Experimental Design in Two-Sided Platforms:
An Analysis of Bias*

Ramesh Johari, Hannah Li, Gabriel Weintraub

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Abstract

We develop an analytical framework to study experimental design in two-sided platforms. In the settings we consider, customers rent listings; rented listings are occupied for some amount of time, then become available. Platforms typically use two common designs to study interventions in such settings: customer-side randomization (CR), and listing-side randomization (LR), along with associated estimators. We develop a stochastic model and associated mean field limit to capture dynamics in such systems, and use our model to investigate how performance of these estimators is affected by interference effects between listings and between customers. Good experimental design depends on market balance: we show that in highly demand-constrained markets, CR is unbiased, while LR is biased; conversely, in highly supply-constrained markets, LR is unbiased, while CR is biased. We also study a design based on two-sided randomization (TSR) where both customers and listings are randomized to treatment and control, and show that appropriate choices of such designs can be unbiased in both extremes of market balance, and also yield low bias in intermediate regimes of market balance.

1 Introduction

We develop a framework to study experiments (also known as A/B tests) that two-sided platform operators routinely employ to improve the platform. Experiments are used to test all types of interventions that affect the interactions between participants in the market; examples include features that change the process by which buyers search for sellers, or interventions that alter the information the platform shares with buyers about sellers. We are particularly motivated

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*Ramesh Johari (rjohari@stanford.edu) and Hannah Li (hannahli@stanford.edu) are with the Department of Management Science and Engineering at Stanford University. Gabriel Weintraub (gweintra@stanford.edu) is with the Graduate School of Business at Stanford University. We would like to thank Inessa Liskovich and Navin Sivanandam of Airbnb for numerous fruitful conversations. The opinions expressed here are those of the authors, and do not necessarily reflect the positions of Airbnb nor of its employees.
by marketplaces where customers do not purchase goods, but rather *rent* (or book) them for some amount of time. This covers a broad array of platforms, e.g., lodging (e.g., Airbnb and Booking.com), freelancing (e.g., Upwork), and many services (tutoring, dogwalking, child care, etc.). While we explicitly model such a rental platform, the model we describe also captures features of a platform where goods are bought, and supply must be replenished for future demand.

Our model consists of a fixed number of *listings*; *customers* arrive sequentially over (continuous) time. For example, on a lodging site, listings include hotel rooms, private rooms, houses for rent, etc.; and customers are travelers looking to book. In online labor platforms, a freelancer offering work is a listing, and a client looking to hire a freelancer is a customer. Naturally, an arriving customer can only rent *available* listings (i.e., those that are not currently rented). The customer forms their consideration set from available listings and then, according to a choice model, chooses which listing to rent from this set (including an outside option). We allow the choice set formation process, the utility of a customer for a listing, and the utility of a customer for the outside option to be heterogeneous across listings and customers. In our paper, we employ the multinomial logit choice model; however, since we allow for arbitrary heterogeneity, this admits a quite a general class of demand models. Once a listing is rented, it is occupied and becomes unavailable until the end of the occupancy time.

In this paper, we consider interventions by the platform that change the parameters governing the choice probability of the customer, such as those described above; we refer to the new choice parameters as the *treatment* model, and the baseline as the *control* model. We assume the platform wants to use an experiment to assess the difference between the rate at which rentals would occur if all choices were made according to the treatment parameters (the *global treatment condition*), and the corresponding rate if all choices were made according to the treatment parameters (the *global control condition*). This is the *global treatment effect* or GTE. In particular, we imagine the quantity of interest is the steady-state (or long-run) GTE, i.e., the long-run average difference in rental rates.\(^1\)

Most platforms employ one of two simple designs for testing such an intervention: either *customer-side randomization* (what we call the CR design) or *listing-side randomization* (what we call the LR design). In the CR design, customers are randomized to treatment or control. All customers in treatment make choices according to the treatment choice model, and all customers in control make choices according to the control choice model. In the LR design, listings are randomized to treatment or control, and the utility of a listing is then determined by its treatment.

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\(^1\)The same framework that we employ in this paper can be used to consider interventions that change other parameters, such as customer arrival rates or the time that listings remain occupied when rented; such application is outside the scope of our current work.

\(^2\)Our framework can also be used to evaluate other metrics of interest based on experimental outcomes; for simplicity we focus on rate of rental in this work.
condition. As a result, in the LR design, in general each arriving customer will consider some listings in the treatment condition and some listings in the control condition. As an example, suppose the platform decides to test an intervention that shows badges for certain listings. In the CR design, all treatment customers see the badges, and no control customers see the badges. In the LR design, all customers see the badges on treated listings, and do not see them on control listings.

Each of these designs are associated with natural estimators. In the CR design, the platform measures the rate of rental by treatment customers, and compares to the rate of rental by control customers; this is what we call the naive CR estimator. In the LR design, the platform measures the rate at which treatment listings are rented, and compares to the rate at which control listings are rented; this is what we call the naive LR estimator.

To develop some intuition for the potential biases, first consider an idealized static setting where listings are instantly replenished upon being rented; in other words, every arriving customer sees the full set of listings as available. As a result, in the CR design there is no interference between treatment and control customers, and consequently the CR estimator is unbiased for the true GTE. On the other hand, in the LR design, every arriving customer considers both treatment and control listings when choosing whether to rent, creating a linkage across listings through customer choice. In other words, in the LR design there is interference between treatment and control, and in general the LR estimator will be biased for the true GTE.

Now return to our dynamic model, where the limited inventory of listings is enforced, i.e., listings remain unavailable for some time after rental. In this case, observe that on top of the preceding discussion, there is a dynamic linkage between customers: the set of listings available for consideration by a customer is dependent on the listings considered and rented by previously arriving customers. This dynamic effect introduces a new form of bias into estimation, and is distinctly unique to our work. In particular, because of this dynamic bias, in general the naive CR estimator will be biased as well.

Our paper develops a dynamic model of two-sided markets with inventory dynamics, and uses this framework to compare and contrast both the designs and estimators above, as well as a novel class of more general designs based on two-sided randomization (of which the two examples above are special cases). In more detail, our contributions and the organization of the paper are as follows.

**Benchmark model and formal mean field limit.** Our first main contribution is to develop a general, flexible theoretical model to capture the dynamics described above. In Section 3, we present a model that yields a continuous-time Markov chain in which the state at any given time is the number of currently available listings of each type. In Section 4, we then suggest a formal mean field analog of this continuous-time Markov chain, by considering a limit where the number of listings in the system approaches infinity. Scaling by the number of listings yields a continuum mass of listings in the limit. In the mean field model, the state at a given time is the mass of
available listings, and this mass evolves via a system of ODEs. Using a Lyapunov argument, we show this system is globally asymptotically stable, and give a succinct characterization of the resulting asymptotic steady state of the system as the solution to an optimization problem.

**Designs and estimators: Two-sided, customer-side, and listing-side randomization.** In Section 5, we develop a more general form of experimental design, called *two-sided randomization* (TSR); an analogous idea was independently proposed recently by [2] (see also Section 2). In a TSR design, *both* customers and *listings* are randomized to treatment and control. However, the intervention is only applied when a treatment customer considers a treatment listing; otherwise, if the customer is in control or the listing is in control, the intervention is not seen by the customer. (In the example above, a customer would see the badge on a listing only if the customer were treated and the listing were treated.) Notably, the CR and LR designs are the special cases of TSR where *all* listings are treated (CR), or *all* customers are treated (LR). We also define natural naive estimators for each design.

**Analysis of bias: The role of market balance.** Finally, in Sections 6 and 7, we study the bias of the different designs and estimators proposed. Our main theoretical results characterize how the bias depends on the relative volume of supply and demand in the market. In particular, in the highly *demand-constrained* regime (where customers arrive slowly and/or listings replenish quickly), the model approaches the static benchmark described above: *the naive CR estimator becomes unbiased, while the naive LR estimator is biased*. On the other hand, in the highly *supply-constrained* regime (where customers arrive rapidly and/or listings replenish slowly), the dynamic bias above suggests a more complicated story. However, we remarkably find that *in fact the naive LR estimator becomes unbiased, while the naive CR estimator is biased*. We show how to interpret these findings via examples in Section 6.

Given these findings, it is natural to ask whether good performance can be achieved in moderately balanced markets by “interpolating” between the naive CR and LR estimators. We show that a naive TSR estimator that achieves this interpolation, and also propose a more sophisticated TSR estimator that exhibits substantially improved performance in numerical examples. This latter estimator explicitly aims to correct for interference in regimes of moderate market balance.

Motivated by the common practice of running short-run experiments, we also study the transient behavior of these different designs and estimators. We note that in highly demand-constrained markets, the naive CR estimator is unbiased even in the transient phase; informally, this is because all arriving customers see the same (full) set of available listings. In general, however, numerical investigation of transient performance reveals that the best design can vary depending on market balance and the time horizon of interest. More generally, when studying these designs and estimators there will be an important tradeoff between reducing bias and increasing variance. We leave these directions for future work.
Taken together, our work sheds light on what experimental designs and associated estimators should be used by two-sided platforms depending on market conditions, to alleviate the biases from interference that arise in such contexts. We view our work as a starting point towards a comprehensive framework for experimental design in two-sided platforms; we discuss some directions for future work in Section 8.

2 Related work

SUTVA. The types of interference described in these experiments are violations of the Stable Unit Treatment Value Assumption (SUTVA) in causal inference [11]. SUTVA requires that the (potential outcome) observation on one unit should be unaffected by the particular assignment of treatments to the other units. A large number of recent works have investigated experiment design in the presence of interference, particularly in the context of markets and social networks.

Interference in marketplaces. Biases from interference can be large: [5] empirically show in an auction experiment that the presence of interference among bidders caused the estimate of the treatment effect to be wrong by a factor of two. This evidence is corroborated by [9], who similarly finds through simulations that a marketplace experiment changing search and recommendation algorithms can be off by a factor of two. Inspired by the goal of reducing such bias, other work has developed approaches to bias characterization and reduction both theoretically (e.g., as in [4] in the context of auctions with budgets), as well as via simulation (e.g., as in [10] who explores the performance of LR designs).

Our work complements this line, by developing a mathematical framework for the study of estimation bias in dynamic platforms. Key to our analysis is the use of a mean field model to model both transient and steady-state behavior of experiments. A related approach is taken in [16], where a mean field analysis is used to study equilibrium effects of an experimental intervention where treatment is incrementally applied in a marketplace (e.g., a small pricing change).

Interference in social networks. A bulk of the literature in experimental design with interference considers an interference that arises through some underlying social network: e.g., [12] studies the identification of treatment responses under interference; [15] introduces a graph cluster based randomization scheme and analyzes the bias and variance of the design; and many other papers, including [1, 3, 13] focus on estimating the spillover effects created by interference. In general, our work is distinct because the interference pattern is endogenous to the experiment, and dynamically evolving over time.

Other experimental designs. In practice, platforms currently mitigate the effects of interference through either clustering techniques that change the unit of observation to reduce spillovers among them [7], similar to some of the works mentioned above (e.g., [10, 15]); or by switchback testing
in which the treatment is turned on and off over time. Both cause a substantial increase in estimation variance due to a reduction in effective sample size, and thus the naive CR and LR designs remain popular workhorses in the platform experimentation toolkit.

Two-sided randomization. Finally, a closely related paper is [2]. Independently of our own work, there the authors propose a more general multiple randomized design of which TSR is a special case. They focus on a static model and provide an elegant and complete statistical analysis under a local interference assumption. By contrast, we focus on a dynamic platform with market-wide interference patterns, and focus on a mean field analysis of bias.

3 A Markov chain model of platform dynamics

In this section, we first introduce the basic dynamic platform model that we study in this paper with a finite number \( N \) of listings. In the next section, we describe a formal mean field limit of our model inspired by the regime where \( N \to \infty \), that we use as the substrate for the remainder of our analysis in the paper. This mean field limit model then serves as the framework within which we study the bias of different experimental designs and associated estimators.

We consider a two-sided platform where we refer to the supply side as listings and the demand side as customers. There are a fixed number of listings in the marketplace. Customers arrive over time and at the time of arrival, the customer can choose from the set of available listings in the market and decide whether to rent the corresponding listing. If the customer chooses a listing, then they rent the listing for a random time period during which it is unavailable for other customers. At the end of this rental, the listing again becomes available for use for other customers.

The formal details of our model are as follows. Note: we use boldface to denote vectors throughout the paper.

**Time.** The system evolves in continuous time \( t \geq 0 \).

**Listings.** The system consists of a fixed number \( N \) of listings. We refer to "the \( N \)'th system" as the instantiation of our model with \( N \) listings present. We use a superscript "\( N \)" to denote quantities in the \( N \)'th system where appropriate. Let \( m^{(N)}(\theta) \) denote the total number of listings of type \( \theta \) in the \( N \)'th system. For each \( \theta \in \Theta \), we assume that \( \lim_{N \to \infty} m^{(N)}(\theta)/N = \rho(\theta) > 0 \). Note that \( \sum_{\theta} \rho(\theta) = 1 \).

We allow for heterogeneity in the listings. Each listing \( \ell \) has a type \( \theta_\ell \in \Theta \), where \( \Theta \) is a finite set (the listing type space). Note that in general, the type may encode both observable and unobservable covariates; in particular, our analysis does not presume that the platform is completely informed about the type of each customer. For example, in a lodging site \( \theta_\ell \) may encode observed characteristics of a house such as the number of bedrooms, but also characteristics
that are unobserved by the platform because they may be difficult or impossible to measure.

**State description.** At each time \( t \), each listing \( \ell \) can be either *available* (i.e., available for rent) or *occupied* (i.e., occupied by a customer who previously rented it). The system state at time \( t \) in the \( N \)'th system is described by \( \sigma_t^{(N)} = (\sigma_t^{(N)}(\theta)) \), where \( \sigma_t^{(N)}(\theta) \) denotes the number of listings of type \( \theta \) available in the system at time \( t \). Let \( S_t^{(N)} = \sum_\theta \sigma_t^{(N)}(\theta) \) be the total number of listings available for rent at time \( t \). In our subsequent development, we develop a model that makes \( \sigma_t^{(N)} \) a continuous-time Markov process.

**Customers.** Customers arrive to the platform sequentially and decide whether or not to rent, and if so, which type of listing to rent. Each customer \( j \) has a type \( \gamma_j \in \Gamma \), where \( \Gamma \) is a finite set (the *customer type space*) that represents customer heterogeneity. As with listings, the type may encode both observable and unobservable covariates, and again, our analysis does not presume that the platform is completely informed about the type of each listing. Customers of type \( \gamma \) arrive according to a Poisson process of rate \( \lambda^{(N)}(\gamma) \); these processes are independent across types. Let \( \lambda^{(N)} = \sum_\gamma \lambda^{(N)}(\gamma) \) be the total arrival rate of customers. Let \( T_j \) denote the arrival time of the \( j \)'th customer.

We assume that \( \lim_{N \to \infty} \lambda^{(N)}/N = \lambda > 0 \), and that for each \( \gamma \in \Gamma \), we have \( \lim_{N \to \infty} \lambda^{(N)}(\gamma)/\lambda^{(N)} = \phi_\gamma > 0 \). Note that \( \sum_\gamma \phi_\gamma = 1 \).

**Consideration sets.** In practice, when customers arrive to a platform, they typically form a *consideration set* of possible listings to rent; the initial formation of the consideration set may depend on various aspects of the search and recommendation algorithms employed by the platform. To simplify the model, we capture this process by assuming that on arrival, each listing of type \( \theta \) available at time \( t \) is included in the arriving customer’s consideration set independently with probability \( \alpha_{\gamma j}(\theta) > 0 \) for a customer of type \( \gamma \). For example, \( \alpha_{\gamma} \) can capture the possibility that the platform’s search ranking is more likely to highlight available listings of type \( \theta \) that are more attractive for a customer of type \( \gamma \), making these listings more likely to be part of the customer’s consideration set; this effect is made clear via our choice model presented below. After the consideration set is formed, a choice model is then applied to the consideration set to determine whether a booking (if any) is made.

Formally, the customer choice process unfolds as follows. Suppose that customer \( j \) arrives at time \( T_j \). For each listing \( \ell \), let \( C_{j \ell} = 0 \) if the listing is unavailable at \( T_j \). Otherwise, if listing \( \ell \) is available, then let \( C_{j \ell} = 1 \) with probability \( \alpha_{\gamma j}(\theta_\ell) \), and let \( C_{j \ell} = 0 \) with probability \( 1 - \alpha_{\gamma j}(\theta_\ell) \), independently of all other randomness. Then the consideration set of customer \( j \) is \( \{ \ell : C_{j \ell} = 1 \} \).

**Customer choice.** Customers choose at most one listing to rent; they can also choose not to rent at all. We assume that customers have a *utility* for each listing that depends on its type: a

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\[^{3}\text{Our analysis does not consider improved estimation via use of observed covariates; this remains an interesting direction for future investigation.}\]
type γ customer has utility $v(\gamma) > 0$ for a type $\theta$ listing. (Note that all utilities are positive.) Let $q_{j\ell}$ denote the probability that arriving customer $j$ of type $\gamma_j$ rents listing $\ell$ of type $\theta_\ell$.

In this paper we assume that customers make choices according to the well-known multinomial logit choice model. In particular, given the realization of $C_j$, we have:

$$q_{j\ell} = \frac{C_{j\ell}v_{\gamma_j}(\theta_\ell)}{\epsilon^{(N)}_{\gamma_j} + \sum_{\ell' = 1}^n C_{j\ell'}v_{\gamma_j}(\theta_{\ell'})}. \quad (1)$$

Here $\epsilon^{(N)}_{\gamma} > 0$ is the value of the outside option for type $\gamma$ customers in the $N$’th system. In particular, the probability that customer $j$ does not book any listing at all grows with $\epsilon^{(N)}_{\gamma}$. We let the outside option scale with $N$; this is motivated by the observation that in practical settings, the probability a customer does not make a rental should remain bounded away from zero even for very large systems. In particular, we assume that $\lim_{N \to \infty} \epsilon^{(N)}_{\gamma}/N = \epsilon_\gamma > 0$.

For later reference, we define:

$$q_j(\theta) = \mathbb{E} \left[ \sum_{\ell : \theta_\ell = \theta} q_{j\ell} \right], \quad (2)$$

where the expectation is over the randomness in $C_j$. With this definition, $q_j(\theta)$ is the probability that customer $j$ rents an available listing of type $\theta$, where the probability is computed prior to realization of the consideration set.

**Dynamics: A continuous-time Markov chain.** The system evolves as follows. Initially all listings are available. Every time a customer arrives, the choice process described above unfolds. Any occupied listing remains occupied (and therefore unavailable for further rental) for an exponential time with parameter $\tau$, independent of all other randomness. Once this time expires, the listing returns to being available.

For simplicity in our analysis, we treat $\tau$ as a constant. We note here that it is straightforward to generalize all our analysis to the case where $\tau$ also depends on listing type, i.e., each listing type has a parameter $\tau(\theta)$ that governs how long the listing is occupied once rented. We omit the details of this generalization in favor of simplicity of presentation. An even more general model might allow $\tau$ to depend on both listing type and the type of the customer who made the rental; such a generalization remains an interesting open direction.

Our preceding specification turns $\sigma^{(N)}$ into a continuous-time Markov process on a finite state space $S^{(N)} = \{ \sigma : 0 \leq \sigma(\theta) \leq m^{(N)}(\theta), \forall \theta \}$. We now describe the transition rates of this Markov process. For a state $\sigma \in S^{(N)}$, $\sigma(\theta)$ represents the number of available listings of type $\theta$.

There are only two types of transitions possible: either (i) a listing that is currently occupied becomes available, or (ii) a customer arrives, and rents a listing that is currently available. (If a

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\[ As \text{ the system we study is irreducible and we analyze its steady state behavior, it would not matter if we chose a different initial condition. }\]
customer arrives but does not rent anything, the state of the system is unchanged.) Let \( e_\theta \) denote the unit basis vector in the direction \( \theta \), i.e., \( e_\theta(\theta) = 1 \), and \( e_\theta(\theta') = 0 \) for \( \theta' \neq \theta \). The rate of the first type of transition is:

\[
R(\sigma, \sigma + e_\theta) = (m^{(N)}(\theta) - \sigma(\theta)) \tau,
\]

since there are \( m^{(N)}(\theta) - \sigma(\theta) \) booked listings of type \( \theta \), and each remains occupied for an exponential time with mean \( 1/\tau \), independently of all other randomness.

The second type of transition requires some more steps to formulate. In principle, our choice model suggests that the identity of both the arriving guest and individual listings affect system dynamics; however, our state description only tracks the aggregate number of listings of each type available at each time \( t \). The key here is that our entire specification depends on guests only through their type, and depends on listings only through their type.

Formally, suppose a customer \( j \) of type \( \gamma_j = \gamma \) arrives to find the system in state \( \sigma \). For each \( \theta \) let \( D_\gamma(\theta) \) be a Binomial(\( \sigma(\theta), \alpha_\gamma(\theta) \)) random variable, independently across \( \theta \). Recall that for each available listing \( \ell \), each \( C_{j\ell} \) is a Bernoulli(\( \alpha_\gamma(\theta) \)) random variable. Recalling \( q_j(\theta) \) as defined in (2), it is straightforward to check that:

\[
q_j(\theta) = r_\gamma(\theta|\sigma) \triangleq \mathbb{E} \left[ \frac{D_\gamma(\theta) v_\gamma(\theta)}{e_\gamma^{(N)} + \sum_{\theta'} D_{\gamma'}(\theta') v_{\gamma'}(\theta')} \right].
\]

In other words, the probability an arriving customer of type \( \gamma \) rents a listing of type \( \theta \) when the state is \( \sigma \) is given by \( r_\gamma(\theta|\sigma) \); and this probability depends on the past history only through the state \( \sigma \) (ensuring the Markov property holds).

With this definition at hand, for states \( \sigma \) with \( \sigma(\theta) > 0 \), the rate of the second type of transition is:

\[
R(\sigma, \sigma - e_\theta) = \sum_\gamma \lambda_\gamma^{(N)} r_\gamma(\theta|\sigma).
\]

Note that the resulting Markov chain is irreducible, since customers have positive probability of sampling into, and renting from, their consideration set, and every listing in the consideration set has positive probability of being rented.

**Steady state.** Since the Markov process defined above is irreducible on a finite state space, there is a unique steady state distribution \( \pi^{(N)} \) on \( S^{(N)} \) for the process. In terms of this steady state distribution note that:

\[
\mathbb{S}^{(N)} = \sum_{\sigma \in S^{(N)}} \pi^{(N)}(\sigma) \sum_\theta \sigma(\theta)
\]

is the steady state expected number of available listings. Thus we refer to \( \mathbb{S}^{(N)}/N \) as the steady state availability in the \( N \)’th system, and we refer to \((N - \mathbb{S}^{(N)})/N \) as the steady state occupancy in the \( N \)’th system.
4 A mean field model of platform dynamics

The continuous-time Markov process described in the preceding section is challenging to analyze directly because the customers’ choices involving consideration sets induce complex dynamics. Instead, to make progress we consider a formal mean field limit of that process, motivated by the regime where \( N \to \infty \), in which the evolution of the system becomes deterministic. We do not prove the mean field limit in this paper, though we conjecture that using relatively standard techniques such a limit can be established. Instead, in this section we formally describe a fluid model via a system of ordinary differential equations (ODEs) that is the analogue of the finite system Markov process.

We consider a continuum model with a unit mass of listings. The total mass of listings of type \( \theta \) in the system is \( \rho(\theta) > 0 \) (recall that \( \sum_\theta \rho(\theta) = 1 \)). We represent the state at time \( t \) by \( s_t = (s(\theta), \theta \in \Theta) \); \( s_t(\theta) \) represents the mass of listings of type \( \theta \) available at time \( t \). The state space for this model is:

\[
S = \{ s : 0 \leq s(\theta) \leq \rho(\theta) \}.
\]

We first present the intuition behind our mean field model. Consider a state \( s \in S \) with \( s(\theta) > 0 \) for all \( \theta \). We view this state as analogous to a state \( \sigma \approx Ns \) in the \( N \)'th system. We consider the system dynamics defined by (3)-(5). Note that the rate at which occupied listings of type \( \theta \) become available is \( (m(N)(\theta) - \sigma(\theta)) \tau \), from (3). If we divide by \( N \), then this rate becomes \( \rho(\theta) - s(\theta) \) as \( N \to \infty \). On the other hand, note that for large \( N \), if \( D_\gamma(\theta) \) is Binomial(\( \sigma(\theta), \alpha_\gamma(\theta) \)), then \( D_\gamma(\theta)/N \) concentrates on \( \alpha_\gamma(\theta)s(\theta) \). Thus the choice probability \( r_\gamma(\theta|\sigma) \) becomes approximately:

\[
p_\gamma(\theta|s) \triangleq \frac{\alpha_\gamma(\theta)v_\gamma(\theta)s(\theta)}{\epsilon_\gamma + \sum_{\theta'} \alpha_\gamma(\theta')v_\gamma(\theta')s(\theta')}.
\]  

(Here we use the fact that \( \epsilon_\gamma(N)/N \to \epsilon_\gamma \) as \( N \to \infty \).) This is the mean field multinomial logit choice model for our system. Now the rate at which listings of type \( \theta \) become occupied is \( \sum_\gamma \lambda_\gamma(N)r_\gamma(\theta|\sigma) \), from (5). If we divide by \( N \), this rate becomes \( \lambda \sum_\gamma \phi_\gamma p_\gamma(\theta|s) \) as \( N \to \infty \).

Inspired by the preceding observations, we define the following system of differential equations for the evolution of \( s_t \):

\[
\frac{d}{dt} s_t(\theta) = (\rho(\theta) - s_t(\theta))\tau - \lambda \sum_\gamma \phi_\gamma p_\gamma(\theta|s_t), \quad \theta \in \Theta.
\]  

This is our formal mean field model. In the remainder of this section, we show that this system has a unique solution for any initial condition; and further, by constructing an appropriate Lyapunov function, we show that there exists a unique limit point to which all trajectories converge (regardless of initial condition). This limiting point is the unique steady state of the mean field.
limit, and can be used as a large system approximation of the steady state of the $N$’th finite system.

4.1 Existence and uniqueness of mean field trajectory

First, we show the straightforward result that the system of ODEs defined in (7) possesses a unique solution. This follows by an elementary application of the Picard-Lindelöf theorem from the theory of differential equations. The proof is in Appendix A.

**Proposition 1.** Fix an initial state $\hat{s} \in S$. The system (7) has a unique solution $\{s_t : t \geq 0\}$ satisfying $0 \leq s_t(\theta) \leq \rho(\theta)$ and for all $t$ and $\theta$, and $s_0 = \hat{s}$.

4.2 Existence and uniqueness of mean field steady state

Next, we show that the system of ODEs in (7) has a unique limit point, to which all trajectories converge regardless of the initial condition. We refer to this as the steady state of the mean field system. We prove the result via the use of a convex optimization problem; the objective function of this problem is a Lyapunov function for the mean field dynamics that guarantees global asymptotic stability of the steady state.

Formally, we have the following result. The proof is in Appendix A.

**Theorem 1.** There exists a unique steady state $s^* \in S$ for (7), i.e., a unique vector $s^* \in S$ solving the following system of equations:

$$(\rho(\theta) - s^*(\theta))\tau = \lambda \sum_{\gamma} \phi_{\gamma} p_{\gamma}(\theta|s^*), \quad \theta \in \Theta. \quad (8)$$

This limit point has the property that $0 < s^*(\theta) < \rho(\theta)$ for all $\theta$, i.e., it is in the interior of $S$. Further, this limit point is globally asymptotically stable, i.e., all trajectories of (7) converge to $s^*$ as $t \to \infty$, for any initial condition $s_0 \in S$.

The limit point $s^*$ is the unique solution to the following optimization problem:

$$\minimize W(s) \triangleq \lambda \sum_{\gamma} \left( \phi_{\gamma} \log \left( \epsilon_{\gamma} + \alpha_{\gamma} \sum_{\theta} s(\theta) v_{\gamma}(\theta) \right) \right) - \tau \sum_{\theta} \rho(\theta) \log s(\theta) + \tau \sum_{\theta} s(\theta)$$

subject to $\quad 0 \leq s(\theta) \leq \rho(\theta), \quad \theta \in \Theta. \quad (9)$$

The function $W$ appearing in the proposition statement is not convex; our proof proceeds by first noting that it suffices to restrict attention to $s$ such that $s(\theta) > 0$ for all $\theta$, then making the transformation $y(\theta) = \log(s(\theta))$. The objective function redefined in terms of these transformed variables is strictly convex, and this allows us to establish the desired result.

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5Establishing such a result rigorously requires showing that steady state can be interchanged with the limit as $N \to \infty$; again, we conjecture such a result can be proven using relatively standard techniques, but we omit any further technical development of such a result in this version of the paper.
5 Experiments: Designs and estimators

In this section, we leverage the framework developed in the previous section to undertake a study of experimental designs a platform might employ to test interventions in the marketplace. For simplicity, we focus on interventions that change the choice probability of one or more types of customers for one or more types of listings, and we assume the platform is interested in estimating the resulting rate at which rentals take place. However, we believe the same approach we employ here can be applied to study other types of interventions and platform objectives as well.

Formally, the platform’s goal is to design experiments with associated estimators to assess the performance of the intervention (the treatment), relative to the status quo (the control). In particular, the platform is interested in determining the steady-state rate of rental when the entire market is in the treatment condition (i.e., global treatment), compared to the steady-state rate of rental when the entire market is in the control condition (i.e., global control). We refer to the difference of these two rates as the global treatment effect. It is important to emphasize that this is a steady-state quantity as typically a platform is interested in the long-run effect of an intervention.

Two types of canonical experimental designs are employed in practice: listing-side randomization (denoted LR) and customer-side randomization (denoted CR). In the former design, listings are randomized to treatment or control; in the latter design, customers are randomized to treatment or control. Each design also has an associated natural "naive" estimator of rental probability. As we discuss, these estimators will typically be biased, due to interference effects.

The LR and CR designs are special cases of a more general two-sided randomization (TSR), where both listings and customers are randomized to treatment and control simultaneously. (TSR designs were also independently introduced and studied in recent work by [2]; see Section 2 for discussion.) In the next subsection we develop the relevant formalism for these designs; we then subsequently define natural "naive" estimators that are commonly used for the LR and CR designs, as well as an interpolation between these two as an estimator for a TSR design. In the remainder of the paper we study the bias of these different designs and estimators under different market conditions.

5.1 Experimental design

Treatment condition. We consider a binary treatment: every customer and listing in the market will either be in treatment or control. (Generalization of our model to more than two treatment conditions is relatively straightforward.) We model the treatment condition by expanding the set of customer and listing types. For every customer type $\gamma \in \Gamma$, we create two new customer types $(\gamma, 0), (\gamma, 1)$; and for every listing type $\theta \in \Theta$, we create a two new listing types $(\theta, 0), (\theta, 1)$. The types $(\gamma, 0), (\theta, 0)$ are control types; the types $(\gamma, 1), (\theta, 1)$ are treatment types.
Two-sided randomization. We assume that a fraction $a_C$ of customers are randomized to treatment, and a fraction $1 - a_C$ are randomized to control, independently; and we assume that a fraction $a_L$ of listings are randomized to treatment, and a fraction $1 - a_L$ are randomized to control, independently. This is the two-sided randomization (TSR) design: randomization takes place on both sides of the market simultaneously.

Treatment as a choice probability shift. Examples of interventions that platforms may wish to test include the introduction of higher quality photos for a hotel listing on a lodging site, or showing previous job completion rates of a freelancer on an online labor market. These interventions change the choice probability of listings by customers. In particular, we continue to assume the multinomial logit choice model, and we assume that for a type $\gamma$ customer and a type $\theta$ listing that have been given the intervention, the utility becomes $\tilde{v}_\gamma(\theta) > 0$; the utility of the outside option becomes $\tilde{\epsilon}_\gamma > 0$; and the probability of inclusion in the consideration set becomes $\tilde{\alpha}_\gamma(\theta) > 0$.

In the TSR designs that we consider, a key feature is that the intervention is applied only when a treated customer interacts with a treated listing. For example, when an online labor marketplace decides to show previous job completion rates of a freelancer as an intervention, only treated customers can see these rates, and they only see them when they consider treated freelancers. We model this by redefining quantities in the experiment as follows:

\begin{align*}
    v_{\gamma,0}(\theta, 0) &= v_{\gamma,1}(\theta, 0) = v_{\gamma,0}(\theta, 1) = v_{\gamma}(\theta); & v_{\gamma,1}(\theta, 1) &= \tilde{v}_\gamma(\theta); \\
    \alpha_{\gamma,0}(\theta, 0) &= \alpha_{\gamma,1}(\theta, 0) = \alpha_{\gamma,0}(\theta, 1) = \alpha_\gamma(\theta); & \alpha_{\gamma,1}(\theta, 1) &= \tilde{\alpha}_\gamma(\theta); \\
    \epsilon_{\gamma,0} &= \epsilon_\gamma; & \epsilon_{\gamma,1} &= \tilde{\epsilon}_\gamma.
\end{align*}

This definition is a natural way to incorporate randomization on each side of the market. However, we remark here that it is not necessarily canonical; for example, an alternate design would be one where the intervention is applied when either the customer has been treated or the listing has been treated. Even more generally, the design might randomize whether the intervention is applied, based on the treatment condition of the customer and the listing. In all likelihood, the relative advantages of these designs would depend not only on the bias they yield in any resulting estimators, but also in the variance characteristics of those estimators. We leave further study and comparison of these designs to future work.

Customer-side and listing-side randomization. Two special cases of the TSR design are as follows. When $a_L = 1$, all listings are in the treatment condition; in this case, randomization only takes place on the customer side of the market. This is the customer-side randomization (CR) design. When $a_C = 1$, all customers are in the treatment condition; in this case, randomization only takes place on the listing side of the market. This is the listing-side randomization (LR) design.

System dynamics. With the specification above, it is straightforward to adapt our mean field system of ODEs, cf. (7), and the associated choice model (6), to this setting. The key changes are
as follows:

1. The mass of control (resp., treatment) listings of type \((\theta, 0)\) (resp., \((\theta, 1)\)) becomes \((1-a_L)\rho(\theta)\) (resp., \(a_L\rho(\theta)\)). In other words, abusing notation, we define \(\rho(\theta, 0) = (1-a_L)\rho(\theta)\), and \(\rho(\theta, 1) = a_L\rho(\theta)\).

2. The arrival rate of control (resp., treatment) customers of type \((\gamma, 0)\) (resp., \((\gamma, 1)\)) becomes \((1-a_C)\lambda\phi(\gamma)\) (resp., \(a_C\lambda\phi(\gamma)\)). Thus we define \(\phi(\gamma, 0) = (1-a_C)\phi(\gamma)\), and \(\phi(\gamma, 1) = a_C\phi(\gamma)\).

3. The choice probabilities are defined as in (6), with the relevant quantities defined according to (11)-(13).

Using Proposition 1 and Theorems 1 we know that there exists a unique solution to the resulting system of ODEs; and that there exists a unique limit point to which all trajectories converge, regardless of initial condition. This limit point is the steady state for a given experimental design. For a TSR experiment with treatment customer fraction \(a_C\), and treatment listing fraction \(a_L\), we use the notation \(s_t(a_C, a_L) = (s_t(\theta, j)|a_C, a_L)\), \(\theta \in \Theta, j \in \{0, 1\}\) to denote the ODE trajectory, and we use \(s^*(a_C, a_L) = (s^*(\theta, j)|a_C, a_L)\), \(\theta \in \Theta, j \in \{0, 1\}\) to denote the steady state.

**Rate of rental.** In our subsequent development, it will be useful to have a shorthand notation for the rate at which rentals of listings of treatment condition \(j \in \{0, 1\}\) are made by customers of treatment condition \(i \in \{0, 1\}\), in the interval \([0, T]\). In particular, we define:

\[
Q_{ij}(T|a_C, a_L) = \frac{\lambda}{T} \int_0^T \sum_\theta \sum_\gamma \phi_{\gamma,i} p_{\gamma,i}(\theta, j|s_t(a_C, a_L)) \, dt. \tag{14}
\]

Further, since \(s_t(a_C, a_L)\) is globally asymptotically stable, bounded, and converges to \(s^*(a_C, a_L)\) as \(t \to \infty\), we have:

\[
Q_{ij}(\infty|a_C, a_L) \triangleq \lim_{T \to \infty} Q_{ij}(T|a_C, a_L) = \lambda \sum_\theta \sum_\gamma \phi_{\gamma,i} p_{\gamma,i}(\theta, j|s^*(a_C, a_L)). \tag{15}
\]

**Global treatment effect.** Recall we assume the *steady-state rate of rental* is the quantity of interest to the platform. In particular, the platform is interested in the change in this rate from the global control condition \((a_C = 0, a_L = 0)\) to the global treatment condition \((a_C = 1, a_L = 1)\).

In the global control condition, the steady state rate at which guests rent is: \(Q^\text{GC} = Q_{00}(\infty|0, 0)\), and in the global treatment condition, the steady state rate at which guests rent is \(Q^\text{GT} = Q_{11}(\infty|1, 1)\). Thus the global treatment effect is \(\text{GTE} = Q^\text{GT} - Q^\text{GC}\).

We remark that the rate of rental decisions made by arriving customers will change over time, even if the market parameters are constant over time (including the arrival rates of different customer types, as well as the utilities that customers have for each listing type). This transient change in rental rates is driven by changes in the state \(s_t\); in general, such fluctuations will lead the
transient rate of rental to differ from the steady-state rate, for all values of $a_C$ and $a_L$ (including global treatment and global control). It is for this reason that we specifically aim to measure the global treatment effect as a comparison of the steady state behavior in the global treatment and global control counterfactual worlds, to capture, informally, the long run change in behavior due to an intervention.

### 5.2 Estimators: Transient and steady state

Thus the goal of the platform is to use the experiment to estimate $GTE$. In this section we consider estimators the platform might use to estimate this quantity. We first consider the CR and LR designs, and we define “naive” estimators that the platform might use to estimate the global treatment effect. These designs and estimators are those most commonly used in practice. We define these estimators during the transient phase of the experiment, as that is the most practically relevant regime (since A/B tests are run for a fixed duration in practice). We then also define associated steady-state versions of these estimators. Finally, we combine these estimation approaches in a natural heuristic that can be employed for any general TSR design.

**Estimators for the CR design.** We start by considering the CR design, i.e., where $a_L = 1$ and $a_C \in (0, 1)$. A simple naive estimate of the rate of rental is to measure the rate at which rentals are made in a given interval of time by control customers, and compare this to the analogous rate for treatment customers. Formally, suppose the platform runs the experiment for the interval $t \in [0, T]$, with a fraction $a_C$ of customers in treatment. The rate at which customers of treatment condition $i \in \{0, 1\}$ rent in this period is $Q_{i1}(T|a_C, 1)$. The naive CR estimator is the difference between treatment and control rates, where we correct for differences in the size of the control and treatment groups, by scaling with the respective masses:

$$\hat{GTE}^{CR}(T|a_C) = \frac{Q_{11}(T|a_C, 1)}{a_C} - \frac{Q_{01}(T|a_C, 1)}{1 - a_C}. \tag{16}$$

We let $\hat{GTE}^{CR}(\infty|a_C) = Q_{11}(\infty|a_C, 1)/a_C - Q_{01}(\infty|a_C, 1)/(1 - a_C)$ denote the steady-state naive CR estimator.

**Estimators for the LR design.** Analogously, we can define a naive estimator for the LR design, i.e., where $a_C = 1$ and $a_L \in (0, 1)$. Formally, suppose the platform runs the experiment for the interval $t \in [0, T]$, with a fraction $a_L$ of listings in treatment. The rate at which listings with treatment condition $j \in \{0, 1\}$ are rented in this period is $Q_{1j}(T|1, a_L)$. The naive LR estimator is the difference between treatment and control rates, again scaled by the mass of listings in each group:

$$\hat{GTE}^{LR}(T|a_L) = \frac{Q_{11}(T|1, a_L)}{a_L} - \frac{Q_{10}(T|1, a_L)}{1 - a_L}. \tag{17}$$
We let \( \hat{GTE}^{LR}(\infty | a_L) = Q_{11}(\infty | 1, a_L)/a_L - Q_{10}(\infty | 1, a_L)/(1 - a_L) \) denote the corresponding steady-state naive LR estimator.

**Estimators for the TSR design.** As with the LR and CR designs, it is possible to design a natural naive estimator for the TSR design as well. In particular, we have the following definition of the naive TSR estimator:

\[
\hat{GTE}^{TSR}(T | a_C, a_L) = \frac{Q_{11}(T | a_C, a_L)}{a_C a_L} - \frac{Q_{01}(T | a_C, a_L) + Q_{10}(T | a_C, a_L) + Q_{00}(T | a_C, a_L)}{1 - a_C a_L}.
\]  

(18)

To interpret this estimator, observe that the first term is the normalized rate at which treatment customers booked treatment listings in the experiment; we normalize this by \( a_C a_L \), since a mass \( a_C \) of customers are in treatment, and a mass \( a_L \) of listings are in treatment. This first term estimates the global treatment rate of rental. The sum \( Q_{01}(T | a_C, a_L) + Q_{10}(T | a_C, a_L) + Q_{00}(T | a_C, a_L) \) is the total rate at which control rentals took place: either because the customer was in the control group, or because the listing was in the control group, or both. (Recall that in the TSR design, the intervention is only seen when treatment customers interact with treatment listings.) This is normalized by the complementary mass, \( 1 - a_C a_L \). This second term estimates the global control rate of rental. As before, we can define a steady-state version of this estimator as \( \hat{GTE}^{TSR}(\infty | a_C, a_L) \), with the steady-state versions of the respective quantities on the right hand side of (18).

It is straightforward to check that as \( a_L \to 1 \), we have \( \hat{GTE}^{TSR}(T | a_C, a_L) \to \hat{GTE}^{CR}(T | a_C) \), the naive CR estimator. Similarly, as \( a_C \to 1 \), we have \( \hat{GTE}^{TSR}(T | a_C, a_L) \to \hat{GTE}^{LR}(T | a_L) \), the naive LR estimator. In this sense, the naive TSR estimator naturally "interpolates" between the naive LR estimator and the naive CR estimator. We exploit this interpolation to choose \( a_C \) and \( a_L \) as a function of market conditions in the next section (in particular, dependent on the imbalance between demand and supply). More generally, inspired by the idea of interpolating between the naive CR estimator and the naive LR estimator, we also explore an alternative, more sophisticated TSR estimator.

6 Analysis of bias: Examples

In the remainder of the paper, we study the behavior of the CR, LR, and TSR designs and associated naive estimators proposed in the previous section. We are particularly interested in characterizing the bias: i.e., the extent to which the estimators we have defined under- or overestimate the true \( GTE^{\text{true}} \).

In this section, we start with a simple discussion via example that illustrates the main effects that cause bias. Throughout the discussion, we assume that there are \( N = 2 \) listings in total in
the market, and that listings are homogenous (i.e., of identical type). Further, we also assume that arriving customers are homogeneous (i.e., of identical type). We let \( v > 0 \) denote the utility of a customer for a listing, and suppose the platform considers an intervention that changes this to \( \tilde{v} > 0 \). Finally, we assume that every arriving customer includes any listing that is available in her consideration set.

An important operational finding of our work is that the market balance \( \lambda/\tau \) has a significant influence in determining which estimator and design is bias-optimal. When \( \lambda/\tau \) becomes large, the market is relatively supply-constrained: customers are arriving much faster than occupied listings become available. When \( \lambda/\tau \) becomes small, the market is demand-constrained, with few customers arriving and many available listings. We divide our discussion of this example into these two extreme cases. Our findings are illustrative of the insights we obtain theoretically in the next section.

6.1 Highly demand-constrained markets

Consider a hypothetical limit where each listing becomes instantly available again after being rented (i.e., \( \tau \to \infty \) but \( \lambda \) remains fixed). This is the demand-constrained extreme, where capacity constraints on listings become irrelevant. Note that on arrival of a customer, both listings are always available, and therefore, in her consideration set. In this limit, observe that the steady-state rate at which customers rent listing \( \ell = 1, 2 \) becomes:

\[
\frac{\lambda v}{\epsilon + 2v}. \tag{19}
\]

(The factor 2 appears in the denominator as there are two listings.) Since the intervention changes \( v \) to \( \tilde{v} \), the GTE is:

\[
GTE = \frac{2\lambda \tilde{v}}{\epsilon + 2\tilde{v}} - \frac{2\lambda v}{\epsilon + 2v}. 
\]

Now suppose we consider a CR design that randomizes a fraction \( a_C \) of arriving customers to treatment. Observe that in this demand-constrained extreme, every arriving control (resp., treatment) customer sees the full global control (resp., treatment) market condition; there is no dynamic influence of one customer’s rental behavior on any other customer. This suggests the naive CR estimator should correctly recover the global treatment effect. Indeed, the steady-state naive CR estimator becomes:

\[
\hat{GTE}^{CR}_{\infty|a_C} = \frac{1}{a_C} \frac{2a_C \lambda \tilde{v}}{\epsilon + 2\tilde{v}} - \frac{1}{1-a_C} \frac{2(1-a_C)\lambda v}{\epsilon + 2v} = GTE. 
\]

In other words, the naive CR estimator is perfectly unbiased.

On the other hand, consider a LR design where listing 1 is (randomly) assigned to treatment, and listing 2 is (randomly) assigned to control. In this design, the steady-state naive LR estimator (with \( a_L = 1/2 \)) becomes:

\[
\hat{GTE}^{LR}_{\infty|a_L} = \frac{1}{a_L} \frac{\lambda \tilde{v}}{\epsilon + v + \tilde{v}} - \frac{1}{1-a_L} \frac{\lambda v}{\epsilon + v + \tilde{v}}. 
\]
It is clear that in general this will not be equal to the GTE, because there is interference between the two listings: every arriving customer sees a market environment that is neither quite global treatment nor global control, and the estimates reflect this imperfection. Even with immediate replenishment, treatment listings compete for customers and “cannibalize” rentals from control listings, causing the naive LR estimator to be biased. (Note that such a violation would arise for virtually any reasonable choice model that could be considered.)

6.2 Highly supply-constrained markets

Now we consider the opposite extreme, where the market is heavily supply constrained; in particular, we consider the hypothetical limit where $\lambda \to \infty$ but $\tau$ remains fixed. Now in this case, note that a listing that becomes available will nearly instantaneously be booked; therefore, virtually every arriving customer will find at most one of the two listings available, and their decision of whether to book will be entirely determined by comparison of that available listing against the outside option. In particular, as a result when $\lambda \to \infty$ the steady-state rate at which listing $\ell = 1, 2$ is rented approaches $\tau$.

We thus require a more refined estimate of this rental rate as $\lambda \to \infty$. Suppose $\lambda$ is large, and suppose listing $\ell$ becomes available. Based on the intuition above, we make the approximation that the listing will be considered in isolation by a succession of customers until it is rented. Customers arrive at rate $\lambda$, and rent an available listing with probability $v/(\epsilon + v)$; in other words, in this regime listings compete only with the outside option, and not with each other. Therefore the mean time until such a rental occurs is $(\epsilon + v)/\lambda v$; and once booked, the listing remains rented for mean time $1/\tau$, at which time it becomes available again. Therefore for large $\lambda$, the steady-state rate at which a listing $\ell = 1, 2$ is rented is approximately:

$$
\left( \frac{\epsilon + v}{\lambda v} + \frac{1}{\tau} \right)^{-1}
$$

As expected, this rate approaches $\tau$ as $\lambda \to \infty$. The GTE is thus:

$$
\text{GTE} = 2 \left( \frac{\epsilon + \tilde{v}}{\lambda \tilde{v}} + \frac{1}{\tau} \right)^{-1} - 2 \left( \frac{\epsilon + v}{\lambda v} + \frac{1}{\tau} \right)^{-1}.
$$

With this observation in hand, suppose we again consider the same LR design where listing 1 is (randomly) assigned to treatment, and listing 2 is (randomly) assigned to control. Since in \cite{20} there is no influence of one listing on the other, observe that the naive LR estimator (with $a_L = 1/2$) becomes:

$$
\hat{\text{GTE}}_{\text{LR}}(\infty | a_L) = \frac{1}{a_L} \left( \frac{\epsilon + \tilde{v}}{\lambda \tilde{v}} + \frac{1}{\tau} \right)^{-1} - \frac{1}{1 - a_L} \left( \frac{\epsilon + v}{\lambda v} + \frac{1}{\tau} \right)^{-1} = \text{GTE}.
$$

18
In other words, the naive LR estimator is perfectly unbiased. This is intuitive: in the limit where \( \lambda \) is large, since listings do not compete with each other for rentals, there is no interference when we implement the LR design.

On the other hand, consider the naive CR design where a fraction \( a_C \) of arriving customers are randomized to treatment. In this case we wish to establish the rate at which rentals are made by treatment and control customers respectively. Suppose listing \( \ell \) was occupied by a treatment customer, and becomes available. Define:

\[
\zeta(a_C) = a_C \tilde{v}/(\epsilon + \tilde{v}) + (1-a_C)v/(\epsilon + v).
\]

This is the probability an arriving customer rents the available listing. Customers arrive at rate \( \lambda \), so a mean time \( 1/(\lambda \zeta(a_C)) \) elapses until a rental is made; the listing then remains occupied for mean time \( 1/\tau \). Conditional on a rental, the rental was made by a treatment guest with probability:

\[
\eta(a_C) = \frac{1}{\zeta(a_C)} \frac{a_C \tilde{v}}{\epsilon + \tilde{v}}.
\]

Thus the mean time between treatment rentals of listing \( \ell \) is a geometrically distributed multiple of \( 1/(\lambda \zeta(a_C)) + 1/\tau \), with parameter \( \eta(a_C) \); in other words, the mean rate at which listing \( \ell \) is rented by treatment customers is:

\[
\left( \frac{1}{\eta(a_C)} \cdot \left( \frac{1}{\lambda \zeta(a_C)} + \frac{1}{\tau} \right) \right)^{-1} = \left( \frac{\epsilon + \tilde{v}}{a_C \lambda \tilde{v}} + \frac{1}{\eta(a_C) \tau} \right)^{-1}.
\]

We can use the same logic for the rental rate of control customers, and so we find the naive CR estimator is:

\[
\text{GTE}^{\text{CR}}(\infty | a_C) = \frac{2}{a_C} \left( \frac{\epsilon + \tilde{v}}{a_C \lambda \tilde{v}} + \frac{1}{\eta(a_C) \tau} \right)^{-1} - \frac{2}{1-a_C} \left( \frac{\epsilon + v}{(1-a_C) \lambda v} + \frac{1}{(1-\eta(a_C)) \tau} \right)^{-1}.
\]

In general, this estimator will be biased, i.e., not equal to GTE. The issue is that in this case, customers have a dynamic influence on each other across the treatment groups: when a listing becomes available, whether or not it is available for booking by a subsequent control customer depends on whether or not a treatment customer had previously booked the listing. In this case, customers compete among each other for listings. This interference across customer groups leads to the biased expression for the naive CR estimator.

We note that the GTE and naive LR estimators converge to zero in the limit where \( \lambda \to \infty \); this is because the rental rate of each listing becomes \( \tau \) in this limit. The naive CR estimator does not converge to zero in general, however, as \( \lambda \to \infty \).

### 6.3 Discussion: Violation of SUTVA

Our simple example illustrates that the naive LR estimator is biased when \( \lambda \to 0 \), and unbiased when \( \lambda \to \infty \); and the naive CR estimator is unbiased when \( \lambda \to 0 \), while it is biased when \( \lambda \to \infty \).
These findings can be interpreted through the lens of the classical potential outcomes model; in that model, an important result is that when the stable unit treatment value assumption (SUTVA) holds, then naive estimators of the sort we consider will be unbiased for the true treatment effect. SUTVA requires that the treatment condition of units other than a given customer or listing should not influence the potential outcomes of that given customer or listing. The discussion above illustrates that in the limit where $\lambda \to 0$, there is no interference across customers in the CR design; this is why the naive CR estimator is unbiased. Similarly, in the limit where $\lambda \to \infty$, there is no interference across listings in the LR design; this is why the naive LR estimator is unbiased. On the other hand, the cases where each estimator is biased involve interference across experimental units.

7 Analysis of bias: Results

We establish two key theoretical results in this section: in the limit of a highly supply-constrained market (where $\lambda/\tau \to \infty$), the naive LR estimator becomes an unbiased estimator of the GTE, while the naive CR estimator is biased. On the other hand, in the limit of a highly demand-constrained market (where $\lambda/\tau \to 0$), the naive CR estimator becomes an unbiased estimator of the GTE, while the naive LR estimator is biased. In other words, each of the two naive designs is respectively optimal in the limits of extreme market imbalance. These results are match the findings in our simple example in the preceding section. At the same time, we find empirically that neither estimator performs well in the region of moderate market balance.

Inspired by this finding, we consider TSR and associated estimators that naturally interpolate between the two naive designs depending on market balance. We first consider the naive TSR estimator. Given the findings above, we show that a simple approach to adjusting $a_C$ and $a_L$ as a function of market balance yields performance that balances between the naive LR estimator and the naive CR estimator. Nevertheless, we show there is significant room for improvement, by adjusting for the types of experimental interference that arise using observations from the TSR experiment. In particular, we propose a heuristic for a novel interpolating estimator for the TSR design that aims to correct these biases, and yields surprisingly good empirical performance. We conclude with a brief discussion of transient performance of the estimators considered, and some insights derived through numerical investigation.

7.1 Theory: Steady-state bias in unbalanced markets

In this section, we theoretically study the bias of the steady-state naive CR and LR estimators in the limits where the market is extremely unbalanced (either demand-constrained or supply-constrained). The key tool we employ is a characterization of the asymptotic behavior of $Q_{ij}(\infty|a_C, a_L)$ as defined in (15) in the limits where $\lambda/\tau \to 0$ and $\lambda/\tau \to \infty$. We use this characterization in turn
to quantify the asymptotic bias of the naive estimators relative to the GTE.

7.1.1 Highly demand-constrained markets

We start by considering the behavior of naive estimators in the limit where \( \lambda/\tau \to 0 \). We start with the following proposition that characterizes behavior of \( Q_{ij}(\infty|a_C,a_L) \) as \( \lambda/\tau \to 0 \). The proof is in Appendix [A].

**Proposition 2.** Fix all system parameters except \( \lambda \) and \( \tau \), and consider a sequence of systems in which \( \lambda/\tau \to 0 \). Then along this sequence,

\[
\frac{1}{\lambda} Q_{ij}(\infty|a_C,a_L) \to \sum_\theta \sum_\gamma \phi_{\gamma,i} p_{\gamma,j}(\theta,j|\rho).
\]

The expression on the right hand side depends on both \( a_C \) and \( a_L \) through \( \phi_{\gamma,i} \) and \( \rho \) respectively. In particular, we recall that \( \phi_{\gamma,1} = a_C \phi_{\gamma} \), \( \phi_{\gamma,0} = (1 - a_C) \phi_{\gamma} \), and \( \rho(\theta,1) = a_L \rho(\theta) \), \( \rho(\theta,0) = (1 - a_L) \rho(\theta) \). In our subsequent discussion in this regime, to emphasize the dependence of \( \rho \) on \( a_L \) below, we will write \( \rho(a_L) = (\rho(\theta,j|a_L), \theta \in \Theta, j = 0,1) \). With this definition, we have \( \rho(\theta,1|a_L) = a_L \rho(\theta) \), \( \rho(\theta,0|a_L) = (1 - a_L) \rho(\theta) \).

The proposition shows that in this limit, the (scaled) rate of rental behaves as if the available listings of type \( (\theta,j) \) was exactly \( \rho(a_L) \) for every \( \theta \) and treatment condition \( j = 0,1 \). It is as if every arriving customer sees the entire mass of listings as being available, as in our simplified example in the previous section; in that example, \( 1/\tau \to 0 \), and so rentals are immediately replenished.

We use the preceding result to study the bias of the steady-state naive CR and LR estimators in the limit where \( \lambda/\tau \to 0 \). Consider a sequence of systems where \( \lambda/\tau \to 0 \). Using the preceding result, we observe that:

\[
\frac{1}{\lambda} \text{GTE} = \frac{1}{\lambda} Q_{11}(\infty|1,1) - \frac{1}{\lambda} Q_{00}(\infty|0,0) \to \sum_\theta \sum_\gamma \phi_{\gamma,1} p_{\gamma,1}(\theta,1|\rho(1)) - \sum_\theta \sum_\gamma \phi_{\gamma,0} p_{\gamma,0}(\theta,0|\rho(0)). \tag{22}
\]

We now use Proposition 2 to show that the steady-state naive CR estimator is unbiased in the limit as \( \lambda/\tau \to 0 \), while the steady-state naive LR estimator remains biased. First we consider a CR experiment paired with the naive CR estimator. Using Proposition 2, it follows that:

\[
\frac{1}{\lambda} \text{GTE}^{\text{CR}}(\infty|a_C) \to \frac{1}{a_C} \sum_\theta \sum_\gamma a_C \phi_{\gamma,1} p_{\gamma,1}(\theta,1|\rho(1)) - \frac{1}{1 - a_C} \sum_\theta \sum_\gamma (1 - a_C) \phi_{\gamma,0} p_{\gamma,0}(\theta,1|\rho(1)).
\]

Now note that \( \rho(\theta,1|1) = \rho(\theta) \) and \( \rho(\theta,0|0) = 1 \) when \( a_L = 1 \); similarly, \( \rho(\theta,0|0) = \rho(\theta) \), and \( \rho(\theta,1|0) = 0 \) when \( a_L = 0 \). Thus, from the definition of the TSR design in (11)-(13) and the definition of the choice probability in (37), the choice probability of a control customer for a
treatment listing at $\rho(1)$ is the same as the choice probability of a control customer for a control listing at $\rho(0)$:

$$p_{\gamma,0}(\theta,1|\rho(1)) = p_{\gamma,0}(\theta,0|\rho(0)).$$

These choice probabilities are the same because (1) all listings are in treatment in the CR design, with the mass of each type $\theta$ equal to $\rho(\theta)$; and (2) control customers have the same choice model parameters for these listings regardless of whether they are in treatment or control. Thus it follows that $\hat{GTE}^{CR}_{LR}(\infty|a_C)/\lambda - \hat{GTE}/\lambda \rightarrow 0$ as $\lambda/\tau \rightarrow 0$, i.e., the steady-state naive CR estimator is asymptotically unbiased.

On the other hand, consider the steady-state naive LR estimator. Observe that:

$$\frac{1}{\lambda} \hat{GTE}^{LR}_{LR}(\infty|a_L) \rightarrow \frac{1}{a_L} \sum_\theta \sum_\gamma \phi_\gamma p_{\gamma,1}(\theta,1|\rho(a_L)) - \frac{1}{1-a_L} \sum_\theta \sum_\gamma \phi_\gamma p_{\gamma,1}(\theta,0|\rho(a_L)).$$

In general, this limit will not be equivalent to the GTE; i.e., the naive LR estimator is asymptotically biased. The reason is that $\rho(a_L)$ is different from both $\rho(1)$ (all listings in treatment) and $\rho(0)$ (all listings in control): in the LR design, there is a positive mass of listings in both treatment and control, and this means the choice probabilities do not match those in either global treatment (in the first term) or global control (in the second term). This is exactly the same interference between listings of different treatment conditions that we saw in the simple example in the preceding section, in which listings compete for customers.

Based on the preceding discussion, we observe that the difference between $\hat{GTE}^{CR}_{LR}(\infty|a_C)$ and the GTE does not converge to zero in general as $\lambda/\tau \rightarrow 0$; i.e., the naive LR estimator is biased. However, the naive CR estimator is unbiased in this limit. We summarize in the following theorem.

**Theorem 2.** Consider a sequence of systems where $\lambda/\tau \rightarrow 0$. Then for all $a_C$ such that $0 < a_C < 1$, $\hat{GTE}^{CR}_{LR}(\infty|a_C)/\lambda - \hat{GTE}/\lambda \rightarrow 0$. However, for $0 < a_L < 1$, generically over parameter values we have $\lim \hat{GTE}^{LR}_{LR}(\infty|a_C)/\lambda - \hat{GTE}/\lambda \neq 0$.

### 7.1.2 Heavily supply-constrained markets

We now characterize the behavior of naive estimators in the limit where $\lambda/\tau \rightarrow \infty$. We start with the next proposition, where we study the behavior of $Q_{ij}$ as $\lambda/\tau \rightarrow \infty$. The proof is in Appendix A.

To state the proposition, we define:

$$g_{\gamma,i}(\theta,j) = \frac{\alpha_{\gamma,i}(\theta,j)v_{\gamma,i}(\theta,j)}{\epsilon_{\gamma,i}}.$$

---

7Here “generically” means for all parameter values, except possibly for a set of parameter values of Lebesgue measure zero.
Proposition 3. Fix all system parameters except $\lambda$ and $\tau$, and consider a sequence of systems in which $\lambda/\tau \to \infty$. Along this sequence, the following limit holds:

$$
\frac{Q_{ij}(\infty|a_C,a_L)}{\tau} \to \sum_\theta \left( \frac{\sum_\gamma \phi_{\gamma,i} g_{\gamma,i}(\theta,j)}{\sum_{i'=0,1} \sum_\gamma \phi_{\gamma,i'} g_{\gamma,i'}(\theta,j)} \right) \rho(\theta,j). \tag{23}
$$

As before, the expression on the right hand side depends on both $a_C$ and $a_L$ through $\phi_{\gamma,i}$ and $\rho$ respectively. In particular, we recall that $\phi_{\gamma,1} = a_C \phi_{\gamma}$, $\phi_{\gamma,0} = (1 - a_C) \phi_{\gamma}$, and $\rho(\theta,1) = a_L \rho(\theta)$, $\rho(\theta,0) = (1 - a_L) \rho(\theta)$.

A key intermediate result we employ is to demonstrate that in the steady-state in this limit, $s^*(\theta,j|a_C,a_L) \to 0$ for all $\theta,j$. We know that in the steady state of the mean field limit, the rate at which occupied listings become available must match the rate at which available listings become occupied (flow conservation). We use this fact to show that to first order in $\lambda/\tau$, in the limit where $\lambda/\tau \to \infty$ we have:

$$
\frac{Q_{11}(\infty|1,1)}{\tau}, \frac{Q_{00}(\infty|0,0)}{\tau} \to \tau \sum_\theta \rho(\theta) = \tau.
$$

Thus as seen in our simple example, the steady-state rate of rental in both global treatment and global control becomes $\tau$, and the global treatment effect $\text{GTE} \to 0$.

We also note that:

$$
Q_{11}(\infty|1,a_L) \to a_L \tau \sum_\theta \rho(\theta) = a_L \tau; \quad Q_{10}(\infty|1,a_L) \to (1 - a_L) \tau \sum_\theta \rho(\theta) = (1 - a_L) \tau.
$$

The preceding two expressions reveal that the steady-state naive LR estimator $\text{GTE}^{\text{LR}}(\infty|a_L)$ in this setting approaches zero, matching the GTE; thus it is asymptotically unbiased.
Finally, it is also now straightforward to see why the CR design will be biased. Note that:

\[ Q_{11}(\infty|a_C, 1) \to a_C \tau \sum_{\theta} \left( \frac{\sum_{\gamma} \phi_{\gamma} g_{\gamma,1}(\theta, 1)}{\sum_{\gamma, \gamma'} \phi_{\gamma}, \gamma' g_{\gamma, \gamma'}(\theta, 1)} \right) \rho(\theta). \]

An analogous expression holds for \( Q_{01}(\infty|a_C, 1) \). We see that the right hand side reflects the dynamic interference created between treatment and control customers: just as in our simple example, whether or not an available listing is seen by, e.g., a control customer depends on whether it has previously been booked by a treatment customer. That is, customers compete for listings. As in the example, the naive CR estimator will remain nonzero in general in the limit, even though the GTE approaches zero.

We summarize our discussion in the following theorem.

**Theorem 3.** Consider a sequence of systems where \( \lambda/\tau \to \infty \). Then GTE/\( \tau \to 0 \), and for all \( a_L \) such that \( 0 < a_L < 1 \), there also holds \( \hat{\text{GTE}}_{LR}(\infty|a_L)/\tau \to 0 \). However, for \( 0 < a_L < 1 \), generically over parameter values we have \( \lim \hat{\text{GTE}}_{CR}(\infty|a_C)/\tau - \text{GTE}/\tau \neq 0 \).

### 7.1.3 An example: Homogeneous customers and listings

In this section, we consider a setting analogous to the examples of Section 6: we assume that both listings and customers are homogeneous, i.e., there is only one type of customer and one type of listing. Further, as in Section 6, we assume that \( v \) is the baseline (i.e., control) utility of a customer for a listing, and \( \tilde{v} \) is the post-intervention (i.e., treatment) utility of a customer for a listing. We assume \( \epsilon \) is the outside option value of both control and treatment customers, and that \( \alpha_0(0) = \alpha_1(1) = 1 \).

In this case, we consider two limits: one where \( \lambda \) is fixed and \( \tau \to \infty \) (the highly demand-constrained regime), and one where \( \tau \) is fixed and \( \lambda \to \infty \) (the highly supply-constrained regime). These match the two limits taken in Section 6.

In the first case, when \( \tau \to \infty \) with \( \lambda \) fixed, if we apply Proposition 2, we obtain:

\[
Q_{00}(\infty|a_C, a_L) \to \lambda \cdot \frac{(1 - a_C)(1 - a_L)\rho v}{\epsilon + \rho v}; \\
Q_{10}(\infty|a_C, a_L) \to \lambda \cdot \frac{a_C(1 - a_L)\rho v}{\epsilon + (1 - a_L)\rho v + a_L\rho \tilde{v}}; \\
Q_{01}(\infty|a_C, a_L) \to \lambda \cdot \frac{(1 - a_C)a_L\rho v}{\epsilon + \rho v}; \\
Q_{11}(\infty|a_C, a_L) \to \lambda \cdot \frac{a_C a_L \rho \tilde{v}}{\epsilon + (1 - a_L)\rho v + a_L\rho \tilde{v}}.
\]

In this limit,

\[
\text{GTE} \to \lambda \cdot \frac{\rho \tilde{v}}{\epsilon + \rho \tilde{v}} - \frac{\rho v}{\epsilon + \rho v}.
\]
From these expressions it is clear that the naive CR estimator is unbiased, while the naive LR estimator is biased. Further, the expressions reveal that listing-side randomization creates interference across listings.

In the second case, when $\lambda \to \infty$ with $\tau$ fixed, if we apply Proposition 3, we obtain:

\begin{align*}
Q_{00}(\infty | a_C, a_L) &\to \tau (1 - a_C)(1 - a_L) \rho; \quad (28) \\
Q_{10}(\infty | a_C, a_L) &\to \tau a_C (1 - a_L) \rho; \quad (29) \\
Q_{01}(\infty | a_C, a_L) &\to \tau \cdot \frac{(1 - a_C) v}{(1 - a_C) v + a_C \bar{v}} a_L \rho; \quad (30) \\
Q_{11}(\infty | a_C, a_L) &\to \tau \cdot \frac{a_C \bar{v}}{(1 - a_C) v + a_C \bar{v}} a_L \rho. \quad (31)
\end{align*}

In this limit, $\text{GTE} \to 0$. From these expressions it is clear that the naive CR estimator is biased, while the naive LR estimator is unbiased. Further, these expressions also reveal that customer-side randomization creates interference across customers.

Interestingly, these expressions highlight a remarkable symmetry. As expected, in the limit of a highly demand-constrained market, customers choose among listings; thus there is competition for customers among listings, and this is the source of potential interference in LR designs. The expressions reveal that in the limit of a highly supply-constrained market, it is as if listings choose among customers; thus there is competition among customers, and this is the source of potential interference in CR designs. Indeed, the expressions in (28)-(31) take the form of a multinomial logit choice model of listings for customers. We believe this type of symmetry provides important insight into the nature of experimental design in two-sided markets, and in particular the roots of the interference typically observed in such settings.

### 7.2 Estimation with the TSR design

The preceding section reveals that each of the naive LR and CR estimators has its virtue, depending on market balance conditions. In this section, we explore whether we can develop TSR designs in which $a_C$ and $a_L$ are chosen as a function of $\lambda/\tau$, to obtain the beneficial asymptotic performance of the naive CR estimator in the highly-demand constrained regime, as well as the LR estimator in the highly-supply-constrained regime.

Recall the naive TSR estimator defined in (18), and in particular the steady-state version of this estimator. Suppose the market observes $\lambda/\tau$; note that this is reasonable from a practical standpoint as this is a measure of market imbalance involving only the overall arrival rate of customers and the average rate at which listings become available. For example, consider the following heuristic choices of $a_C$ and $a_L$ for the TSR design, for some fixed values of $\bar{a}_C$ and $\bar{a}_L$:

\begin{align*}
    a_C(\lambda/\tau) &= \left(1 - e^{-\lambda/\tau}\right) + \bar{a}_C e^{-\lambda/\tau}; \quad a_L(\lambda/\tau) = \bar{a}_L \left(1 - e^{-\lambda/\tau}\right) + e^{-\lambda/\tau}. \quad (32)
\end{align*}
Then as $\lambda/\tau \to 0$, we have $a_C(\lambda/\tau) \to \overline{a}_C$ and $a_L(\lambda/\tau) \to 1$, while as $\lambda/\tau \to \infty$ we have $a_C(\lambda/\tau) \to 1$ and $a_L(\lambda/\tau) \to \overline{a}_L$. With these choices, it follows that in the highly demand-constrained limit ($\lambda/\tau \to 0$), the naive TSR estimator becomes equivalent to the naive CR estimator, while in the highly supply-constrained limit ($\lambda/\tau \to \infty$), the naive TSR estimator becomes equivalent to the naive LR estimator. In particular, using Propositions 2 and 3, it is straightforward to show that the steady-state naive TSR estimator is unbiased in both limits; we state this as the following theorem, and omit the proof.

**Corollary 1.** For each $\lambda/\tau$, consider the TSR design with $a_C$ and $a_L$ defined as in (32). Consider a sequence of systems where either $\lambda/\tau \to 0$, or $\lambda/\tau \to \infty$. Then in either limit:

$$\hat{GTE}_{TSR}(\infty|a_C(\lambda/\tau), a_L(\lambda/\tau)) - GTE \to 0.$$  

Figure 1 reveals the steady-state performance of the different naive estimators (CR, LR, and TSR with the preceding scaling of $a_C$ and $a_L$), as a function of market balance $\lambda/\tau$. The figures show the difference between each estimator and $GTE$; the estimators are all upward-biased because of the parameter values chosen, though the qualitative findings in the figure are robust across parameter choices. As we see, the naive TSR estimator performs well in each asymptotic regime.

We are also led to ask whether we can improve upon the naive TSR estimator when the market is moderately balanced. Note that the naive TSR estimator does not explicitly correct for either the fact that there is interference across listings, or the fact that there is interference across customers. We now suggest a heuristic for correction of these effects that leads to an improved interpolating TSR estimator; this is the fourth estimator that appears in Figure 1.

First, abusing notation, let $\hat{GTE}_{CR}(T|a_C, a_L)$ denote the estimator in (16) using the same terms from a TSR design, and dividing through by $a_L$ on both terms as normalization. Similarly abusing notation, let $\hat{GTE}_{LR}(T|a_C, a_L)$ denote the estimator in (16) using the same terms from a TSR design, and dividing through by $a_C$ on both terms as normalization. Motivated by these naive estimators, we explicitly consider an interpolation between the LR and CR estimators of the form:

$$\beta \hat{GTE}_{CR}(T|a_C, a_L) + (1 - \beta) \hat{GTE}_{LR}(T|a_C, a_L).$$  

Now, consider the quantity $Q_{00}(T|a_C, a_L)/((1 - a_C)(1 - a_L)) - Q_{10}(T|a_C, a_L)/((1 - a_C)a_L)$ in a TSR design. This is the (appropriately normalized) difference between the rate at which control customers book control listings, and the rate at which treatment customers book control listings. Note that the difference between control customers and treatment customers is that the latter are exposed to treatment listings, while the former are not. Hence, the difference in steady-state rates of rental among these two groups on control listings must be driven by the fact that

---

8Our choice of exponent here is somewhat arbitrary; the same analysis follows even if we replace $e^{-\lambda/\tau}$ with $e^{-c\lambda/\tau}$ for any value of $c > 0$.  

---

26
treatment customers substitute rentals from control listings to treatment listings (or vice versa). This difference is precisely the "cannibalization" effect (i.e., interference) that was found in LR designs in the highly demand-constrained regime.

Thus motivated, we subtract an appropriately weighted "correction term" for the LR design from our interpolating TSR estimator in (33). (This argument is of course heuristic: it ignores dynamic effects in regimes of intermediate market balance.) Using a symmetric argument we also subtract an appropriately weighted correction term associated to interference across customers in a CR design: $Q_{00}(T|a_C, a_L)/(1 - a_L) - Q_{01}(T|a_C, a_L)/(1 - a_L)$. (Similar correction terms were also studied in [2]; see the related work for further details on this work.) We weight these correction terms in a market-balance-dependent fashion, based on the direction of market balance in which we have seen that the respective interference grows. Combining these insights, for $\beta \in (0, 1)$ our improved TSR estimator is given by:

$$
\hat{GTE}_{TSRI}^{C}(T|a_C, a_L) = 
\beta \left[ \frac{Q_{11}(T|a_C, a_L)}{a_C a_L} - \frac{Q_{01}(T|a_C, a_L)}{(1 - a_C) a_L} - (1 - \beta) \left( \frac{Q_{00}(T|a_C, a_L)}{(1 - a_C)(1 - a_L)} - \frac{Q_{01}(T|a_C, a_L)}{(1 - a_C) a_L} \right) \right] 
+ (1 - \beta) \left[ \frac{Q_{11}(T|a_C, a_L)}{a_C a_L} - \frac{Q_{10}(T|a_C, a_L)}{a_C(1 - a_L)} - \beta \left( \frac{Q_{00}(T|a_C, a_L)}{(1 - a_C)(1 - a_L)} - \frac{Q_{10}(T|a_C, a_L)}{a_C(1 - a_L)} \right) \right],
$$

(34)

Given market balance $\lambda/\tau$, we set $\beta = e^{-\lambda/\tau}$, and we choose $a_C$ and $a_L$ as in (32).

In the limit where $\lambda/\tau \to 0$, note that $\hat{GTE}_{TSRI}^{C}(T|a_C(\lambda/\tau), a_L(\lambda/\tau)) \to \hat{GTE}_{CR}^{C}(T|\bar{a}_C)$ as expected. Similarly, in the limit where $\lambda/\tau \to \infty$, we have $\hat{GTE}_{TSRI}^{C}(T|a_C(\lambda/\tau), a_L(\lambda/\tau)) \to \hat{GTE}_{LR}^{C}(T|\bar{a}_L)$. In particular, it is straightforward to show as a result that this new estimator is also unbiased in both the highly demand-constrained and highly supply-constrained regimes. In these limits, the correction terms play no role. However, for moderate values of market balance, both the cannibalization correction terms kick in.

### 7.3 Transient behavior

For practical implementation, it is important to consider the relative bias in the candidate estimators in the transient system. Theoretically, we can provide some insight when $\tau \to \infty$: in this case, the dominant term in the right hand side of (7) is $(\rho(\theta) - s_t(\theta))\tau$. Using this fact, it can be shown that as $\tau \to \infty$, for each $t > 0$, there holds $s_t^*(a_C, a_L) \to \rho(a_L)$ (where we define $\rho(a_L)$ as in Section 7.1). In other words, the state remains at $\rho(a_L)$ at all times. As a result in this limit the transient estimators $\hat{GTE}_{LR}^{C}(T|a_C)$ and $\hat{GTE}_{LR}^{L}(T|a_L)$ are equivalent to $\hat{GTE}_{CR}^{C}(\infty|a_C)$ and $\hat{GTE}_{CR}^{L}(\infty|a_L)$, respectively. In particular, asymptotically as $\tau \to \infty$, the transient estimator $\hat{GTE}_{CR}^{C}(T|a_C, a_L)$ will be an unbiased estimate of $GTE$ at all times $T > 0$. (The same is true if $\lambda \to 0$, provided the initial state is $s_0 = \rho(a_L)$.)
Figure 1: Left: Difference between estimator and GTE in steady state. We consider variation in $\lambda/\tau$ by fixing $\tau = 1$ and varying $\lambda$; analogous results are obtained if $\lambda$ is fixed and $\tau$ is varied. Right: transient behavior of estimators. Again we fix $\tau = 1$. For both plots, we set $\epsilon_{\gamma,i} = 1, \alpha_{\gamma,i} = 0.5$ for all $\gamma, i$. In the CR design, $a_C = 1/2$. In the LR design, $a_L = 1/2$. There are three listing types and two customer types. Utilities for $\gamma_1$ are $v_{\gamma_1}(\theta_1) = 1.5, v_{\gamma_1}(\theta_2) = 3, v_{\gamma_1}(\theta_3) = 6, \tilde{v}_{\gamma_1}(\theta_1) = 1, \tilde{v}_{\gamma_1}(\theta_2) = 8, \tilde{v}_{\gamma_1}(\theta_3) = 12$. Utilities for $\gamma_2$ are $v_{\gamma_2}(\theta_1) = 1, v_{\gamma_2}(\theta_2) = 2, v_{\gamma_2}(\theta_3) = 4, \tilde{v}_{\gamma_2}(\theta_1) = 0.5, \tilde{v}_{\gamma_2}(\theta_2) = 6, \tilde{v}_{\gamma_2}(\theta_3) = 8$.

More generally, Figure 1 numerically investigates how the time horizon of the experiment affects the performance of the estimators; there we plot an example where $\lambda/\tau$ takes a moderate value. The relative performance of estimators depends on the time horizon of interest as well as market balance conditions. In Appendix B we present two other examples as well that illustrate this phenomenon. Of course, as the time horizon shrinks, the experimenter will also be concerned about increased variance of estimation; this is especially relevant in regimes where different estimators have similar bias. A thorough study of variance of these estimators, though outside the scope of this paper, remains an important direction for future work.

8 Conclusion

This paper has proposed a general mean field framework to study the dynamics of inventory rental in two-sided platforms, and we have leveraged this framework to study the design and analysis of a number of different experimental designs and estimators. We study both commonly used designs and estimators (CR, LR), as well as more general two-sided randomization designs and estimators (TSR). Our work sheds light on the market conditions in which each approach to estimation performs best, based on the relative supply and demand balance in the marketplace.

We suggest two significant directions for future work. First, by virtue of our deterministic
mean field model, we have focused on analysis of bias in this paper. Of course, from a practical standpoint it is natural to characterize the variance of the different designs and estimators studied (e.g., to construct confidence intervals); this requires stochastic analysis of the limit of our finite Markov chain model as the number of listings grows. Further, we have proposed two natural TSR estimators; more generally, however, it is worth asking which TSR designs and estimators are optimal. In particular, studying the bias and variance of different designs can provide a principled approach to two-sided randomization inference in online platforms.

References


Proof of Proposition \[ \text{[7]} \]

For each \( \theta \in \Theta \), define \( f_\theta(s) \) to be the right hand side of \[(7)\]:

\[
    f_\theta(s) = (\rho(\theta) - s(\theta))\tau - \lambda \sum_\gamma \phi_\gamma p_\gamma(\theta|s).
\]

Let \( f \) denote the \(|\Theta|\)-dimensional vector-valued function where each component is defined by \( f_\theta \).

For a fixed \( c > 0 \) define the set \( I = (-c, \infty) \) of times \( t \) for which we wish to show the solution is unique. We require that \( s_t \in S \) for all \( t \in I \) and \( s_0 = \hat{s} \).

By the Picard-Lindelöf theorem \[8\], if \( f(s) \) is Lipschitz continuous for all \( s \in S \), then there exists a unique solution \( \{s_t : t \in I\} \) on the entire time interval \( I \) with the desired initial condition. We will show that each component \( f_\theta(s) \) satisfies the Lipschitz condition, which then implies that the vector-valued function \( f(s) \) satisfies the condition.

Consider the partial derivatives of \( f_\theta(s) \) with respect to each \( s(\theta') \):

\[
    \frac{\partial f_\theta(s)}{\partial s(\theta')} = \begin{cases} 
        -\lambda \sum_\gamma \phi_\gamma \cdot \frac{\epsilon_\gamma + \alpha_\gamma \sum_{\theta'} s(\theta')v_\gamma(\theta') - \alpha_\gamma^2 s(\theta) v_\gamma(\theta)\alpha_\gamma^2 s(\theta) v_\gamma(\theta)}{\epsilon_\gamma + \alpha_\gamma \sum_{\theta'} s(\theta') v_\gamma(\theta')} - \tau, & \hat{\theta} = \theta; \\
        \lambda \sum_\gamma \phi_\gamma \cdot \frac{\alpha_\gamma^2 s(\theta) v_\gamma(\theta) s(\theta') v_\gamma(\theta')}{\epsilon_\gamma + \alpha_\gamma \sum_{\theta'} s(\theta') v_\gamma(\theta')} - \tau, & \hat{\theta} \neq \theta.
    \end{cases}
\]

The partial derivatives of \( f_\theta(s) \) are continuous and (since \( \epsilon_\gamma > 0 \) for all \( \gamma \)) are bounded on \( S \), and so \( f_\theta(s) \) is Lipschitz continuous on \( S \). It follows then that \( f(s) \) is Lipschitz on \( S \) and so there exists a unique solution \( \{s_t : t \geq 0\} \) to \[(7)\] in \( S \) such that \( s_0 = \hat{s} \).
Proof of Theorem 1. Let \( \mathcal{Y} = \{ y : y(\theta) \leq \log \rho(\theta) \} \). For \( y \in \mathcal{Y} \), define \( \omega_y(\theta) = e^{y(\theta)} \); note that \( \omega_y \in \mathcal{S} \). For \( y \in \mathcal{Y} \), we define \( V(y) = W(\omega_y) \), i.e.:

\[
V(y) = \lambda \sum_{\gamma} \phi_{\gamma} \log(e_{\gamma} + \alpha_{\gamma} \sum_{\theta} e^{y(\theta)v_{\gamma}(\theta)}) - \tau \sum_{\theta} \rho(\theta)y(\theta) + \tau \sum_{\theta} e^{y(\theta)}. \quad (35)
\]

We prove the theorem in a sequence of steps.

**Step 1:** \( V(y) \) is strictly convex for \( y \in \mathcal{Y} \). The first term is the log-sum-exp function, which is strictly convex (recall \( \epsilon_{\gamma} > 0 \) for all \( \gamma \)); the second term is linear; and the last term is strictly convex.

**Step 2:** \( V(y) \) possesses a unique minimum \( y^* \) on \( \mathcal{Y} \). Note that as \( y(\theta) \to -\infty \), we have \( V(y) \to \infty \) (recall that \( \epsilon_{\gamma} > 0 \) for all \( \gamma \)). Therefore \( V \) must possess a minimizer on \( \mathcal{Y} \); since \( V \) is strictly convex, this minimizer is unique.

**Step 3:** Define \( s^* = \omega_{y^*} \), i.e., \( s^*(\theta) = e^{y^*(\theta)} \). Then \( s^* \) is the unique solution to \((9)-(10)\). Since \( V(y) = W(\omega_y) \), and the mapping \( y \mapsto \omega_y \) maps \( \mathcal{Y} \) to \( \{ s : 0 < s(\rho) \leq \rho(\theta) \} \subset \mathcal{S} \), it suffices to show that \((9)-(10)\) cannot be minimized at any \( s \) such that \( s(\rho) = 0 \) for some \( s(\theta) \). To see this, note that since \( V(y) \to \infty \) as \( y(\theta) \to -\infty \), it follows that \( W(s) \to \infty \) as \( s(\theta) \to 0 \). It follows that \( s^* \) is the unique solution to \((9)-(10)\).

**Step 4:** \( y^* \) lies in the interior of \( \mathcal{Y} \), and thus \( s^* \) lies in the interior of \( \mathcal{S} \). We have already shown that \( s^*(\theta) > 0 \) for all \( \theta \). It is straightforward to check that if \( y(\theta) = \log \rho(\theta) \), the derivative of \( V(y) \) becomes positive, because the derivative of the first term of \( V(y) \) with respect to \( y(\theta) \) is always positive, and the derivatives of the last two terms cancel when \( y(\theta) = \log \rho(\theta) \). Therefore we must have \( y^*(\theta) < \rho(\theta) \) for all \( \theta \), which suffices to establish the claim.

For the next step, fix an initial condition \( s_0 \in \mathcal{S} \) with \( s_0(\theta) > 0 \) for all \( \theta \), and let \( s_t \) be the resulting trajectory of \((7)\). We first observe that the right hand side of \((7)\) is equal to \( \tau \rho(\theta) \) when \( s(\theta) = 0 \), and this is positive; therefore, we must have \( s_t(\theta) > 0 \) for all \( t \geq 0 \). Define \( y_t(\theta) = \log s_t(\theta) \), and let \( y_t = (y_t(\theta), \theta \in \Theta) \).

**Step 4:** \( V \) is a Lyapunov function for \( \{ y_t : t \geq 0 \} \). Further, \( y^* \) is the unique limit point of \( \{ y_t : t \geq 0 \} \), and it is globally asymptotically stable over all \( y_0 \in \mathcal{Y} \). We consider the function.
\[ V(y_t) \text{ as a function of } t. \text{ By the chain rule, we have:} \]
\[
\frac{d}{dt} V(y_t) = \sum_{\theta} \frac{\partial V(y_t)}{\partial y(\theta)} \cdot \frac{dy_t(\theta)}{ds_t(\theta)} \cdot \frac{ds_t(\theta)}{dt}
\]
\[
= \sum_{\theta} \left( \sum_{\gamma} \left( \frac{\lambda \phi_{\gamma,i} a_{ij} c_{ij}(\theta)}{c_{ij} + \alpha_i \sum c_{ij}(\theta) v_{ij}(\theta)} - \tau \rho(\theta) + \tau c_{ij}(\theta) \right) - \tau \rho(\theta) - \tau c_{ij}(\theta) \right) \cdot \frac{1}{s_t(\theta)}
\]
\[
= -\sum_{\theta} \frac{1}{s_t(\theta)} \left( \sum_{\gamma} \left( \frac{\lambda \phi_{\gamma,i} a_{ij} c_{ij}(\theta)}{c_{ij} + \alpha_i \sum c_{ij}(\theta) v_{ij}(\theta)} - \tau \rho(\theta) + \tau c_{ij}(\theta) \right) \right)
\]
\[
= -\sum_{\theta} \frac{1}{s_t(\theta)} \left( \frac{\partial V(y_t)}{\partial y(\theta)} \right)^2.
\]

It follows that \( \frac{dV(y_t)}{dt} < 0 \) whenever \( y_t \neq y^* \), and \( \frac{dV(y_t)}{dt} = 0 \) if and only if \( y_t = y^* \). \( V \) is clearly positive definite, since it is strictly convex; and as shown, it is minimized at \( y^* \). Thus it is a Lyapunov function for \( y_t \), as required.

**Step 5:** \( s^* \) is the unique limit point of \( \{ s_t : t \geq 0 \} \), and it is globally asymptotically stable over all \( s_0 \in S \). This follows from the preceding observation, as long as \( s_0(\theta) > 0 \) for all \( \theta \). If \( s_0(\theta) = 0 \) for some \( \theta \), then again because the right hand side of (38) is positive when \( s_0(\theta) = 0 \), we must have \( s_t(\theta) \geq 0 \) for all \( t > 0 \). In this case we need only define \( y_t(\theta) = \log s_t(\theta) \) for \( t > 0 \), the desired result follows from the preceding step. This completes the proof of the theorem. □

**Proof of Proposition 2** Throughout the proof, to simplify notation we fix \( a_C, a_L \), and then suppress them throughout the proof (e.g., instead of \( s^*(\theta, j|a_C, a_L) \), we simply write \( s^*(\theta, j) \)). We also assume for simplicity that \( 0 < a_C < 1 \) and \( 0 < a_L < 1 \). This assumption can be made without loss of generality: if one or more of these inequalities fails, we can reduce the type space by eliminating one or more of the treatment conditions, and then replicate the argument below.

We recall from (35) that \( Q_{ij}(\infty) \) is:
\[
Q_{ij}(\infty) = \lambda \sum_{\theta} \sum_{\gamma} \phi_{\gamma,i} p_{\gamma,i}(\theta, j|s^*),
\]
(36)
where the choice probability is:
\[
p_{\gamma,i}(\theta, j|s^*) = \frac{\alpha_{\gamma,i}(\theta, j)v_{\gamma,i}(\theta, j)s^*(\theta, j)}{c_{ij} + \sum_{\theta'} \sum_{j'=0,1} \alpha_{\gamma,i}(\theta', j')v_{\gamma,i}(\theta', j')s^*(\theta', j')}.
\]
(37)

We also make use of the following flow conservation condition cf. (8), which we rewrite here for the experimental setting:
\[
(\rho(\theta, j) - s^*(\theta, j)) \tau = \lambda \sum_{\gamma} \sum_{i=0,1} \phi_{\gamma,i} p_{\gamma,i}(\theta, j|s^*).
\]
(38)
Step 1: We have \( s^*(\theta, j) \to \rho(\theta, j) \) for all \( \theta, j \). Divide both sides of (38) by \( \tau s^*(\theta, j) \). The left hand side of the equation becomes \( \rho(\theta, j)/s^*(\theta, j) - 1 \). The right hand side of the equation becomes

\[
\frac{\lambda}{\tau} \sum_{i=0,1} \epsilon_{\gamma,i} + \sum_{\theta'} \sum_{j'=0,1} \alpha_{\gamma,i}(\theta', j') v_{\gamma,i}(\theta', j') s^*(\theta', j')
\]

where we used the definition of the choice probability. Note that each term in the sum is bounded by one, and there are finitely many terms, so the entire expression approaches zero as \( \lambda/\tau \to 0 \). Thus we have \( \rho(\theta, j)/s^*(\theta, j) \to 1 \).

Step 2: For all \( \gamma, \theta, i, j \), we have \( p_{\gamma,i}(\theta, j | s^*) \to p_{\gamma,i}(\theta, j | \rho) \). This follows because the choice probabilities \( p_{\gamma,i}(\theta, j | s^*) \) are continuous in \( s^* \).

Step 3: Completing the proof. The limit in (21) follows immediately from Step 2 and the definition of \( Q_{ij}(\infty) \).

Proof of Proposition 3. We prove the proposition in a sequence of steps. We adopt the same conventions as in the proof of Proposition 2 to simplify notation we fix \( a_C, a_L \), and then suppress them throughout the proof (e.g., instead of \( s^*(\theta, j | a_C, a_L) \), we simply write \( s^*(\theta, j) \)). We also again assume that \( 0 < a_C < 1 \) and \( 0 < a_L < 1 \); as before, this assumption is without loss of generality.

Step 1: We have \( s^*(\theta, j) \to 0 \) for all \( \theta, j \). Suppose instead that for some \( \theta, j \) pair, the limit inferior of \( s^*(\theta, j) \) is positive along the sequence of systems considered. Divide both sides of (8) by \( \lambda \), and take the limit inferior of each side. The left hand side approaches zero. On the other hand, the right hand side remains positive (because \( \phi, \epsilon, \alpha, \text{ and } v \) are all positive). Thus we have a contradiction, establishing the claim.

Step 2: The following limit holds:

\[
\frac{s^*(\theta, j)}{1/(\lambda/\tau)} \to \frac{\rho(\theta, j)}{\sum_{\gamma} \sum_{i=0,1} \phi_{\gamma,i} g_{\gamma,i}(\theta, j)}.
\]

To prove this, divide both sides of (38) by \( \lambda s^*(\theta, j) \). The left hand side becomes \( (\rho(\theta, j) - s^*(\theta, j)) \cdot (1/(\lambda/\tau)) \cdot (1/s^*(\theta, j)) \); the left hand side will then have the same limit as:

\[
\frac{\rho(\theta, j)}{\lambda/\tau} \cdot \frac{1}{s^*(\theta, j)}.
\]

On the other hand, the limit of the right hand side becomes \( \sum_{\gamma} \sum_{i=0,1} \phi_{\gamma,i} g_{\gamma,i}(\theta, j) \), establishing the desired result.

Step 3: For all \( \gamma, \theta, i, j \), the following limit holds:

\[
\frac{p_{\gamma,i}(\theta, j | s^*)}{1/(\lambda/\tau)} \to \frac{\rho(\theta, j) g_{\gamma,i}(\theta, j)}{\sum_{\gamma'} \sum_{i'=0,1} \phi_{\gamma',i'} g_{\gamma',i'}(\theta, j)}.
\]

This follows by the definition of \( p_{\gamma,i} \), and the previous step.

Step 4: Completing the proof. Given the definition of \( Q_{ij} \) in (36), the preceding step completes the proof.
B Additional numerics

In this section we present two additional figures that depict behavior of the different estimators considered in the transient phase. In Figure 2 the market is relatively demand-constrained, i.e., \( \lambda/\tau \) is small; in Figure 3 the market is relatively supply-constrained, i.e., \( \lambda/\tau \) is large.

![Figure 2: Transient dynamics of estimators with \( \lambda/\tau = 0.3 \). Parameters are as in Figure 1.](image)

![Figure 3: Transient dynamics of estimators with \( \lambda/\tau = 3 \). Parameters are as in Figure 1.](image)