\Sigma^2 + (1 - L)^2) J_{\rho,\rho} + r^2 \Sigma^2 J_{\rho,\rho}^2} \\
\leq 1 + J_\pi(\hat{\pi}, \rho).
\]

By the Feynman-Kac theorem,\(^{20}\) solving the SDE for \( J_\rho(\hat{\pi}_t, \rho_t) \) forward in time gives

\[
J_\rho(\hat{\pi}_t, \rho_t) = \mathbb{E}_t \left[ \int^\infty_t e^{-(r+k)(s-t)} ks ds \right] = \frac{k}{r + k}.
\]

Due to the full commitment assumption, \( \rho_0 \) is a free parameter that the principal sets at the outset of the contract. Using the fact that \( J_\rho(\hat{\pi}_0, \rho_0) = 0 \), fixing \( t = 0 \), and getting rid of the terms independent of \( \rho \) in the HJB equation (19), the principal chooses \( \rho_0 \) to solve

\[
\rho_0 \in \arg\max_{\rho} \left\{ \frac{rc\Sigma^2(\psi)L_0\rho (2 (1 + J_\pi) - L\rho)}{2 \left( (1 - L_0)^2 + r\Sigma^2(\psi) \right)} \right\}.
\]

\(^{20}\)See Theorem 1.3.17 and Remark 3.5.6 in Pham (2009).
since \( \rho_0 \) serves as an absorbing boundary. Solving this initial value problem gives the optimal value of \( \rho_0 \).

**Proof of Theorem 3**

*Proof*. Using the first-order condition of the principal’s HJB equation (19) with respect to the recognition rule \( \psi \), \( \Sigma(\psi) \) must satisfy

\[
\Sigma(\psi) = 0, \quad \Sigma'(\psi) = 0,
\]

or

\[
0 = (1 - L)^2 J_{\rho\rho} \left[ (1 - L)^2 (J_{\pi\pi} - J_{\pi\pi}J_{\rho\rho}) + 2rc(1 - L)J_{\pi\rho}J_{\rho} \left( 1 + J_\pi + kJ_\rho - L\rho \right) \\
+rc \left( rcJ_\rho^2 + J_{\rho\rho} \right) \left( 1 + J_\pi + kJ_\rho - L\rho \right)^2 \right] \\
+ rc\Sigma^2(\psi) \left( J_{\pi\pi}^2 - J_{\pi\pi} \left( rcJ_\rho^2 + J_{\rho\rho} \right) \right) \left[ 2 (1 - L)^2 J_{\rho\rho} + rc \left( rcJ_\rho^2 + J_{\rho\rho} \right) \Sigma^2(\psi) \right].
\]

The first condition cannot hold by construction. Solving for \( \psi \) in the second condition yields the stationary policy \( \bar{\psi} \) as, by Lemma 2, all of the elements of \( \Sigma(\psi) \) are time-independent. The third condition is a fourth-order polynomial in \( \Sigma(\psi) \). Retaining the positive roots,

\[
\Sigma^2(\psi) = -\frac{(1 - L)^2 J_{\rho\rho}}{rc \left( rcJ_\rho^2 + J_{\rho\rho} \right)} \\
\pm \frac{(1 - L) J_{\rho\rho} \left[ (1 - L) J_\rho J_{\pi\rho} + \left( rcJ_\rho^2 + J_{\rho\rho} \right) \left( 1 + J_\pi + kJ_\rho - L\rho \right) \right]}{\sqrt{rcJ_{\rho\rho}(J_{\pi\pi} \left( rcJ_\rho^2 + J_{\rho\rho} \right) - J_{\rho\rho}^2)}}.
\]

By construction, the solutions derived in (31) must be strictly positive. In (31), call

\[
\frac{-(1 - L)^2 J_{\rho\rho} + (1 - L) J_{\rho\rho} \left( 1 + J_\rho + kJ_\rho - L\rho \right)}{rc \left( rcJ_\rho^2 + J_{\rho\rho} \right) \sqrt{rcJ_{\rho\rho}(J_{\pi\pi} \left( rcJ_\rho^2 + J_{\rho\rho} \right) - J_{\rho\rho}^2)}}
\]

the positive solution and

\[
\frac{-(1 - L)^2 J_{\rho\rho} - (1 - L) J_{\rho\rho} \left( 1 + J_\rho + kJ_\rho - L\rho \right)}{rc \left( rcJ_\rho^2 + J_{\rho\rho} \right) \sqrt{rcJ_{\rho\rho}(J_{\pi\pi} \left( rcJ_\rho^2 + J_{\rho\rho} \right) - J_{\rho\rho}^2)}}
\]

the negative solution. By the assumption of joint concavity, the Hessian matrix of the principal’s value function must be negative semidefinite: \( J_{\pi\pi} \leq 0 \), \( J_{\rho\rho} \leq 0 \), and \( J_{\pi\pi} J_{\rho\rho} - J_{\rho\rho}^2 \geq 0 \). The positive solution in (31) is strictly positive if and only if

\[
1 + J_\pi + kJ_\rho - L\rho < \frac{(1 - L) \left( rcJ_\rho J_{\pi\rho} - \sqrt{rcJ_{\rho\rho}(J_{\pi\pi} \left( rcJ_\rho^2 + J_{\rho\rho} \right) - J_{\rho\rho}^2)} \right)}{rc \left( rcJ_\rho^2 + J_{\rho\rho} \right)},
\]

(32)
On the other hand, the negative solution in (31) is strictly positive if and only if

\[ 1 + J_{\pi} + k J_{\rho} - L_{\rho} > -\frac{(1 - L) \left( rcJ_{\rho}J_{\pi\rho} + \sqrt{rcJ_{\rho\rho}(J_{\pi\pi}(rcJ_{\rho}^2 + J_{\rho\rho}) - J_{\pi\rho}^2)} \right)}{rc \left( rcJ_{\rho}^2 + J_{\rho\rho} \right)}. \] (33)

Consider the left-hand sides of conditions (32) and (33). By the proof of Theorem 2,

\[-k J_{\rho}(\hat{\pi}_0, \rho_0) + L\rho_0 = L\rho_0 = -\frac{J_{\pi}(\hat{\pi}_0, \rho_0)J_{\rho}(\hat{\pi}_0, \rho_0)}{1 - J_{\rho\rho}(\hat{\pi}_0, \rho_0)} < J_{\pi}(\hat{\pi}_0, \rho_0),\]

since \( J_{\pi} < 1 \). Then, since \( J_{\pi} \) is stationary, by the comparison theorem for SDEs,\(^{21}\)

\[ 1 + J_{\pi} + k J_{\rho} - L_{\rho} < 0. \] (34)

It is useful to note that the negative semidefiniteness condition outlined above implies that

\[ \sqrt{rcJ_{\rho\rho}(J_{\pi\pi}(rcJ_{\rho}^2 + J_{\rho\rho}) - J_{\pi\rho}^2)} \geq rcJ_{\rho}J_{\pi\rho} \].

We have to consider two cases for the nature of \( rcJ_{\rho}^2 + J_{\rho\rho} \). First, suppose \( rcJ_{\rho}^2 + J_{\rho\rho} < 0 \). Then, (33) yields \( 1 + J_{\pi} + k J_{\rho} - L_{\rho} > 0 \), a contradiction to (34). On the other hand, (32) satisfies (34), meaning the positive solution to (31) is the appropriate one in this case. Next, suppose \( rcJ_{\rho}^2 + J_{\rho\rho} > 0 \). Then, \( rcJ_{\rho}J_{\pi\rho} + \sqrt{rcJ_{\rho\rho}(J_{\pi\pi}(rcJ_{\rho}^2 + J_{\rho\rho}) - J_{\pi\rho}^2)} > 0 \). Condition (32) always holds in this case. Suppose \( J_{\pi\rho} > 0 \). Then, condition (33) is violated if \( rcJ_{\rho}J_{\pi\rho} + \sqrt{rcJ_{\rho\rho}(J_{\pi\pi}(rcJ_{\rho}^2 + J_{\rho\rho}) - J_{\pi\rho}^2)}) < 0 \), which always holds. Next, suppose \( J_{\pi\rho} < 0 \). Then, condition (33) is violated if \( rcJ_{\rho}J_{\pi\rho} - \sqrt{rcJ_{\rho\rho}(J_{\pi\pi}(rcJ_{\rho}^2 + J_{\rho\rho}) - J_{\pi\rho}^2)}) < 0 \), which must hold by the negative semidefiniteness condition. In either case, condition (33) and again, the positive solution to (31) is the appropriate one. The solution for \( \psi \) follows by noting that

\[ (\Sigma^2)^{-1}(x) = \frac{2\sigma_x^2(\sigma_\rho^2 + \sigma_\lambda^2)}{\sigma_x^2 + 2\sigma_\rho^2 - \sigma + \sqrt{2(\sigma_\rho^2 + \sigma_\lambda^2)(\sigma_\rho^2 + \sigma_\lambda^2)x/(\sigma_\rho^2 + \sigma_\lambda^2)^2} - (\sigma^2 + \sigma_\rho\sigma_\lambda)^2} \]

for any \( x \in \mathbb{R}_+ \), where \( \sigma = \sqrt{\sigma_x^2 + \sigma_\rho^2 + \sigma_\lambda^2} \). Substituting the positive solution in (31) gives the result.

Now let \( \phi : \mathbb{R}_+ \to \mathbb{R} \) be a twice continuously differentiable function such that \( J(\hat{\pi}, \rho) - \phi(\rho) \) has a local minimum at the threshold \( \rho = \rho^* \). Thus, \( \phi \) is a viscosity supersolution of \( J \) at \( \rho^* \).\(^{22}\) By Assumption 1, the kink at \( \rho^* \) must be convex, meaning \( \phi \) is indeed the viscosity

\(^{21}\)See Proposition 5.2.18 in Karatzas and Shreve (1998).

\(^{22}\)Definition II.4.1 in Fleming and Soner (2006).
solution of \( J \) at \( \rho^* \). The uniqueness of \( \phi \) follows from Theorem II.9.1 in Fleming and Soner (2006) and Theorem 4.6.1 in Yong and Zhou (1999).

\[ \square \]

**Proof of Corollary 5**

**Proof.** For the stationary accrual policy,

\[
\frac{\partial \psi}{\partial \sigma_\alpha^2} \bigg|_{\sigma_\alpha^2 \leq 1} = \frac{\sigma_\alpha^2 (\sigma_\alpha^2 (\sigma_\alpha^2 - \sigma_\beta^2 + 3) - \sigma_\alpha^2 (\sigma_\alpha^2 + \sigma_\beta^2))}{\sigma \left( \sigma_\alpha^2 (2\sigma_\alpha^2 + \sigma_\beta^2 - \sigma) + \sigma_\pi (\sigma_\alpha^2 + \sigma_\beta^2) \right)^2},
\]

\[
\frac{\partial \psi}{\partial \sigma_\beta^2} \bigg|_{\sigma_\alpha^2 \leq 1} = \frac{\sigma_\beta^4 (\sigma_\beta^2 + 2\sigma) - \sigma^2}{\sigma \left( \sigma^2 + \sigma_\beta^2 (2\sigma^2 - \sigma) \right)^2} > 0,
\]

\[
\frac{\partial \psi}{\partial \sigma_\lambda^2} \bigg|_{\sigma_\lambda^2 \leq 1} = \frac{\sigma_\alpha^2 (\sigma_\alpha^2 + \sigma_\beta^2) (\sigma_\alpha^2 - 2\sigma)}{\sigma \left( \sigma^2 - \sigma_\beta^2 \sigma + 2\sigma_\beta^2 \right)^2} < 0,
\]

where the inequalities follow because \( \sigma_\lambda^2 \) satisfies the threshold (23). Consider then the relation between \( \psi \) and \( \sigma_\alpha^2 \). If \( \sigma_\alpha^2 \geq \sigma_\beta^2 (1 + \frac{1}{\sqrt{2}}) \), then \( \psi \) is always decreasing in \( \sigma_\alpha^2 \) since \( \sigma_\lambda^2 \) satisfies the threshold (23). Otherwise, if \( \sigma_\alpha^2 < \sigma_\beta^2 (1 + \frac{1}{\sqrt{2}}) \), then \( \psi \) is decreasing in \( \sigma_\alpha^2 \) if

\[
\sigma_\alpha^2 > \frac{\sigma_\beta^4 (3 + 2\sqrt{2}) - \sigma_\alpha^2}{\sigma_\alpha^2}.
\]

which exceeds the threshold (23). Next, consider the nonstationary accrual policy \( \psi^* \). Differentiating \( \xi \),

\[
\frac{\partial \xi}{\partial (rc)} = -\frac{J_{\rho} \xi}{rcJ_{\rho}^2 + J_{\rho p}} + \frac{(1-L)J_{\rho p}}{rcJ_{\rho}^2 + J_{\rho p}} \left( \frac{1-L}{r^2c^2} + \frac{J_{\rho}^2 (1 + J_{\rho} + kJ_{\rho} - L\rho)}{\sqrt{rcJ_{\rho p} (J_{\rho}^2 + J_{\rho p})}} - \frac{J_{\rho p} \left( 2rcJ_{\rho}^2 + J_{\rho p} \right) \left[ (1-L)J_{\rho}J_{\rho p} + (rcJ_{\rho}^2 + J_{\rho p}) (1 + J_{\rho} + kJ_{\rho} - L\rho) \right]}{\left( \sqrt{rcJ_{\rho p} (J_{\rho}^2 + J_{\rho p})} - J_{\rho p}^2 \right)^3} \right) < 0,
\]

\[
\frac{\partial \xi}{\partial k} = \frac{(1-L)J_{\rho}J_{\rho p}}{\sqrt{rcJ_{\rho p} (J_{\rho}^2 + J_{\rho p})} - J_{\rho p}^2} > 0,
\]

where the first inequality follows from the identities derived in the proof of Theorem 3. Since the relation between \( \psi^* \) and any \( \{r, c, k\} \) takes the opposite sign as the corresponding relation between \( \xi, \psi^* \) is indeed increasing in \( (r, c) \) and decreasing in \( k \).
Figure 1: Belief distortions at time $t \geq s$ between the principal and the agent with and without accounting accruals when the agent shirks for an instant of time at time $s = 1.0$. 
Figure 2: Monte-Carlo simulation of 1,000 sample paths of the long-term and short-term incentive processes $(\rho, \delta)$ over time for high and low accrual policies.

Figure 3: The general shape of the principal’s value function over the state space $(\hat{s}, \rho)$. 
Figure 4: Left Panel: Principal’s value functions using the stationary accrual policy $J(\overline{\psi})$ and the non-stationary accrual policy $J(\psi^*_t)$ over $\rho_t$ for a given $\hat{\pi}_t$. Right Panel: Short-term pay-performance sensitivity under the two accrual policies.

Figure 5: Difference in the principal’s value functions across the agency-specific parameters (the common discount rate $r$ and the agent’s risk aversion $c$) and depreciation rate of firm fundamentals $k$. 

48
Figure 6: Comparative statics for the stationary accrual policy $\bar{\psi}$ with respect to timing error volatility $\sigma_\theta$, measurement volatility $\sigma_\lambda$, and fundamental volatility $\sigma_\pi$. 