Optimal Investment Strategies in Inefficient Markets

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Abstract

We determine optimal investment strategies in a calibrated equilibrium model where value
and momentum anomalies arise because capital moves slowly from underperforming to over-
performing market segments. Over long horizons, value’s Sharpe ratio increases with horizon,
momentum’s stays flat, and the value-momentum correlation turns positive. Momentum’s op-
timal weight relative to value’s declines significantly as horizon increases. Value’s conditional
Sharpe ratio switches from negative to positive during the capital-flow cycle, while momentum’s
is negative at the cycle’s intermediate stage. The value-momentum correlation predicts posi-
tively value’s and momentum’s short-horizon Sharpe ratios, while the value spread becomes a
better predictor for long-horizon ones.

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1 Introduction

How should investors choose their portfolio of financial assets? According to the CAPM, investors should hold the market portfolio, which weighs assets according to their market capitalization. In particular, investors should hold stocks through a capitalization-weighted stock index. Many models of dynamic portfolio choice use the CAPM’s basic insight and simplify portfolio choice between stocks and cash to between a stock index and cash. They determine the optimal investment in the index as function of investors’ horizon and of variables predicting the distribution of stock returns.\(^1\)

The CAPM fails to describe well asset prices and portfolio allocations, as both empirical research and market practice reveal. Two of the most prominent violations of the CAPM, and arguably of the Efficient Market Hypothesis (Fama (1970)), are value and momentum. Value strategies, which buy assets that trade at a low price relative to measures of fundamental value, such as book equity or earnings, and sell assets that trade at a high price, earn abnormally high returns. The same is true for momentum strategies, which buy assets that trend up in price and sell assets that trend down.\(^2\) Value and momentum underlie much of asset management. Most active mutual funds are classified as value or growth, and are evaluated against value and growth indices. A large number of passive mutual funds and ETFs have been created to track value, growth and momentum indices.

Despite the empirical relevance of CAPM violations, academic guidance on how to incorporate them into portfolio choice is scarce beyond the general tools provided by portfolio theory. The questions are similar in spirit to those addressed in the literature on how to invest in stocks versus cash. For example, should investors pursue value and momentum strategies? Should long-horizon investors use a different mix of these strategies than short-horizon ones? Are the returns of value and momentum predictable, and should investors time their investment in each strategy to exploit the predictability? Guidance on these questions can improve investors’ financial returns and risk allocations, while also making markets more efficient. In this paper we develop a model to address these questions.

Unlike most papers on dynamic portfolio choice, our model does not take prices as given but derives them within market equilibrium. An equilibrium approach is particularly important when studying portfolio choice in a non-CAPM world. Indeed, dynamic portfolio choice depends on the dynamic evolution of asset prices, and specifying that evolution in the presence of market anomalies involves many degrees of freedom, e.g., how each anomaly is reflected in the entire cross-section

\(^1\)References are provided in the literature review section at the end of the Introduction.

of assets, which variables predict each anomaly’s return, and how the anomalies correlate with each other. Inferring the corresponding moments from the data can involve a prohibitively large amount of noise. Data limitations are particularly severe when inferring moments of long-horizon returns, which are, however, crucial for long-horizon investing. Deriving the moments from an equilibrium model can provide a tight and internally consistent specification. Our model delivers such a specification. It also delivers, as a by-product, a rich set of empirical predictions for value and momentum returns.

Our equilibrium model, described in Section 2 and solved in Section 3, is based on Vayanos and Woolley (2013, VW). Momentum and value arise from performance-driven flows across investment funds. Suppose that a negative shock hits the fundamental value of some assets. Investment funds holding those assets realize low returns, triggering outflows by investors who infer that fund managers’ ability is likely to be low. Because of the outflows, funds sell assets they own, and this depresses further the prices of the assets hit by the original shock. Momentum arises because the outflows are assumed to be gradual and because, despite their predictability, they impact negatively future expected returns. Value arises because outflows push prices below fundamental values, so expected returns eventually rise. Delegated asset management is not key to VW’s core mechanism: what is key is that capital moves from poorly performing market segments to well-performing ones, and does so slowly. Such flows of capital are ubiquitous in financial markets.

Section 4 defines value and momentum strategies as well as performance measures for general strategies, and calibrates the model. An asset’s value weight is assumed linear in the difference between the present value of the asset’s expected dividends discounted at the riskless rate, and the asset’s price. An asset’s momentum weight is assumed linear in the asset’s cumulative return over a given lookback window. Weights change continuously, implying continuous rebalancing of the strategies. We measure a strategy’s performance by the annualized Sharpe ratio (expected return over standard deviation) over a given horizon. A strategy maximizing the utility of an investor with mean-variance preferences over wealth over that horizon maximizes the Sharpe ratio. The linear structure of our model makes it possible to compute Sharpe ratios in closed form, even over long horizons and even for strategies that rebalance continuously. We calibrate the model using moments of asset returns, fund flows and fund holdings.

Using the equilibrium prices generated by the calibrated model, we compute the performance of trading strategies and show our main results. This is done in Sections 5-7. Section 5 evaluates strategies over an infinitesimal horizon. Section 6 considers all horizons longer than infinitesimal. Section 7 performs a sensitivity analysis to different parameter values.

Our first result concerns the performance of value and momentum in isolation as function of the investment horizon. Over short horizons of up to two years, the strategies’ Sharpe ratios decrease
with horizon. This reflects the short-horizon positive autocorrelation of strategies’ returns, driven by asset-level momentum. Because of that autocorrelation, the annualized variance of returns increases with horizon, and Sharpe ratios decrease. Over longer horizons, the Sharpe ratio of momentum becomes approximately independent of horizon, while that of value increases significantly, overtaking momentum’s for horizons longer than thirteen years in our main calibration. Intuitively, momentum has short memory because it weighs assets based only on recent performance. As a consequence, its returns are approximately independent over time when evaluated over longer horizons, and its annualized variance is constant. By contrast, value has long memory because it loads up on assets that have underperformed over a long period. If the assets held by value experience a further long period of underperformance, then their expected returns increase and so does the weight given to them by value. This boosts value’s expected return, resulting in strong negative long-horizon autocorrelation of value returns.

Our second result concerns the diversification gains of combining value and momentum as function of horizon. Over short horizons, the strategies are negatively correlated, as has been documented empirically (Asness, Moskowitz, and Pedersen (2013)). This is because value loads up on assets that have underperformed over a long period, while momentum tends to short those assets as they have been trending down in the recent past. Over horizons longer than one year, by contrast, the correlation turns positive. This is partly because of a positive lead-lag effect from momentum to value. Indeed, assets with poor recent performance are expected to continue underperforming because of flows out of funds holding those assets. Shorting those assets, which is a momentum strategy, offers a particularly high return when outflows are larger than expected, which is when the assets become highly underpriced and value’s expected returns are high.

Our third result concerns the weights of value and momentum in their optimal (mean-variance maximizing) combination, and the performance of that combination. The optimal combination tilts away from momentum and towards value as horizon increases. Momentum’s weight is 169-199% that of value for horizons up to two years. It then decreases with horizon, becoming equal to value’s weight for thirteen years, and to 57% of value’s weight for forty years. The optimal value-momentum combination achieves a high Sharpe ratio: 30% higher than the aggregate market’s for horizons ranging from two to five years, and up to 93% higher for shorter or longer horizons. This Sharpe ratio is market-adjusted, i.e., represents compensation for risk orthogonal to the market. Value and momentum achieve most of the available gains in our model: the Sharpe ratio of their optimal combination is above 90% of the fully optimal strategy’s, and above 95% for horizons longer than 3.5 years.

Our final set of results concern the performance of value and momentum conditional on predictor variables. These results can be understood in terms of the “flow cycle,” which describes
how capital moves across funds. Following a negative shock to the fundamentals of some assets, capital moves slowly out of funds holding those assets. Since those assets are expected to continue underperforming in the near term, and momentum goes short in them, it has high conditional short-horizon Sharpe ratio at the cycle’s early stage. By contrast, value has negative Sharpe ratio because it goes long. Value’s Sharpe ratio rises at the cycle’s intermediate stage, when most capital has moved out: the assets are then severely undervalued, with high expected returns. It remains high at the cycle’s late stage, when the undervalued assets begin to accumulate a history of good performance. At the late stage, momentum’s Sharpe ratio is also high because it goes long in the undervalued assets. It is instead low at the intermediate stage, when the return history of the undervalued assets has not yet caught up with their high expected returns.

The variation of conditional short-horizon Sharpe ratios over the flow cycle is reflected into their relationship with two key predictors: the value spread, whose predictive power for value returns has been documented empirically (Cohen, Polk, and Vuolteenaho (2003)), and the short-horizon value-momentum correlation. Value and momentum are negatively correlated at the cycle’s early stage, since value longs the assets that momentum shorts, and are positively correlated at the late stage, since they long the same assets. Therefore, their correlation is strongly positively related to value’s Sharpe ratio. A positive relationship exists with momentum’s Sharpe ratio as well. The value spread is positively related to value’s Sharpe ratio, but the relationship is weaker than those involving the correlation. This is because at the cycle’s early stage the value spread is wide but value’s Sharpe ratio is negative. The predictive relationships change for long-horizon Sharpe ratios. The value spread becomes strongly positively related to value’s and momentum’s Sharpe ratios, and the correlation becomes weakly related.

A common thread running through our results is that long- and short-horizon investors should hold very different portfolios in a non-CAPM world. In particular, Sharpe ratios computed using time-series of monthly or annual returns, as is commonly done in practice, can give poor guidance for the investment of retirement assets. Our analysis also identifies new predictors and statistical properties of value and momentum returns. In companion work we find support for the new implications of our model. We also develop a methodology to estimate long-horizon Sharpe ratios of value and momentum.


Brennan, Schwartz, and Lagnado (1997) and Barberis (2000) incorporate the empirically doc-
umented positive relationship between the aggregate stock market’s expected return and dividend yield—a value effect for the aggregate market—in their numerical analysis of portfolio choice between a stock index and cash. Campbell and Viceira (1999, 2002) and Chacko and Viceira (2005) analyze portfolio choice between a stock index and cash using log-linear approximations. When the market’s expected return is positively related to the dividend yield, the market’s Sharpe ratio increases with horizon and long-horizon investors allocate a larger fraction of their wealth to stocks than short-horizon investors. Koijen, Rodriguez, and Sbuelz (2009) allow for both a value and a momentum effect for the aggregate market, and show that the fraction of wealth allocated to stocks initially decreases and then increases with horizon. These papers focus on the allocation between stocks and cash in a CAPM world, while we focus on portfolio choice over the cross-section of stocks (or other assets) in a non-CAPM world. Moreover, while these papers emphasize time-variation in the market’s expected return, that return is constant in our model and is thus not driving the variation in value and momentum Sharpe ratios with investment horizon.

Jurek and Viceira (2011) study portfolio choice between a value index, a growth index and cash, and show that long-horizon investors invest less in value than short-horizon investors. In the spirit of Brennan, Schwartz, and Lagnado (1997), Campbell and Viceira (1999) and subsequent papers, they estimate moments of value and growth returns from the data, using vector auto-regressions. Because they do not include the value spread as a predictor, they do not find the negative long-horizon autocorrelation of value returns, which makes value attractive for long-horizon investors in our model.

Barberis and Shleifer (2003, BS) develop a behavioral theory of momentum and value in which capital moves slowly from poorly performing investment styles to well-performing ones. Investors in BS are assumed not to anticipate future flows. This makes momentum in BS more profitable than in VW and our model, in which future flows are rationally anticipated. Both BS and VW compute Sharpe ratios of value and momentum, but only unconditionally and over short investment horizons—one period in BS and infinitesimal in VW. Our results, which concern performance measures over general investment horizons and conditionally on predictor variables, indicate that short-horizon measures are misleading for longer horizons. Our calibration also improves on VW’s because we choose parameter values through a better mapping with target moments.

2 Model

Time $t$ is continuous and goes from zero to infinity. There are $N + 1$ assets. Asset zero is riskless and has an exogenous, continuously compounded return $r$. Assets $n = 1, ..., N$ are risky and their prices are determined endogenously in equilibrium. We interpret the risky assets as stocks or as
industry-sector portfolios. We denote by $D_{nt}$ the cumulative dividend per share of asset $n = 1, \ldots, N$, by $S_{nt}$ the asset’s price, by $dR_{nt} = dD_{nt} + dS_{nt} - rS_{nt}dt$ the asset’s return per share in excess of the riskless asset, and by $\pi_n$ the asset’s supply in terms of number of shares. We refer to $dR_{nt}$ simply as return.

There are three agents: a representative investor, a representative active-fund manager, and a representative noise trader. The investor can invest in the riskless asset. She can also invest in the risky assets through a passive fund that tracks mechanically a market index and through an active fund. The index includes $\eta_n$ shares of asset $n$. We assume that $\eta_n$ is proportional to the supply $\pi_n$ of asset $n$. The index is thus capitalization-weighted.

The index is not an optimal portfolio because the noise trader holds a portfolio other than the index. We denote by $\pi_n - \theta_n$ the number of shares of risky asset $n$ held by the noise trader. The number of shares held by the other agents is thus $\theta_n$. We refer to $\theta_n$ as asset $n$’s residual supply.

Our assumption that the noise trader does not hold the index amounts to the vectors $\eta = (\eta_1, \ldots, \eta_N)$ and $\theta = (\theta_1, \ldots, \theta_N)$ being linearly independent. We set

$$
\Delta \equiv \theta \Sigma \theta' \eta \Sigma \eta' - (\eta \Sigma \theta')^2 > 0.
$$

The assumption that the noise trader does not hold the index ensures that the active fund can add value over the index fund. The same result would hold even without a noise trader provided that the index is not capitalization-weighted.

The investor determines how to allocate her wealth between the riskless asset, the index fund, and the active fund. She maximizes expected utility of intertemporal consumption. Utility is exponential, i.e.,

$$
-E \int_0^\infty \exp(-\alpha c_t - \beta t) dt, \quad (2.1)
$$

where $\alpha$ is the coefficient of absolute risk aversion, $c_t$ is consumption, and $\beta$ is the discount rate. The investor’s control variables are consumption $c_t$ and the number of shares $x_t$ and $y_t$ of the index and active fund, respectively.

The active-fund manager runs the fund and can invest his personal wealth in it. He thus determines the active portfolio and the allocation of his wealth between the riskless asset and the fund. He maximizes expected utility of intertemporal consumption. Utility is exponential, i.e.,

$$
-E \int_0^\infty \exp(-\bar{\alpha} \bar{c}_t - \bar{\beta} t) dt, \quad (2.2)
$$
where $\bar{\alpha}$ is the coefficient of absolute risk aversion, $\bar{c}_t$ is consumption, and $\bar{\beta}$ is the discount rate. The manager’s control variables are consumption $\bar{c}_t$, the number of shares $\bar{y}_t$ of the active fund, and the active portfolio $z_t \equiv (z_{1t}, ..., z_{Nt})$, where $z_{nt}$ denotes the number of shares of asset $n$ included in one share of the active fund.

The assumption that the manager can invest his personal wealth in the active fund is a convenient modelling device. It generates an objective that the fund maximizes and a counterparty to the investor’s flows. The fund’s objective is implied by the manager’s: the manager chooses the fund’s portfolio to maximize the utility that he derives from his stake in the fund. The manager is also the counterparty to the investor’s flows: when the investor flows out of the active fund, the manager finds it optimal to increase his stake in the fund. The manager can be interpreted as the aggregate of all agents absorbing the investor’s flows.

To ensure that the investor has a motive to move between the index and the active fund, we assume that she suffers a time-varying cost from investing in the active fund. This cost drives a wedge between the investor’s net return from the active fund, and the gross return made of the dividends and capital gains of the stocks held by the fund. We interpret the cost as a reduced form for managerial ability, with high cost corresponding to low ability. The index fund entails no cost, so its gross and net returns coincide.

We model the cost as a flow (i.e., the cost between $t$ and $t + dt$ is of order $dt$), and assume that the flow cost is proportional to the number of shares $y_t$ that the investor holds in the active fund. We denote the coefficient of proportionality by $C_t$ and assume that it follows the process

$$dC_t = \kappa(\bar{C} - C_t)dt + s dB^C_t\) \quad (2.3)$$

where $\kappa$ is a mean-reversion parameter, $\bar{C}$ is a long-run mean, $s$ is a positive scalar, and $B^C_t$ is a Brownian motion.

We define one share of the fund by the requirement that its market value equals the equilibrium market value of the entire fund. Under this definition, the number of fund shares held by the investor and the manager in equilibrium sum to one, i.e.,

$$y_t + \bar{y}_t = 1. \quad (2.4)$$

We define one share of the index fund to coincide with the market index $\eta$.

We denote the vector of the risky assets’ cumulative dividends by $D_t \equiv (D_{1t}, ..., D_{Nt})'$ and the vector of the risky assets’ prices by $S_t \equiv (S_{1t}, ..., S_{Nt})'$, where $v'$ denotes the transpose of the vector...
v. We assume that $D_t$ follows the process

$$dD_t = F_t dt + \sigma dB_t^D,$$

(2.5)

where $F_t \equiv (F_{1t}, \ldots, F_{Nt})'$ is a time-varying drift equal to the expected dividend rate, $\sigma$ is a constant matrix of diffusion coefficients, and $B_t^D$ is a $d$-dimensional Brownian motion independent of $B_t^C$. We model time-variation in $F_t$ through the process

$$dF_t = \kappa(\bar{F} - F_t) dt + \phi \sigma dB_t^F,$$

(2.6)

where the mean-reversion parameter $\kappa$ is the same as for $C_t$ for simplicity, $\bar{F}$ is a long-run mean, $\phi$ is a positive scalar, and $B_t^F$ is a $d$-dimensional Brownian motion independent of $B_t^C$ and $B_t^D$. The diffusion matrices for $D_t$ and $F_t$ are proportional for simplicity. We set $\Sigma \equiv \sigma \sigma'$.

We assume that the investor can adjust her active-fund holdings $y_t$ only gradually. Gradual adjustment can result from limited attention or institutional decision lags. We model these frictions as a flow transaction cost $\frac{1}{2} \psi \left( \frac{dy_t}{dt} \right)^2$ that the investor must incur when changing $y_t$.

The manager observes all the variables in the model. The investor observes the returns and share prices of the index and active funds, but not the same variables for individual stocks. She does not observe $C_t$ and $F_t$.

3 Equilibrium

In equilibrium, the prices $S_t$ of the risky assets and the rate $v_t \equiv \frac{dy_t}{dt}$ at which the investor changes her active-fund holdings are linear in the state variables. The state variables are the expected dividend rate $F_t$, the cost $C_t$ of investing in the active fund, the investor’s expectation $\hat{C}_t$ of that cost, and the investor’s active-fund holdings $y_t$.

Asset prices take the form

$$S_t = \frac{F}{r} + \frac{F_t - \bar{F}}{r + \kappa} - \frac{\alpha \bar{f}}{\alpha + \bar{\alpha} \Sigma \eta'} \Sigma \eta' - (\gamma_0 + \gamma_1 \hat{C}_t + \gamma_2 C_t + \gamma_3 y_t) \Sigma p_f',$$

(3.1)

where $(\gamma_0, \gamma_1, \gamma_2, \gamma_3)$ are constants and

$$p_f \equiv \theta - \frac{\eta \Sigma \theta'}{\eta \Sigma \eta} \eta$$

(3.2)
is a “flow portfolio” describing the flows that the investor generates when moving across funds. The first two terms in (3.1) are the present value of expected dividends discounted at the riskless rate \( r \). The third term is a risk discount proportional to the covariance \( \Sigma \eta' \) with the index. This discount is constant over time and reflects an adjustment for index risk. The fourth term is a risk discount proportional to the covariance \( \Sigma p_f' \) with the flow portfolio. This discount is time-varying and reflects the price impact of flows.

The flow portfolio \( p_f \) is equal to the residual-supply portfolio \( \theta \) plus a short position in the index \( \eta \) such that the overall position has zero covariance with the index (\( \eta \Sigma p_f' = 0 \)). Long positions in \( p_f \) are in risky assets that the active fund overweights relative to the index. These assets are sold when the investor moves from the active to the index fund. Conversely, short positions in \( p_f \) are in assets that the active fund underweights, and these assets are bought when the investor moves from the active to the index fund.

Changes in \((C_t, \hat{C}_t, y_t)\) affect asset prices through the covariance with \( p_f \). Following an increase in \( \hat{C}_t \), the investor gradually moves from the active to the index fund, selling gradually over time a slice of \( p_f \). As a consequence, assets covarying positively with \( p_f \) experience a price decline, while assets covarying negatively experience a price rise. Hence, the constant \( \gamma_1 \) in (3.1) is positive. A similar argument implies that \( \gamma_2 \) is positive and \( \gamma_3 \) is negative. We confirm these signs, as well as all other signs mentioned from now on, in our calibrated example.

The rate at which the investor changes her active-fund holdings takes the form

\[
vt \equiv \frac{dy_t}{dt} = b_0 - b_1 \hat{C}_t - b_2 y_t , \tag{3.3}
\]

where \((b_0, b_1, b_2)\) are constants and \((b_1, b_2)\) are positive. The investor’s active-fund holdings \( y_t \) evolve towards the time-varying target \( \frac{b_0 - b_1 \hat{C}_t}{b_2} \), which is decreasing in \( \hat{C}_t \). The long-run mean of \( y_t \) is \( \bar{y} \equiv \frac{b_0 - b_1 \bar{C}}{b_2} \).

The investor determines her expectation \( \hat{C}_t \) of the cost \( C_t \) by observing the return and price of the index and the active fund. The dynamics of \( \hat{C}_t \), derived using recursive filtering, are

\[
d\hat{C}_t = \kappa (\bar{C} - \hat{C}_t) dt - \beta_1 \left[ p_f [dD_t - E_t (dD_t)] - (C_t - \hat{C}_t) dt \right] - \beta_2 p_f [dS_t - E_t (dS_t)] , \tag{3.4}
\]

where \((\beta_1, \beta_2)\) are positive constants. The investor raises her estimate of \( C_t \) if the dividends from the active fund are low relative to those from the index fund, or if the active fund’s price is low.
relative to the index fund’s. Indeed, high $C_t$ lowers the dividends paid out by the active fund. High $C_t$ also forecasts future outflows by the fund, thus lowering the price of the active portfolio.

The properties of equilibrium prices that are key for our subsequent analysis of value and momentum strategies concern the impact of flows on (i) the covariance between asset returns and (ii) the predictability of returns in the cross section and the time series. The covariance matrix of returns is

$$\text{Cov}_t(dR_t, dR_t') = \left( f\Sigma + k\Sigma p_f'p_f \right) \Lambda_t,$$

(3.5)

where $f \equiv 1 + \frac{\sigma^2}{(\tau + \kappa)^2}$ and $k$ are positive constants. It is the sum of a fundamental covariance, $f\Sigma dt$, driven purely by cashflows, and a non-fundamental covariance, $k\Sigma p_f'p_f \Sigma dt$, introduced by fund flows. The non-fundamental covariance between an asset pair is proportional to the product of the covariances between each asset in the pair and the flow portfolio $p_f$. It is thus positive for asset pairs whose covariance with $p_f$ has the same sign, and negative otherwise. Intuitively, all assets whose covariance with $p_f$ has the same sign move in the same direction in response to flows. Following an increase in $C_t$, which triggers gradual outflows from the active fund, all assets covarying positively with $p_f$ experience a price decline and all assets covarying negatively experience a price rise.

Expected returns in the cross-section are described by the two-factor model

$$\mathbb{E}_t(dR_t) = \frac{r\bar{\alpha}}{\alpha + \bar{\alpha} \eta \Sigma q'} \text{Cov}_t(dR_t, \eta dR_t) + \Lambda_t \text{Cov}_t(dR_t, p_f dR_t),$$

(3.6)

with the factors being the index $\eta$ and the flow portfolio $p_f$. The risk premium associated to $\eta$ is constant over time. The risk premium $\Lambda_t$ associated to $p_f$ is time-varying and equal to

$$\Lambda_t = r\bar{\alpha} + \frac{1}{f + \frac{k\Delta}{\eta \Sigma q'}} \left( \gamma_1^R C_t + \gamma_2^R C_t + \gamma_3^R y_t - k_1\bar{q}_1 - k_2\bar{q}_2 \right),$$

(3.7)

where $(\gamma_1^R, \gamma_2^R, \gamma_3^R, k_1, k_2, \bar{q}_1, \bar{q}_2)$ are constants, $\gamma_1^R$ and $\gamma_2^R$ are negative, and $\gamma_3^R$ is positive. Equations (3.6) and (3.7) imply that expected returns follow a cycle with a cross-sectional and a time-series dimension. In the cross-section, assets are divided into two segments according to the sign of their covariance with $p_f$. That covariance reflects the pattern of fund holdings: assets overweighted by the active fund belong to one segment, and underweighted assets belong to the other. In the time-series, the expected returns of assets in each segment exhibit common variation depending on fund flows. When the investor begins to reallocate from one fund to the other, assets
in the losing segment are expected to earn low returns. After some flows occur, the expected returns of assets in the losing segment become high. These patterns repeat when the investor reallocates in the opposite direction. The initial decline in expected returns gives rise to short-run momentum, while the subsequent increase gives rise to long-run reversal.

To illustrate the patterns, consider an increase in $\hat{C}_t$, which triggers gradual outflows from the active fund. Assets covarying positively with $p_f$ experience a price decline. Since $\gamma_t R < 0$, these assets experience also a decline in their expected returns. Over time, as the outflows from the active fund materialize, $y_t$ drops. Since $\gamma_3 R > 0$, assets covarying positively with $p_f$ experience an increase in their expected returns, and that effect eventually dominates.

The initial decline in expected returns, which gives rise to short-run momentum, is surprising. Indeed, as the investor flows out of the active fund, the manager increases his holdings in the fund, absorbing the investor’s flows. Why should the manager buy assets that the fund overweights, knowing that these assets’ expected returns have declined? Why shouldn’t instead those assets drop immediately to a level from which they are expected to earn higher future returns? The answer lies in the manager’s intertemporal hedging demand, whose effect in this setting VW term bird-in-the-hand effect. The anticipation of outflows from the active fund causes assets covarying positively with $p_f$ to be underpriced and to earn an attractive return over a long horizon (one bird in the hand). The manager could earn an even more attractive return on average (two birds in the bush) by buying these assets after the outflows occur, or even by riding on momentum and shorting the assets. This exposes him, however, to the risk that the outflows might not occur, in which case the assets would cease to be underpriced and future investment opportunities would become unprofitable.

The predictability patterns can be described more formally using the conditional covariance between current and future returns. This covariance is

$$\text{Cov}_t(dR_t, dR_{t'}) = \left( \chi_1 e^{-\kappa(t' - t)} + \chi_2 e^{-\kappa(t' - t)} + \chi_3 e^{-b_2(t' - t)} \right) \Sigma p_f p_f \Sigma dtdt',$$  \hspace{1cm} (3.8)

where $(\chi_1, \chi_2, \chi_3, \rho)$ are constants. The term in square brackets in (3.8) is positive for small values of $t' - t$ and switches to negative for larger values. The switch reflects the cyclical pattern described previously. Return shocks trigger the cycle because they affect $\hat{C}_t$. Suppose that at time $t$ a negative shock hits an asset that the active fund overweights. Since the shock lowers the active fund’s return relative to the index fund, it raises $\hat{C}_t$ and triggers the cycle.
4 Trading Strategies and Performance Measures

4.1 Value and Momentum

We define a trading strategy by a vector of weights $w_t = (w_{1t}, ..., w_{Nt})$, where $w_{nt}$ is the number of shares invested in risky asset $n$ at time $t$. We include in the strategy a position $-\sum_{n=1}^{N} w_{nt} S_{nt}$ in the riskless asset, so that the value of the combined position is zero. The strategy rebalances continuously if the weights $w_t$ change continuously over time. Any gains are paid out and any losses are covered continuously so that the value of the combined position remains zero.

We consider the value strategy

$$w_t^V \equiv \left( \frac{\bar{F}_t}{r} + \epsilon (F_t - \bar{F}) - S_t \right)',$$  

(4.1)

where $\epsilon \in \{0, 1\}$. A risky asset’s value weight increases linearly in the difference between the asset’s fundamental value and price. We measure fundamental value by the present value of expected dividends discounted at $r$, and use two measures of expected dividends: the optimal forecast, which depends on the expected dividend rate $F_t$ and corresponds to $\epsilon = 1$ in (4.1), and the crude forecast, which sets expected dividends equal to their unconditional mean $\bar{F}$ and corresponds to $\epsilon = 0$.

We consider the momentum strategy

$$w_t^M \equiv \left( \int_{t-\tau}^{t} dR_u \right)'.$$  

(4.2)

A risky asset’s momentum weight increases linearly in the asset’s cumulative past return over the interval $[t - \tau, t]$. We refer to the length $\tau > 0$ of that interval as the lookback window.

4.2 Performance Measures

We measure the performance of a trading strategy $w_t$ by the Sharpe ratio of its index-adjusted version

$$\hat{w}_t \equiv w_t - \frac{\text{Cov}_t(w_t dR_t, \eta dR_t)}{\text{Var}_t(\eta dR_t)} \eta.$$  

(4.3)
The index-adjusted strategy \( \hat{w}_t \) is constructed by combining \( w_t \) with a position in the index such that the covariance between the overall position and the index is zero. (We also adjust the position in the riskless asset, so that the value of the combined position remains zero). The Sharpe ratio of the index-adjusted strategy represents compensation for risk orthogonal to the index.

The Sharpe ratio over an infinitesimal horizon \( dt \) is

\[
SR_{w,t} \equiv \frac{\mathbb{E}_{\mathcal{I}_t}(\hat{w}_t dR_t)}{\sqrt{\mathbb{V}
\text{ar}_{\mathcal{I}_t}(\hat{w}_t dR_t)dt}}.
\]

It is derived by dividing the expected excess return of \( \hat{w}_t \) by the return’s standard deviation, and expressing the ratio in annualized terms by dividing by \( \sqrt{dt} \). The return moments are conditional on an information set \( \mathcal{I}_t \) that depends on \( t \). We use the subscript \( \mathcal{I}_t \) for moments conditional on \( \mathcal{I}_t \), the subscript \( t \) for moments conditional on all information available at time \( t \) (as in Section 3), and no subscript for unconditional moments. We likewise omit the subscript \( t \) from the unconditional Sharpe ratio. We refer to \( SR_{w,t} \) interchangeably as the Sharpe ratio of \( w_t \) or of \( \hat{w}_t \).

Our use of \( SR_{w,t} \) to measure performance measure can be motivated based on portfolio optimization. Suppose that an investor with horizon \( dt \) has mean-variance preferences, and can invest in the riskless asset, the index \( \eta \) and the strategy \( w_t \). In Appendix C (Lemma C.1), we show that the investor’s maximum utility is proportional to the sum of the squared Sharpe ratio of the index \( \eta \) and of \( w_t \). In particular, it depends on \( w_t \) only through \( SR_{w,t} \).

We extend our use of the Sharpe ratio over a general finite horizon \( T \). The Sharpe ratio, expressed in annualized terms, is

\[
SR_{w,t,T} \equiv \frac{\mathbb{E}_{\mathcal{I}_t} \left( \int_t^{t+T} \hat{w}_u dR_u \right)}{\sqrt{\mathbb{V}
\text{ar}_{\mathcal{I}_t} \left( \int_t^{t+T} \hat{w}_u dR_u \right) T}}.
\]

and can be motivated based on portfolio optimization, as in the case of an infinitesimal horizon \( dt \). An orthogonality condition on the strategy weights \( w_u \) is required to ensure that \( \eta dR_u \) is uncorrelated with \( \hat{w}_u dR_u \) conditionally on \( \mathcal{I}_t \) for \( t < u < u' \). That condition is met for the strategies that we examine in the rest of the paper. The derivations are in Appendix C (Lemma C.2).

We define and calculate Sharpe ratios using returns per share rather than per dollar invested. This is because our CARA-normal model is better suited for calculating per share returns and their moments: the calculations are simplified by the properties that prices are linear in the state.
variables and that state variables are normally distributed. The same properties complicate a calculation of dollar returns because prices can become zero or negative. Because Sharpe ratios are unit-free, they are relatively insensitive to whether returns per share or per dollar are used. The same is true for other unit-free moments that we use in our analysis and that involve returns, such as return correlation or fraction of return variance driven by flows.

4.3 Calibration

We next calibrate our model. The parameter values are summarized in Table I. The model-implied moments are calculated in Appendix C.

Table I: Calibration of model parameters.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Symbol</th>
<th>Value</th>
<th>Target</th>
</tr>
</thead>
<tbody>
<tr>
<td>Investor’s risk-aversion coefficient</td>
<td>α</td>
<td>1</td>
<td>Normalization</td>
</tr>
<tr>
<td>Manager’s risk-aversion coefficient</td>
<td>ō</td>
<td>29</td>
<td>Fraction of return variance generated by flows</td>
</tr>
<tr>
<td>Number of assets</td>
<td>N</td>
<td>10</td>
<td>Industry-sector portfolios</td>
</tr>
<tr>
<td>Number of shares of each asset in the index</td>
<td>ηn</td>
<td>1</td>
<td>Normalization</td>
</tr>
<tr>
<td>Average residual supply across assets</td>
<td>θ ≡ ΣNn=1 θn / N</td>
<td>1</td>
<td>Normalization</td>
</tr>
<tr>
<td>Standard deviation of residual supply across assets</td>
<td>( \sqrt{\frac{ΣNn=1(θ_n - θ)^2}{N}} )</td>
<td>0.2</td>
<td>Industry-sector level active share of aggregate portfolio of mutual funds</td>
</tr>
<tr>
<td>Expected dividends per share of each asset</td>
<td>( \bar{F}_n )</td>
<td>0.33</td>
<td>Expected excess return of index</td>
</tr>
<tr>
<td>Variance of dividends per share of each asset</td>
<td>Σnn</td>
<td>0.47</td>
<td>Sharpe ratio of index</td>
</tr>
<tr>
<td>Covariance of dividends per share of each asset pair</td>
<td>Σnn'</td>
<td>0.41</td>
<td>Correlation between average industry-sector portfolio and index</td>
</tr>
<tr>
<td>Shocks to expected dividends</td>
<td>φ</td>
<td>0.05</td>
<td></td>
</tr>
<tr>
<td>Standard deviation of shocks to ( C_t )</td>
<td>s</td>
<td>1.2</td>
<td>Standard deviation of return gap</td>
</tr>
<tr>
<td>Mean-reversion coefficient of ( C_t ) and ( F_t )</td>
<td>κ</td>
<td>0.3</td>
<td>Mean-reversion of shocks to return gap</td>
</tr>
<tr>
<td>Long-run mean of ( C_t )</td>
<td>( \bar{C} )</td>
<td>-0.22</td>
<td>Investor’s share in active fund</td>
</tr>
<tr>
<td>Transaction cost</td>
<td>ψ</td>
<td>0.65</td>
<td>Volume generated by fund flows</td>
</tr>
<tr>
<td>Riskless rate</td>
<td>r</td>
<td>0.04</td>
<td></td>
</tr>
</tbody>
</table>

We set some parameters to one using appropriate normalizations. By redefining the units of
the consumption good, we set the investor’s risk-aversion coefficient \( \alpha \) to one. By redefining the dividend per share of each asset \( n \), we set the asset’s supply \( \pi_n \) to a value that is common across assets and such that the average residual supply \( \bar{\theta} \equiv \frac{\sum_{n=1}^{N} \theta_n}{N} \) is equal to one. Since assets are supplied in the same number of shares, the index includes the same number of shares \( \eta_n = \bar{\eta} \) of each asset \( n \). By rescaling the index, we set \( \bar{\eta} \) to one.

We interpret assets as industry-sector portfolios and set their number \( N \) to ten. We assume that all sector portfolios have the same expected dividends, the same standard deviation of dividends and the same pairwise correlation. We denote the vector of expected dividends per share by \( \bar{F} = \mathcal{F} \mathbf{1} \) and the covariance matrix of dividends per share by \( \Sigma = \hat{\sigma}^2 (I + \omega \mathbf{1} \mathbf{1}') \), where \( \mathbf{1} \) is the \( N \times 1 \) vector of ones, \( I \) is the \( N \times N \) identity matrix, and \( \mathcal{F}, \hat{\sigma}, \omega \) are scalars. We choose \( \mathcal{F} \) so that the index’s expected return per dollar in excess of the riskless rate is 4%. We choose \( \Sigma_{nn} = \hat{\sigma}^2 (1 + \omega) \) so that the annualized Sharpe ratio of the index is 30%. (The index’s Sharpe ratio is horizon-independent in our model.) We choose \( \Sigma_{nn'} = \hat{\sigma}^2 \omega \) for \( n' \neq n \) so that the return correlation between industry-sector portfolios and the index is 87%. This is the average correlation between sector portfolios and the index in Ang and Chen (2002). The remaining parameter describing dividends is \( \phi \). It is the size of shocks to the process \( F_t \) that drives expected dividends relative to shocks to the process \( D_t \) that drives dividends. Shocks to expected dividends render prices not fully revealing about \( C_t \), and induce a causal link from fund performance to fund flows as the investor uses performance to learn about \( C_t \). We set \( \phi \) to 0.05, a value that maximizes the investor’s uncertainty about \( C_t \). Even under that value, uncertainty is small: the investor’s conditional standard deviation of \( C_t \) is 18% of the unconditional standard deviation. Changing \( \phi \) has a small effect on Sharpe ratios.

With a symmetric covariance matrix of dividends, the only characteristic of residual supply \( \theta_n \) that affects Sharpe ratios, beyond the average \( \bar{\theta} = 1 \) across assets, is the standard deviation \( \sigma(\theta) \equiv \sqrt{\frac{\sum_{n=1}^{N} (\theta_n - \bar{\theta})^2}{N}} \). We choose \( \sigma(\theta) \) based on the active share of the residual-supply portfolio (Cremers and Petajisto (2009)). Buffa, Vayanos, and Woolley (2021) find that the active share of the aggregate portfolio of all active equity mutual funds, computed at the industry-sector level, is 10.81%. Defining the residual supply portfolio to also include index funds, and taking index fund assets to be 10% of total fund assets (active and index), the active share of the residual supply portfolio is 9.73% (=90% \times 10.81%). We set \( \sigma(\theta) = 0.2 \). Under the assumption that \( \theta_n \) is equal to \( \bar{\theta} + \sigma(\theta) = 1.2 \) for half of the assets and to \( \bar{\theta} - \sigma(\theta) = 0.8 \) for the other half, the active share of the residual supply portfolio is 10%.

We choose the manager’s coefficient of absolute risk aversion \( \bar{\alpha} \) based on the fraction of asset return variance generated by fund flows. Intuitively, when the manager is more risk-averse, the
investor’s flows have larger price impact and account for a larger fraction of price movements. Greenwood and Thesmar (2011) find that fund flows explain 8% of the variance of individual stocks. Gabaix and Koijen (2020) find that flows explain up to 50% of the volatility of the aggregate market. This amounts to 25% of the variance if flows are independent of fundamentals. We assume that the effect for industry-sector portfolios lies in-between, and use 15% as our target. The corresponding value of $\bar{\alpha}$ is 29, i.e., the manager is 29 times more risk-averse than the investor. In our sensitivity analysis in Section 7 we consider a target of 10%.

We choose the mean-reversion coefficient $\kappa$ and the diffusion coefficient $s$ of the cost $C_t$ by identifying $C_t$ with the return gap in Kacperczyk, Sialm, and Zheng (2008, KSZ). KSZ define the return gap as the difference between a mutual fund’s return over a given quarter and the return of a hypothetical portfolio invested in the stocks that the fund holds at the beginning of the quarter. We set $\kappa$ to 0.3, to match KSZ’s finding that shocks to the return gap shrink to about one-third of their size within four years. We set $s$ to 1.2 to match KSZ’s finding that the spread in monthly CAPM alpha between top and bottom fund deciles sorted based on lagged one-year return gap is 0.704% ($= 0.273\% - (-0.431\%)$). Since in our model there is only one active fund, we interpret cross-sectional properties of the return gap as time-series ones.

We choose the long-run mean $\bar{C}$ of $C_t$ based on the long-run mean $\bar{y}$ of the investor’s share $y_t$ in the active fund. The share $y_t$ can be interpreted as the extent to which non-expert investors participate in trades that require financial expertise, which in our model consists in identifying noise-trader induced mispricing. In the absence of frictions ($C_t = 0$ for all $t$), $y_t$ would be determined by perfect risk-sharing and be always equal to $\bar{y} = \frac{\bar{\alpha}}{\alpha + \bar{\alpha}} = 96.8\%$. Positive values of $\bar{C}$ lower $\bar{y}$. Even when $\bar{C} = 0$, $\bar{y}$ is lower than under perfect risk-sharing because the investor is averse to the risk that $C_t$ might increase in the future. The latter effect is quantitatively important: when $\bar{C} = 0$, $\bar{y}$ is only 14%. To generate larger values for $\bar{y}$, we must allow $\bar{C}$ to be negative. Negative values of $\bar{C}$ can be interpreted as an exaggerated belief by investors in managerial ability. We set $\bar{C}$ to -0.22, which yields $\bar{y} = 90\%$. In our sensitivity analysis in Section 7 we consider the value $\bar{C} = 0$, which is consistent with KSZ’s finding that the average return gap in the cross-section is zero.

We choose the transaction cost $\psi$ based on the volume generated by fund flows. Intuitively, a high transaction cost results in low volume. Lou (2012) computes flow-induced trading (FIT) for each stock by aggregating the trades that all mutual funds perform on that stock in response to inflows or outflows they experience, and dividing by the funds’ aggregate holdings of the stock. We set $\psi$ to 0.65, to match Lou’s finding that the spread in quarterly FIT between top and bottom stock deciles sorted based on FIT is 22.27% ($= 16.76\% - (-5.51\%)$). In our sensitivity analysis in Section 7 we consider a target 50% higher ($=22.27\%\times 1.5$). We set the riskless rate to 4%.
The flows in Lou (2012) arise between active and index funds, as in our model, and within active funds, e.g., from value to growth. Since we choose parameters to match the volume generated by all flows, our calibration generates unrealistically large flows between active and index funds. The flows in our calibration should instead be interpreted as including flows between active funds. Since differences between active funds can be larger than between an aggregate of active funds and of index funds, we allow the portfolios of the two funds in our model to be more different. In our sensitivity analysis in Section 7 we consider an active share of the residual supply portfolio of 20%.

As a robustness check for our parameter choices, we compute the horizon over which fund flows respond to performance. Following a positive (negative) shock to the active fund’s return, the fund experiences inflows (outflows) for 22 months, with 90% of the effect occurring within the first 13 months. This is similar to the estimate in Coval and Stafford (2007) that flows into a mutual fund during quarter $t$ increase in the fund’s return during quarters $t-1$ to $t-4$, and are roughly independent of the return during quarters $t-5$ to $t-8$.

5 Performance over an Infinitesimal Horizon

5.1 Optimal Strategy

In Appendix D (Lemma D.1) we show that the Sharpe ratio of a strategy $w_t$ over an infinitesimal horizon $dt$ is

$$SR_{w,t} = \frac{\left( f + \frac{\Delta \eta}{\eta \Sigma \eta'} \right) E_{Z_t} \left( \Lambda_t \Sigma p_f' \right)}{\sqrt{f \left[ E_{Z_t} \left( w_t \Sigma w'_t \right) - \frac{E_{Z_t} \left( w_t \Sigma w'_t \right)^2}{\eta \Sigma \eta'} \right] + kE_{Z_t} \left( (w_t \Sigma p_f')^2 \right)}},$$

and is maximized for $w_t = \Lambda_t p_f$. The intuition why the strategy $w_t = \Lambda_t p_f$ is optimal comes from the two-factor model (3.6) for expected returns. The two factors are the index $\eta$, with a constant risk premium, and the flow portfolio $p_f$, with a time-varying risk premium $\Lambda_t$. Since the Sharpe ratio $SR_{w,t}$ concerns the index-adjusted version $\hat{w}_t$ of $w_t$, it reflects compensation for the risk corresponding to the second factor $p_f$ only. Therefore, it is maximized for a strategy that invests only in $p_f$: risk that is uncorrelated with $p_f$ (and $\eta$) is not compensated. The size of the investment in $p_f$ is proportional to that factor’s risk premium $\Lambda_t$. 

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The Sharpe ratio of the optimal strategy \( w_t = \Lambda_t p_f \) is (Proposition D.1):

\[
SR_{w,t}^* \equiv \sqrt{\left( f + \frac{k \Delta}{\eta \Sigma \eta'} \right) \frac{\Delta}{\eta \Sigma \eta'} E_{I_t}(\Lambda_t^2)}.
\] (5.2)

The unconditional Sharpe ratio is proportional to \( \sqrt{E(\Lambda_t^2)} \). The conditional Sharpe ratio is proportional to the absolute value \( |\Lambda_t| \) when the conditioning set \( I_t \) includes the time-\( t \) values of the state variables \((\hat{C}_t, C_t, y_t)\). Indeed, since \( \Lambda_t \) is a function of \((\hat{C}_t, C_t, y_t)\) only, \( E_{I_t}(\Lambda_t^2) = \Lambda_t^2 \). Since \( \Lambda_t \) is affine in \((\hat{C}_t, C_t, y_t)\), the conditional Sharpe ratio is high when \( \hat{C}_t, C_t \) or \( y_t \) are large in absolute value.

In our calibrated example, the unconditional Sharpe ratio of the optimal strategy is 70.21%. It is thus 2.34 times higher than the index’s Sharpe ratio, which is 30%. The Sharpe ratio of the optimal strategy is high especially since it represents risk orthogonal to the index.

The optimal strategy’s conditional Sharpe ratio on \((\hat{C}_t, C_t, y_t)\) has mean 56.02% and standard deviation 42.32%. It thus varies significantly over time. It is lower on average than the unconditional Sharpe ratio because it tends to be high when the optimal strategy has high conditional standard deviation.

5.2 Value

In Appendix D (Proposition D.2) we derive a closed-form solution for the unconditional Sharpe ratio of the value strategy (4.1). In our calibrated example, the unconditional Sharpe ratio is 26.88% when the present value of expected dividends is computed using the optimal forecast \((\epsilon = 1)\), and 27.05% when the crude forecast \((\epsilon = 0)\) is used. The intuition why the crude forecast enhances the value strategy’s performance is as follows. Following an adverse shock to an asset’s expected dividends, the asset’s price drops. This triggers outflows from funds that overweight the asset, and causes the asset’s expected return to rise. Since the price drops, the asset’s value weight in the crude-forecast strategy rises, enhancing the strategy’s performance. By contrast, the asset’s value weight in the optimal-forecast strategy does not change because the present value of expected dividends drops by the same amount as the price.

While the value strategy offers a Sharpe ratio comparable to that of the index, and with orthogonal risk, it achieves less than 40% of the optimal strategy’s Sharpe ratio \((\frac{27.05\%}{70.21\%} = 38.53\%)\). This is because the value strategy fails to account for short-run momentum. Consider an increase in \( \hat{C}_t \), which triggers gradual outflows from the active fund. Assets covarying positively with the
flow portfolio $p_f$ experience an immediate price decline, and thus an increase in their value weight. Since these assets also experience a decline in their expected return, the value strategy earns a low expected return at this stage of the cycle. Over time, as the outflows from the active fund materialize and $y_t$ drops, the value weight of these assets increases further. Their expected return switches to being high, and so does the expected return of the value strategy.

To characterize the performance of the value strategy at different stages of the cycle, we compute in Appendix D (Proposition D.3) the strategy’s conditional Sharpe ratio. We condition on $(\hat{C}_t, y_t)$ only, because these are the key variables describing the cycle and are observed by the investor in our model (while $C_t$ is not). Equations (3.1) and (4.1) imply that the expected weights of the value strategy conditional on $I_t = (\hat{C}_t, y_t)$ and when $\epsilon = 1$ are

$$
E_{I_t}(w^V_t) = \frac{\alpha \bar{f}}{\alpha + \bar{\alpha} \eta \Sigma \eta^\prime \Sigma} \left( \gamma_0 + (\gamma_1 + \gamma_2) \hat{C}_t + \gamma_3 y_t \right) p_f \Sigma \tag{5.3}
$$

Likewise, (3.7) implies that the conditional expectation of $\Lambda_t$ is

$$
E_{I_t}(\Lambda_t) = r\bar{\alpha} + \frac{1}{f + \frac{k\Delta}{\eta \Sigma \eta^\prime}} \left( (\gamma_1^R + \gamma_2^R) \hat{C}_t + \gamma_3^R y_t - k_1 \bar{q}_1 - k_2 \bar{q}_2 \right). \tag{5.4}
$$

An increase in $\hat{C}_t$ raises value weights of assets covarying positively with $p_f$ because $(\gamma_1, \gamma_2)$ are positive. It lowers $\Lambda_t$, thus lowering the expected returns of those assets, because $\gamma_1^R + \gamma_2^R$ is negative. By contrast, a decrease in $y_t$ raises those assets’ value weights and expected returns because $(\gamma_3, \gamma_3^R)$ are negative. In line with these observations, the conditional Sharpe ratio of the value strategy is negative and large when $\hat{C}_t$ becomes large in absolute value (early stage of the cycle). It switches to being positive and large when $y_t$ adjusts to the change in $\hat{C}_t$ by becoming large with opposite sign to $\hat{C}_t$ (late stage).

The mean and standard deviation of the value strategy’s conditional Sharpe ratio are 20.60% and 65.03%, respectively, when $\epsilon = 1$, and 21.46% and 63.25%, respectively, when $\epsilon = 0$. The Sharpe ratio of the value strategy varies more than that of the optimal strategy, reflecting the sharply different performance of value at different stages in the cycle.

We next examine how the conditional Sharpe ratio of the value strategy correlates with the value spread. We define the value spread as the standard deviation of the market-to-book ratio in the cross-section of assets. We assume that all assets have the same book value, which we take to be the average price in the cross-section of assets and over time. We compute the value spread
conditional on \((\hat{C}_t, y_t)\) in Appendix D (Proposition D.4). The value spread is high when \(\hat{C}_t\) is large in absolute value because the price discrepancies between assets that covary positively and assets that covary negatively with the flow portfolio \(p_f\) are large. The value spread is even higher when \(y_t\) adjusts to the change in \(\hat{C}_t\) because the price discrepancies become even larger.

The value spread correlates positively but imperfectly with the conditional Sharpe ratio of the value strategy. This is because \(\hat{C}_t\) moves the two variables in opposite directions, while \(y_t\) moves them in the same direction and has a dominant effect. When \(\hat{C}_t\) becomes large in absolute value (early stage of the cycle), the value spread is large and the conditional Sharpe ratio of value is negative. When \(y_t\) adjusts to the change in \(\hat{C}_t\) by becoming large with opposite sign to \(\hat{C}_t\) (late stage), the value spread is even larger and the conditional Sharpe ratio of value is positive. The correlation between the value spread and the conditional Sharpe ratio of the value strategy is 26.31% when \(\epsilon = 1\) and 26.00% when \(\epsilon = 0\). For the remainder of our analysis, we focus on the value strategy with \(\epsilon = 0\).

### 5.3 Momentum

In Appendix D (Proposition D.5) we derive a closed-form solution for the unconditional Sharpe ratio of the momentum strategy \((4.2)\). Figure 1 plots the unconditional Sharpe ratio in our calibrated example as function of the lookback window \(\tau\) over which past returns are calculated. The unconditional Sharpe ratio reaches its maximum value 53.66% for a window of seven months, and exceeds 50% for windows ranging from four to eleven months. When the window goes to zero, the Sharpe ratio does too because performance over a very short interval is almost uninformative about future flows. Conversely, when the window becomes large, the Sharpe ratio becomes negative because momentum turns into reversal.

The momentum strategy with the 4-12 month lookback window performs significantly better than the value strategy because it is better aligned with movements in expected returns. Consider an increase in \(\hat{C}_t\), which triggers gradual outflows from the active fund. Assets covarying positively with the flow portfolio \(p_f\) experience an immediate price decline, and thus a decrease in their momentum weight. Since these assets also experience a decline in their expected return, the momentum strategy earns a high expected return at the early stage of the cycle. It also earns a moderately high expected return at the late stage. Indeed, after the outflows materialize, the expected return of assets covarying positively with \(p_f\) is high. As a consequence, these assets’ return history improves, and their momentum weight rises. Momentum’s underperformance occurs at intermediate stages of the cycle. Indeed, the expected return of assets covarying positively with \(p_f\) has increased but
their return history has not caught up with that increase.

To characterize the performance of the momentum strategy at different stages of the cycle, we compute in Appendix D (Proposition D.6) the strategy’s conditional Sharpe ratio. As with the value strategy, we condition on \((\hat{C}_t, y_t)\) only. Using (4.2) and (A.3), we show in the Appendix that the expected weights of the momentum strategy conditional on \(I_t = (\hat{C}_t, y_t)\) are

\[
E_{I_t}(w_t^M) = \frac{r\alpha \hat{f} \tau}{\alpha + \bar{\alpha}} \frac{\eta' \theta'}{\eta' \Sigma} \eta \Sigma + (\delta_0^M + \delta_{12}^M \hat{C}_t + \delta_3^M y_t) p_f \Sigma, \tag{5.5}
\]

where \((\delta_0^M, \delta_{12}^M, \delta_3^M)\) are constants. For the remainder of our analysis, we focus on the optimal momentum strategy with the seven month lookback window, for which \((\delta_{12}^M, \delta_3^M)\) are positive and the ratio \(\frac{\delta_{12}^M}{\delta_3^M}\) is smaller than \(\frac{\gamma_R^R}{\gamma_R^L + \gamma_R^L}\). An increase in \(\hat{C}_t\) lowers momentum weights of assets covarying positively with \(p_f\) because \(\delta_{12}^M\) is negative. Because it also lowers \(\Lambda_t\), the conditional Sharpe ratio of the momentum strategy is positive and large when \(\hat{C}_t\) is large in absolute value (early stage of the cycle). A decrease in \(y_t\) raises the momentum weights of assets covarying positively with \(p_f\) because \(\delta_3^M\) is negative. Because it also raises \(\Lambda_t\), the conditional Sharpe ratio of the momentum strategy is positive when \(y_t\) adjusts to the change in \(\hat{C}_t\) by becoming large with opposite sign to \(\hat{C}_t\) (late stage). The conditional Sharpe ratio is instead negative for a range of intermediate values of \(y_t\). Indeed, because \(\frac{\delta_{12}^M}{\delta_3^M} < \frac{\gamma_L^R}{\gamma_L^L + \gamma_L^L}\), \(\Lambda_t\) changes sign before momentum weights do during the process of \(y_t\)’s adjustment.
The mean and standard deviation of the conditional Sharpe ratio of the momentum strategy are 40.74% and 46.58%, respectively. The Sharpe ratio of the momentum strategy varies less than that of the value strategy, reflecting the more limited variation in momentum’s performance over the cycle.

Since momentum performs well at the early stage of the cycle, moderately well at the late stage, and poorly at intermediate stages, it is weakly correlated with the value spread. The correlation between the value spread and the conditional Sharpe ratio of momentum is -8.13%. The correlation between the conditional Sharpe ratios of momentum and value is 28.22%.

Figure 2 plots the dynamics following a shock that moves the state variables $(\hat{C}_t, y_t)$ away from their long-run means $(\bar{C}, \bar{y})$. The shock is a decline to the flow portfolio’s return at time zero, equal to one standard deviation of the portfolio’s annual return. The left panel plots the shock’s effect on $(\hat{C}_t, y_t)$, as function of time $t$. Following the shock, the investor’s expectation $\hat{C}_t$ of the active fund’s cost jumps up and declines gradually to $\bar{C}$. The investor’s share $y_t$ in the active fund declines gradually for 22 months after the shock. After that time, it increases gradually to $\bar{y}$.

The right panel plots the Sharpe ratios conditional on $(\hat{C}_t, y_t)$ for the optimal strategy (black solid line), the value strategy with $\epsilon = 0$ (blue dashed line) and the momentum strategy (red dashed-dotted line), as function of time $t$. The value spread is also plotted (green dotted line). The units for the Sharpe ratios are shown in the left $y$-axis, and the units for the value spread are shown in the right $y$-axis.

When $(\hat{C}_t, y_t)$ are equal to their long-run means $(\bar{C}, \bar{y})$, the Sharpe ratios of momentum and value are close to zero: -2.48% for momentum and 0.38% for value. Following the shock, the Sharpe
ratios experience large movements in opposite directions. The Sharpe ratio of momentum jumps up to 114.99%. It then declines rapidly, becomes negative twelve months after the shock, becomes positive again twenty-two months after the shock, and finally declines to its value for \((\hat{C}_t, y_t) = (\bar{C}, \bar{y})\). The Sharpe ratio of value jumps down to -116.96%. It then rises rapidly, becomes positive thirteen months after the shock, reaches a maximum of 36.71% three years after the shock, and finally declines to its value for \((\hat{C}_t, y_t) = (\bar{C}, \bar{y})\). The value spread jumps up following the shock, and keeps increasing as the mispricing worsens. It reaches a maximum thirteen months after the shock, and then declines to its value for \((\hat{C}_t, y_t) = (\bar{C}, \bar{y})\).

The Sharpe ratio of the optimal strategy is remarkably close to that of value or of momentum. The difference between the optimal strategy’s Sharpe ratio and the larger of the value and the momentum Sharpe ratios is smaller than 6% for a long period after the shock, which is composed of two sub-periods. During the first sub-period which lasts for the first five months after the shock, momentum is approximately optimal and its Sharpe ratio lies within 6% of the optimal strategy’s. During the second sub-period which lasts from fourteen months to 10.5 years after the shock, value is instead approximately optimal and its Sharpe ratio lies within 6% of the optimal strategy’s. (That difference shrinks to 2.5% for the period starting 1.5 year and ending six years after the shock.) Hence, within the context of our model, value and momentum span well the set of trading strategies, with each of them being approximately optimal at a different stage of the cycle.

The shock in Figure 2 is a decline to the flow portfolio’s return. Under the opposite shock, \((\hat{C}_t - \bar{C}, y_t - \bar{y})\) would change sign, and the right panel would flip around the \(x\)-axis. The left panel would remain approximately the same, however. This is because the Sharpe ratios depend almost exclusively on the stage of the flows cycle and not on whether the flows occur from the active to the index fund or vice-versa. The conditional correlation between value and momentum, plotted in Figure 3, would also remain approximately the same.

### 5.4 Combining Value and Momentum

We next compute the gains from combining value and momentum. This requires computing the correlation between the two strategies’ returns, an exercise of independent interest since it reveals how the strategies relate to each other. For two general strategies \((w^A_t, w^B_t)\), the Sharpe ratio of their optimal (mean-variance maximizing) combination is (Appendix D, Lemma D.5)

\[
SR_{w^A,w^B,t} \equiv \sqrt{\frac{SR^2_{w^A,t} + SR^2_{w^B,t} + 2SR_{w^A,t}SR_{w^B,t}\text{Corr}_{\mathcal{I}_t}(\hat{w}^A_t dR_t, \hat{w}^B_t dR_t)}{1 - \text{Corr}_{\mathcal{I}_t}(\hat{w}^A_t dR_t, \hat{w}^B_t dR_t)^2}}. \tag{5.6}
\]
The Sharpe ratio of the optimal combination depends only on the two strategies’ Sharpe ratios and on the correlation between the strategies’ returns. The calculations in (5.6) are conditional on an information set $\mathcal{I}_t$, in the sense that strategy weights, Sharpe ratios and the correlation can depend on $\mathcal{I}_t$. The correlation in (5.6) is between the strategies’ index-adjusted versions, but as with the Sharpe ratios we refer to it as pertaining to the strategies as well.

In Appendix D (Proposition D.7) we derive a closed-form solution for the unconditional correlation between value and momentum, and for the Sharpe ratio of the optimal unconditional combination of the two strategies ($\mathcal{I}_t = \emptyset$). In our calibrated example, this Sharpe ratio is 63.45%. It is 10% larger than the unconditional Sharpe ratio of momentum (53.66%) and 36% larger than that of value (27.05%). Thus, combining value and momentum improves significantly over using one or the other strategy and yields a Sharpe ratio close to that of the optimal strategy (70.21%). The improvement is due to the low unconditional correlation between value and momentum, which is -12.20%. The correlation is negative because value loads up on assets that have underperformed over a long period, while momentum tends to short those assets as they have been trending down in the recent past.

The low unconditional correlation between value and momentum masks large variation in the conditional correlation. We compute the value-momentum correlation conditionally on $(\hat{C}_t, y_t)$ in Appendix D (Proposition D.7). The conditional correlation has mean -8.13% and standard deviation 73.02%. Figure 3 illustrates the large variation in the conditional correlation by plotting its dynamics following the same shock as in Figure 2.

![Figure 3: Conditional correlation between value and momentum following a one standard deviation drop in the return of the flow portfolio $p_f$ at time zero.](image)

When $(\hat{C}_t, y_t)$ are equal to their long-run means $(\bar{C}, \bar{y})$, momentum and value are approximately independent, with a correlation of -4.53%. Following the shock, the correlation jumps down to
-97.06%. The two strategies thus become approximately perfectly negatively correlated. The correlation remains below -80% for the ten months after the shock. It then rises gradually, becomes positive twenty months after the shock, reaches a maximum of 52.30% three years and four months after the shock, and finally declines to its value for \((\hat{C}_t, y_t) = (\bar{C}, \bar{y})\).

During the period of high negative correlation, the momentum strategy shorts assets that covary positively with the flow portfolio \(p_f\) because the shock drives down these assets’ returns. By contrast, the value strategy longs these assets because their price is low. During the period of high positive correlation, value continues longing these assets, and momentum switches in expectation to longing them as well.

Combining value and momentum conditionally on \((\hat{C}_t, y_t)\) yields small gains over using one of the two strategies only. In the dynamics shown in Figure 2, the Sharpe ratio of the optimal value-momentum combination never exceeds the larger of the individual Sharpe ratios by more than 6%. (This can be anticipated from the closeness between the Sharpe ratio of the optimal strategy and the larger of the individual Sharpe ratios.) The improvement from combining value and momentum conditionally is smaller than unconditionally because of the variation in the relative performance of the two strategies. Momentum is the much better strategy for a set of values of \((\hat{C}_t, y_t)\) and value is for another set. Within either set, combining the strategies yields small gains relative to using the better strategy. When, however, information on \((\hat{C}_t, y_t)\) is not used, the identity of the better strategy is unknown. Combining the strategies then yields larger gains because the weight on the worse strategy is reduced and so is the scope for under-performance.

The conditional correlation between value and momentum is informative about the Sharpe ratio of each strategy. That information is particularly precise for value: the correlation between the conditional value-momentum correlation and the conditional Sharpe ratio of value is 86.02%. This can be anticipated from Figures 2 and 3, as the conditional correlation and the conditional Sharpe ratio of value respond similarly to the shock. While Figures 2 and 3 suggest a negative correlation between the conditional value-momentum correlation and the conditional Sharpe ratio of momentum, that correlation is positive and equal to 41.34%. Thus, a positive value-momentum correlation indicates high conditional Sharpe ratio of both value and momentum.
Performance over a General Finite Horizon

6.1 Optimal Strategy

We next allow the investment horizon to take any finite value. To determine how horizon influences the choice of strategy, we begin with an optimization exercise. Consider an investor who has horizon $T$ and maximizes the unconditional Sharpe ratio $SR_{w,T}$. Suppose that the investor must follow a strategy of the form $w_t = (\delta_0 + \delta_1 \hat{C}_t + \delta_2 C_t + \delta_3 y_t)p_f$ and can optimize over the coefficients $(\delta_0, \delta_1, \delta_2, \delta_3)$. The optimal coefficients depend on the horizon $T$, and we denote them by $(\delta_{0,T}^*, \delta_{1,T}^*, \delta_{2,T}^*, \delta_{3,T}^*)$. We determine in Appendix E the Sharpe ratio $SR_{w,T}^*$ of the optimal strategy (Proposition E.1) and the strategy’s correlation with value and momentum (Propositions E.5 and E.6).

The optimization is not over the full set of strategies: the strategies are assumed to invest only in the flow portfolio $p_f$; the investment in $p_f$ is assumed linear in the state variables $(\hat{C}_t, C_t, y_t)$; and the coefficients in the linear function are assumed constant over time. The first and second assumptions are shown as results for the short-horizon optimal strategy $w_t = \Lambda_t p_f$ (Section 5.1), and we conjecture that these results extend to the long-horizon optimization. In particular, since the Sharpe ratio $SR_{w,T}$ reflects compensation only for risk corresponding to $p_f$, it should be maximized for a strategy that invests only in $p_f$. The third assumption is restrictive. Indeed, since the optimal coefficients $(\delta_{0,T}^*, \delta_{1,T}^*, \delta_{2,T}^*, \delta_{3,T}^*)$ depend on the horizon $T$, the investor may want to change them as time passes and the end of the horizon approaches. Restricting the coefficients to be time-independent simplifies the calculation of the (constrained) optimal strategy and of its closeness to value and momentum (both of which are defined to be time-independent).

Figure 4 illustrates properties of the optimal strategy. The left panel plots the unconditional return correlation of the optimal strategy with the value strategy (blue dashed line) and with the momentum strategy (red dashed-dotted line), as function of the investment horizon. The correlation concerns returns computed over the horizon corresponding to the optimal strategy (e.g., one-year returns for the one-year optimal strategy, and ten-year returns for the ten-year optimal strategy). When the horizon is short, the optimal strategy correlates more highly with momentum than with value. The higher correlation with momentum is consistent with the finding in Section 5 that momentum has a higher Sharpe ratio than value over an infinitesimal horizon.

The main new observation from the figure concerns the variation of the correlation with the investment horizon. As horizon increases, the correlation of the optimal strategy with momentum decreases, while that with value increases and overtakes momentum’s for horizons longer than thir-
Figure 4: Unconditional correlation of the optimal strategy with value and momentum (left panel) and Sharpe ratio of the optimal strategy, of value and of momentum (right panel), as function of the investment horizon.

teen years. Similar conclusions follow when measuring closeness by weights in a tracking portfolio. When value and momentum are combined into a portfolio whose return is the closest to the optimal strategy’s, as measured by unconditional variance, their weights have a similar dependence on horizon as the correlations. We defer a fuller discussion of optimal weights to Section 6.4.

The effects of investment horizon on the correlation that the optimal strategy has with value and momentum are related to the effects on the strategies’ Sharpe ratios. The right panel of Figure 4 plots the unconditional Sharpe ratio of the optimal strategy (black line), of value (blue dashed line) and of momentum (red dashed-dotted line), as function of horizon. We compute the unconditional Sharpe ratios of value and momentum in Appendix E (Propositions E.2 and E.3, respectively). The Sharpe ratio of the optimal strategy is closer to that of momentum for short horizons and to that of value for long horizons. This is consistent with the results on the correlations. Consistent with those results is also that value’s Sharpe ratio overtakes momentum’s for horizons longer than thirteen years. The main new observation from the figure concerns the variation of Sharpe ratios with horizon. The Sharpe ratio of the optimal strategy is an inverse hump-shaped function of horizon, as is the Sharpe ratio of value. The Sharpe ratio of momentum initially decreases with horizon and then stays essentially flat.

The effects of horizon on Sharpe ratios are driven by the autocorrelation of strategy returns. In Appendix E (Lemma E.2) we show that the unconditional Sharpe ratio $SR_{w,T}$ of a strategy $w_T$ over horizon $T$ can be expressed in terms of the unconditional Sharpe ratio $SR_w$ over an infinitesimal
horizon and the autocovariance of returns over all horizons from zero to $T$:

$$SR_{w,T} = \frac{SR_w}{\sqrt{1 + 2 \int_t^{t+T} \left(1 - \frac{u-t}{T}\right) \frac{\text{Cov}(\hat{w}_t dR_t, \hat{w}_u dR_u)}{\text{Var}(\hat{w}_t dR_t)}}}$$

(6.1)

The Sharpe ratio is independent of horizon when strategy returns are serially uncorrelated. This is because expected returns are horizon-independent when expressed in annualized terms (dividing by $T$), and lack of serial correlation implies that the same is true for standard deviation (dividing by $\sqrt{T}$). When instead strategy returns are positively autocorrelated, annualized variance increases with horizon, and $SR_{w,T}$ is smaller than $SR_w$. The converse is true when returns are negatively autocorrelated. Thus, the effects of horizon shown in the right panel of Figure 4 reflect variation in return autocorrelation. We examine that variation next, in the context of value and momentum.

### 6.2 Value

We compute autocorrelations of value and momentum returns in Appendix E (Proposition E.8), and plot them in Figure 5. The left panel plots correlations between value (blue dashed line) or momentum (red dashed-dotted line) returns over a lookback window $\tau$ ending at time $t$ with the same strategy’s returns over the year starting at time $t$. The right panel replaces returns one year ahead by returns ten years ahead. Value returns are positively autocorrelated over short horizons and lookback windows, reflecting asset-level momentum. They become negatively autocorrelated over long horizons or lookback windows, reflecting partly asset-level reversal and mostly the nature of the value strategy. Value loads up on assets that have performed poorly, and has low turnover because it holds assets based on slow-moving signals. Suppose that the poorly performing assets held by value experience a further long period of underperformance, lowering value returns. The expected returns of those assets increase, and so does the weight given to them by the value strategy. This boosts value’s expected return, resulting in negative autocorrelation of value returns over long lookback windows. Lengthening the horizon (moving from the left to the right panel) renders autocorrelations more negative, and negative for all lookback windows. This is because the effect of momentum is small over long horizons.

The autocorrelation pattern of value returns is reflected into the inverse hump-shaped pattern of value’s Sharpe ratio. For short horizons, the relevant autocorrelations are those over short lookback windows in the left panel of Figure 5. Since these are positive, annualized variance increases with horizon and the Sharpe ratio decreases. For long horizons instead, the autocorrelations over long lookback windows become relevant, lowering the annualized variance and raising the Sharpe ratio. The long-window autocorrelations are larger in absolute terms than the short-horizon ones, and
Figure 5: Autocorrelations of value and momentum, between returns over a lookback window \( \tau \) ending at time \( t \) and returns over the year (left panel) and the ten years (right panel) starting at time \( t \). Autocorrelations are plotted as function of \( \tau \).

die off to zero slowly when the window increases. Consequently, their effect on the Sharpe ratio is quantitatively important. While the Sharpe ratio of value is 27.05% over an infinitesimal horizon (Section 5) and drops to 19.78% for a two-year horizon, it rises to 29.66% for ten years, 38.43% for twenty years, and 47.42% for forty years.

We next turn to the conditional Sharpe ratio of value, which we compute in Appendix E (Proposition E.4). The conditional Sharpe ratio can intuitively be thought of as an average of the current and future expected Sharpe ratios during the investment horizon, adjusted for the effects of autocorrelation. Being an average, it varies less than its infinitesimal-horizon counterpart. The standard deviation of the conditional Sharpe ratio drops from 63.25% for an infinitesimal horizon to 39.39% for a one-year horizon, 40.88% for five years, 31.33% for ten years, and 18.16% for twenty years.

Lengthening the horizon changes drastically the predictability of the conditional Sharpe ratio based on the value spread and on the correlation between value and momentum. The value spread becomes a better predictor. Its correlation with the conditional Sharpe ratio of value rises from 26.00% for an infinitesimal horizon to 77.24% for a one-year horizon, 97.81% for five years, 96.92% for ten years, and 96.45% for twenty years. Conversely, the (instantaneous) correlation becomes a worse predictor. Its correlation with the conditional Sharpe ratio of value drops from 86.02% for an infinitesimal horizon (Section 5) to 47.63% for a one-year horizon, -3.56% for five years, -7.72% for ten years, and -6.92% for twenty years.

The effect of horizon on the predictive relationships can be understood by plotting the response of the conditional Sharpe ratios to a one standard deviation drop in the flow portfolio’s return.
This exercise is performed for infinitesimal Sharpe ratios in Figure 2, and we repeat it for five-year Sharpe ratios in Figure 6. While the infinitesimal Sharpe ratio drops substantially in response to the shock, its five-year counterpart rises. This is because the five-year Sharpe ratio reflects an average of future expected Sharpe ratios, which rise in response to the shock. As a consequence, the five-year Sharpe ratio moves more in sync with the value spread, resulting in a higher correlation. It also moves less in sync with the value-momentum correlation, resulting in a lower correlation.

![Figure 6: Conditional five-year Sharpe ratios and the value spread following a one standard deviation drop in the return of the flow portfolio $p_f$ at time zero.](image)

### 6.3 Momentum

Similar to value returns, momentum returns are positively autocorrelated over short horizons and lookback windows, reflecting asset-level momentum. In contrast to value returns, the autocorrelation does not become negative for long horizons or lookback windows but instead drops to zero. The autocorrelation vanishes because the momentum strategy has high turnover, holding assets based only on their recent performance. The autocorrelation does not become negative because momentum loads up on assets that have performed well. Suppose that the well-performing assets held by momentum experience a long period of underperformance, lowering momentum returns. The expected returns of these assets increase but momentum shorts these assets. This lowers momentum’s expected return, resulting in positive autocorrelation of momentum returns.

The positive autocorrelation of momentum returns is reflected into momentum’s Sharpe ratio, which decreases with investment horizon. The Sharpe ratio of momentum is 53.66% over an infinitesimal horizon (Section 5), and drops to 37.06% for an one-year horizon. It then stays essentially flat, equal to 34.28% for five years, 33.49% for ten years, 33.04% for twenty years and 32.82% for forty years. Value overtakes momentum for horizons longer than thirteen years.
We next turn to the conditional Sharpe ratio of momentum, which we compute in Appendix E (Proposition E.4). Momentum’s conditional Sharpe ratio varies significantly less than value’s over the same long horizon, reflecting its smaller variation over an infinitesimal horizon. The standard deviation of momentum’s conditional Sharpe ratio drops from 46.58% for an infinitesimal horizon to 11.93% for a one-year horizon, 6.52% for five years, 4.52% for ten years, and 2.35% for twenty years.

Since momentum’s conditional Sharpe ratio exhibits small variation over long horizons, its predictability matters less than value’s. Its predictability becomes similar to value’s as horizon increases. It is well predicted by the value spread, with the correlation rising from -8.13% for an infinitesimal horizon to 27.13% for a one-year horizon, 87.99% for five years, 91.75% for ten years, and 91.53% for twenty years. Conversely, it is not well predicted by the correlation between value and momentum. Its correlation with that variable rises from 41.34% for an infinitesimal horizon (Section 5) to 53.91% for a one-year horizon, but subsequently drops to 18.47% for five years, 8.01% for ten years, and 6.64% for twenty years.

6.4 Combining Value and Momentum

We compute the unconditional correlation between value and momentum returns over a general investment horizon in Appendix E (Proposition E.7). Figure 7 plots this correlation as function of horizon. The correlation is negative for returns computed over horizons up to seven months and turns positive for longer horizons. It is relatively small in absolute value, rising from -12.20% for an infinitesimal horizon (Section 6.4) to 19.92% for a forty-year horizon. As a consequence, combining value and momentum improves significantly over using one or the other strategy and yields a Sharpe ratio close to that of the optimal strategy. The difference between the Sharpe ratio of the optimal strategy and of the optimal value-momentum combination drops from 6.76% for an infinitesimal horizon to 3.20% for an one-year horizon, 1.76% for five years, 1.30% for ten years, 0.94% for twenty years, and 0.64% for forty years.

The change in sign of the value-momentum correlation from negative to positive as horizon increases reflects the cross-autocorrelations between the strategies, known also as lead-lag effects. We compute lead-lag effects for value and momentum returns in Appendix E (Proposition E.8), and plot them in Figure 8. The left panel plots correlations between value (blue dashed line) or momentum (red dashed-dotted line) returns over a lookback window $\tau$ ending at time $t$ with the other strategy’s returns over the year starting at time $t$. The right panel replaces returns one year ahead by returns ten years ahead.

Lead-lag effects differ for short- and long-horizon returns. Over short horizons (left panel),
they are mainly driven by the joint variation of the strategies’ expected returns during the flow cycle, and they are present only from value to momentum and only over short lookback windows.\footnote{In Appendix E (Equations (E.6) and (E.7)), we show that the correlation \(\text{Cov}_{\tau} \left( \hat{w}_j^u dR_u, \hat{w}_k^{u'} dR_{u'} \right)\) between instantaneous returns of strategy \(j\) at time \(u\) and strategy \(k\) at time \(u' > u\) can be written as}

\[
\text{Cov}_{\tau} \left[ \mathbb{E}_u(\hat{w}_j^u dR_u), \mathbb{E}_{u'}(\hat{w}_k^{u'} dR_{u'}) \right] + \mathbb{E}_t \left[ \hat{w}_j^u \text{Cov}_u(dR_u, \hat{w}_k^{u'} \mathbb{E}_{u'}(dR_{u'})) \right].
\]

The first term is the correlation between expected returns at \(u\) and \(u'\), and drives lead-lag effects over short horizons in our calibration. The second term describes the response of expected returns at \(u'\) to shocks at \(u\), and drives lead-lag effects over long horizons.
momentum starts earning high expected returns as well. By contrast, momentum returns do not predict short-horizon returns on value. This is because momentum earns high expected returns at the beginning or at the end of the flow cycle, and these are followed by low expected returns of value in the former case and by high expected returns in the latter case.

Lead-lag effects over long horizons (right panel) are mainly driven by the response of the strategies’ expected returns to shocks, and they are present from both value to momentum and from momentum to value. A long period of underperformance by the assets held by value indicates that those assets will earn high future expected returns. Hence, those assets will be included in momentum portfolios, which will perform well on average (negative lead-lag effect). A long period of overperformance by momentum indicates that flows out of funds holding poorly performing assets and into funds holding well performing ones are larger than expected. This indicates high mispricing and thus high future expected returns by value (positive lead-lag effect).

The cross-autocorrelation from value to momentum changes sign to positive as horizon increases because the positive lead-lag effects dominate the negative ones. Therefore, the drivers of the switch in sign of the value-momentum correlation are the short-horizon lead-lag effect from value to momentum, and the long-horizon one from momentum to value.

Using the unconditional Sharpe ratios and correlation of value and momentum, we compute in Appendix E (Proposition E.9) the investment in these strategies in their optimal (mean-variance maximizing) combination. We express the investment in normalized terms by rescaling strategy weights in the assets so that strategy standard deviation times investor risk aversion is equal to one. We refer to the normalized investment as the weight given by the investor in a strategy. The weights for value and momentum are plotted as function of horizon in the left panel of Figure 9. Momentum’s weight is 169% that of value for an infinitesimal horizon, and rises to 199% for a two-year horizon. It then decreases with horizon, becoming equal to value’s weight for thirteen years, and to 57% of value’s weight for forty years. Momentum’s weight declines with horizon relative to value’s weight because value’s Sharpe ratio increases while momentum’s stays essentially flat. Momentum’s weight declines with horizon in absolute terms because value’s weight rises and because the value-momentum correlation turns positive.

The right panel of Figure 9 plots value and momentum weights in the combination that best approximates the optimal strategy derived in Section 6.1. We construct that combination by minimizing the unconditional variance of the difference in returns between that combination and the optimal strategy. The weights of value and momentum in that combination are computed in Appendix E (Proposition E.10). Figure 9 shows that they are nearly identical to those in the mean-variance maximizing value-momentum combination.
7 Sensitivity Analysis

We next examine the sensitivity of the quantitative analysis in Sections 5 and 6 to changes in parameter values. The results are in Table II. The table reports moments derived in Sections 5 and 6 in the following cases: baseline, where parameter values are as in Table I; lookback window for momentum equal to one year instead of seven months; fraction of asset return variance generated by fund flows equal to 10% instead of 15%; volume generated by fund flows larger by 50% than in the baseline; and active share of residual supply portfolio equal to 20% instead of 10%. The case where \( \bar{C} \) is equal to zero instead of -0.22 yields results similar to the baseline and is not reported in the table. When deviating from the baseline to meet a calibration target, we choose parameter values to meet all remaining targets in Table I.

Table II indicates that many of the patterns shown in Sections 5 and 6 are robust across cases. In particular: (i) the Sharpe ratio of value, which is remarkably stable across cases when horizon is short (infinitesimal), drops somewhat when horizon increases (to five years), and rises significantly when horizon increases further (to twenty years); (ii) the Sharpe ratio of momentum, which is less stable than value’s across cases when horizon is short, drops significantly when horizon increases, and becomes essentially flat when horizon increases further; (iii) the Sharpe ratio of value is more volatile than that of momentum, especially for long horizons; (iv) value and momentum are modestly negatively correlated for short horizons and modestly positively for long horizons; (v) the value spread is positively correlated with value’s Sharpe ratio for short horizons and strongly so for long horizons; (vi) the value spread is slightly negatively correlated with momentum’s Sharpe ratio.
for short horizons but strongly positively correlated for long horizons; (vii) the value-momentum correlation is strongly positively correlated with value’s Sharpe ratio for short horizons but slightly negatively correlated for long horizons; and (viii) the value-momentum correlation is positively correlated with momentum’s Sharpe ratio, especially for short horizons.

When the lookback window of momentum increases to one year, momentum’s short-horizon correlations with value and the value spread become more negative. This is because momentum with a long lookback window becomes more similar to the opposite of a value strategy: it buys assets with a long history of good performance, which trade on average at a high price relative to
fundamental value.

When flows account for a smaller fraction of asset return variance, momentum’s Sharpe ratio decreases significantly. The intuition goes back to the momentum-generating mechanism in the model. Long-horizon investors buy assets with poor recent and expected future performance because they do not want to run the risk that by waiting and buying later the assets cease to be underpriced. Since mispricing is caused by flows, it becomes less volatile when flows generate smaller price variation. Therefore, long-horizon investors bear less risk by waiting, causing prices of assets with poor recent performance to drop fast rather than more gradually, and momentum to become less profitable. With momentum becoming less profitable at the beginning of the flow cycle, value becomes less unprofitable at that stage of the cycle, and the short-horizon correlation of its Sharpe ratio with the value spread increases.

When flows account for a larger fraction of volume, momentum’s Sharpe ratio drops slightly. Intuitively, since the fraction of asset return variance generated by flows is held constant, the manager’s coefficient of absolute risk aversion must be smaller. As a consequence, the manager becomes less averse to waiting to buy underpriced assets, and momentum becomes less profitable. Momentum becomes also less profitable when the residual supply portfolio has higher active share.

8 Conclusion

We study dynamic portfolio choice in an economy where value and momentum arise endogenously because of performance-driven flows. By grounding these well-known strategies in a dynamic model, our approach provides valuable structure for characterizing how these strategies perform across different horizons either in isolation or when optimally combined.

Specifically, in our model, the Sharpe Ratio of value increases significantly at horizons of two years or longer while the Sharpe Ratio of momentum remains flat at those horizons. We also find that the correlation between value and momentum flips from negative to positive at horizons longer than one year. As a consequence, we show that value becomes an increasingly larger share of the optimal composite portfolio for long-horizon investors.

Our results reflect the dynamics of the flow cycle, and thus highlight the important role played by two natural time-varying signals that arise from our theory and characterize the variation in value-momentum opportunities through time. We show how both the value spread and the short-horizon correlation between value and momentum contain useful information about the time-varying Sharpe Ratio on value, momentum, and their optimal combination.

We calibrate our theory to the data, and in companion work, confirm that our theoretically
motivated predictors do forecast time-varying expected returns in the way our theory predicts. By imposing structure on optimal dynamic portfolio choice across value and momentum, we hope to provide better out-of-sample properties for strategies exploiting these patterns in real-world applications.
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Appendix

A Proofs of Results in Section 3

Equation (3.1) follows from combining VW equation (28) with (29) and (B34). VW equation (B34) implies

\[ a_0 = \frac{\alpha \tilde{\alpha} f}{\bar{\alpha} + \tilde{\alpha}} \frac{\eta \Sigma \theta'}{\eta \Sigma \eta'} \Sigma \eta' + \gamma_0 \Sigma p_f' \]

with

\[ \gamma_0 = \frac{\kappa(\gamma_1 + \gamma_2)\bar{C} + b_0 \gamma_3 - k_1 \tilde{q}_1 - k_2 \tilde{q}_2}{r} + \tilde{\alpha} \left( f + \frac{k \Delta}{\eta \Sigma \eta'} \right). \] (A.1)

Equation (3.3) is VW equation (30). Equation (3.4) is VW equation (31), shown in Proposition 4.

Equation (3.5) for the covariance matrix of returns is VW equation (37) shown in Corollary 8. Equations (A.3) and (3.7) for the cross-section of expected returns are VW equations (22) and (38), shown in Corollary 9. Equation (3.8) for the predictability of returns is VW equation (40), shown in Corollary 11.

Additional equations from VW that we use in subsequent proofs are those describing the properties of the flow portfolio (stated between VW equations (A28) and (A29))

\[ \eta \Sigma p_f' = 0, \]
\[ \theta \Sigma p_f' = p_f \Sigma p_f' = \frac{\Delta}{\eta \Sigma \eta'}, \]

the investor's stock holdings (stated just before VW equation (A63))

\[ x_t \eta + y_t z_t = y_t p_f + \frac{\tilde{\alpha}}{\alpha + \tilde{\alpha}} \frac{\eta \Sigma \theta'}{\eta \Sigma \eta'} \eta, \] (A.2)

stock returns (VW equation (B7))

\[ dR_t = \left[ \frac{r \alpha \tilde{\alpha} f}{\alpha + \tilde{\alpha}} \frac{\eta \Sigma \theta'}{\eta \Sigma \eta'} \Sigma \eta' + \left( r \gamma_0 + \gamma_1 R \bar{C}_t + \gamma_2 R C_t + \gamma_3 R y_t - \kappa(\gamma_1 + \gamma_2)\bar{C} - b_0 \gamma_3 \right) \Sigma p_f' \right] dt \]
\[ + \left( \sigma + \beta_1 \gamma_1 \Sigma p_f \rho_f \sigma \right) dB_t^D + \frac{\phi}{r + \kappa} \left( \sigma + \beta_2 \gamma_1 \Sigma p_f \rho_f \sigma \right) dB_t^F - s\gamma_2 \left( 1 + \frac{\beta_2 \gamma_1 \Delta}{\eta \Sigma \eta'} \right) \Sigma p_f dB_t^C, \tag{A.3} \]

where
\[ \gamma_1^R \equiv (r + \kappa + \rho) \gamma_1 + b_1 \gamma_3, \tag{A.4} \]
\[ \gamma_2^R \equiv (r + \kappa) \gamma_2 - \rho \gamma_1, \tag{A.5} \]
\[ \gamma_3^R \equiv (r + b_2) \gamma_3, \tag{A.6} \]
\[ \rho \equiv \beta_1 \left( 1 - \frac{(r + \kappa) \gamma_2 \Delta}{\eta \Sigma \eta'} \right), \tag{A.7} \]

and the dynamics of \( \hat{C}_t \) (VW equation (B6) combined with (B8))
\[ d\hat{C}_t = \kappa (\widehat{C} - \hat{C}_t) dt + \rho (C_t - \hat{C}_t) dt - \beta_1 p_f \sigma dB_t^D - \beta_2 \left( \frac{\phi p_f \sigma dB_t^F}{r + \kappa} - \frac{s\gamma_2 \Delta dB_t^C}{\eta \Sigma \eta'} \right), \tag{A.8} \]

where
\[ \beta_1 \equiv T \left[ 1 - (r + k) \frac{\gamma_2 \Delta}{\eta \Sigma \eta'} \right] \frac{\eta \Sigma \eta'}{\Delta}, \tag{A.9} \]
\[ \beta_2 \equiv \frac{s^2 \gamma_2}{(r + \kappa)^2} - \frac{s^2 \gamma_2 \Delta}{\eta \Sigma \eta'} \tag{A.10} \]

and \( T \), the investor’s conditional variance of \( C_t \), is the positive solution to
\[ T^2 \left( 1 - (r + \kappa) \frac{\gamma_2 \Delta}{\eta \Sigma \eta'} \right)^2 \frac{\eta \Sigma \eta'}{\Delta} + 2 \kappa T - \frac{s^2 \phi^2}{(r + \kappa)^2} + \frac{k^2 \Delta}{\eta \Sigma \eta'} = 0. \tag{A.11} \]

We focus on the steady state reached when \( t \) goes to infinity, where the coefficients \( (\beta_1, \beta_2, T) \) are time-independent, and thus so are all other coefficients describing the equilibrium.

Substituting \( \gamma_0 \) from (A.1) and using (3.7), we can write (A.3) as
\[ dR_t = \left[ \frac{r \alpha \bar{\alpha} f}{\alpha + \bar{\alpha}} \eta \Sigma \theta' \Sigma \eta' + \left( f + \frac{k \Delta}{\eta \Sigma \eta'} \right) \Lambda t \Sigma p_f' \right] dt \]
\[ + (\sigma + \beta_1 \gamma_1 \Sigma p_f' p_f \sigma) \, dB_t^D + \frac{\phi}{r + \kappa} \left( \sigma + \beta_2 \gamma_1 \Sigma p_f' p_f \sigma \right) \, dB_t^F - s \gamma_2 \left( 1 + \frac{\beta_2 \gamma_1 \Delta}{\eta \Sigma \eta'} \right) \Sigma p_f' dB_t^C. \]

(A.12)

The term in square brackets in (A.12) is $E_t(dR_t)$ and maps to the two-factor model (3.6).

**B Additional Background Notation and Results**

Lemma B.1 determines the state variables $(F_t, \hat{C}_t, C_t, y_t)$ in steady state as function of all past Brownian shocks.

**Lemma B.1.** The values of $(F_t, \hat{C}_t, C_t, y_t)$ in the steady state reached when $t \to \infty$ are

\[ F_t = \bar{F} + \int_{-\infty}^{t} e^{-\kappa(t-u)} \phi \sigma dB_u^F, \]  \hspace{1cm} (B.1)

\[ \hat{C}_t = \bar{C} + \int_{-\infty}^{t} e^{-\kappa(t-u)} dB_u^C - \int_{-\infty}^{t} e^{-(\kappa+\rho)(t-u)} \left[ \beta_1 p_f \sigma dB_u^D + \frac{\phi \beta_2 p_f \sigma dB_u^F}{r + \kappa} + s \left( 1 - \frac{\beta_2 \gamma_2 \Delta}{\eta \Sigma \eta'} \right) dB_u^C \right], \]  \hspace{1cm} (B.2)

\[ C_t = \bar{C} + \int_{-\infty}^{t} e^{-\kappa(t-u)} dB_u^C, \]  \hspace{1cm} (B.3)

\[ y_t = \bar{y} + \int_{-\infty}^{t} \left( \frac{b_1}{\kappa - b_2} \right) \left[ e^{-\kappa(t-u)} - e^{-b_2(t-u)} \right] dB_u^C \]

\[ - \int_{-\infty}^{t} \left( \frac{b_1}{\kappa + \rho - b_2} \right) \left[ e^{-(\kappa+\rho)(t-u)} - e^{-b_2(t-u)} \right] \left[ \beta_1 p_f \sigma dB_u^D + \frac{\phi \beta_2 p_f \sigma dB_u^F}{r + \kappa} + s \left( 1 - \frac{\beta_2 \gamma_2 \Delta}{\eta \Sigma \eta'} \right) dB_u^C \right]. \]  \hspace{1cm} (B.4)

**Proof:** The dynamics of $F_t$ are given by the stochastic differential equation (2.6). Integrating that equation with initial condition $F_0$, and letting $t \to \infty$, we find (B.1). The dynamics of $(\hat{C}_t, C_t, y_t)$ are given by the system of stochastic differential equations (A.8) and (2.3), and ordinary differential equation (3.3). Integrating that system with initial conditions $(\hat{C}_0, C_0, y_0)$, and letting $t \to \infty$, we find (B.2)-(B.4).

We next introduce some notation, which we use together with Lemma B.1 to compute autocovariances of $(F_t, \hat{C}_t, C_t, y_t, dR_t)$ in Lemma B.3. For scalars $(\psi_1, \psi_2, \psi_3, \hat{\psi}_1, \hat{\psi}_2, \hat{\psi}_3)$ and a function $\nu(\omega, T)$, we define the function $G(\psi_1, \psi_2, \psi_3, T, \nu)$ by

\[ G(\psi_1, \psi_2, \psi_3, T, \nu) \equiv \]
the function $H(\psi_1, \psi_2, \psi_3, \hat{\psi}_1, \hat{\psi}_2, \hat{\psi}_3, T, \nu)$ by

$$H(\psi_1, \psi_2, \psi_3, \hat{\psi}_1, \hat{\psi}_2, \hat{\psi}_3, T, \nu) \equiv$$

$$- \left[ \psi_1 \nu(\kappa + \rho, T) + \frac{\psi_3 b_1}{\kappa + \rho - b_2} (\nu(\kappa + \rho, T) - \nu(b_2, T)) \right] \beta_1 \left( 1 + \frac{\beta_1 \gamma_1 \Delta}{\eta \Sigma \eta'} \right)$$

$$- \left[ (\psi_1 + \psi_2) \nu(\kappa, T) + \frac{\psi_3 b_1}{\kappa - b_2} (\nu(\kappa, T) - \nu(b_2, T)) \right] s^2 \gamma_2 \left( 1 + \frac{\beta_2 \gamma_1 \Delta}{\eta \Sigma \eta'} \right),$$

and the functions $K_1(\psi_1, \psi_3, T, \nu)$ and $K_2(\psi_1, \psi_3, T, \nu)$ by

$$K_1(\psi_1, \psi_3, T, \nu) \equiv - \frac{1}{2\kappa + \rho} \left( \psi_1 - \frac{\psi_3 b_1}{\kappa + b_2} \right) \nu(\kappa, T) \frac{\phi^2 \beta_2}{r + \kappa},$$

$$K_2(\psi_1, \psi_3, T, \nu) \equiv - \left[ \frac{1}{2\kappa + \rho} \left( \psi_1 + \frac{\psi_3 b_1}{\kappa + \rho - b_2} \right) \nu(\kappa + \rho, T) - \frac{\psi_3 b_1}{(\kappa + b_2)(\kappa + \rho - b_2)} \nu(b_2, T) \right] \frac{\phi^2 \beta_2}{r + \kappa}.$$
We define the functions \( \nu_0(t) \) and \( \{ \nu_i(\omega, T) \}_{i=1, \ldots, 4} \) for \( T = (t, \Delta t) \) and \( \Delta t > 0 \) by
\[
\nu_0(\omega, t) \equiv e^{-\omega t},
\]
\[
\nu_1(\omega, T) \equiv \int_{t-\Delta t}^{t} \nu_0(\omega, |u|)1_{\{u \geq 0\}}du,
\]
\[
\nu_2(\omega, T) \equiv \int_{t-\Delta t}^{t} \nu_0(\omega, |u|)1_{\{u \leq 0\}}du,
\]
\[
\nu_3(\omega, T) \equiv \int_{w'=-\Delta t}^{w'=-\Delta t} \nu_0(\omega, |u' - u|)1_{\{u \leq w'\}}dudu',
\]
\[
\nu_4(\omega, T) \equiv \int_{w'=-\Delta t}^{w'=-\Delta t} \nu_0(\omega, |u' - u|)1_{\{u \geq w'\}}dudu'.
\]

We define the scalars \((L_1, L_2)\) by
\[
L_1 \equiv \frac{r \alpha \bar{\alpha} f \eta \Sigma \theta'}{\alpha + \alpha \eta \Sigma \eta'}, \tag{B.5}
\]
\[
L_2 \equiv r \bar{\alpha} \left( f + \frac{k \Delta}{\eta \Sigma \eta'} \right) + (\gamma_1 R + \gamma_2 R) \bar{C} + \gamma_3 R \bar{y} - k_1 \bar{q}_1 - k_2 \bar{q}_2 = \left( f + \frac{k \Delta}{\eta \Sigma \eta'} \right) \mathcal{E}(\Lambda t), \tag{B.6}
\]
and the scalars \((\Delta_1, \Delta_2, \Delta_3, \Delta_4)\) by
\[
\Delta_1 \equiv f \left( \eta \Sigma^3 \eta' - \frac{(\eta \Sigma^2 \eta')^2}{\eta \Sigma \eta'} \right) + k \left( \eta \Sigma^2 \eta' f \right)^2,
\]
\[
\Delta_2 \equiv f \left( \eta \Sigma^3 p_f' - \frac{\eta \Sigma^2 \eta' \eta \Sigma^2 p_f'}{\eta \Sigma \eta'} \right) + k \eta \Sigma^2 \eta' p_f \Sigma^2 p_f',
\]
\[
\Delta_3 \equiv f \left( p_f \Sigma^3 p_f' - \frac{(\eta \Sigma^2 \eta' p_f')^2}{\eta \Sigma \eta'} \right) + k \left( p_f \Sigma^2 p_f' \right)^2,
\]
\[
\Delta_4 \equiv f \left( \text{Tr}(\Sigma^2) - \frac{\eta \Sigma^3 \eta'}{\eta \Sigma \eta'} \right) + k p_f \Sigma^3 p_f',
\]
where \(\text{Tr}(M)\) denotes the trace of the matrix \(M\).

Lemma B.2 derives closed-form solutions for the functions \( \{ \nu_i(\omega, T) \}_{i=1,2,3,4} \).

**Lemma B.2.** The functions \( \{ \nu_i(\omega, T) \}_{i=1,2,3,4} \) are equal to
\[
\nu_1(\omega, T) = \frac{e^{-\omega \max\{t-\Delta t, 0\}} - e^{-\omega \max\{t, 0\}}}{\omega},
\]
\[ \nu_2(\omega, T) = \nu_1(\omega, (-t + \Delta t, \Delta t)) = \frac{e^{\omega \min\{t,0\}} - e^{\omega \min\{t-\Delta t,0\}}}{\omega}, \]

\[ \nu_3(\omega, T) = \frac{e^{-\omega \max\{t+\Delta t,0\}} + e^{-\omega \max\{t-\Delta t,0\}} - 2e^{-\omega \max\{t,0\}}}{\omega^2} \]
\[ + \frac{\min\{\max\{t, -\Delta t\},0\} - \min\{\max\{t - \Delta t, -\Delta t\},0\}}{\omega}, \]

\[ \nu_4(\omega, T) = \nu_3(\omega, (-\Delta t)) = \frac{e^{\omega \min\{t+\Delta t,0\}} + e^{\omega \min\{t-\Delta t,0\}} - 2e^{\omega \min\{t,0\}}}{\omega^2} \]
\[ + \frac{\max\{\min\{t, 0\}, -\Delta t\} - \max\{\min\{t - \Delta t, 0\}, -\Delta t\}}{\omega}. \]

**Proof:** We first compute \((\nu_1(\omega, T), \nu_2(\omega, T))\). Since the variable \(u\) in \(\nu_1(\omega, T)\) is non-negative, we can drop the indicator function \(1\{u \geq 0\}\) and change the integration bounds and the argument of \(\nu_0(\omega, t)\) as follows:

\[ \nu_1(\omega, T) = \int_{\max\{t-\Delta t,0\}}^{\max\{t,0\}} e^{-\omega u} du = \frac{e^{-\omega \max\{t-\Delta t,0\}} - e^{-\omega \max\{t,0\}}}{\omega}. \]

This is the expression in the lemma. To compute \(\nu_2(\omega, T)\), we make the change of variable from \(u\) to \(-u\):

\[ \nu_2(\omega, T) = \int_{-t}^{-t+\Delta t} \nu_0(\omega, |u|)1\{u \geq 0\} du \]
\[ = \int_{-(t-\Delta t)-\Delta t}^{-(t-\Delta t)} \nu_0(\omega, |u|)1\{u \geq 0\} du \]
\[ = \nu_1(\omega, (-t + \Delta t, \Delta t)) \]
\[ = \frac{e^{-\omega \max\{-t+\Delta t, -\Delta t\}} - e^{-\omega \max\{-t+\Delta t, 0\}}}{\omega} \]
\[ = \frac{e^{-\omega \max\{-t,0\}} - e^{-\omega \max\{-(t-\Delta t),0\}}}{\omega} \]
\[ = \frac{e^{\omega \min\{t,0\}} - e^{\omega \min\{t-\Delta t,0\}}}{\omega}, \]

which is the expression in the lemma.

We next compute \((\nu_3(\omega, T), \nu_4(\omega, T))\). Since the difference \(u' - u\) in \(\nu_3(\omega, T)\) is non-negative and \(u \geq -\Delta t\), \(u'\) must also exceed \(-\Delta t\). Proceeding as for \((\nu_1(\omega, T), \nu_2(\omega, T))\), we drop the indicator
function $1_{\{u \leq u'\}}$ and change the integration bounds and the argument of $\nu_0(\omega, t)$ as follows:

\[
\nu_3(\omega, T) = \int_{u' = \max\{t - \Delta t, -\Delta t\}}^{\max\{t, -\Delta t\}} \int_{u = -\Delta t}^{\min\{u', 0\}} e^{\omega(u - u')} du \, du' = \int_{\max\{t - \Delta t, -\Delta t\}}^{\max\{t, -\Delta t\}} \frac{e^{\omega(u - u')} - e^{\omega(-\Delta t - u')}}{\omega} du'. \tag{B.7}
\]

Integrating the second term inside the integral yields

\[
\int_{\max\{t - \Delta t, -\Delta t\}}^{\max\{t, -\Delta t\}} \frac{e^{\omega(-\Delta t - u')}}{\omega} du' = \frac{e^{-\omega \max\{t, 0\}} - e^{-\omega \max\{t + \Delta t, 0\}}}{\omega^2} \tag{B.8}
\]

To integrate the first term inside the integral, we separate it into two using indicator functions, and then change the integration bounds:

\[
\int_{\max\{t - \Delta t, -\Delta t\}}^{\max\{t, -\Delta t\}} \frac{e^{\omega(u' \max\{u', 0\} - u')}}{\omega} du' = \int_{\max\{t - \Delta t, -\Delta t\}}^{\max\{t, -\Delta t\}} \frac{e^{\omega(u' \max\{u', 0\} - u')} 1_{\{u' \geq 0\}}}{\omega} du' + \int_{\max\{t - \Delta t, -\Delta t\}}^{\max\{t, -\Delta t\}} \frac{e^{\omega(u' \max\{u', 0\} - u')} 1_{\{u' \leq 0\}}}{\omega} du'
\]

\begin{align*}
&= \int_{\max\{t, -\Delta t\}}^{\max\{t, -\Delta t\}} \frac{e^{-\omega u'}}{\omega} du' + \int_{\max\{t - \Delta t, -\Delta t\}, 0}^{\max\{t, -\Delta t\}, 0} \frac{1}{\omega} du' \\
&= \int_{\max\{t, -\Delta t\}, 0}^{\max\{t, -\Delta t\}, 0} \frac{e^{-\omega u'}}{\omega} du' + \int_{\min\{\max\{t, -\Delta t\}, -\Delta t\}, 0}^{\min\{\max\{t, -\Delta t\}, -\Delta t\}, 0} \frac{1}{\omega} du' \\
&= \frac{e^{-\omega \max\{t - \Delta t, 0\}} - e^{-\omega \max\{t, 0\}}}{\omega^2} + \min\{\max\{t, -\Delta t\}, 0\} - \min\{\max\{t - \Delta t, -\Delta t\}, 0\} \tag{B.9}
\end{align*}

Substituting (B.8) and (B.9) into (B.7) yields

\[
\nu_3(\omega, T) = \frac{e^{-\omega \max\{t - \Delta t, 0\}} - e^{-\omega \max\{t, 0\}}}{\omega^2} - \frac{e^{-\omega \max\{t, 0\}} - e^{-\omega \max\{t + \Delta t, 0\}}}{\omega^2} + \frac{\min\{\max\{t, -\Delta t\}, 0\} - \min\{\max\{t - \Delta t, -\Delta t\}, 0\}}{\omega},
\]

which is the expression in the lemma. To compute $\nu_4(\omega, T)$, we revert the order of integration, and
add \(-t\) to both integrands:

\[
\nu_4(\omega, T) = \int_{u=-\Delta t}^{0} \int_{u'=t-\Delta t}^{t} \nu_0(\omega, |u' - u|)1_{\{u' \geq u\}} du' du
\]

\[
= \int_{u=-\Delta t}^{0} \int_{u=t-\Delta t}^{t} \nu_0(\omega, |u' - u|)1_{\{u' \leq u\}} du du'
\]

\[
= \int_{u=-\Delta t}^{-t} \int_{u'=-\Delta t}^{0} \nu_0(\omega, |u' - u|)1_{\{u' \leq u\}} du du'
\]

\[
= \nu_3(\omega, (-t, Dt))
\]

\[
= \frac{e^{-\omega \max\{t+\Delta t, 0\}} + e^{-\omega \max\{-t-\Delta t, 0\}} - 2e^{-\omega \max\{-t, 0\}}}{\omega^2}
\]

\[
+ \frac{\min\{\max\{-t, -\Delta t\}, 0\} - \min\{\max\{-t - \Delta t, -\Delta t\}, 0\}}{\omega}
\]

\[
= \frac{e^{-\omega \max\{-(t-\Delta t), 0\}} + e^{-\omega \max\{-t+\Delta t, 0\}} - 2e^{-\omega \max\{-t, 0\}}}{\omega^2}
\]

\[
+ \frac{\min\{- \min\{t, \Delta t\}, 0\} - \min\{- \min\{t + \Delta t, \Delta t\}, 0\}}{\omega}
\]

\[
= \frac{e^{\omega \min\{t-\Delta t, 0\}} + e^{\omega \min\{t+\Delta t, 0\}} - 2e^{\omega \min\{t, 0\}}}{\omega^2}
\]

\[
+ \frac{- \max\{\min\{t, \Delta t\}, 0\} + \max\{\min\{t + \Delta t, \Delta t\}, 0\}}{\omega}
\]

Adding \(-\Delta t\) to both terms inside each maximum in the last line, we can write it as

\[
- \max\{\min\{t - \Delta t, \Delta t - \Delta t\}, -\Delta t\} + \max\{\min\{t + \Delta t - \Delta t, \Delta t - \Delta t\}, -\Delta t\}
\]

\[
= - \max\{\min\{t - \Delta t, 0\}, -\Delta t\} + \max\{\min\{t, 0\}, -\Delta t\}
\]

and thus obtain the expression in the lemma.

Lemma B.3. For \(t' > t\),

\[
\text{Cov}_t(dR_t, \psi_1 \hat{C}_{t'} + \psi_2 C_{t'} + \psi_3 y_{t'}) = G(\psi_1, \psi_2, \psi_3, t' - t, \nu_0)\Sigma p_{t'} dt,
\]

(B.10)

\[
\text{Cov}_t(dR_t, F_{t'}) = \frac{\phi^2}{r + \kappa} (\Sigma + \beta_2 \gamma_1 \Sigma p_{t'} p_{t'} \Sigma) \nu_0(\kappa, t' - t) dt,
\]

(B.11)
\[ \text{Cov}(dR_t, dR_t') = G(\gamma_1^R, \gamma_2^R, \gamma_3^R, t' - t, \nu_0) \Sigma_{pf}^t \sigma \Sigma dt dt', \]  

(B.12)

and for \( t' \geq t, \)

\[ \text{Cov} \left( \psi_1 \dot{C}_t + \psi_2 C_t + \psi_3 y_t, \dot{y}_t \right) = H(\psi_1, \psi_2, \psi_3, \dot{y}_t, t' - t, \nu_0), \]  

(B.13)

\[ \text{Cov} \left( \psi_1 \dot{C}_t + \psi_2 C_t + \psi_3 y_t, F_{t'} \right) = K_1(\psi_1, \psi_2, \psi_3, \dot{y}_t, t' - t, \nu_0) \Sigma_{pf}^t, \]  

(B.14)

\[ \text{Cov} \left( F_t, \psi_1 \dot{C}_t + \psi_2 C_t + \psi_3 y_t \right) = K_2(\psi_1, \psi_2, \psi_3, \dot{y}_t, t' - t, \nu_0) \Sigma_{pf}^t, \]  

(B.15)

\[ \text{Cov}(F_t, F_{t'}) = \frac{\phi^2 \Sigma}{2\kappa} \nu(\kappa, t' - t). \]  

(B.16)

**Proof:** We first show (B.10). Since the covariance is conditional as of time \( t, \) it involves only the Brownian terms in \( dR_t \) and not the drift terms. Using (A.3) and (B.2)-(B.4), and noting that the only non-zero covariances are between Brownian increments of the same process as of time \( t, \) we find

\[ \text{Cov}(dR_t, \psi_1 \dot{C}_t + \psi_2 C_t + \psi_3 y_t) \]

\[ = - \left( \sigma + \beta_1 \gamma_1 \Sigma_{pf}^t \sigma \right) \left[ \psi_1 e^{- (\kappa + \rho)(t' - t)} + \frac{\psi_3 b_1}{\kappa + \rho - b_2} (e^{- (\kappa + \rho)(t' - t)} - e^{- b_2(t' - t)}) \right] \beta_1 \sigma \Sigma_{pf}^t dt \]

\[ - \frac{\phi}{r + \kappa} \left[ \sigma + \beta_2 \gamma_1 \Sigma_{pf}^t \sigma \right] \left[ \psi_1 e^{- (\kappa + \rho)(t' - t)} + \frac{\psi_3 b_1}{\kappa + \rho - b_2} (e^{- (\kappa + \rho)(t' - t)} - e^{- b_2(t' - t)}) \right] \frac{\phi \beta_2}{r + \kappa} \sigma \Sigma_{pf}^t dt \]

\[ - s \gamma_2 \left[ 1 + \frac{\beta_2 \gamma_1 \Delta}{\eta \Sigma_{n'}} \right] \Sigma_{pf}^t \left[ (\psi_1 + \psi_2) e^{- \kappa(t' - t)} + \frac{\psi_3 b_1}{\kappa - b_2} (e^{- \kappa(t' - t)} - e^{- b_2(t' - t)}) \right] \left( 1 - \frac{\beta_2 \gamma_2 \Delta}{\eta \Sigma_{n'}} \right) s dt \]

\[ = \left\{ - \left[ \psi_1 e^{- (\kappa + \rho)(t' - t)} + \frac{\psi_3 b_1}{\kappa + \rho - b_2} (e^{- (\kappa + \rho)(t' - t)} - e^{- b_2(t' - t)}) \right] \right\} \right. \]

\[ \times \left[ \beta_1 \left( 1 + \frac{\beta_1 \gamma_1 \Delta}{\eta \Sigma_{n'}} \right) + \left( \frac{\phi \beta_2}{(r + \kappa)^2} - s \gamma_2 \left( 1 - \frac{\beta_2 \gamma_2 \Delta}{\eta \Sigma_{n'}} \right) \right) \left( 1 + \frac{\beta_2 \gamma_1 \Delta}{\eta \Sigma_{n'}} \right) \right] \]

\[ - \left[ (\psi_1 + \psi_2) e^{- \kappa(t' - t)} + \frac{\psi_3 b_1}{\kappa - b_2} (e^{- \kappa(t' - t)} - e^{- b_2(t' - t)}) \right] s^2 \gamma_2 \left( 1 + \frac{\beta_2 \gamma_1 \Delta}{\eta \Sigma_{n'}} \right) \left\{ \Sigma_{pf}^t dt \right\}. \]
\[- \left[ (\psi_1 + \psi_2)e^{-\kappa (t'-t)} + \frac{\psi_3 b_1}{\kappa - b_2} \left( e^{-\kappa (t'-t)} - e^{-b_2(t'-t)} \right) \right] s^2 \gamma_2 \left( 1 + \frac{\beta_2 \gamma_1 \Delta}{\eta \Sigma y'} \right) \right\} \Sigma p'_\beta dt, \quad (B.17)\]

where the third step follows from (A.10). Equation (B.17) yields (B.10).

We next show (B.11). Using (A.3) and (B.1), and noting that the conditional covariance involves only the Brownian terms in $dR_t$, and that the only non-zero covariances are between the Brownian increments of the process $F_t$ as of time $t$, we find

\[
\text{Cov}_t(dR_t, F'_t) = \frac{\phi}{\tau + \kappa} \left( \sigma + \beta_2 \gamma_1 \Sigma p'_\beta \Sigma p' \right) \phi \sigma' e^{-\kappa (t'-t)} dt,
\]

which yields (B.11).

We next show (B.12). Using (A.3), and noting that the conditional covariance involves only the Brownian terms in $dR_t$ and only the drift terms in $dR'_t$, we find

\[
\text{Cov}_t(dR_t, dR'_t) = \text{Cov}_t(dR_t, E_t'(dR'_t)) \\
= \text{Cov}_t(dR_t, \gamma_1^R \phi' \psi + \gamma_2^R \phi' \psi + \gamma_3^R \gamma \phi' \psi) \Sigma dt', \quad (B.18)
\]

where the second step follows from (A.3). Combining (B.18) with (B.10) yields (B.12).

We next show (B.13). Using (B.2)-(B.4) and noting that the only non-zero covariances are between Brownian increments of the same process as of the same time $u \in (-\infty, t]$, we find

\[
\text{Cov} \left( \psi_1 \phi' \psi + \psi_2 \phi' \psi + \psi_3 \phi' \psi + \hat{\psi}_3 \phi' \psi \right) \quad (B.19)
\]

\[
= \int_{-\infty}^t \left[ \psi_1 e^{-(\kappa + \rho)(t-u)} + \frac{\psi_3 b_1}{\kappa + \rho - b_2} \left( e^{-(\kappa + \rho)(t-u)} - e^{-b_2(t-u)} \right) \right] \left[ \hat{\psi}_1 e^{-(\kappa + \rho)(t-u)} + \frac{\hat{\psi}_3 b_1}{\kappa + \rho - b_2} \left( e^{-(\kappa + \rho)(t-u)} - e^{-b_2(t-u)} \right) \right] \left( \beta_1^2 + \frac{\phi^2 \beta_2^2}{(r + \kappa)^2} \right) \frac{\Delta}{\eta \Sigma y'} du \\
+ \int_{-\infty}^t \left[ (\psi_1 + \psi_2) e^{-\kappa (t-u)} + \frac{\psi_3 b_1}{\kappa - b_2} \left( e^{-\kappa (t-u)} - e^{-b_2(t-u)} \right) \right] \left( \psi_1 e^{-(\kappa + \rho)(t-u)} + \frac{\psi_3 b_1}{\kappa + \rho - b_2} \left( e^{-(\kappa + \rho)(t-u)} - e^{-b_2(t-u)} \right) \right] \left( 1 - \frac{\beta_2 \gamma_2 \Delta}{\eta \Sigma y'} \right) \left( \beta_1^2 + \frac{\phi^2 \beta_2^2}{(r + \kappa)^2} \right) \frac{\Delta}{\eta \Sigma y'} du \\
- \left[ \psi_1 e^{-(\kappa + \rho)(t-u)} + \frac{\psi_3 b_1}{\kappa + \rho - b_2} \left( e^{-(\kappa + \rho)(t-u)} - e^{-b_2(t-u)} \right) \right] \left( \hat{\psi}_1 e^{-(\kappa + \rho)(t-u)} + \frac{\hat{\psi}_3 b_1}{\kappa + \rho - b_2} \left( e^{-(\kappa + \rho)(t-u)} - e^{-b_2(t-u)} \right) \right] \left( 1 - \frac{\beta_2 \gamma_2 \Delta}{\eta \Sigma y'} \right) \left( \beta_1^2 + \frac{\phi^2 \beta_2^2}{(r + \kappa)^2} \right) \frac{\Delta}{\eta \Sigma y'} du \\
\times \left[ \hat{\psi}_1 e^{-(\kappa + \rho)(t-u)} + \frac{\hat{\psi}_3 b_1}{\kappa + \rho - b_2} \left( e^{-(\kappa + \rho)(t-u)} - e^{-b_2(t-u)} \right) \right]
\]
\[-\left[ \psi_1 e^{-(\kappa+\rho)(t'-u)} + \frac{\psi_3 b_1}{\kappa + \rho - b_2} \left( e^{-(\kappa+\rho)(t'-u)} - e^{-b_2(t'-u)} \right) \left( 1 - \frac{\beta_2 \gamma_2 \Delta}{\eta \Sigma \eta'} \right) \right] s^2 du. \quad (B.20)\]

Integrating all products of exponentials in (B.20) and summing, yields (B.13). To perform the algebra, we separate the terms in \( 1 - \frac{\beta_2 \gamma_2 \Delta}{\eta \Sigma \eta'} \) into quadratic terms in \( \frac{\beta_2 \gamma_2 \Delta}{\eta \Sigma \eta'} \), linear terms and constant terms.

We next show (B.14) and (B.15). Using (B.1)-(B.4) and noting that the only non-zero covariances are between Brownian increments of the same process as of the same time \( u \in (-\infty, t] \), we find

\[
\text{Cov}\left( \psi_1 \hat{C}_t + \psi_2 C_t + \psi_3 y_t, F_t' \right) = -\int_{-\infty}^{t} \left[ \psi_1 e^{-(\kappa+\rho)(t-u)} + \frac{\psi_3 b_1}{\kappa + \rho - b_2} \left( e^{-(\kappa+\rho)(t-u)} - e^{-b_2(t-u)} \right) \right] e^{-\kappa(t'-u)} \frac{\phi^2 \beta_2 \Sigma p'_f}{r + \kappa} du. \quad (B.21)
\]

and

\[
\text{Cov}\left( F_t, \psi_1 \hat{C}_t' + \psi_2 C_t' + \psi_3 y_t' \right) = -\int_{-\infty}^{t} e^{-\kappa(t-u)} \left[ \psi_1 e^{-(\kappa+\rho)(t'-u)} + \frac{\psi_3 b_1}{\kappa + \rho - b_2} \left( e^{-(\kappa+\rho)(t'-u)} - e^{-b_2(t'-u)} \right) \right] \frac{\phi^2 \beta_2 \Sigma p'_f}{r + \kappa} du. \quad (B.22)
\]

Integrating all products of exponentials in (B.21) and (B.22) and summing, yields (B.14) and (B.15), respectively.

We finally show (B.16). Using (B.1) and noting that the only non-zero covariances are between Brownian increments of the same process as of the same time \( u \in (-\infty, t] \), we find

\[
\text{Cov}(F_t, F_t') = \int_{-\infty}^{t} \phi^2 \Sigma e^{-\kappa(t-u)} e^{-\kappa(t'-u)} du. \quad (B.23)
\]

Integrating (B.23), we find (B.16).
C Proofs of Results in Section 4

The portfolio optimization problem corresponding to $SR_{w,t}$ is as follows. Consider an investor at time $t$ with infinitesimal horizon $dt$, who can invest in the riskless asset, the index $\eta$ and the strategy $w_t$. The investor has mean-variance preferences

$$
\mathbb{E}_{\mathcal{I}_t}(dW_t) - \frac{a}{2} \text{Var}_{\mathcal{I}_t}(dW_t).
$$

(C.1)

She chooses an overall exposure $\hat{x}_t$ to the index and a position $\hat{y}_t$ in the strategy. These positions can depend on information in $\mathcal{I}_t$. The investor’s overall exposure to the index at time $t$ is

$$
\hat{x}_t = \hat{x}_t + \hat{y}_t \frac{\text{Cov}_t(w_t dR_t, \eta dR_t)}{\text{Var}_t(\eta dR_t)},
$$

the sum of a position $\hat{x}_t$ in the index and an exposure resulting from the strategy. The investor’s budget constraint is

$$
dW_t = rW_t dt + \hat{x}_t \eta dR_t + \hat{y}_t w_t dR_t
$$

$$
= rW_t dt + \left( \hat{x}_t + \hat{y}_t \frac{\text{Cov}_t(w_t dR_t, \eta dR_t)}{\text{Var}_t(\eta dR_t)} \right) \eta dR_t + \hat{y}_t \hat{w}_t dR_t
$$

$$
= rW_t dt + \hat{x}_t \eta dR_t + \hat{y}_t \hat{w}_t dR_t.
$$

(C.2)

Lemma C.1. The investor’s maximum utility is

$$
\frac{1}{2a} \left( SR^2_{\eta} + SR^2_{w,t} \right) dt.
$$

(C.3)

Proof: Substituting $dW_t$ from (C.2), and noting that $(\eta dR_t, \hat{w}_t dR_t)$ are uncorrelated, we can write (C.1) as

$$
\hat{x}_t \mathbb{E}_{\mathcal{I}_t}(\eta dR_t) + \hat{y}_t \mathbb{E}_{\mathcal{I}_t}(\hat{w}_t dR_t) - \frac{a}{2} \left( \hat{x}_t^2 \text{Var}_{\mathcal{I}_t}(\eta dR_t) + \hat{y}_t^2 \text{Var}_{\mathcal{I}_t}(\hat{w}_t dR_t) \right).
$$

(C.4)

Maximizing (C.4) over $(\hat{x}_t, \hat{y}_t)$ yields the utility

$$
\frac{1}{2a} \left( SR^2_{\eta,t} + SR^2_{w,t} \right) dt.
$$

(C.5)
Since $\eta \Sigma p'_f = 0$ and (A.3) imply

$$\eta dR_t = \frac{r\alpha f}{\alpha + \alpha} \eta \Sigma \theta' dt + \eta \sigma \left( dB_t^D + \frac{\phi dB_t^F}{r + \kappa} \right), \quad (C.6)$$

$\mathbb{E}_t(\eta dR_t)$ and $\text{Var}_t(\eta dR_t)$ coincide with their unconditional values. Therefore, $SR_{\eta,t} = SR_{\eta}$, and (C.5) coincides with (C.3).

The portfolio optimization problem corresponding to $SR_{w,t,T}$ is as follows. Consider an investor at time $t$ with horizon $T$, who can invest in the riskless asset, the index $\eta$ and the strategy $w_t$. The investor has mean-variance preferences

$$\mathbb{E}_t(\Delta W_{t+T}) - \frac{a}{2} \text{Var}_t(\Delta W_{t+T}) \quad (C.7)$$

over the increment $\Delta W_{t+T} \equiv W_{t+T} e^{-rT} - W_t$ in discounted wealth at the riskless rate $r$. She chooses an overall exposure $\hat{x}_t$ to the index and a position $\hat{y}_t$ in the strategy at time $t$. These positions can depend on information in $\mathcal{I}_t$. The investor is assumed to scale up these positions over time at the riskless rate $r$, to $\hat{x}_u = \hat{x}_t e^{(u-t)}$ and $\hat{y}_u = \hat{y}_t e^{(u-t)}$, respectively, at time $u$. The investor’s overall exposure to the index at time $u$ is

$$\hat{x}_u = \hat{x}_u + \hat{y}_u \frac{\text{Cov}_u(w_u dR_u, \eta dR_u)}{\text{Var}_u(\eta dR_u)};$$

the sum of a position $\hat{x}_u$ in the index and an exposure resulting from the strategy. The investor’s budget constraint is

$$dW_u = r W_u dt + \hat{x}_u \eta dR_u + \hat{y}_u w_u dR_u$$

$$= r W_u dt + \left( \hat{x}_u + \hat{y}_t \frac{\text{Cov}_u(w_u dR_u, \eta dR_u)}{\text{Var}_u(\eta dR_u)} \right) \eta dR_u + \hat{y}_t \hat{w}_u dR_u \quad (C.8)$$

$$= r W_u dt + \hat{x}_t e^{(u-t)} \eta dR_u + \hat{y}_t e^{(u-t)} \hat{w}_u dR_u, \quad (C.9)$$

and integrates to

$$\Delta W_{t+T} = \hat{x}_t \int_t^{t+T} \eta dR_u + \hat{y}_t \int_t^{t+T} \hat{w}_u dR_u \quad (C.10)$$

from time $t$ to $t + T$. 

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Lemma C.2. Suppose $\text{Cov}_{I_t}(\eta dR_u, \hat{w}_u' dR_{u'}) = 0$ for $t < u < u'$. The investor’s maximum utility is
\[
\frac{1}{2a} \left( SR^2_{\eta} + SR^2_{w,t,T} \right) T. \tag{C.11}
\]

Proof: Substituting $\Delta W_{t+T}$ from (C.10), we can write (C.7) as
\[
\hat{x}_t E_{I_t} \left( \int_t^{t+T} \eta dR_u \right) + \hat{y}_t E_{I_t} \left( \int_t^{t+T} \hat{w}_u dR_u \right) - \frac{a}{2} \left[ \hat{x}_t^2 \text{Var}_{I_t} \left( \int_t^{t+T} \eta dR_u \right) + \hat{y}_t^2 \text{Var}_{I_t} \left( \int_t^{t+T} \hat{w}_u dR_u \right) + 2 \hat{x}_t \hat{y}_t \text{Cov}_{I_t} \left( \int_t^{t+T} \eta dR_u, \int_t^{t+T} \hat{w}_u dR_u \right) \right]. \tag{C.12}
\]

To compute the covariance in (C.12), we write it as
\[
\int_t^{t+T} \text{Cov}_{I_t} (\eta dR_u, \hat{w}_u' dR_{u'}) + \int_{u=t}^{t+T} \int_{u'=t}^{t+T} \text{Cov}_{I_t} (\eta dR_u, \hat{w}_{u'}' dR_{u'}) , \tag{C.13}
\]
where the first term in (C.13) is the covariance between contemporaneous returns and the second term is the covariance between lagged returns. The first term is zero because of the definition (4.3) of $\hat{w}_t$. Since (C.6) implies that the covariance $\text{Cov}_{I_t} (\eta dR_u, \hat{w}_{u'}' dR_{u'})$ for $u > u'$ is zero, the second term is
\[
\int_{u=t}^{t+T} \int_{u'=t}^{t+T} \text{Cov}_{I_t} (\eta dR_u, \hat{w}_{u'}' dR_{u'}) ,
\]
and is zero because of the assumption $\text{Cov}_{I_t}(\eta dR_u, \hat{w}_{u'}' dR_{u'}) = 0$ for $t < u < u'$. In the proof of Proposition E.10 we show that this assumption is satisfied for the strategies that we examine in this paper. With a zero covariance in (C.12), maximization over $(\hat{x}_t, \hat{y}_t)$ yields the maximum utility
\[
\frac{1}{2a} \left( SR^2_{\eta,t,T} + SR^2_{w,t,T} \right) T. \tag{C.14}
\]
Since (C.6) implies that the conditional moments $E_{I_t} \left( \int_t^{t+T} \eta dR_u \right)$ and $\text{Var}_{I_t} \left( \int_t^{t+T} \eta dR_u \right)$ coincide with their unconditional values, $SR_{\eta,t,T} = SR_{\eta,T}$. Since, in addition, $\text{Cov}_{I_t} (\eta dR_u, \eta dR_{u'}) = \text{Cov} (\eta dR_u, \eta dR_{u'}) = 0$ for $u \neq u'$, Lemma E.2 implies $SR_{\eta,T} = SR_{\eta}$. Therefore, (C.14) coincides
with (C.11).

We next move to the calibration. We compute model-implied moments for general asset payoffs, and specialize them to symmetric assets, with \( \eta = 1' \), \( \bar{F} = F1 \) and \( \Sigma = \sigma^2(I + \omega 11') \), in Lemma C.6.

The calculations of Sharpe ratios and correlations in Appendices D and E also concern general asset payoffs, except when symmetry is explicitly mentioned. Lemma C.3 computes the Sharpe ratio of the index, the correlation between an asset and the index, and the fraction of an asset’s variance that is generated by fund flows.

**Lemma C.3.** The Sharpe ratio of the index \( \eta \) is

\[
SR_{\eta} = SR_{\eta,T} = \frac{r\alpha\sqrt{T}}{\eta \Sigma \theta'} \frac{\eta \Sigma \theta'}{\sqrt{\eta \Sigma \theta'}}
\]  
(C.15)

The correlation between asset \( n \) and the index is

\[
\text{Corr}(dR_{nt}, \eta dR_t) = \frac{\sqrt{T}(\eta \Sigma)_n}{\sqrt{\eta \Sigma \theta'}[f \Sigma_{nn} + k[(p_f \Sigma)_n]^2]}
\]  
(C.16)

The fraction of asset \( n \)'s variance that is generated by fund flows is

\[
\frac{k[(p_f \Sigma)_n]^2}{f \Sigma_{nn} + k[(p_f \Sigma)_n]^2}
\]  
(C.17)

**Proof:** Equation (C.6) implies

\[
E(\eta dR_t) = \frac{r\alpha\bar{\alpha}}{\alpha + \bar{\alpha}} \eta \Sigma \theta' dt,
\]  
(C.18)

\[
\text{Var}(\eta dR_t) = f \Sigma \eta' dt.
\]  
(C.19)

Substituting (C.18) and (C.19) into (4.4), and noting that \( SR_{\eta,T} = SR_{\eta} \), we find (C.15). The correlation between asset \( n \) and the index is

\[
\text{Corr}(dR_{nt}, \eta dR_t) = \frac{\text{Cov}(dR_{nt}, \eta dR_t)}{\sqrt{\text{Var}(dR_{nt}) \text{Var}(\eta dR_t)}} = \frac{(f \Sigma \eta)_n}{\sqrt{f \eta \Sigma \eta' (f \Sigma + k \Sigma p_f^1 p_f \Sigma)_{nn}}},
\]  
(C.20)

where the second step follows from (3.5), (C.19) and \( \eta \Sigma p_f' = 0 \). Equation (C.20) implies (C.16). Equation (C.17) follows from (3.5).
We approximate the calculation of the index’s expected return per dollar by dividing the index’s expected return per share by the index’s expected price

\[
\frac{\mathbb{E}(\eta dR_t)}{\mathbb{E}(\eta S_t)} = \frac{\frac{r \alpha f}{\alpha + \tilde{\alpha}} \eta \Sigma \theta' dt}{\eta \tilde{F} - \frac{r \alpha f}{\alpha + \tilde{\alpha}} \eta \Sigma \eta'}.
\] (C.21)

The active-share calculation is as follows. The active share of the residual supply portfolio is

\[
AS_\theta = \frac{1}{2} \sum_{n=1}^{N} \left| \frac{\theta_n S_n}{\sum_{m=1}^{N} \theta_m S_m} - \frac{\eta_n S_n}{\sum_{m=1}^{N} \eta_m S_m} \right|.
\] (C.22)

Since asset prices vary over time, active share does too. We use expected active share, and approximate its calculation by replacing prices \(S_n\) in the numerator and denominator of (C.22) by their expectations. Our approximation for expected active share thus is

\[
\overline{AS}_\theta = \frac{1}{2} \sum_{n=1}^{N} \left| \frac{\theta_n \mathbb{E}(S_n)}{\sum_{m=1}^{N} \theta_m \mathbb{E}(S_m)} - \frac{\eta_n \mathbb{E}(S_n)}{\sum_{m=1}^{N} \eta_m \mathbb{E}(S_m)} \right|.
\] (C.23)

Using (3.1) and Lemma B.1, we find that expected prices are

\[
\mathbb{E}(S_t) = \tilde{F} - \frac{\alpha \tilde{f}}{\alpha + \tilde{\alpha}} \eta \Sigma \theta' \Sigma \eta' - (\gamma_0 + (\gamma_1 + \gamma_2) \tilde{C} + g \tilde{y}) \Sigma \eta' .
\]

Lemma C.4 replicates within the context of our model KSZ’s calculation of the spread in monthly CAPM alpha between top and bottom fund deciles sorted based on lagged one-year return gap. The lemma computes the difference in the active fund’s expected index-adjusted return over a time interval \([t + \Delta t_1, t + \Delta t_2]\) between top and bottom deciles sorted based on the cost \(C_t\) at time \(t\).

**Lemma C.4.** The difference in the active fund’s expected index-adjusted return over the time interval \([t + \Delta t_1, t + \Delta t_2]\) between top and bottom deciles sorted based on \(C_t\) at time \(t\) is

\[
H \left( 0, 1, 0, \gamma_1 \frac{\Delta}{\eta \Sigma \eta'}, \gamma_2 \frac{\Delta}{\eta \Sigma \eta'} - 1, \gamma_3 \frac{\Delta}{\eta \Sigma \eta'}, \mathcal{T}, \nu_1 \right) \frac{\sqrt{2 \kappa}}{s} \left[ \mathbb{E}(z | z > z_0) - \mathbb{E}(z | z < z_1) \right],
\] (C.24)

where \(\mathcal{T} \equiv (\Delta t_2, \Delta t_2 - \Delta t_1)\), \(z\) is a standardized normal variable, and \(z_i, i = 1, .., 9\), is the boundary between deciles \(i\) and \(i + 1\) of \(z\).
Proof: Since the active fund’s return in equilibrium is \((\theta - x_t\eta) dR_t - C_t dt\), the fund’s index-adjusted return is

\[
(\theta - x_t\eta) dR_t - C_t dt - \frac{\text{Cov}_t[(\theta - x_t\eta) dR_t - C_t dt, \eta dR_t]}{\text{Var}(\eta dR_t)} \eta dR_t
\]

\[
= (\theta - x_t\eta) dR_t - C_t dt - \frac{\text{Cov}_t[(\theta - x_t\eta) dR_t, \eta dR_t]}{\text{Var}(\eta dR_t)} \eta dR_t
\]

\[
= (\theta - x_t\eta) dR_t - C_t dt - \left( \frac{\theta \Sigma \eta'}{\eta \Sigma \eta'} - x_t \right) \eta dR_t
\]

\[
= p_f dR_t - C_t dt,
\]

where the second step follows from (3.5). The fund’s cumulative index-adjusted return over the time interval \([t + \Delta t_1, t + \Delta t_2]\) is

\[
\int_{t+\Delta t_1}^{t+\Delta t_2} (p_f dR_u - C_u du).
\]

Because of normality, the expectation of that return conditional on \(C_t\) is \(Z C_t\), where

\[
Z \equiv \frac{\text{Cov}[C_t, \int_{t+\Delta t_1}^{t+\Delta t_2} (p_f dR_u - C_u du)]}{\text{Var}(C_t)} = \frac{\int_{t+\Delta t_1}^{t+\Delta t_2} \text{Cov}[C_t, p_f dR_u - C_u du]}{\text{Var}(C_t)}.
\]  \hspace{1cm} \text{(C.25)}

The difference in expected index-adjusted return between top and bottom deciles of \(C_t\) is

\[
Z \sqrt{\text{Var}(C_t)} \left[ E(z > z_0) - E(z < z_1) \right].
\]  \hspace{1cm} \text{(C.26)}

The covariance inside the integral in (C.25) is

\[
\text{Cov}[C_t, p_f dR_u - C_u du]
\]

\[
= \text{Cov}[C_t, p_f E_u d(R_u) - C_u du]
\]

\[
= \text{Cov} \left[ C_t, (\gamma_1^R \tilde{C}_u + \gamma_2^R C_u + \gamma_3^R y_u) \Delta \frac{\Delta}{\eta \Sigma \eta'} - C_u \right] du
\]

\[
= H \left( 0, 1, 0, \frac{\Delta}{\eta \Sigma \eta'}, \gamma_2^R \frac{\Delta}{\eta \Sigma \eta'}, \gamma_3^R \frac{\Delta}{\eta \Sigma \eta'}, -1, \gamma_1^R \frac{\Delta}{\eta \Sigma \eta'}, u - t, \nu_0 \right) du,
\]  \hspace{1cm} \text{(C.27)}

where the second step follows from (A.3) and the third from (B.13). Substituting (C.27) into
(C.25), integrating, substituting into (C.26), and noting that (2.3) implies $\text{Var}(C_t) = \frac{s^2}{2\kappa}$, we find (C.24).

The return difference in Lemma C.4 is computed in the time-series and is expressed per share of the active fund, with one share coinciding with the entire fund. To map it to KSZ’s calculation, suppose that the return gap in KSZ concerns an aggregate active fund, whose assets are 90% of total fund assets, with the remainder held by an aggregate index fund. The spread in monthly CAPM alpha between top and bottom fund deciles, computed in the time-series and expressed per dollar invested across all funds is then 0.634\% (= 90\% \times 0.704\%). The counterpart quantity in our model is the difference (C.24) divided by the value of the residual-supply portfolio $\theta S_t$. We divide by the expectation of that value $\theta E(S_t)$, which we compute as in the active-share calculation. To complete the mapping to KSZ, we set $(\Delta t_1, \Delta t_2) = (3/12, 4/12)$, and note that the term $[E(z|z > z_9) - E(z|z < z_1)]$ is approximately 3.4. Lemma C.5 computes the standard deviation of flow-induced trading for asset $n$.

**Lemma C.5.** The standard deviation of the change in the investor’s holdings of asset $n$ between $t$ and $t + \Delta t$ is

$$\sqrt{2[H(0,0,1,0,0,1,0,\nu_0) - H(0,0,1,0,1,\Delta t,\nu_0)] |(p_f)_n|}. \tag{C.28}$$

**Proof:** Equation (A.2) implies that the change in the investor’s holdings of asset $n$ between $t$ and $t + \tau$ is

$$(x_{t+\tau} + y_{t+\Delta t z_{t+\Delta t}})_n - (x_t + y_{t z_t})_n = (y_{t+\Delta t} - y_t)(p_f)_n.$$  

The standard deviation of that change is

$$\sqrt{\text{Var}(y_{t+\Delta t} - y_t) |(p_f)_n|. \tag{C.29}$$

Since

$$\text{Var}(y_{t+\Delta t} - y_t) = \text{Var}(y_{t+\Delta t}) + \text{Var}(y_t) - 2\text{Cov}(y_t, y_{t+\Delta t})$$  

$$= 2\text{Var}(y_t) - 2\text{Cov}(y_t, y_{t+\Delta t}),$$  

where the second step follows in steady state, (B.13) and (C.29) imply (C.28).

The standard deviation in Lemma C.5 is computed for a given asset $n$ over time and is expressed per share of the asset. When assets are symmetric and $\theta_n$ is equal to $\bar{\theta} + \sigma(\theta)$ for half of the assets
and to $\bar{\theta} - \sigma(\theta)$ for the other half, $|(p_f)_n|$ is the same for all $n$, and the standard deviation (C.28) of the change in asset holdings is the same across assets. Hence, changes in asset holdings are drawn from the same distribution for all assets, and the standard deviation (C.28) describes both the cross-section and the time-series. Lou (2012) computes a spread in changes in asset holdings between top and bottom deciles of 22.27%. This translates to a standard deviation of 6.55% ($=22.27%/3.4$). The counterpart quantity in our model is the standard deviation (C.28) divided by the number of shares held by the active and the index funds. That number is $\bar{\theta} = 1$ for the average asset.

The ratio of the investor’s conditional standard deviation of $C_t$ to the unconditional standard deviation is $\sqrt{\frac{T}{2\sigma_t^2}}$, where $T$ is given from (A.11). Equation (B.10) implies that the response of the investor’s share $y_t$ in the active fund at time $t'$ to a shock $dR_t$ at time $t$ is proportional to $G(0, 0, 1, t' - t, \nu_0)$.

Lemma C.6 derives formulas for symmetric assets. We use these formulas to simplify the model-implied moments computed in Appendix C, and the Sharpe ratios and correlations computed in Appendices D and E.

**Lemma C.6.** Suppose $\eta = 1'$ and $\Sigma = \hat{\sigma}^2(I + \omega 11')$. For all $i \in \mathbb{N}$,

\[ \eta \Sigma^i 1' = \hat{\sigma}^2(1 + \omega N)^i N, \tag{C.30} \]

\[ \eta \Sigma^i p_f' = 0, \tag{C.31} \]

\[ p_f \Sigma^i 1' = \hat{\sigma}^2 \sigma(\theta)^2 N, \tag{C.32} \]

\[ T\text{r}(\Sigma^i) = \hat{\sigma}^2 i [(1 + \omega N)^i + N - 1], \tag{C.33} \]

\[ \Sigma^i p_f' = \hat{\sigma}^2 (\theta' - \bar{\theta} I). \tag{C.34} \]

**Proof:** Using the binomial formula and $\eta = 1'$, we find

\[ \Sigma^i = \hat{\sigma}^2 i \left( \sum_{i' = 0}^{i} C(i, i') \omega^{i'} (11')^{i'} \right) = \hat{\sigma}^2 i \left( I + \sum_{i' = 1}^{i} C(i, i') \omega^{i'} N^{i'-1} 11' \right). \tag{C.35} \]

Post-multiplying (C.35) by $\eta'$ and $\theta'$ yields

\[ \Sigma^i \eta' = \hat{\sigma}^2 i \left( 1 + \sum_{i' = 1}^{i} C(i, i') \omega^{i'} N^{i'} \right) 1 = \hat{\sigma}^2 (1 + \omega N)^i 1, \tag{C.36} \]
\[ \Sigma^t \theta' = \hat{\sigma}^2 \left[ \theta' + \left( \sum_{i'=1}^t C(i, i') \omega_{i'} N_{i'} \right) \bar{\theta} \right] = \hat{\sigma}^2 \left[ \theta' - \bar{\theta} + (1 + \omega N_i) \bar{\theta} \right]. \]  \hfill (C.37)

respectively. Pre-multiplying (C.36) and (C.37) by \( \eta \), and setting \( i = 1 \), yields

\[ \eta \Sigma \theta' = \bar{\theta} \eta \Sigma \eta'. \]  \hfill (C.38)

which in turn implies

\[ p_f = \theta - \bar{\theta} \eta = \theta - \bar{\theta} \eta'. \]  \hfill (C.39)

Pre-multiplying (C.36) by \( \eta \) yields (C.30). Post-multiplying \( \Sigma^t \) by \( p_f' \) and using (C.36)-(C.39) yields (C.34). Pre-multiplying (C.34) by \( \eta \) yields (C.31). Pre-multiplying (C.34) by \( p_f' \) and using (C.34) yields (C.32). Summing the diagonal terms in (C.35) yields (C.33).

The model-implied moments computed in this section depend on \( \theta \) through the aggregate quantities in Lemma C.6 and the components of the vector \( p_f = \theta - \bar{\theta} \eta' = \theta - 1' \). The aggregate quantities depend on \( \theta \) only through \( \bar{\theta} = 1 \) and \( \sigma(\theta) \). To compute the components of \( p_f \), we assume that \( \theta_n \) is equal to \( \bar{\theta} + \sigma(\theta) \) for half of the assets and to \( \bar{\theta} - \sigma(\theta) \) for the other half.

D  Proofs of Results in Section 5

Lemma D.1 computes the Sharpe ratio of a general strategy \( w_t \) over an infinitesimal horizon \( dt \). It also characterizes the optimal strategy and its Sharpe ratio.

**Lemma D.1.** The Sharpe ratio of a strategy \( w_t \) over horizon \( dt \) is \( \text{SR}_{w,t} = \frac{N_{w,t}}{D_{w,t}} \), where

\[ N_{w,t} \equiv \frac{1}{dt} \mathbb{E}_t(\hat{w}_t dR_t) = \left( f + \frac{k \Delta}{\eta \Sigma \eta'} \right) \mathbb{E}_t (\Lambda_t w_t \Sigma p_f'), \]  \hfill (D.1)

\[ D_{w,t} \equiv \frac{1}{dt} \text{Var}_t(\hat{w}_t dR_t) = f \left[ \mathbb{E}_t(w_t \Sigma w_t') - \frac{\mathbb{E}_t [(w_t \Sigma p_f')^2]}{\eta \Sigma \eta'} \right] + k \mathbb{E}_t [(w_t \Sigma p_f')^2]. \]  \hfill (D.2)

It is maximized for the strategy \( w_t = \Lambda_t p_f \). The Sharpe ratio of the optimal strategy is given by (5.2).

**Proof:** Lemma D.1 coincides with VW Proposition 8 in the case of the unconditional Sharpe ratio. The arguments in that proposition extend to the case of the conditional Sharpe ratio.
Proposition D.1 computes the optimal strategy’s unconditional Sharpe ratio, as well as its Sharpe ratios conditional on different information sets.

**Proposition D.1.** The unconditional Sharpe ratio of the optimal strategy is

\[
SR^*_w = \sqrt{\frac{\Delta}{f + \frac{k\Delta}{\eta\Sigma^\eta}}} \left[ L_2^2 + H(\gamma_1 R, \gamma_2 R, \gamma_3 R, \gamma_1 L, \gamma_2 L, \gamma_3 L, 0, \nu_0) \right].
\] (D.3)

When \( \mathcal{I}_t \) includes \((\hat{C}_t, C_t, y_t)\), the Sharpe ratio of the optimal strategy is

\[
SR^*_{w,t} = \sqrt{\frac{\Delta}{f + \frac{k\Delta}{\eta\Sigma^\eta}}} \left[ L_2 \left( \gamma_1^R(\hat{C}_t - \bar{C}) + \gamma_2^R(C_t - \bar{C}) + \gamma_3^R(y_t - \bar{y}) \right) \right]
\] (D.4)

and has unconditional expectation

\[
\mathbb{E}(SR^*_{w,t}) = \sqrt{\frac{\Delta}{f + \frac{k\Delta}{\eta\Sigma^\eta}}} \frac{H(\gamma_1 R, \gamma_2 R, \gamma_3 R, \gamma_1 L, \gamma_2 L, \gamma_3 L, 0, \nu_0)}{\eta\Sigma^\eta} \times \left[ \sqrt{\frac{2}{\pi}} e^{-\frac{R(\Lambda_t)^2}{2}} + R(\Lambda_t) \left[ 1 - 2N(-R(\Lambda_t)) \right] \right]
\] (D.5)

and unconditional variance

\[
\text{Var}(SR^*_{w,t}) = \frac{\Delta}{f + \frac{k\Delta}{\eta\Sigma^\eta}} \frac{H(\gamma_1 R, \gamma_2 R, \gamma_3 R, \gamma_1 L, \gamma_2 L, \gamma_3 L, 0, \nu_0)}{\eta\Sigma^\eta} \left[ R(\Lambda_t)^2 + 1 \right] - \mathbb{E}(SR^*_{w,t})^2,
\] (D.6)

where \( R(\Lambda_t) \equiv \frac{L_2}{\sqrt{H(\gamma_1 R, \gamma_2 R, \gamma_3 R, \gamma_1 L, \gamma_2 L, \gamma_3 L, 0, \nu_0)}} \). When instead \( \mathcal{I}_t = (\hat{C}_t, y_t) \), the Sharpe ratio of the optimal strategy is

\[
SR^*_{w,t} = \sqrt{\frac{\Delta}{f + \frac{k\Delta}{\eta\Sigma^\eta}}} \left[ L_2 \left( \gamma_1^R + \gamma_2^R \right)(\hat{C}_t - \bar{C}) + \gamma_3^R(y_t - \bar{y}) \right]^2 + (\gamma_2^R)^2 T.
\] (D.7)

**Proof:** To derive (D.3), we set \( \mathcal{I}_t = \emptyset \) in (5.2) and note that

\[
\mathbb{E}(\Lambda_t^2) = \mathbb{E}(\Lambda_t)^2 + \text{Var}(\Lambda_t)
\]
\[ \frac{1}{(f + \frac{k\Delta}{\eta\Sigma\eta'})^2} \left[ L_2^2 + H(\gamma_1^R, \gamma_2^R, \gamma_3^R, \gamma_1^R, \gamma_2^R, \gamma_3^R, 0, \nu_0) \right], \]

where the second step follows because the definition (B.6) of \( L_2 \) implies

\[ \mathbb{E}(\Lambda_t) = \frac{1}{f + \frac{k\Delta}{\eta\Sigma\eta'}} L_2. \tag{D.8} \]

and because (3.7) and (B.13) imply

\[ \text{Var}(\Lambda_t) = \frac{1}{(f + \frac{k\Delta}{\eta\Sigma\eta'})^2} H(\gamma_1^R, \gamma_2^R, \gamma_3^R, \gamma_1^R, \gamma_2^R, \gamma_3^R, 0, \nu_0). \tag{D.9} \]

To derive (D.4), we note that when \( \mathcal{I}_t \) includes \((\hat{C}_t, C_t, y_t)\), \( \mathbb{E}(\Lambda_t^2) = \Lambda_t^2 \). Substituting into (5.2), we find

\[ SR_{w,t}^* = \left( f + \frac{k\Delta}{\eta\Sigma\eta'} \right) \frac{\Delta}{\eta\Sigma\eta'} |\Lambda_t|. \tag{D.10} \]

Equation (D.4) follows from (D.10) and because (B.6) and (B.3)-(B.4) imply

\[ \Lambda_t = \frac{1}{f + \frac{k\Delta}{\eta\Sigma\eta'}} \left[ L_2 + \gamma_1^R(\hat{C}_t - \bar{C}) + \gamma_2^R(C_t - \bar{C}) + \gamma_3^R(y_t - \bar{y}) \right]. \tag{D.11} \]

Since \( \Lambda_t \) is normally distributed,

\[ \mathbb{E}(|\Lambda_t|) = \sqrt{\text{Var}(\Lambda_t)} \left[ \sqrt{\frac{2}{\pi}} e^{-\frac{R(\Lambda_t)^2}{2}} + R(\Lambda_t) \left[ 1 - 2N(-R(\Lambda_t)) \right] \right] \tag{D.12} \]

and unconditional variance

\[ \text{Var}(|\Lambda_t|) = \text{Var}(\Lambda_t) \left[ R(\Lambda_t)^2 + 1 \right] - \mathbb{E}(SR_{w,t}^*)^2, \tag{D.13} \]

where \( R(\Lambda_t) = \frac{\mathbb{E}(\Lambda_t)}{\sqrt{\text{Var}(\Lambda_t)}} \) and \( N(.) \) is the cumulative distribution function of the standard normal.

Equations (D.5) and (D.6) follow from (D.8)-(D.10), (D.12) and (D.13). To derive (D.7), we set
\( \mathcal{I}_t = (\hat{C}_t, y_t) \) in (5.2) and note that

\[
\mathbb{E}_{\mathcal{I}_t}(\Lambda_t^2) = \mathbb{E}_{\mathcal{I}_t}(\Lambda_t)^2 + \text{Var}_{\mathcal{I}_t}(\Lambda_t)
\]

\[
= \frac{1}{(f + \frac{k\Delta}{\eta T})^2} \left[ (L_2 + (\gamma_1^R + \gamma_2^R)(\hat{C}_t - \bar{C}) + \gamma_3^R(y_t - \bar{y}))^2 + (\gamma_2^R)^2 T \right],
\]

where the second step follows from (D.11) and because conditionally on \((\hat{C}_t, y_t), C_t\) is normal with mean \(\hat{C}_t\) and variance \(T\).

Lemma D.2 specializes the Sharpe ratio formula that Lemma D.1 derives for a general strategy to a class of strategies whose moments have a specific form. Value and momentum strategies belong to that class.

**Lemma D.2.** Suppose that for a strategy \(w_t\) and information set \(\mathcal{I}_t\),

\[
\mathbb{E}_{\mathcal{I}_t}(w_t) = \Phi_{1t}\eta\Sigma + \Phi_{2t}p_f\Sigma, \quad (D.14)
\]

\[
\text{Cov}_{\mathcal{I}_t}(\Lambda_t, w_t) = \frac{1}{f + \frac{k\Delta}{\eta T}} \Phi_i^\Sigma p_f \Sigma, \quad (D.15)
\]

\[
\text{Cov}_{\mathcal{I}_t}(w_t', w_t) = \hat{\Phi}_{1t}^\Sigma \Sigma + \hat{\Phi}_{2t}^\Sigma p_f \Sigma, \quad (D.16)
\]

Then, the Sharpe ratio of \(w_t\) conditional on \(\mathcal{I}_t\) is \(\frac{N_{w,t}}{\sqrt{D_{w,t}}}\), with

\[
N_{w,t} = \left( f + \frac{k\Delta}{\eta T} \right) \mathbb{E}_{\mathcal{I}_t}(\Lambda_t) \left( \Phi_{1t}\eta\Sigma^2 p_f' + \Phi_{2t}p_f^2 \Sigma^2 p_f' \right) + \Phi_i^\Sigma p_f^2 \Sigma^2 p_f',
\]

\[
D_{w,t} = \Phi_{1t}^2 \Delta_1 + 2\Phi_{1t}\Phi_{2t} \Delta_2 + \left( \Phi_{2t}^2 + \Phi_{1t}^2 \right) \Delta_3 + \Phi_i^\Sigma \Delta_4.
\]

**Proof:** We can write the numerator in (5.1) as

\[
\left( f + \frac{k\Delta}{\eta T} \right) [\mathbb{E}_{\mathcal{I}_t}(\Lambda_t) \mathbb{E}_{\mathcal{I}_t}(w_t) + \text{Cov}_{\mathcal{I}_t}(\Lambda_t, w_t)] \Sigma p_f'
\]

\[
= \left( f + \frac{k\Delta}{\eta T} \right) \mathbb{E}_{\mathcal{I}_t}(\Lambda_t) \left( \Phi_{1t}\eta\Sigma^2 p_f' + \Phi_{2t}p_f^2 \Sigma^2 p_f' \right) + \Phi_i^\Sigma p_f^2 \Sigma^2 p_f', \quad (D.17)
\]

where the second step follows by substituting \(\mathbb{E}_{\mathcal{I}_t}(w_t)\) and \(\text{Cov}_{\mathcal{I}_t}(\Lambda_t, w_t)\) from (D.14) and (D.15),
where the third step follows by substituting $\mathbb{E}_{\mathcal{I}_t}(w_t)$ and $\text{Cov}_{\mathcal{I}_t}(w'_t, w_t)$ from (D.14) and (D.16), respectively. The lemma follows from (D.17) and (D.18).

Proposition D.2 computes the unconditional Sharpe ratio of the value strategy.

**Proposition D.2.** The unconditional Sharpe ratio of the value strategy (4.1) is $\text{SR}_{w^V} = \frac{N_{w^V}}{\sqrt{D_{w^V}}}$ where

\[
N_{w^V} = \frac{L_1L_2}{r} \eta \Sigma^2 p'_f + \left( \frac{L_2^2}{r} - \frac{1 - \epsilon}{r + \kappa} K_1(\gamma_1 R, \gamma_3, 0, \nu_0) + H(\gamma_1 R, \gamma_2 R, \gamma_3, \gamma_1, \gamma_2, \gamma_3, 0, \nu_0) \right) p_f \Sigma^2 p'_f,
\]

\[
D_{w^V} = \frac{L_1}{r^2} \Delta_1 + \frac{2L_1L_2}{r^2} \Delta_2 + \left( \frac{L_2^2}{r^2} - \frac{2(1 - \epsilon)}{r + \kappa} K_1(\gamma_1, \gamma_3, 0, \nu_0) + H(\gamma_1, \gamma_2, \gamma_3, \gamma_1, \gamma_2, \gamma_3, 0, \nu_0) \right) \Delta_3 + \frac{(1 - \epsilon)^2 \phi^2}{2(r + \kappa)^2 \kappa} \Delta_4.
\]
**Proof:** Substituting (3.1) into (4.1), we can write the value weights as

\[ w^V_t = -\frac{(1 - \epsilon)(F_t - \bar{F})'}{r + \kappa} + \frac{\alpha \hat{f} \eta \Sigma' \eta}{\alpha + \hat{\alpha} \eta \Sigma' \eta} \eta \Sigma + (\gamma_0 + \gamma_1 \hat{C}_t + \gamma_2 C_t + \gamma_3 y_t) p_t \Sigma. \]  

(D.19)

Taking unconditional expectations in (D.19), we find

\[ \mathbb{E}(w^V_t) = \frac{\alpha \hat{f}}{\alpha + \hat{\alpha} \eta \Sigma' \eta} \eta \Sigma + (\gamma_0 + (\gamma_1 + \gamma_2) \hat{C} + \gamma_3 \bar{y}) p_t \Sigma. \]

(D.20)

where the second step follows from (B.5) and because (A.1), (A.4)-(A.6) and (B.6) imply

\[ \frac{L_2}{r} = \gamma_0 + (\gamma_1 + \gamma_2) \hat{C} + \gamma_3 \bar{y}. \]  

(D.21)

Taking the unconditional covariance of (D.19) with (3.7), and using (B.13) and (B.15), we find

\[ \text{Cov}(\Lambda_t, w^V_t) = \frac{1}{f} + \frac{k\Delta}{\eta \Sigma' \eta} \left( -\frac{1 - \epsilon}{r + \kappa} K_1(\gamma_1^R, \gamma_3^R, 0, \nu_0) + H(\gamma_1^R, \gamma_2^R, \gamma_3^R, \gamma_1, \gamma_2, \gamma_3, \gamma_3, 0, \nu_0) \right) p_t \Sigma. \]  

(D.22)

Taking the unconditional covariance of (D.19) with the transpose of (D.19), and using (B.13), (B.15) and (B.16), we find

\[ \text{Cov}\left((w^V_t)', w^V_t\right) = \frac{(1 - \epsilon)^2 \phi^2}{2(r + \kappa)^2} \Sigma + \left( -\frac{2(1 - \epsilon)}{r + \kappa} K_1(\gamma_1, \gamma_3, 0, \nu_0) + H(\gamma_1, \gamma_2, \gamma_3, \gamma_1, \gamma_2, \gamma_3, \gamma_3, 0, \nu_0) \right) \Sigma p_t p_t \Sigma. \]

(D.23)

Equations (D.20), (D.22) and (D.23) imply that the unconditional Sharpe ratio of the value strategy can be deduced from Lemma D.2 by setting \( \mathcal{I}_t = \emptyset \),

\[ \Phi_{1t} = \frac{L_1}{r}, \]

\[ \Phi_{2t} = \frac{L_2}{r}, \]

\[ \Phi^\Lambda_t = -\frac{1 - \epsilon}{r + \kappa} K_1(\gamma_1^R, \gamma_3^R, 0, \nu_0) + H(\gamma_1^R, \gamma_2^R, \gamma_3^R, \gamma_1, \gamma_2, \gamma_3, 0, \nu_0), \]

65
\[ \hat{\Phi}_t^\Sigma = \frac{(1 - \epsilon)^2 \phi^2}{2(r + \kappa)^2 \kappa}. \]

\[ \hat{\Phi}_t = \frac{2(1 - \epsilon)}{r + \kappa} K_1(\gamma_1, \gamma_2, 0, \nu_0) + H(\gamma_1, \gamma_2, \gamma_3, 0, \nu_0). \]

The proposition follows from this observation and (D.8).

Lemma D.3. For \( t'' \geq t' \geq t \),

\[ E_{\mathcal{I}_t}(w_{t''}^V) = \frac{L_1}{r} \eta \Sigma + \left( \frac{L_2}{r} + \delta_{12,t',t}^V(\hat{C}_t - \bar{C}) + \delta_{3,t',t}^V(y_t - \bar{y}) \right) p_f \Sigma, \]  

(D.24)  

\[ E_{\mathcal{I}_t}(\Lambda_{t''}) = \frac{1}{f + \frac{kA}{\eta \Sigma}} \left( \frac{L_2}{r} + \delta_{12,t',t}^\Lambda(\hat{C}_t - \bar{C}) + \delta_{3,t',t}^\Lambda(y_t - \bar{y}) \right), \]  

(D.25)  

\[ \text{Cov}_{\mathcal{I}_t}(w_{t''}^V, \Lambda_{t''}) = \frac{1}{f + \frac{kA}{\eta \Sigma}} C_{t'',t''-t}^V \Sigma + C_{t'',t''-t}^\Lambda p_f \Sigma, \]  

(D.26)  

\[ \text{Cov}_{\mathcal{I}_t}(\Lambda_{t''}, w_{t''}^V) = \frac{1}{f + \frac{kA}{\eta \Sigma}} C_{t'',t''-t}^\Lambda \Sigma + C_{t'',t''-t}^V p_f \Sigma, \]  

(D.27)  

\[ \text{Cov}_{\mathcal{I}_t}(\hat{w}_{t''}^V, \hat{w}_{t''}^V) = \text{Cov}_{t''-t}^\Sigma \Sigma + \text{Cov}_{t''-t}^\Lambda p_f \Sigma, \]  

(D.28)  

\[ \text{Cov}_{\mathcal{I}_t}(\Lambda_{t''}, \Lambda_{t''}) = C_{t''-t}^\Lambda, \]  

(D.29)

where

\[ \left( \begin{array}{c} \delta_{12,t',t}^V \\ \delta_{3,t',t}^V \end{array} \right) = \Sigma_{C,y}^{-1} \left( \begin{array}{c} -\frac{(1-\epsilon)K_1(0,1,t',t''-t)\eta}{r+\kappa} + H(1,0,0,\gamma_1,\gamma_2,\gamma_3,0,\nu_0) \\ -\frac{(1-\epsilon)K_1(0,1,t',t''-t)\eta}{r+\kappa} + H(0,0,1,\gamma_1,\gamma_2,\gamma_3,0,\nu_0) \end{array} \right), \]  

(D.30)  

\[ \left( \begin{array}{c} \delta_{12,t',t}^\Lambda \\ \delta_{3,t',t}^\Lambda \end{array} \right) = \Sigma_{C,y}^{-1} \left( \begin{array}{c} H(1,0,0,\gamma_1^R,\gamma_2^R,\gamma_3^R,0,\nu_0) \\ H(0,0,1,\gamma_1^R,\gamma_2^R,\gamma_3^R,0,\nu_0) \end{array} \right), \]  

(D.31)
\[
C_{t-t',t''-t}^{V} \equiv -\frac{(1-\epsilon)K_2(\gamma_1^R, \gamma_3^R, t''-t', \nu_0)}{r+\kappa} + H(\gamma_1, \gamma_2, \gamma_3, \gamma_1^R, \gamma_3^R, t''-t', \nu_0)
- (\delta_{12,t''-t}, \delta_{3,t'-t})\Sigma \hat{C}_y \left( \begin{array}{c} \delta_{12,t''-t}^V \\ \delta_{3,t'-t}^V \end{array} \right),
\]

\[
C_{t-t',t''-t}^{AV} \equiv -\frac{(1-\epsilon)K_1(\gamma_1^R, \gamma_3^R, 0, \nu_0)}{r+\kappa} + H(\gamma_1^R, \gamma_2^R, \gamma_3^R, \gamma_1, \gamma_2, \gamma_3, 0, \nu_0)
- (\delta_{12,t''-t}, \delta_{3,t'-t})\Sigma \hat{C}_y \left( \begin{array}{c} \delta_{12,t''-t}^V \\ \delta_{3,t'-t}^V \end{array} \right),
\]

\[
C_{t-t',t''-t}^{V\Sigma} \equiv \frac{(1-\epsilon)^2\phi^2}{2(r+\kappa)^2}\nu_0(\kappa, t''-t')
\]

\[
C_{t-t',t''-t}^{V'} \equiv -\frac{1-\epsilon}{r+\kappa} \left[ K_1(\gamma_1, \gamma_3^R, t''-t', \nu_0) + K_2(\gamma_1, \gamma_3, t''-t', \nu_0) \right]
+ H(\gamma_1, \gamma_2, \gamma_3, \gamma_1, \gamma_2, \gamma_3, t''-t', \nu_0)
- (\delta_{12,t''-t}, \delta_{3,t'-t})\Sigma \hat{C}_y \left( \begin{array}{c} \delta_{12,t''-t}^V \\ \delta_{3,t'-t}^V \end{array} \right),
\]

\[
C_{t-t',t''-t}^{\Lambda} \equiv H(\gamma_1^R, \gamma_2^R, \gamma_3^R, \gamma_1, \gamma_2, \gamma_3, t''-t', \nu_0)
- (\delta_{12,t''-t}, \delta_{3,t'-t})\Sigma \hat{C}_y \left( \begin{array}{c} \delta_{12,t''-t}^\Lambda \\ \delta_{3,t'-t}^\Lambda \end{array} \right).
\]

**Proof:** Using the joint normality of \( \left( (w_t^V)' , \Lambda_t , \hat{C}_t , y_t \right) \), (D.8) and (D.20), we can set

\[
w_t^V - \left( \frac{L_1}{r} \Sigma + \frac{L_2}{r} p_f \Sigma \right) = \Delta_{12,t'-t}^V (\hat{C}_t - \bar{C}) + \Delta_{3,t'-t}^V (y_t - \bar{y}) + \zeta_t^V, \tag{D.32}
\]

\[
\Lambda_t - \frac{1}{f + \frac{\kappa A}{\eta^2 \gamma^2}} L_2 = \frac{1}{f + \frac{\kappa A}{\eta^2 \gamma^2}} \left[ \delta_{12,t'-t}^\Lambda (\hat{C}_t - \bar{C}) + \delta_{3,t'-t}^\Lambda (y_t - \bar{y}) + \zeta_t^\Lambda \right], \tag{D.33}
\]

where the error terms \((\zeta_t^V, \zeta_t^\Lambda)\) have mean zero and are independent of \((\hat{C}_t, y_t)\).

Taking covariances of both sides of (D.32) with \( \hat{C}_t \) and \( y_t \), and using (B.13), (B.14), (D.19) and the independence of \( \zeta_t^V \) from \((\hat{C}_t, y_t)\), we find

\[
\left( -\frac{(1-\epsilon)K_1(1,0,t'-t, \nu_0)}{r+\kappa} + H(1,0,0, \gamma_1, \gamma_2, \gamma_3, t'-t, \nu_0) \right) p_f \Sigma = \Delta_{12,t'-t}^V \Sigma \hat{C}_y + \Delta_{3,t'-t}^V \Sigma \hat{C}_y, \tag{D.34}
\]
Equation (D.26) follows from (D.38) by noting that the independence of $\hat{V}$ and is independent of $(\hat{C}_t, y_t)$ from (D.33) because $\zeta_{t'}^V$ has mean zero and is independent of $(\hat{C}_t, y_t)$.

Taking covariances of both sides of (D.33) with $\hat{C}_t$ and $y_t$, and using (3.7), (B.13), (B.14) and the independence of $\zeta_{t'}^\Lambda$ from $(\hat{C}_t, y_t)$, we find
\begin{align*}
H(1, 0, 0, \gamma_1^R, \gamma_2^R, \gamma_3^R, t' - t, \nu_0) &= \delta_{12, t' - t}^\Lambda \hat{C}_y + \delta_{3, t' - t}^\Lambda \hat{C}_y, \tag{D.36} \\
H(0, 0, 1, \gamma_1^R, \gamma_2^R, \gamma_3^R, t' - t, \nu_0) &= \delta_{12, t' - t}^\Lambda \hat{C}_y + \delta_{3, t' - t}^\Lambda \hat{C}_y, \tag{D.37}
\end{align*}
respectively. Solving (D.36) and (D.37) for $(\delta_{12, t' - t}^\Lambda, \delta_{3, t' - t}^\Lambda)$, we find (D.31). Equation (D.25) follows from (D.32) because $\zeta_{t'}^V$ has mean zero and is independent of $(\hat{C}_t, y_t)$.

Writing that the covariance between the left-hand side of (D.33) evaluated at $t'$ and the right-hand side of (D.33) evaluated at $t''$ is equal to the covariance between the corresponding right-hand sides, and using (3.7), (B.13), (B.14), (D.19), $\Delta_{12}^{V, t' - t} = \delta_{12, t' - t}^V p_f \Sigma$, $\Delta_{3}^{V, t' - t} = \delta_{3, t' - t}^V p_f \Sigma$ and the independence of $(\zeta_{t'}^V, \zeta_{t''}^\Lambda)$ from $(\hat{C}_t, y_t)$, we find
\begin{align*}
\left( -\frac{(1 - \epsilon) K_1(0, 1, t' - t, \nu_0)}{r + \kappa} + H(0, 0, 1, \gamma_1, \gamma_2, \gamma_3, t' - t, \nu_0) \right) p_f \Sigma &= \left( \begin{pmatrix} \delta_{12, t' - t}^V \\ \delta_{3, t' - t}^V \end{pmatrix} \right) \left( \begin{pmatrix} \hat{C}_y \\ \hat{C}_y \end{pmatrix} \right) p_f \Sigma + \text{Cov}(\zeta_{t'}^V, \zeta_{t''}^\Lambda). \tag{D.38}
\end{align*}

Equation (D.26) follows from (D.38) by noting that $\frac{1}{f + \frac{4\pi^2}{\nu_0^2}} \text{Cov}(\zeta_{t'}^V, \zeta_{t''}^\Lambda) = \text{Cov}_\zeta(\nu_t^V, \Lambda_{t''})$.

Writing that the covariance between the left-hand side of (D.33) evaluated at $t'$ and the right-hand side of (D.32) evaluated at $t''$ is equal to the covariance between the corresponding right-hand sides, and using (3.7), (B.13), (B.14), (D.19), $\Delta_{12}^{V, t' - t} = \delta_{12, t' - t}^V p_f \Sigma$, $\Delta_{3}^{V, t' - t} = \delta_{3, t' - t}^V p_f \Sigma$ and the
independence of \((\zeta^V_t, \zeta_t^V)\) from \((\hat{C}_t, y_t)\), we find

\[
\begin{aligned}
\left( -\frac{(1 - \epsilon) K_1(\gamma_1^R, \gamma_2^R, t'' - t', \nu_0)}{r + \kappa} + H(\gamma_1^R, \gamma_2^R, \gamma_3^R, \gamma_1, \gamma_2, \gamma_3, t'' - t', \nu_0) \right) p_f \Sigma \\
= (\delta^V_{12,t''-t}, \delta^V_{3,t''-t}) \left( \begin{array}{c}
\sum \hat{C}_y \\
\sum \hat{C}_y
\end{array} \right) \left( \begin{array}{c}
\delta^V_{12,t''-t} \\
\delta^V_{3,t''-t}
\end{array} \right) p_f \Sigma \right)
\end{aligned}
\]  

\tag{D.39}

Equation (D.27) follows from (D.39) by noting that \(\frac{1}{f + \frac{1}{2} \sigma^2} \text{Cov}(\zeta^A_t, \zeta^V_t) = \text{Cov}_Y(\Lambda_t, w^V_t).\)

Writing that the covariance between the left-hand side of (D.32) evaluated at \(t''\) and the transpose of the left-hand side of (D.32) evaluated at \(t'\) is equal to the covariance between the corresponding right-hand sides, and using (B.13)-(B.16), (D.19), \(\Delta^V_{12,t''-t} = \delta^V_{12,t''-t} p_f \Sigma, \Delta^V_{3,t''-t} = \delta^V_{3,t''-t} p_f \Sigma\) and the independence of \((\zeta^V_t, \zeta^V_t)\) from \((\hat{C}_t, y_t)\), we find

\[
\begin{aligned}
\frac{(1 - \epsilon)^2 \sigma^2}{2\kappa} v_0(\kappa, t'' - t') \Sigma + \left( -\frac{1 - \epsilon}{r + \kappa} \left[ K_1(\gamma_1, \gamma_3, t'' - t', \nu_0) + K_2(\gamma_1, \gamma_3, t'' - t', \nu_0) \right] \\
+ H(\gamma_1, \gamma_2, \gamma_3, \gamma_1, \gamma_2, \gamma_3, t'' - t', \nu_0) \right) \Sigma p_f p_f \Sigma \\
= (\delta^V_{12,t''-t}, \delta^V_{3,t''-t}) \left( \begin{array}{c}
\sum \hat{C}_y \\
\sum \hat{C}_y
\end{array} \right) \left( \begin{array}{c}
\delta^V_{12,t''-t} \\
\delta^V_{3,t''-t}
\end{array} \right) \Sigma p_f p_f \Sigma + \text{Cov} \left( \left( \zeta^V_t \right)', \zeta^V_t \right).
\end{aligned}
\]  

\tag{D.40}

Equation (D.26) follows from (D.38) by noting that \(\text{Cov} \left( \left( \zeta^V_t \right)', \zeta^V_t \right) = \text{Cov}_Y \left( \Lambda_t, w^V_t \right).\)

Writing that the covariance of the left-hand side of (D.33) evaluated at \(t''\) and the left-hand side of (D.33) evaluated at \(t'\) is equal to the covariance between the corresponding right-hand sides, and using (3.7), (B.13) and the independence of \((\zeta^A_t, \zeta^A_t)\) from \((\hat{C}_t, y_t)\), we find

\[
\begin{aligned}
H(\gamma_1^R, \gamma_2^R, \gamma_3^R, \gamma_1^R, \gamma_2^R, \gamma_3^R, t'' - t', \nu_0) = (\delta^A_{12,t''-t}, \delta^A_{3,t''-t}) \left( \begin{array}{c}
\sum \hat{C}_y \\
\sum \hat{C}_y
\end{array} \right) \left( \begin{array}{c}
\delta^A_{12,t''-t} \\
\delta^A_{3,t''-t}
\end{array} \right) + \text{Cov}(\zeta^A_t, \zeta^A_t).
\end{aligned}
\]  

\tag{D.41}

Equation (D.29) follows from (D.41) by noting that \(\text{Cov}(\zeta^A_t, \zeta^A_t) = \text{Cov}_X(\Lambda_t, \Lambda_t).\)

Proposition D.3 computes the Sharpe ratio of the value strategy conditional on \((\hat{C}_t, y_t)\).

**Proposition D.3.** The Sharpe ratio of the value strategy (4.1) conditional on \((\hat{C}_t, y_t)\) is \(SR_{w,v,t} = \)
\[ \frac{N_{wV,t}}{\sqrt{D_{wV,t}}}, \text{ where} \]

\[ N_{wV,t} = \left[ L_2 + \delta_{12,0}(\hat{C}_t - \bar{C}) + \delta_{3,0}(y_t - \bar{y}) \right] \]
\[ \times \left[ \frac{L_1}{r} \eta \Sigma^2 p_f' + \left( \frac{L_2}{r} + \delta_{12,0}(\hat{C}_t - \bar{C}) + \delta_{3,0}(y_t - \bar{y}) \right) p_f \Sigma^2 p_f' \right] + C^V \Sigma \Delta. \]

\[ D_{wV,t} = \frac{L_1^2}{r^2} \Delta_1 + 2 \frac{L_1}{r} \left( \frac{L_2}{r} + \delta_{12,0}(\hat{C}_t - \bar{C}) + \delta_{3,0}(y_t - \bar{y}) \right) \Delta_2 \]
\[ + \left[ \left( \frac{L_2}{r} + \delta_{12,0}(\hat{C}_t - \bar{C}) + \delta_{3,0}(y_t - \bar{y}) \right)^2 + C^V \right] \Delta_3 + C^V \Sigma. \]

**Proof:** Lemma D.3 implies that the Sharpe ratio of the value strategy conditional on \((\hat{C}_t, y_t)\) can be deduced from Lemma D.2 by setting \(\mathcal{I}_t = \{\hat{C}_t, y_t\}\),

\[ \Phi_{1t} = \frac{L_1}{r}, \]
\[ \Phi_{2t} = \frac{L_2}{r} + \delta_{12,0}(\hat{C}_t - \bar{C}) + \delta_{3,0}(y_t - \bar{y}), \]
\[ \Phi^\Lambda = C^V \Lambda, \]
\[ \hat{\Phi}^\Sigma = C^V \Sigma, \]
\[ \hat{\Phi}_t = C^V. \]

The proposition follows from this observation and (D.25).

We next compute the value spread. We define the value spread as the standard deviation of the market-to-book ratio in the cross-section of assets,

\[ V S_t = \sqrt{\frac{\sum_{n=1}^N \left( \frac{S_{nt}}{B_{nt}} - \frac{\sum_{n'=1}^N S_{n't}}{N B_{n't}} \right)^2}{N}}, \]

and assume that all assets have the same book value, which we take to be the average price in the
cross-section of assets and over time,

$$B_{nt} = \frac{1}{N} \sum_{n=1}^{N} \mathbb{E}(S_{nt}) \equiv B.$$  \hfill (D.43)

Proposition D.4 computes the variance of the market-to-book ratio conditional on \((\hat{C}_t, y_t)\) in the cross-section of symmetric assets. We take the square root of that quantity as our measure of the value spread conditional on \((\hat{C}_t, y_t)\).

**Proposition D.4.** Suppose \(\eta = 1'\), \(\bar{F} = F1\) and \(\Sigma = \hat{\sigma}^2(I + \omega 11')\). The value spread conditional on \(I_t = (\hat{C}_t, y_t)\) is

$$\sqrt{\mathbb{E}_t V S_t^2} = \frac{\sqrt{\left[ \left( \frac{L_2}{r} + \delta_{12,0}^{V}(\hat{C}_t - \bar{C}) + \delta_{3,0}^{V}(y_t - \bar{y}) \right)^2 + C_{0,0}^{V} \right] \bar{\sigma}^4 \sigma(\theta)^2 + C_{\Sigma}^{V}(N-1)\hat{\sigma}^2}}{\bar{\rho} - \frac{\alpha_{0}f}{\alpha_{0} + \hat{\theta}} \hat{\sigma}^2(1 + \omega N)},$$  \hfill (D.44)

where \((\delta_{12,0}^{V}, \delta_{3,0}^{V}, C_{0,0}^{V}, C_{\Sigma}^{V})\) are derived in Lemma D.3 for \(\epsilon = 0\).

**Proof:** Using \(B_{nt} = B\), we can write (D.42) as

$$V S_t = \frac{1}{B} \sqrt{\frac{\sum_{n=1}^{N} \left( S_{nt} - \frac{\sum_{n'}=1}{} N S_{nt} \right)^2}{N}} = \frac{1}{B} \sqrt{\frac{\sum_{n=1}^{N} \left( w_{nt}^{V} - \frac{\sum_{n'}=1}{} N w_{nt}^{V} \right)^2}{N}},$$  \hfill (D.45)

where the second step follows by using (4.1) and setting \(\epsilon = 0\) and \(\bar{F} = F1\). Equation (D.32) implies

$$w_{nt}^{V} - \frac{\sum_{n'}=1}{} N w_{nt}^{V} = \frac{L_1}{r} \left( (\eta \Sigma)_{n} - \frac{\sum_{n'=1}^{N} (\eta \Sigma)_{n'}}{N} \right) + \left( \frac{L_2}{r} + \delta_{12,0}^{V}(\hat{C}_t - \bar{C}) + \delta_{3,0}^{V}(y_t - \bar{y}) \right) \left( p_{f} \Sigma \right)_n - \frac{\sum_{n'=1}^{N} (p_f \Sigma)_{n'}}{N} + (\zeta_{t}^{V})_n - \frac{\sum_{n'}=1}{} N (\zeta_{t}^{V})_{n'},$$  \hfill (D.46)

where the second step follows from \(\eta = 1'\) and \(\Sigma = \hat{\sigma}^2(I + \omega 11')\). Squaring both sides of (D.46), taking expectations conditional on \(I_t = (\hat{C}_t, y_t)\), and denoting by \(e_n\) the \(N \times 1\) vector with \(n\)'th
element equal to one and all other elements equal to zero, we find

\[
\mathbb{E}_t \left( w_{nt} V - \frac{\sum_{n'=1}^N w_{nt}}{N} \right)^2 \\
= \left( \frac{L_2}{r} + \delta_{12,0} \left( \hat{C}_t - \bar{C} \right) + \delta_{3,0} (y_t - \bar{y}) \right) (p_f \Sigma_n)^2 + \mathbb{E}_t \left( \zeta V_t - \frac{\sum_{n'=1}^N (\zeta_{n'})_{n'}}{N} \right)^2 \\
= \left( \frac{L_2}{r} + \delta_{12,0} \left( \hat{C}_t - \bar{C} \right) + \delta_{3,0} (y_t - \bar{y}) \right) (p_f \Sigma_n)^2 \\
+ \left( e_n - \frac{1}{N} \mathbf{1} \right)' \left( C^{V} \Sigma + C^{V} \Sigma p'_f p_f \Sigma \right) \left( e_n - \frac{1}{N} \mathbf{1} \right), \\
= \left[ \left( \frac{L_2}{r} + \delta_{12,0} \left( \hat{C}_t - \bar{C} \right) + \delta_{3,0} (y_t - \bar{y}) \right) (p_f \Sigma_n)^2 + C^{V} \Sigma_{0,0} \right] (p_f \Sigma_n)^2 + C^{V} \Sigma \frac{(N - 1) \sigma^2}{N}, \\
\text{(D.47)}
\]

where the second step follows from (D.28) and the third step follows from

\[
\left( e_n - \frac{1}{N} \mathbf{1} \right)' \Sigma p'_f p_f \Sigma \left( e_n - \frac{1}{N} \mathbf{1} \right) = (p_f \Sigma (e_n - \frac{1}{N} \eta'))^2 = (p_f \Sigma e_n)^2 = (p_f \Sigma_n)^2
\]

and

\[
\left( e_n - \frac{1}{N} \mathbf{1} \right)' \Sigma \left( e_n - \frac{1}{N} \mathbf{1} \right) = \left( e_n - \frac{1}{N} \mathbf{1} \right)' \sigma^2 (I + \omega \mathbf{1} \mathbf{1}') \left( e_n - \frac{1}{N} \mathbf{1} \right) \\
= \left( e_n - \frac{1}{N} \mathbf{1} \right)' \sigma^2 I \left( e_n - \frac{1}{N} \mathbf{1} \right) \\
= \sigma^2 \left[ \left( \frac{N - 1}{N} \right)^2 + (N - 1) \left( \frac{1}{N} \right)^2 \right] = \sigma^2 \frac{N - 1}{N}.
\]

Summing (D.47) across assets and using (C.34), we find the term inside the square root in (D.44). To compute \( B \), we note that since \( \eta = \mathbf{1}' \), \( B = \eta \mathbb{E}(S_t) \frac{1}{N} = \mathbb{E}(\eta S_t) \frac{1}{N} \). Using the expression for \( \mathbb{E}(\eta S_t) \) in the denominator of (C.21), together with \( \eta = \mathbf{1}' \), (C.30) and (C.38), we find the denominator of (D.44).

Proposition D.5 computes the unconditional Sharpe ratio of the momentum strategy.

**Proposition D.5.** The unconditional Sharpe ratio of the momentum strategy (4.2) is \( \text{SR}_{u,M} = \)
\[
\frac{N_{w,M}}{\sqrt{\delta_{w,M}}} \text{ where} \nabla
\]

\[
N_{w,M} = L_1 L_2 \tau \eta \Sigma^2 p_f' + \left[ L_2^2 \tau + H(\gamma_1^R, \gamma_2^R, \gamma_3^R, \gamma_1^R, \gamma_2^R, \gamma_3^R, T, \nu_2) + G(\gamma_1^R, \gamma_2^R, \gamma_3^R, T, \nu_2) \right] p_f \Sigma^2 p_f',
\]

\[
D_{w,M} = L_1^2 \tau^2 \Delta_1 + 2 L_1 L_2 \tau^2 \Delta_2 \nabla
\]

\[
+ \left[ L_2^2 \tau^2 + 2H(\gamma_1^R, \gamma_2^R, \gamma_3^R, \gamma_1^R, \gamma_2^R, \gamma_3^R, T, \nu_4) + 2G(\gamma_1^R, \gamma_2^R, \gamma_3^R, T, \nu_4) + k\tau \right] \Delta_3 + f\tau \Delta_4,
\]

and \( T = (0, \tau) \).

**Proof:** Substituting (3.1) into (4.1), we can write the momentum weights as

\[
w^M_t = \frac{\kappa \alpha f}{\alpha + \kappa \eta \Sigma \eta'} \tau \eta \Sigma + \left( f + \frac{k\Delta}{\eta \Sigma \eta'} \right) \left( \int_{t-\tau}^t \Lambda_u du \right) p_f \Sigma + \int_{t-\tau}^t \left[ dR_u - E_u(dR_u) \right]' (D.48)
\]

Taking unconditional expectations in (D.48), we find

\[
E(w^M_t) = \frac{\kappa \alpha f}{\alpha + \kappa \eta \Sigma \eta'} \tau \eta \Sigma + \left( f + \frac{k\Delta}{\eta \Sigma \eta'} \right) \left( \int_{t-\tau}^t E(\Lambda_u) du \right) p_f \Sigma \nabla
\]

\[
= L_1 \tau \eta \Sigma + L_2 \tau p_f \Sigma, \quad (D.49)
\]

where the second step follows from (B.5) and (D.8). Taking the unconditional covariance of (D.48) with (3.7), we find

\[
\text{Cov}(\Lambda_t, w^M_t) = \left( f + \frac{k\Delta}{\eta \Sigma \eta'} \right) \left( \int_{t-\tau}^t \text{Cov}(\Lambda_u, \Lambda_u) du \right) p_f \Sigma + \int_{t-\tau}^t \text{Cov}(\Lambda_t, [dR_u - E_u(dR_u)]'). \quad (D.50)
\]

To compute the second term in (D.50), we note that for a random variable \( X_t \) that depends on information up to time \( t \)

\[
\text{Cov}(X_t, dR_u - E_u(dR_u)) = E(X_t[dR_u - E_u(dR_u)]) \nabla
\]

\[
= E[E_u(X_t[dR_u - E_u(dR_u)])] \nabla
\]

\[
= E[Cov_u(X_t, dR_u)]. \quad (D.51)
\]
Using (3.7), (B.10), (B.13) and (D.51), we can write (D.50) as

$$\text{Cov} (\Lambda_t, w^M_t) = \frac{1}{f + \frac{k\Delta}{\eta\Sigma}} [H(\gamma_1^R, \gamma_2^R, \gamma_3^R, \gamma_1^R, \gamma_2^R, \gamma_3^R, \tau, \nu_2) + G(\gamma_1^R, \gamma_2^R, \gamma_3^R, \tau, \nu_2)] pf \Sigma. \quad \text{(D.52)}$$

Taking the unconditional covariance of (D.48) with the transpose of (D.48), we find

$$\text{Cov} \left( \left( w^M_t \right)^T, w^M_t \right) = \left( f + \frac{k\Delta}{\eta\Sigma} \right)^2 \left( \frac{\int_{u'=t-\tau}^{t} \text{Cov} (\Lambda_u, \Lambda_{u'}) d\nu u' \right) \Sigma f' pf \Sigma$$
$$\quad + \left( f + \frac{k\Delta}{\eta\Sigma} \right) \Sigma f' \left( \int_{u'=t-\tau}^{t} \text{Cov} (\Lambda_u, [dR_{u'} - E_{u'}(dR_{u'})]) du' \right)$$
$$\quad + \left( f + \frac{k\Delta}{\eta\Sigma} \right) \left( \int_{u'=t-\tau}^{t} \text{Cov} (\Lambda_{u'}, dR_u - E_u(dR_u)) du' \right) pf \Sigma$$
$$\quad + \int_{t-\tau}^{t} \text{Cov} (dR_u - E_u(dR_u), [dR_{u'} - E_{u'}(dR_{u'})]) du'. \quad \text{(D.53)}$$

To compute the second and third terms in (D.53), we note that the covariance in (D.51) is zero for \( t < u \). To compute the fourth term in (D.53), we note that it is equal to \( \mathbb{E}[\text{Cov}_u(dR_u, dR_u')] \). We can thus write (D.53) as

$$\text{Cov} \left( \left( w^M_t \right)^T, w^M_t \right) = 2 \left( f + \frac{k\Delta}{\eta\Sigma} \right)^2 \left( \frac{\int_{u'=t-\tau}^{t} \text{Cov} (\Lambda_u, \Lambda_{u'}) d\nu u' \right) \Sigma f' pf \Sigma$$
$$\quad + \left( f + \frac{k\Delta}{\eta\Sigma} \right) \Sigma f' \left( \int_{u'=t-\tau}^{t} \text{Cov} (\Lambda_u, [dR_{u'} - E_{u'}(dR_{u'})]) du' \right)$$
$$\quad + \left( f + \frac{k\Delta}{\eta\Sigma} \right) \left( \int_{u'=t-\tau}^{t} \text{Cov} (\Lambda_{u'}, dR_u - E_u(dR_u)) du' \right) pf \Sigma$$
$$\quad + \int_{t-\tau}^{t} \mathbb{E}[\text{Cov}_u(dR_u, dR_u')]. \quad \text{(D.54)}$$

Using (3.5), (3.7), (B.10), (B.13) and (D.54), we find

$$\text{Cov} \left( \left( w^M_t \right)^T, w^M_t \right) = \left[ 2H(\gamma_1^R, \gamma_2^R, \gamma_3^R, \gamma_1^R, \gamma_2^R, \gamma_3^R, \tau, \nu_4) + 2G(\gamma_1^R, \gamma_2^R, \gamma_3^R, \tau, \nu_4) \right] \Sigma f' pf \Sigma$$
$$\quad + \tau \left( f \Sigma + k \Sigma f' pf \Sigma \right). \quad \text{(D.55)}$$

Equations (D.49), (D.52) and (D.55) imply that the unconditional Sharpe ratio of the momentum
strategy can be deduced from Lemma D.2 by setting $\mathcal{I}_t = \emptyset$,

$$
\Phi_{1t} = L_1 \tau,
$$

$$
\Phi_{2t} = L_2 \tau,
$$

$$
\Phi^A_t = H(\gamma^R_1, \gamma^R_2, \gamma^R_3, \gamma^R_1, \gamma^R_2, \gamma^R_3, T, \nu_2) + G(\gamma^R_1, \gamma^R_2, \gamma^R_3, T, \nu_2),
$$

$$
\hat{\Phi}^*_t = f \tau,
$$

$$
\hat{\Phi}_t = 2H(\gamma^R_1, \gamma^R_2, \gamma^R_3, \gamma^R_1, \gamma^R_2, \gamma^R_3, T, \nu_4) + 2G(\gamma^R_1, \gamma^R_2, \gamma^R_3, T, \nu_4) + k \tau.
$$

The proposition follows from this observation and (D.8).

Lemma D.4 computes moments of momentum weights conditional on $\mathcal{I}_t = (\hat{C}_t, y_t)$.

**Lemma D.4.** For $t'' \geq t' \geq t$,

$$
E_{\mathcal{I}_t}(w^M_{t''}) = L_1 \tau \Sigma + \left( L_2 \tau + \delta^{M}_{12,t'-t}(\hat{C}_t - \bar{C}) + \delta^{M}_{3,t'-t}(y_t - \bar{y}) \right) p_f \Sigma, \quad (D.56)
$$

$$
\text{Cov}_{\mathcal{I}_t}(w^M_{t'}, \Lambda_{t''}) = \frac{1}{f} + \frac{k_\Lambda}{\eta \Sigma} C^{M\Lambda}_{t'-t''-t} p_f \Sigma, \quad (D.57)
$$

$$
\text{Cov}_{\mathcal{I}_t}(\Lambda_{t'}, w^M_{t''}) = \frac{1}{f} + \frac{k_\Lambda}{\eta \Sigma} C^{M\Lambda}_{t'-t''-t} p_f \Sigma, \quad (D.58)
$$

$$
\text{Cov}_{\mathcal{I}_t}\left( (w^M_{t'})', w^M_{t''} \right) = C^{M\Sigma}_{t'-t''-t} \Sigma + C^{M}_{t'-t''-t} \Sigma p_f p_f \Sigma, \quad (D.59)
$$

where

$$
\begin{pmatrix}
\delta^{M}_{12,t'-t} \\
\delta^{M}_{3,t'-t}
\end{pmatrix}
\equiv
\left( \Sigma^{Cy} \right)^{-1}
\begin{pmatrix}
H(1, 0, 0, \gamma^R_1, \gamma^R_2, \gamma^R_3, T, \nu_1) \\
+ H(\gamma^R_1, \gamma^R_2, \gamma^R_3, 1, 0, 0, T, \nu_2) + G(1, 0, 0, T, \nu_2) \\
H(0, 0, 1, \gamma^R_1, \gamma^R_2, \gamma^R_3, T, \nu_1) \\
+ H(\gamma^R_1, \gamma^R_2, \gamma^R_3, 0, 0, 1, T, \nu_2) + G(0, 0, 1, T, \nu_2)
\end{pmatrix}, \quad (D.60)
$$

$$
C^{M \Lambda}_{t'-t''-t} \equiv H(\gamma^R_1, \gamma^R_2, \gamma^R_3, \gamma^R_1, \gamma^R_2, \gamma^R_3, T', \nu_1) + H(\gamma^R_1, \gamma^R_2, \gamma^R_3, T', \nu_1)
$$

$$
- 2 \left( \delta^{M}_{12,t'-t}, \delta^{M}_{3,t'-t} \right) \Sigma^{Cy} \left( \begin{pmatrix}
\delta^{A}_{12,t''-t} \\
\delta^{A}_{3,t''-t}
\end{pmatrix}
\right),
$$

$$
C^{\Lambda M}_{t'-t''-t} \equiv H(\gamma^R_1, \gamma^R_2, \gamma^R_3, \gamma^R_1, \gamma^R_2, \gamma^R_3, T', \nu_1) + H(\gamma^R_1, \gamma^R_2, \gamma^R_3, T', \nu_1)
$$

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\[ + G(\gamma_1^R, \gamma_2^R, \gamma_3^R, T', \nu_2) - (\delta_{12,t'-t}, \delta_{3, t'-t}) \Sigma \begin{pmatrix} \delta_{12,t''-t}^M \\ \delta_{3, t''-t}^M \end{pmatrix}, \]

\[ C_{Mt'}^M \equiv f \max\{\tau + t' - t'', 0\}, \quad (D.61) \]

\[ C_{M,t'-t''-t}^M \equiv H(\gamma_1^R, \gamma_2^R, \gamma_3^R, T', \nu_3) + H(\gamma_1^R, \gamma_2^R, \gamma_3^R, T', \nu_4) \]

\[ + G(\gamma_1^R, \gamma_2^R, \gamma_3^R, T', \nu_3) + G(\gamma_1^R, \gamma_2^R, \gamma_3^R, T', \nu_4) \]

\[ + k \max\{\tau + t' - t'', 0\} - (\delta_{12,t'-t}, \delta_{3, t'-t}) \Sigma \begin{pmatrix} \delta_{12,t''-t}^M \\ \delta_{3, t''-t}^M \end{pmatrix}, \]

\[ \mathcal{T} \equiv (t' - t, \tau), \quad \mathcal{T}' \equiv (t'' - t', \tau) \text{ and } \mathcal{T}'' \equiv (t' - t'', \tau). \]

**Proof:** Using the joint normality of \((w_{it}', \hat{C}_t, y_t)\) and \((D.49)\), we can set

\[ w_{it}^M - (L_1\tau \eta + L_2\tau p_f \Sigma) = \Delta_{M, t'-\tau}^M(\hat{C}_t - \bar{C}) + \Delta_{M, t'-\tau}^M(y_t - \bar{y}) + \zeta_{t'}^M, \quad (D.62) \]

where the error term \(\zeta_{t'}^M\) has mean zero and is independent of \((\hat{C}_t, y_t)\).

Taking covariances of both sides of \((D.62)\) with \(\hat{C}_t\) and \(y_t\), and using \((D.48)\) and the independence of \(\zeta_{t'}^M\) from \((\hat{C}_t, y_t)\), we find

\[ \left( f + \frac{k_{\Delta}}{\eta \Sigma \eta} \right) \left[ \int_{t' - \tau}^{t'} \text{Cov}(\hat{C}_t, \Lambda_u) \, du \right] p_f \Sigma + \int_{t' - \tau}^{t'} \text{Cov}(\hat{C}_t, [dR_u - E_u(dR_u)]') \]

\[ = \Delta_{12, t'-\tau}^M \Sigma_y + \Delta_{3, t'-\tau}^M \Sigma_y, \quad (D.63) \]

\[ \left( f + \frac{k_{\Delta}}{\eta \Sigma \eta} \right) \left[ \int_{t' - \tau}^{t'} \text{Cov}(y_t, \Lambda_u) \, du \right] p_f \Sigma + \int_{t' - \tau}^{t'} \text{Cov}(y_t, [dR_u - E_u(dR_u)]') \]

\[ = \Delta_{12, t'-\tau}^M \Sigma_y + \Delta_{3, t'-\tau}^M \Sigma_y. \quad (D.64) \]

Noting that the covariances in the second term of \((D.63)\) and \((D.64)\) are zero for \(t < u\), and using \((3.7)\), \((B.10)\), \((B.13)\) and \((D.51)\), we can write \((D.63)\) and \((D.64)\) as

\[ [H(1, 0, 0, \gamma_1^R, \gamma_2^R, \gamma_3^R, T, \nu_1) + H(1, 0, 0, \gamma_2^R, \gamma_3^R, T, \nu_2) + G(1, 0, 0, T, \nu_2)] p_f \Sigma \]

\[ = \Delta_{12, t'-\tau}^M \Sigma_y + \Delta_{3, t'-\tau}^M \Sigma_y. \quad (D.65) \]
Equation (D.57) follows from (D.68) by noting that

$$\Delta s$$

and using the right-hand side of (D.62) evaluated at $$t$$ respectively. Equations (D.65) and (D.66) imply $$\Delta M$$ two scalars ($$\hat{s} M$$) and is independent of ($$\hat{s} C$$)

Writing that the covariance between the left-hand side of (D.33) evaluated at $$t$$, the right-hand side of (D.33) evaluated at $$t$$, and the left-hand side of (D.33) evaluated at $$t$$, we find (D.60). Equation (D.56) follows from (D.62) because $$\zeta^M$$ has mean zero and is independent of ($$\hat{C} t, y_t$$).

Writing that the covariance between the left-hand side of (D.62) evaluated at $$t'$$ and the left-hand side of (D.33) evaluated at $$t''$$ is equal to the covariance between the corresponding right-hand sides, and using

$$\text{Cov} (w^M_{t'}, \Lambda_{t''}) = \left( f + \frac{k\Delta}{\eta \Sigma \eta'} \right) \left( \int_{t' - \tau}^{t'} \text{Cov} (\Lambda_u, \Lambda_{t''}) \, du \right) p_f \Sigma + \int_{t' - \tau}^{t'} \text{Cov} \left( [dR_u - E_u(dR_u)]', \Lambda_{t''} \right)$$

which generalizes (D.50) from ($$\Lambda_t, w^M_t$$) to ($$\Lambda_{t''}, w^M_{t''}$$), together with (3.7), (B.10), (B.13), (D.51), (D.66)

$$\Delta M_{12,t' - t} = \delta M_{12,t' - t} p_f \Sigma$$

and the independence of ($$\zeta^M_{t'}, \zeta^M_{t''}$$) from ($$\hat{C} t, y_t$$), we find

$$\text{Cov} (w^M_{t'}, \Lambda_{t''}) = \left( \delta^M_{12,t' - t}, \delta^M_{3,t' - t} \right) \left( \Sigma_{11} \hat{C}^y \left( \int_{t' - \tau}^{t''} \text{Cov} (\Lambda_{t'}, \Lambda_u) \, du \right) p_f \Sigma + \int_{t' - \tau}^{t''} \text{Cov} (\Lambda_{t'}, [dR_u - E_u(dR_u)]') \right)$$

Equation (D.57) follows from (D.68) by noting that $$(\frac{1}{f + \frac{k\Delta}{\eta \Sigma \eta'}} \text{Cov} (\zeta^M_{t'}, \zeta^M_{t''}) = \text{Cov}_{t} (w^M_{t'}, \Lambda_{t''})$$.

Writing that the covariance between the left-hand side of (D.33) evaluated at $$t'$$ and the left-hand side of (D.62) evaluated at $$t''$$ is equal to the covariance between the corresponding right-hand sides, and using

$$\text{Cov} (\Lambda_{t'}, w^M_{t''}) = \left( f + \frac{k\Delta}{\eta \Sigma \eta'} \right) \left( \int_{t' - \tau}^{t''} \text{Cov} (\Lambda_{t'}, \Lambda_u) \, du \right) p_f \Sigma + \int_{t' - \tau}^{t''} \text{Cov} (\Lambda_{t'}, [dR_u - E_u(dR_u)]')$$
which generalizes (D.50) from \((\Lambda_t, w^M_t)\) to \((\Lambda_{t'}, w^M_{t'})\), together with (3.7), (B.10), (B.13), (D.51),

\[
\Delta^M_{12,t' - t} = \delta^M_{12,t' - t} \Sigma_{M}, \Delta^M_{3,t' - t} = \delta^M_{3,t' - t} \Sigma_{M}
\]

and the independence of \((\zeta^\Lambda_{t'}, \zeta^M_{t'})\) from \((\tilde{C}_t, y_t)\), we find

\[
[H(\gamma_1^R, \gamma_2^R, \gamma_3^R, \gamma_1^R, \gamma_2^R, \gamma_3^R, T', \nu_3) + H(\gamma_1^R, \gamma_2^R, \gamma_3^R, \gamma_1^R, \gamma_2^R, \gamma_3^R, T', \nu_3)] \Sigma_{M} p_f \Sigma
\]

\[
= (\delta^M_{12,t' - t}, \delta^M_{3,t' - t}) \left( \frac{\Sigma_{C_t} + \Sigma_{C_{t'}}}{\Sigma_{M}} \right) \left( \frac{\Sigma_{M} p_f \Sigma + \text{Cov}(\zeta^\Lambda_{t'}, \zeta^M_{t'})}{\Sigma_{M}} \right) .
\]

Equation (D.26) follows from (D.38) by noting that

\[
\frac{1}{f + \frac{k}{\eta \Sigma_{M}}} \text{Cov}(\zeta^\Lambda_{t'}, \zeta^M_{t'}) = \text{Cov}_{t'}(\Lambda_{t'}, w^M_{t'}). \]

Writing that the covariance between the left-hand side of (D.62) evaluated at \(t''\) and the transpose of the left-hand side of (D.62) evaluated at \(t'\) is equal to the covariance between the corresponding right-hand sides, and using

\[
\text{Cov} \left( (w^M_{t'})', (w^M_{t''})' \right) = \left( f + \frac{k \Delta}{\eta \Sigma_{M}} \right)^2 \left( \int_{u''=t''-\tau}^{t''} \int_{u'=t'-\tau}^{t'} \text{Cov} (\Lambda_{u'}, \Lambda_{u''}) \, du' \, du'' \right) \Sigma_{M} p_f \Sigma
\]

\[
+ \left( f + \frac{k \Delta}{\eta \Sigma_{M}} \right) \Sigma_{M} p_f \left( \int_{u''=t''-\tau}^{t''} \int_{u'=t'-\tau}^{t'} \text{Cov} (\Lambda_{u'}, [dR_{u''} - E_{u''}(dR_{u''})]) \, du' \right)
\]

\[
+ \left( f + \frac{k \Delta}{\eta \Sigma_{M}} \right) \left( \int_{u''=t''-\tau}^{t''} \int_{u'=t'-\tau}^{t'} \text{Cov} (dR_{u'} - E_{u'}(dR_{u'}), \Lambda_{u''}) \, du'' \right) p_f \Sigma
\]

\[
+ 1_{\{\tau + t' - t'' > 0\}} \int_{t''-\tau}^{t'} \text{Cov} (dR_u - E_u(dR_u), [dR_u - E_u(dR_u)])
\]

which generalizes (D.53) from \((w^M_t, w^M_t)\) to \((w^M_{t'}, w^M_{t'})\), together with (3.5), (3.7), (B.10), (B.13),

(D.51), \(\Delta^M_{12,t' - t} = \delta^M_{12,t' - t} \Sigma_{M}, \Delta^M_{3,t' - t} = \delta^M_{3,t' - t} \Sigma_{M}\) and the independence of \(\zeta^M_{t'}\) from \((\tilde{C}_t, y_t)\), we find

\[
[H(\gamma_1^R, \gamma_2^R, \gamma_3^R, \gamma_1^R, \gamma_2^R, \gamma_3^R, T', \nu_3) + H(\gamma_1^R, \gamma_2^R, \gamma_3^R, \gamma_1^R, \gamma_2^R, \gamma_3^R, T', \nu_3)] \Sigma_{M} p_f \Sigma
\]

\[
+ k \Sigma_{M} p_f \Sigma \]

\[
= (\delta^M_{12,t' - t}, \delta^M_{3,t' - t}) \left( \frac{\Sigma_{C_t} + \Sigma_{C_{t'}}}{\Sigma_{M}} \right) \left( \frac{\Sigma_{M} p_f \Sigma + \text{Cov}(\zeta^\Lambda_{t'}, \zeta^M_{t'})}{\Sigma_{M}} \right) .
\]
Equation (D.59) follows from (D.72) by noting that $\text{Cov}\left((\zeta_t^M)' , \zeta_t^M\right) = \text{Cov}_{\mathcal{I}_t}\left((w_t^M)' , w_t^M\right)$. 

Proposition D.6 computes the Sharpe ratio of the momentum strategy conditional on $(\hat{C}_t, y_t)$.

**Proposition D.6.** The Sharpe ratio of the momentum strategy (4.2) conditional on $(\hat{C}_t, y_t)$ is

$$SR_{w_t^M,t} = \frac{N_{w_t^M,t}}{\sqrt{D_{w_t^M,t}}},$$

where

$$N_{w_t^M,t} = \left[ L_2 + \delta_{12,0}^M(\hat{C}_t - \bar{C}) + \delta_{3,0}^M(y_t - \bar{y}) \right] \times \left[ L_1 \tau \eta \Sigma_f p_f' + \left( L_2 \tau + \delta_{12,0}^M(\hat{C}_t - \bar{C}) + \delta_{3,0}^M(y_t - \bar{y}) \right) p_f \Sigma_f^2 p_f' \right] + c_{0,0}^{MA} p_f \Sigma_f^2 p_f',
$$

$$D_{w_t^M,t} = L_1^2 \tau^2 \Delta_1 + 2 L_1 \tau \left( L_2 \tau + \delta_{12,0}^M(\hat{C}_t - \bar{C}) + \delta_{3,0}^M(y_t - \bar{y}) \right) \Delta_2 + \left[ \left( L_2 \tau + \delta_{12,0}^M(\hat{C}_t - \bar{C}) + \delta_{3,0}^M(y_t - \bar{y}) \right)^2 + c_{0,0}^M \right] \Delta_3 + c_{0,0}^{MS} \Delta_4.$$

**Proof:** Lemma D.4 implies that the Sharpe ratio of the momentum strategy conditional on $(\hat{C}_t, y_t)$ can be deduced from Lemma D.2 by setting $\mathcal{I}_t = \{\hat{C}_t, y_t\}$,

$$\Phi_{1t} = L_1 \tau,$n $$\Phi_{2t} = L_2 \tau + \delta_{12,0}^M(\hat{C}_t - \bar{C}) + \delta_{3,0}^M(y_t - \bar{y}),$$

$$\Phi^A_t = c_{0,0}^{MA},$$

$$\Phi^\Sigma_t = c_{0,0}^{MS},$$

$$\hat{\Phi}_t = c_{0,0}^M.$$

The proposition follows from this observation and (D.25).

Lemma D.5 computes the Sharpe ratio of the optimal (mean-variance maximizing) combination of two strategies $(w_t^A, w_t^B)$.

**Lemma D.5.** The maximum Sharpe ratio of a combination of $(w_t^A, w_t^B)$ is given by (5.6).

**Proof:** Consider an investor at time $t$ with infinitesimal horizon $dt$, who can invest in the riskless asset, the index $\eta$ and the strategies $(w_t^A, w_t^B)$. The investor’s optimization problem is as in Lemma
The term inside the square root in (D.76) has numerator

\[ dW_t = rW_t dt + \hat{x}_t \eta dR_t + \hat{y}_t^A \hat{w}_t^A dR_t^A + \hat{y}_t^M \hat{w}_t^B dR_t^B. \]  

Substituting \( dW_t \) from (D.73), and noting that \( \eta dR_t \) is uncorrelated with \( (\hat{w}_t^A dR_t, \hat{w}_t^B dR_t) \), we can write the investor’s objective (C.1) as

\[
\hat{x}_t E_{\mathcal{I}_t} (\eta dR_t) + \hat{y}_t^A E_{\mathcal{I}_t} (\hat{w}_t^A dR_t) + \hat{y}_t^M E_{\mathcal{I}_t} (\hat{w}_t^B dR_t) - \frac{a}{2} \left( \hat{x}_t^2 \text{Var}_{\mathcal{I}_t} (\eta dR_t) 
+ (\hat{y}_t^A)^2 \text{Var}_{\mathcal{I}_t} (\hat{w}_t^A dR_t) + (\hat{y}_t^M)^2 \text{Var}_{\mathcal{I}_t} (\hat{w}_t^B dR_t) + 2\hat{y}_t^A \hat{y}_t^M \text{Cov}_{\mathcal{I}_t} (\hat{w}_t^A dR_t, \hat{w}_t^B dR_t) \right). \tag{D.74}
\]

Maximizing (D.74) over \( (\hat{x}_t, \hat{y}_t^A, \hat{y}_t^M) \) yields the utility

\[
\frac{1}{2a} \left( SR_{\eta dR_t}^2 dt + (E_{\mathcal{I}_t}^{AB})' (\text{Cov}_{\mathcal{I}_t}^{AB})^{-1} E_{\mathcal{I}_t}^{AB} \right), \tag{D.75}
\]

where \( E_{\mathcal{I}_t}^{AB} \equiv (E_{\mathcal{I}_t} (\hat{w}_t^A dR_t), E_{\mathcal{I}_t} (\hat{w}_t^B dR_t))' \) and

\[
\text{Cov}_{\mathcal{I}_t}^{AB} \equiv \begin{pmatrix}
\text{Var}_{\mathcal{I}_t} (\hat{w}_t^A dR_t) & \text{Cov}_{\mathcal{I}_t} (\hat{w}_t^A dR_t, \hat{w}_t^B dR_t) \\
\text{Cov}_{\mathcal{I}_t} (\hat{w}_t^A dR_t, \hat{w}_t^B dR_t) & \text{Var}_{\mathcal{I}_t} (\hat{w}_t^B dR_t)
\end{pmatrix}.
\]

Comparison of (C.5) and (D.75) yields

\[
SR_{\omega^{AB}, t} = \sqrt{\frac{1}{dt} (E_{\mathcal{I}_t}^{AB})' (\text{Cov}_{\mathcal{I}_t}^{AB})^{-1} E_{\mathcal{I}_t}^{AB}}. \tag{D.76}
\]

The term inside the square root in (D.76) has numerator

\[
(E_{\mathcal{I}_t}(\hat{w}_t^A dR_t))^2 \text{Var}_{\mathcal{I}_t} (\hat{w}_t^B dR_t) + (E_{\mathcal{I}_t}(\hat{w}_t^B dR_t))^2 \text{Var}_{\mathcal{I}_t} (\hat{w}_t^A dR_t) \\
- 2E_{\mathcal{I}_t}(\hat{w}_t^A dR_t)E_{\mathcal{I}_t}(\hat{w}_t^B dR_t)\text{Cov}_{\mathcal{I}_t} (\hat{w}_t^A dR_t, \hat{w}_t^B dR_t)
\]

and denominator

\[
\left[ \text{Var}_{\mathcal{I}_t} (\hat{w}_t^A dR_t) \text{Var}_{\mathcal{I}_t} (\hat{w}_t^B dR_t) - \text{Cov}_{\mathcal{I}_t} (\hat{w}_t^A dR_t, \hat{w}_t^B dR_t) \right]^2 dt.
\]

Dividing numerator and denominator by \( \text{Var}_{\mathcal{I}_t} (\hat{w}_t^A dR_t) \text{Var}_{\mathcal{I}_t} (\hat{w}_t^B dR_t) dt \), we can write the term
inside the square root in (D.76) as the term inside the square root in (5.6).

Lemma D.5 computes the covariance between the returns of (the index-adjusted versions of) two strategies \( (w^A_t, w^B_t) \).

**Lemma D.6.** The covariance between the returns of \( (w^A_t, w^B_t) \) conditional on \( I_t \) is given by

\[
G_{w^A, w^B, t} \equiv \frac{1}{dt} \text{Cov}_{I_t} (\dot{w}^A_t dR_t, \dot{w}^B_t dR_t) \]

\[= f \left[ \text{E}_{I_t} \left( w^A_t \Sigma (w^B_t) \right) \right] - \frac{\text{E}_{I_t} \left( w^A_t \Sigma \eta w^B_t \Sigma \eta' \right)}{\eta \Sigma \eta'} + k \text{E}_{I_t} \left( w^A_t \Sigma p f \Sigma p f \right). \tag{D.77} \]

Suppose that for \( i = A, B \)

\[
\text{E}_{I_t} (w^i_t) = \Phi^i_{1t} \eta \Sigma + \Phi^i_{2t} p f \Sigma \tag{D.78} \]

\[
\text{Cov}_{I_t} (w^A'_t, w^B'_t) = \Phi^A_{1B} \Sigma + \Phi^A_{2B} p f \Sigma. \tag{D.79} \]

Then, the covariance between the returns of \( (w^A_t, w^B_t) \) conditional on \( I_t \) is given by

\[
G_{w^A, w^B, t} = \Phi^A_{11} \Phi^B_{11} \Delta_1 + \left( \Phi^A_{12} \Phi^B_{21} + \Phi^A_{21} \Phi^B_{12} \right) \Delta_2 + \left( \Phi^A_{22} \Phi^B_{22} + \Phi^A_{2B} \right) \Delta_3 + \Phi^{AB} \Sigma \Delta_4. \tag{D.80} \]

**Proof:** The covariance between the returns of \( (w^A_t, w^B_t) \) conditional on \( I_t \) is

\[
\text{Cov}_{I_t} (\dot{w}^A_t dR_t, \dot{w}^B_t dR_t) = \text{E}_{I_t} (\dot{w}^A_t dR_t \dot{w}^B_t dR_t) - \text{E}_{I_t} (\dot{w}^A_t dR_t) \text{E}_{I_t} (\dot{w}^A_t dR_t) 
\]

\[= \text{E}_{I_t} (\dot{w}^A_t dR_t \dot{w}^B_t dR_t) \]

\[= \text{E}_{I_t} \text{E}_t (\dot{w}^A_t dR_t \dot{w}^B_t dR_t) \]

\[= \text{E}_{I_t} \left( \dot{w}^A_t \text{E}_t (dR_t dR_t') \left( \dot{w}^B_t \right) \right) \]

\[= \text{E}_{I_t} \left( \dot{w}^A_t \text{Cov}_t (dR_t dR_t') + E_t (dR_t) E_t (dR_t') \right) (\dot{w}^B_t) \]

\[= \text{E}_{I_t} \left( \dot{w}^A_t \text{Cov}_t (dR_t dR_t') \left( \dot{w}^B_t \right) \right) \]

\[= \text{E}_{I_t} \left( \dot{w}^A_t (f \Sigma + k \Sigma p f \Sigma) (\dot{w}^B_t) \right) dt, \tag{D.81} \]

where the second and sixth steps follow because the term that is dropped is of order \( (dt)^2 \) while the term that is kept is of order \( dt \), and the last step follows from (3.5).
Using (4.3), we can write (D.81) divided by \( dt \) as

\[
G_{w^A,w^B,t} = \mathbb{E}_{\mathcal{I}_t} \left[ \left( w^A_t - \frac{\text{Cov}_t(w^A_t dR_t, \eta dR_t)}{\text{Var}_t(\eta dR_t)} \eta \right) \left( f \Sigma + k \Sigma p^f_j p^f \Sigma \right) \left( w^B_t - \frac{\text{Cov}_t(w^B_t dR_t, \eta dR_t)}{\text{Var}_t(\eta dR_t)} \eta \right) \right]
\]

\[
\mathbb{E}_{\mathcal{I}_t} \left[ \left( w^A_t - \frac{\eta \Sigma \eta'^f}{\eta \Sigma \eta'} \right) \left( f \Sigma + k \Sigma p^f_j p^f \Sigma \right) \left( \frac{w^B_t - \eta \Sigma \eta'^f}{\eta \Sigma \eta'} \right) \right],
\]

(D.82)

where the second step follows from (3.5) and \( \eta \Sigma p'_f = 0 \). Expanding the products in (D.82) and using \( \eta \Sigma p'_f = 0 \) yields (D.77).

We can write the right-hand side of (D.77) as

\[
f \left[ \mathbb{E}_{\mathcal{I}_t} \left( w^A_t \right) \mathbb{E}_{\mathcal{I}_t} \left( (w^B_t)' \right) \right] - \frac{\mathbb{E}_{\mathcal{I}_t} \left( w^A_t \right) \mathbb{E}_{\mathcal{I}_t} \left( (w^B_t)' \right) \eta' \eta}{\eta' \eta} + k \mathbb{E}_{\mathcal{I}_t} \left( w^A_t \right) \mathbb{E}_{\mathcal{I}_t} \left( w^B_t \right) \mathbb{E}_{\mathcal{I}_t} \left( \eta'^f \right) + \frac{\mathbb{E}_{\mathcal{I}_t} \left( w^A_t \right) \mathbb{E}_{\mathcal{I}_t} \left( \eta' \right)}{\eta' \eta} + k \mathbb{E}_{\mathcal{I}_t} \left( w^A_t \right) \mathbb{E}_{\mathcal{I}_t} \left( \eta'^f \right) \left( \eta'^f \right). \quad (D.83)
\]

Using (D.78), (D.79) and (D.83), and proceeding as in the derivation of (D.18), we can derive (D.80).

Lemma D.7 computes covariances between the weights of value and momentum strategies conditional on \( \mathcal{I}_t = (\hat{C}_t, \eta_t) \).

**Lemma D.7.** For \( t'' \geq t' \geq t \),

\[
\text{Cov}_{\mathcal{I}_t} \left( (w^V_{t''})', w^M_{t''} \right) = C^V_{t' \rightarrow t''} \Sigma + C^V_{t' \rightarrow t''} \Sigma p^f_j p^f \Sigma,
\]

(D.84)

\[
\text{Cov}_{\mathcal{I}_t} \left( (w^M_{t''})', w^V_{t''} \right) = C^V_{t' \rightarrow t''} \Sigma + C^V_{t' \rightarrow t''} \Sigma p^f_j p^f \Sigma,
\]

(D.85)

where

\[
C^V_{t' \rightarrow t''} \equiv - \frac{1 - \epsilon}{r + \kappa} \left[ K_2(\gamma_1, R, \gamma_3, \tau', \nu_1) + K_1(\gamma_1, R, \gamma_3, \tau', \nu_2) \right] - \frac{(1 - \epsilon) \phi^2}{(r + \kappa)^2} \beta_2 \gamma_1 \nu_2(\kappa, \tau') + H(\gamma_1, R, \gamma_2, R, \gamma_3, \tau', \nu_1) + H(\gamma_1, R, \gamma_2, R, \gamma_3, \tau', \nu_2) + G(\gamma_1, \gamma_2, \gamma_3, \tau', \nu_2) - (\delta^V_{12,t''-t}, \delta^V_{3,t''-t}) \Sigma \delta^V_g \begin{pmatrix} \tilde{\delta}^M_{12,t''-t} \\ \tilde{\delta}^M_{3,t''-t} \end{pmatrix},
\]

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\[
C^V_{t''-t'} \equiv -\frac{(1-\epsilon)\phi^2}{(r+\kappa)^2} \nu_2(\kappa, T'),
\]
\[
C^M_{t''-t'} \equiv \frac{1-\epsilon}{r+\kappa}K_1(\gamma_1^R, \gamma_3^R, T''), \nu_2) - \frac{(1-\epsilon)\phi^2}{(r+\kappa)^2} \beta_2 \gamma_1 \nu_2(\kappa, T')
\]
\[
+ H(\gamma_1^R, \gamma_2^R, \gamma_3^R, \gamma_1, \gamma_2, \gamma_3, T''), \nu_2) + G(\gamma_1, \gamma_2, \gamma_3, T''), \nu_2)
\]
\[
- (\delta_{12,t''-t}, \delta_{3, t''-t}) \Sigma_y \left( \begin{array}{c}
\delta_{12,t''-t}^V \\
\delta_{3, t''-t}^V
\end{array} \right),
\]
\[
C^3_{t''-t'} \equiv -\frac{(1-\epsilon)\phi^2}{(r+\kappa)^2} \nu_2(\kappa, T'),
\]

\[T' \equiv (t'' - t', \tau), \quad T'' \equiv (t' - t'', \tau), \quad \text{and} \ (\delta_{12,u}, \delta_{3,u}) \quad \text{and} \ (\delta_{12,u}, \delta_{3,u}) \quad \text{are defined in Lemmas D.3 and D.4, respectively.}
\]

**Proof:** Writing that the covariance between the left-hand side of (D.62) evaluated at \(t''\) and the transpose of the left-hand side of (D.32) evaluated at \(t'\) is equal to the covariance between the corresponding right-hand sides, and using (D.19), (D.48), \(\Delta^V_{12,t''-t} = \delta^V_{12,t''-tp_f}\Sigma, \Delta^V_{3,t''-t} = \delta^V_{3,t''-tp_f}\Sigma, \Delta^M_{12,t''-t} = \delta^M_{12,t''-tp_f}\Sigma, \Delta^M_{3,t''-t} = \delta^M_{3,t''-tp_f}\Sigma\) and the independence of \((\zeta^V_t, \zeta^M_t)\) from \((\hat{C}_t, y_t)\), we find

\[
\left( f + \frac{k\Delta}{\eta\Sigma f} \right) \left( -\frac{1-\epsilon}{r+\kappa} \int_{u''=t''-\tau}^{t''} \text{Cov}(\Lambda_{u''}, F_{t'}) \text{d}u''p_f\Sigma \right)
\]
\[
+ \int_{u''=t''-\tau}^{t''} \text{Cov} \left( \Lambda_{u''}, \gamma_1 \hat{C}_{t'} + \gamma_2 C_{t'} + \gamma_3 y_{t'} \right) \text{d}u'' \Sigma p_f p_f \Sigma
\]
\[
+ \left( -\frac{1-\epsilon}{r+\kappa} \int_{u''=t''-\tau}^{t''} \text{Cov} \left( F_{t'}, [dR_{u''} - E_{u''}(dR_{u''})]'' \right) \text{d}u' \right)
\]
\[
+ \Sigma p_f \int_{u''=t''-\tau}^{t''} \text{Cov} \left( \gamma_1 \hat{C}_{t'} + \gamma_2 C_{t'} + \gamma_3 y_{t'}, [dR_{u''} - E_{u''}(dR_{u''})]'' \right) \text{d}u''
\]
\[
= (\delta_{12,t''-t}, \delta_{3, t''-t}) \left( \begin{array}{c}
\Sigma_{11} \Sigma_{12} \\
\Sigma_{12} \Sigma_{21}
\end{array} \right) \left( \begin{array}{c}
\delta_{12,t''-t}^V \\
\delta_{3, t''-t}^V
\end{array} \right) \Sigma p_f p_f \Sigma + \left( \left( \zeta^V_{t'} \right) ', \zeta^M_{t'} \right) \right),
\]
Using (3.7), (B.10), (B.11), (B.13), (B.14), and (D.51), we can write (D.87) as

\[
- \frac{(1 - \epsilon) \phi^2}{(r + \kappa)^2} \nu_2(\kappa, T') \Sigma + \left[ - \frac{1 - \epsilon}{r + \kappa} [K_2(\gamma_1^R, \gamma_3^R, T', \nu_1) + K_1(\gamma_1^R, \gamma_3^R, T'^-, \nu_2)] + H(\gamma_1, \gamma_2, \gamma_3, \gamma_1^R, \gamma_2^R, \gamma_3^R, T', \nu_1) + H(\gamma_1, \gamma_2, \gamma_3, \gamma_1^R, \gamma_2^R, \gamma_3^R, T'^-, \nu_2) \right. \\
+ \frac{(1 - \epsilon) \phi^2}{(r + \kappa)^2} \beta_2 \gamma_1 \nu_2(\kappa, T') + G(\gamma_1, \gamma_2, \gamma_3, T', \nu_2) \left. \right] \Sigma p_f^p p_f \Sigma \\
= (\delta_{12,t'-t}^V, \delta_{3,t'-t}^M) \left( \begin{array}{c} \Sigma y_{11} \\ \Sigma y_{21} \end{array} \right) \left( \begin{array}{c} \delta_{12,t'-t}^M \\ \delta_{3,t'-t}^M \end{array} \right) \Sigma p_f^p p_f \Sigma + \text{Cov} \left( (\zeta_{t'}^V, \zeta_{t'}^M) \right). \tag{D.88}
\]

Equation (D.84) follows from (D.88) by noting that \( \text{Cov} \left( (\zeta_{t'}^V, \zeta_{t'}^M) \right) = \text{Cov}_{\Sigma_t} \left( (w_{t'}^V), (w_{t'}^M) \right) \).

Writing that the covariance between the left-hand side of (D.32) evaluated at \( t'' \) and the transpose of the left-hand side of (D.62) evaluated at \( t' \) is equal to the covariance between the corresponding right-hand sides, we likewise find

\[
\left( f + \frac{k \Delta}{\eta \Sigma \eta'} \right) \left( - \frac{1 - \epsilon}{r + \kappa} \Sigma p_f^p \int_{t' - \tau}^{t'} \text{Cov} \left( \Lambda_w, F_{t''} \right) du' \right) \tag{D.89}
\]

\[
+ \int_{t' - \tau}^{t'} \text{Cov} \left( \Lambda_w, \gamma_1 \hat{C}_{t''} + \gamma_2 C_{t''} + \gamma_3 y_{t''} \right) du' \Sigma p_f^p p_f \Sigma \\
+ \left( - \frac{1 - \epsilon}{r + \kappa} \int_{t' - \tau}^{t'} \text{Cov} \left( dR_{w'} - E_{w'}(dR_{w'}), F_{t''} \right) du' \right) \\
+ \int_{t' - \tau}^{t'} \text{Cov} \left( dR_{w'} - E_{w'}(dR_{w'}), \gamma_1 \hat{C}_{t''} + \gamma_2 C_{t''} + \gamma_3 y_{t''} \right) du' p_f \Sigma \\
= (\delta_{12,t'-t}^M, \delta_{3,t'-t}^M) \left( \begin{array}{c} \Sigma y_{11} \\ \Sigma y_{21} \end{array} \right) \left( \begin{array}{c} \delta_{12,t''-t}^M \\ \delta_{3,t''-t}^M \end{array} \right) \Sigma p_f^p p_f \Sigma + \text{Cov} \left( (\zeta_{t'}^V, \zeta_{t'}^M) \right). \tag{D.90}
\]

Using (3.7), (B.10), (B.11), (B.13), (B.14), and (D.51), we can write (D.90) as

\[
- \frac{(1 - \epsilon) \phi^2}{(r + \kappa)^2} \nu_2(\kappa, T'^-) \Sigma + \left[ - \frac{1 - \epsilon}{r + \kappa} K_1(\gamma_1^R, \gamma_3^R, T'^-, \nu_2) + H(\gamma_1, \gamma_2, \gamma_3, T'^-, \nu_2) \right. \\
+ \frac{(1 - \epsilon) \phi^2}{(r + \kappa)^2} \beta_2 \gamma_1 \nu_2(\kappa, T'^-) + G(\gamma_1, \gamma_2, \gamma_3, T'^-, \nu_2) \left. \right] \Sigma p_f^p p_f \Sigma
\]


\begin{equation}
= (\delta_{12,t''-t}^M, \delta_{3, t''-t}^M) \left( \begin{array}{c}
\sum_{C_{11}} \hat{C}_t \delta_{21}^V \\
\sum_{C_{21}} \hat{C}_t \delta_{32}^V
\end{array} \right) \Sigma p^p p \Sigma + \text{Cov} \left( (\zeta_t^M)' , \zeta_t^V \right). \tag{D.91}
\end{equation}

Equation (D.85) follows from (D.91) by noting that \( \text{Cov} \left( (\zeta_t^M)' , \zeta_t^V \right) = \text{Cov}_{\mathcal{I}_t} \left( (w_t^M)' , w_t^V \right) \). \qed

Proposition D.7 computes the correlation between the returns of value and momentum strategies, both unconditionally and conditionally on \((\hat{C}_t, y_t)\).

**Proposition D.7.** The unconditional correlation between the returns of the value strategy (4.1) and the momentum strategy (4.2) is \( \text{Corr}(\hat{w}_t^V dR_t, \hat{w}_t^M dR_t) = \frac{G_{w^V, w^M}}{\sqrt{D_{w^V} D_{w^M}}} \), where

\[
G_{w^V, w^M} = \frac{L_2}{r} \tau \Delta_1 + 2 \frac{L_1 L_2}{r} \tau \Delta_2 + \left( \frac{L_2}{r} - \frac{1 - \epsilon}{r + \kappa} K_1(\gamma_1^R, \gamma_2^R, \tau, \nu_2) - \frac{1 - \epsilon}{(r + \kappa)^2} \beta_2 \gamma_1 \nu_2(\kappa, \tau) \right) \Delta_3 - \frac{(1 - \epsilon) \phi^2}{(r + \kappa)^2} \nu_2(\kappa, \tau) \Delta_4,
\]

\( \mathcal{T} = (0, \tau) \), and \( D_{w^V} \) and \( D_{w^M} \) are defined in Propositions D.2 and D.5, respectively. The correlation between the returns of the value and momentum strategies conditional on \( \mathcal{I}_t = (\hat{C}_t, y_t) \) is \( \text{Corr}_{\mathcal{I}_t}(\hat{w}_t^V dR_t, \hat{w}_t^M dR_t) = \frac{G_{w^V, w^M,t}}{\sqrt{D_{w^V,t} D_{w^M,t}}} \), where

\[
G_{w^V, w^M,t} = \frac{L_2}{r} \tau \Delta_1 + \frac{L_1}{r} \left( L_2 \tau + \delta_{12,0}(\hat{C}_t - \bar{C}) + \delta_{3,0}(y_t - \bar{y}) \right) \Delta_2 + L_1 \tau \left( \frac{L_2}{r} + \delta_{12,0}(\hat{C}_t - \bar{C}) + \delta_{3,0}(y_t - \bar{y}) \right) \Delta_3 + \left( \frac{L_2}{r} + \delta_{12,0}(\hat{C}_t - \bar{C}) + \delta_{3,0}(y_t - \bar{y}) \right) \left( L_2 \tau + \delta_{12,0}(\hat{C}_t - \bar{C}) + \delta_{3,0}(y_t - \bar{y}) \right) \Delta_3 + \frac{C_{0,0}^{MV}}{C_{0,0}^{MV}} \Delta_4,
\]

and \( D_{w^V,t} \) and \( D_{w^M,t} \) are defined in Propositions D.3 and D.6, respectively.

**Proof:** To show the equation for the unconditional correlation, we need to show that the unconditional covariance between the returns of the value and momentum strategies is \( G_{w^V, w^M} dt \). Since the unconditional expectation of value weights is given by (D.20) and of momentum weights is given by (D.49), the result follows from Lemma D.6 provided that

\[
\text{Cov} \left( (w_t^M)' , w_t^V \right) = -\frac{(1 - \epsilon) \phi^2}{(r + \kappa)^2} \nu_2(\kappa, \tau) \Sigma + \left( \frac{1 - \epsilon}{r + \kappa} K_1(\gamma_1^R, \gamma_2^R, \tau, \nu_2) - \frac{(1 - \epsilon) \phi^2}{(r + \kappa)^2} \beta_2 \gamma_1 \nu_2(\kappa, \tau) \right) \Delta_3 - \frac{(1 - \epsilon) \phi^2}{(r + \kappa)^2} \nu_2(\kappa, \tau) \Delta_4.
\]

(D.92)
Equation (D.92) follows by noting that \( \text{Cov}\left((w_t^M)' , w_t^V\right) \) is equal to the left-hand side of (D.91) for \( t'' = t' = t \).

To show the equation for the correlation conditional on \( \mathcal{I}_t = (\hat{C}_t, y_t) \), we need to show that the conditional covariance between the returns of the value and momentum strategies is \( \mathcal{G}_{w^V, w^M, t} dt \).

Since the conditional expectation of value weights is given by (D.24) and of momentum weights is given by (D.56), the result follows from Lemma D.6 provided that

\[
\text{Cov}_{\mathcal{I}_t}\left((w_t^M)' , w_t^V\right) = C_{0}^{MV} \Sigma + C_{0,0}^{MV} \rho_{f} \rho_{f} \Sigma.
\]

Equation (D.93) follows from Lemma D.7, by setting \( t'' = t' = t \) in (D.85).

The unconditional expectations and standard deviations of functions of \( (\hat{C}_t, y_t) \) are calculated using the unconditional distribution of \( (\hat{C}_t, y_t) \), which is normal with mean \( (\bar{C}, \bar{y}) \) and covariance matrix \( \Sigma_{\hat{C}y} \).

## E Proofs of Results in Section 6

Lemma E.1 expresses the Sharpe ratio over investment horizon \( T \) of a general strategy \( w_t \) in terms of expectations, variances, and autocovariances of instantaneous returns.

**Lemma E.1.** The Sharpe ratio of a strategy \( w_t \) over investment horizon \( T \) is

\[
SR_{w,t,T} = \frac{N_{w,t,T}}{\sqrt{D_{w,t,T} + D_{w,t,T}^{\text{Cov1}} + D_{w,t,T}^{\text{Cov2}}}},
\]

where

\[
N_{w,t,T} = \frac{1}{T} \int_{t}^{t+T} \mathcal{E}_t(\hat{w}_u dR_u), \tag{E.1}
\]

\[
D_{w,t,T} = \frac{1}{T} \int_{t}^{t+T} \mathcal{V}_t(\hat{w}_u dR_u), \tag{E.2}
\]

\[
D_{w,t,T}^{\text{Cov1}} = \frac{2}{T} \int_{u=t}^{t+T} \int_{u'=u}^{t+T} \mathcal{Cov}_t[\hat{w}_u \mathcal{E}_u(dR_u), \hat{w}_u' \mathcal{E}_{u'}(dR_{u'})], \tag{E.3}
\]

\[
D_{w,t,T}^{\text{Cov2}} = \frac{2}{T} \int_{u=t}^{t+T} \int_{u'=u}^{t+T} \mathcal{E}_t[\hat{w}_u \mathcal{Cov}_u(dR_u), \hat{w}_u' \mathcal{E}_{u'}(dR_{u'})]. \tag{E.4}
\]
Proof: The Lemma will follow from the definition (4.5) of the Sharpe ratio provided that

\[
\frac{1}{T} \text{Var}_{\mathcal{I}_t} \left( \int_t^{t+T} \hat{w}_u dR_u \right) = D_{w,t,T} + D_{w,t,T}^{\text{cov}_1} + D_{w,t,T}^{\text{cov}_2}.
\] (E.5)

We can write the left-hand side of (E.5) as

\[
\frac{1}{T} \text{Cov}_{\mathcal{I}_t} \left( \int_t^{t+T} \hat{w}_u dR_u, \int_t^{t+T} \hat{w}_u dR_u \right)
\]

\[
= \frac{1}{T} \int_t^{t+T} \text{Cov}_{\mathcal{I}_t} (\hat{w}_u dR_u, \hat{w}_u dR_u) + \frac{1}{T} \int_{u=t}^{t+T} \int_{u'=t}^{t+T} \text{Cov}_{\mathcal{I}_t} (\hat{w}_u dR_u, \hat{w}_{u'} dR_{u'}),
\] (E.6)

where the second step follows by separating the covariance between contemporaneous returns and the covariance between lagged returns. We can write the second term in (E.6) as

\[
\frac{2}{T} \int_{u=t}^{t+T} \int_{u'=u}^{t+T} \{ \text{Cov}_{\mathcal{I}_t} [\hat{w}_u dR_u, \mathbb{E}_{u'}(\hat{w}_{u'} dR_{u'})] + \text{Cov}_{\mathcal{I}_t} [\hat{w}_u dR_u, \hat{w}_{u'} dR_{u'} - \mathbb{E}_{u'}(\hat{w}_{u'} dR_{u'})] \}
\]

\[
= \frac{2}{T} \int_{u=t}^{t+T} \int_{u'=u}^{t+T} \text{Cov}_{\mathcal{I}_t} [\hat{w}_u dR_u, \mathbb{E}_{u'}(\hat{w}_{u'} dR_{u'})]
\]

\[
= \frac{2}{T} \int_{u=t}^{t+T} \int_{u'=u}^{t+T} \{ \text{Cov}_{\mathcal{I}_t} [\mathbb{E}_u(\hat{w}_u dR_u), \mathbb{E}_{u'}(\hat{w}_{u'} dR_{u'})] + \text{Cov}_{\mathcal{I}_t} [\hat{w}_u dR_u - \mathbb{E}_u(\hat{w}_u dR_u), \mathbb{E}_{u'}(\hat{w}_{u'} dR_{u'})] \}
\]

\[
= \frac{2}{T} \int_{u=t}^{t+T} \int_{u'=u}^{t+T} \{ \text{Cov}_{\mathcal{I}_t} [\mathbb{E}_u(\hat{w}_u dR_u), \mathbb{E}_{u'}(\hat{w}_{u'} dR_{u'})] + \mathbb{E}_{\mathcal{I}_t} [\text{Cov}_u(\hat{w}_u dR_u, \mathbb{E}_{u'}(\hat{w}_{u'} dR_{u'}))] \}
\]

\[
= \frac{2}{T} \int_{u=t}^{t+T} \int_{u'=u}^{t+T} \{ \text{Cov}_{\mathcal{I}_t} [\hat{w}_u \mathbb{E}_u(dR_u), \hat{w}_{u'} \mathbb{E}_{u'}(dR_{u'})] + \mathbb{E}_{\mathcal{I}_t} [\hat{w}_u \text{Cov}_u(dR_u, \mathbb{E}_{u'}(dR_{u'}))] \}
\]

\[
= D_{w,t,T}^{\text{cov}_1} + D_{w,t,T}^{\text{cov}_2},
\] (E.7)

where the second step follows from writing \( \text{Cov}_{\mathcal{I}_t} [\hat{w}_u dR_u, \hat{w}_{u'} dR_{u'} - \mathbb{E}_{u'}(\hat{w}_{u'} dR_{u'})] \) as

\[
\mathbb{E}_{\mathcal{I}_t} [\hat{w}_u dR_u(\hat{w}_{u'} dR_{u'} - \mathbb{E}_{u'}(\hat{w}_{u'} dR_{u'}))] - \mathbb{E}_{\mathcal{I}_t}(\hat{w}_u dR_u)\mathbb{E}_{\mathcal{I}_t} [\hat{w}_{u'} dR_{u'} - \mathbb{E}_{u'}(\hat{w}_{u'} dR_{u'})]
\]

and noting that each term is zero because of the Law of Iterative Expectations, and the fourth step follows from (D.51). Combining (E.6) and (E.7), we find (E.5).

Lemma E.2 specializes Lemma E.1 to the unconditional Sharpe ratio (\( \mathcal{I}_t = \emptyset \)).
Lemma E.2. The unconditional Sharpe ratio of a strategy $w_t$ over investment horizon $T$ is

$$SR_{w,T} = \frac{N_w}{\sqrt{D_w + D_{w,T} + D_{w,T}^{\text{Cov}}}},$$

where

$$N_w \equiv \frac{1}{dt} \mathbb{E}(\hat{w}_t dR_t) = \left( f + \frac{k \Delta}{\eta \Sigma \eta'} \right) \mathbb{E} \left( \Delta_t w_t \Sigma p'_j \right),$$

$$D_w \equiv \frac{1}{dt} \mathbb{Var}(\hat{w}_t dR_t) = f \left[ \mathbb{E}(w_t \Sigma w'_t) - \frac{\mathbb{E}[(w_t \Sigma \eta')^2]}{\eta \Sigma \eta'} \right] + k \mathbb{E}[(w_t \Sigma p'_j)^2],$$

are the unconditional versions of $(N_{w,t}, D_{w,t})$ defined in Lemma D.1, and

$$D_{w,T}^{\text{Cov}} = \frac{2}{dt} \int_t^{t+T} \left( 1 - \frac{u - t}{T} \right) \text{Cov} \left[ \hat{w}_t \mathbb{E}_t(dR_t), \hat{w}_u \mathbb{E}_u(dR_u) \right],$$

(E.8)

$$D_{w,T}^{\text{Cov}} = \frac{2}{dt} \int_t^{t+T} \left( 1 - \frac{u - t}{T} \right) \mathbb{E} \left\{ \hat{w}_t \text{Cov}_t \left[ dR_t, \hat{w}_u \mathbb{E}_u(dR_u) \right] \right\}. \tag{E.9}$$

Proof: The lemma follows from Lemma E.1 by noting that when $\mathcal{I}_t = 0$:

$$N_{w,t,T} = \frac{1}{T} \int_t^{t+T} \mathbb{E}(\hat{w}_u dR_u) = \frac{1}{T} \int_t^{t+T} \frac{\mathbb{E}(\hat{w}_u dR_u)}{du} du = \frac{1}{T} \int_t^{t+T} \frac{\mathbb{E}(\hat{w}_t dR_t)}{dt} dt = N_w,$$

where the third step follows because the expectation is unconditional;

$$D_{w,t,T} = \frac{1}{T} \int_t^{t+T} \mathbb{Var}(\hat{w}_u dR_u) = \frac{1}{T} \int_t^{t+T} \frac{\mathbb{Var}(\hat{w}_u dR_u)}{du} du = \frac{1}{T} \int_t^{t+T} \frac{\mathbb{Var}(\hat{w}_t dR_t)}{dt} dt = D_w,$$

where the third step follows because the variance is unconditional;

$$D_{w,T}^{\text{Cov}} = \frac{2}{T} \int_{u=t}^{t+T} \int_{s=0}^{t+T-u} \text{Cov} \left[ \hat{w}_u \mathbb{E}_u(dR_u), \hat{w}_{u+s} \mathbb{E}_{u+s}(dR_{u+s}) \right]$$

$$= \frac{2}{T} \int_s^T \int_{u=t}^{t+T-s} \text{Cov} \left[ \hat{w}_u \mathbb{E}_u(dR_u), \hat{w}_{u+s} \mathbb{E}_{u+s}(dR_{u+s}) \right] \frac{du}{du}$$

$$= \frac{2}{T} \int_s^T \int_{t-s}^{T-s} \text{Cov} \left[ \hat{w}_t \mathbb{E}_t(dR_t), \hat{w}_{t+s} \mathbb{E}_{t+s}(dR_{t+s}) \right] \frac{du}{dt}$$

$$= \frac{2}{T} \int_s^T (T - s) \text{Cov} \left[ \hat{w}_t \mathbb{E}_t(dR_t), \hat{w}_{t+s} \mathbb{E}_{t+s}(dR_{t+s}) \right] \frac{du}{dt}$$

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\[
\frac{1}{T} \int_{u=t}^{t+T} \left( T - (u - t) \right) \frac{\text{Cov} \left[ \hat{w}_t \text{E}_t (dR_t), \hat{w}_u \text{E}_u (dR_u) \right]}{dt} = D_{w,T}^{\text{Cov}_1},
\]

where the first and sixth steps follow from the change of variable \( s = u - t \), the second step follows by changing the order of the integrals, and the fourth step follows because the covariance is unconditional and depends only on \( s \); and

\[
D_{w,t,T}^{\text{Cov}_2} = \frac{2}{T} \int_{u=t}^{t+T} \int_{s=0}^{t+T-s} \mathbb{E} \left\{ \hat{w}_t \text{Cov}_t [dR_t, \hat{w}_{u+s} \text{E}_{u+s} (dR_{u+s})] \right\}
\]

\[
= \frac{2}{T} \int_{s=0}^{T} \int_{u=t}^{t+T-s} \mathbb{E} \left\{ \hat{w}_u \text{Cov}_u [dR_u, \hat{w}_{u+s} \text{E}_{u+s} (dR_{u+s})] \right\} du
\]

\[
= \frac{2}{T} \int_{s=0}^{T} \int_{u=t}^{t+T-s} \mathbb{E} \left\{ \hat{w}_t \text{Cov}_t [dR_t, \hat{w}_{u+s} \text{E}_{u+s} (dR_{u+s})] \right\} du
\]

\[
= \frac{2}{T} \int_{s=0}^{T} \left( T - s \right) \mathbb{E} \left\{ \hat{w}_t \text{Cov}_t [dR_t, \hat{w}_{u+s} \text{E}_{u+s} (dR_{u+s})] \right\} dt
\]

\[
= \frac{2}{T} \int_{u=t}^{t+T} \left( T - (u - t) \right) \mathbb{E} \left\{ \hat{w}_t \text{Cov}_t [dR_t, \hat{w}_{u+s} \text{E}_{u+s} (dR_{u+s})] \right\} dt = D_{w,T}^{\text{Cov}_2},
\]

where the first and sixth steps follow from the change of variable \( s = u - t \), and the fourth step follows because the expectation is unconditional and depends only on \( s \).

Since the same argument as in the derivation of (E.7) implies

\[
D_{w,T}^{\text{Cov}_1} + D_{w,T}^{\text{Cov}_2} = \frac{2}{t} \int_{t}^{t+T} \left( 1 - \frac{u - t}{T} \right) \text{Cov} (\hat{w}_t dR_t, \hat{w}_u dR_u),
\]

we can write the Sharpe ratio \( SR_{w,T} \) as

\[
SR_{w,T} = \frac{1}{\frac{1}{dt} \text{Var}(\hat{w}_t dR_t) + \frac{2}{t} \frac{1}{dt} \int_{t}^{t+T} \left( 1 - \frac{u - t}{T} \right) \text{Cov} (\hat{w}_t dR_t, \hat{w}_u dR_u)}.
\]

Dividing numerator and denominator by \( \sqrt{\frac{1}{dt} \text{Var}(\hat{w}_t dR_t) + \frac{2}{t} \frac{1}{dt} \int_{t}^{t+T} \left( 1 - \frac{u - t}{T} \right) \text{Cov} (\hat{w}_t dR_t, \hat{w}_u dR_u)} \)

we find (6.1).

The terms \( \{D_{w,t,T}^{\text{Cov}_i}\}_{i=1,2} \) in Lemma E.1 and \( \{D_{w,T}^{\text{Cov}_i}\}_{i=1,2} \) in Lemma E.2 involve covariances.
between products of random variables, such as $\hat{w}_u \mathbb{E}_u (dR_u)$ and $\hat{w}_{u'} \mathbb{E}_{u'} (dR_{u'})$. Lemma E.3 computes covariances between products of normal random variables.

**Lemma E.3.** If the random variables $\{X_i\}_{i=1,2,3,4}$ are jointly normal, then

\[
\text{Cov}(X_1X_2, X_3) = \mathbb{E}(X_1) \text{Cov}(X_2, X_3) + \mathbb{E}(X_2) \text{Cov}(X_1, X_3) \tag{E.10}
\]

\[
\text{Cov}(X_1X_2, X_3X_4) = \mathbb{E}(X_1) \mathbb{E}(X_3) \text{Cov}(X_2, X_4) + \mathbb{E}(X_1) \mathbb{E}(X_4) \text{Cov}(X_2, X_3) + \mathbb{E}(X_2) \mathbb{E}(X_3) \text{Cov}(X_1, X_4) + \mathbb{E}(X_2) \mathbb{E}(X_4) \text{Cov}(X_1, X_3) \tag{E.11}
\]

**Proof:** We first show (E.10) and (E.11) in the special case where $\{X_i\}_{i=1,2,3,4}$ are mean zero. Since these variables are normal, we can set

\[
X_i = \frac{\text{Cov}(X_i, X_3)}{\text{Var}(X_3)} X_3 + \epsilon_i, \tag{E.12}
\]

for $i = 1, 2, 4$, where $\epsilon_i$ is normal, mean zero and independent of $X_3$.

Using (E.12), we can write the left-hand side of (E.10) as

\[
\frac{\text{Cov}(X_1, X_3)}{\text{Var}(X_3)^2} \text{Cov}(X_2, X_3) + \frac{\text{Cov}(X_1, X_3)}{\text{Var}(X_3)} \text{Cov}(X_3 \epsilon_2, X_3) + \frac{\text{Cov}(X_2, X_3)}{\text{Var}(X_3)} \text{Cov}(X_3 \epsilon_1, X_3) + \text{Cov}(\epsilon_1 \epsilon_2, X_3). \tag{E.13}
\]

The first term in (E.13) is zero because

\[
\text{Cov}(X_3^2, X_3) = \mathbb{E}(X_3^3) - \mathbb{E}(X_3^2) \mathbb{E}(X_3) = 0,
\]

where the second step follows because the normality and mean zero properties of $X_3$ imply $\mathbb{E}(X_3^3) = \mathbb{E}(X_3) = 0$. The second and third terms in (E.13) are zero because

\[
\text{Cov}(X_3 \epsilon_i, X_3) = \mathbb{E}(X_3^2 \epsilon_i) - \mathbb{E}(X_3) \mathbb{E}(X_3 \epsilon_i) = \left[ \mathbb{E}(X_3^2) - \mathbb{E}(X_3)^2 \right] \mathbb{E}(\epsilon_i) = 0,
\]

for $i = 1, 2$, where the second step follows because $\epsilon_i$ is independent of $X_3$, and the third step follows because $\epsilon_i$ is mean zero. The fourth term in (E.13) is zero because $(\epsilon_1, \epsilon_2)$ are independent of $X_3$. Therefore, (E.13) is equal to zero, which implies (E.10).
Using (E.12), we can write the left-hand side of (E.11) as

\[
\frac{\text{Cov}(X_3, X_4)}{\text{Var}(X_3)} \text{Cov}(X_1 X_2, X_3^2) + \text{Cov}(X_1 X_2, X_3 \epsilon_4) = \\
\frac{\text{Cov}(X_3, X_4)}{\text{Var}(X_3)} \left[ \frac{\text{Cov}(X_1, X_3) \text{Cov}(X_2, X_3)}{\text{Var}(X_3)^2} \text{Cov}(X_3^2, X_3^2) + \frac{\text{Cov}(X_1, X_3)}{\text{Var}(X_3)} \text{Cov}(X_3 \epsilon_2, X_3^2) \right. \\
+ \frac{\text{Cov}(X_2, X_3)}{\text{Var}(X_3)} \text{Cov}(X_3 \epsilon_1, X_3^2) + \text{Cov}(\epsilon_1 \epsilon_2, X_3^2) \left] + \frac{\text{Cov}(X_1, X_3) \text{Cov}(X_2, X_3)}{\text{Var}(X_3)^2} \text{Cov}(X_3^2, X_3 \epsilon_4) \\
+ \frac{\text{Cov}(X_1, X_3)}{\text{Var}(X_3)} \text{Cov}(X_3 \epsilon_2, X_3 \epsilon_4) + \frac{\text{Cov}(X_2, X_3)}{\text{Var}(X_3)} \text{Cov}(X_3 \epsilon_1, X_3 \epsilon_4) + \text{Cov}(\epsilon_1 \epsilon_2, X_3 \epsilon_4). \tag{E.14}
\]

To compute the first term in (E.14), we note that

\[
\text{Cov}(X_3^2, X_3^2) = \mathbb{E}(X_3^4) - \mathbb{E}(X_3^2)^2 = 2\mathbb{E}(X_3^2)^2 = 2\text{Var}(X_3)^2, \tag{E.15}
\]

where the second step follows because the mean-zero property of $X_3$ implies that $\frac{\mathbb{E}(X_3^4)}{\mathbb{E}(X_3^2)^2}$ is the kurtosis of $X_3$ and because the normality of $X_3$ implies that $X_3$ has a kurtosis of three. The second and third terms in (E.14) are zero because

\[
\text{Cov}(X_3 \epsilon_i, X_3^2) = \mathbb{E}(X_3^3 \epsilon_i) - \mathbb{E}(X_3 \epsilon_i) \mathbb{E}(X_3^2) = \left[ \mathbb{E}(X_3^3) - \mathbb{E}(X_3) \mathbb{E}(X_3^2) \right] \mathbb{E}(\epsilon_i) = 0,
\]

for $i = 1, 2$, where the second step follows because $\epsilon_i$ is independent of $X_3$, and the third step follows because $\epsilon_i$ is mean zero. The fourth term in (E.14) is zero because $(\epsilon_1, \epsilon_2)$ are independent of $X_3$. The fifth term in (E.14) is zero because

\[
\text{Cov}(X_3^2, X_3 \epsilon_4) = \mathbb{E}(X_3^3 \epsilon_4) - \mathbb{E}(X_3^2) \mathbb{E}(X_3 \epsilon_4) = \left[ \mathbb{E}(X_3^3) - \mathbb{E}(X_3^2) \mathbb{E}(X_3) \right] \mathbb{E}(\epsilon_4) = 0,
\]

where the second step follows because $\epsilon_4$ is independent of $X_3$, and the third step follows because $\epsilon_4$ is mean zero. To compute the sixth and seventh terms in (E.14), we note that

\[
\text{Cov}(X_3 \epsilon_i, X_3 \epsilon_4) = \mathbb{E}(X_3^2 \epsilon_i \epsilon_4) - \mathbb{E}(X_3 \epsilon_4) \mathbb{E}(X_3 \epsilon_i) = \\
\mathbb{E}(X_3^2) \mathbb{E}(\epsilon_i \epsilon_4) - \mathbb{E}(X_3)^2 \mathbb{E}(\epsilon_i) \mathbb{E}(\epsilon_4) = \\
\mathbb{E}(X_3^2) \mathbb{E}(\epsilon_i \epsilon_4) = \\
\text{Var}(X_3^2) \text{Cov}(\epsilon_i, \epsilon_4)
\]
\[ = \mathbb{V} \text{ar}(X^2_3) \text{Cov}(X_i, \epsilon_4), \quad \text{(E.16)} \]

for \( i = 1, 2 \), where the second step follows because \( (\epsilon_i, \epsilon_4) \) are independent of \( X_3 \), the third and fourth step follow because \( (X_3, \epsilon_i, \epsilon_4) \) are mean zero, and the fifth step follows from (E.12) and the independence of \( (X_3, \epsilon_4) \). The eighth term in (E.14) is zero because

\[
\text{Cov}(\epsilon_1 \epsilon_2, X_3 \epsilon_4) = \mathbb{E}(\epsilon_1 \epsilon_2 X_3 \epsilon_4) - \mathbb{E}(\epsilon_1 \epsilon_2) \mathbb{E}(X_3 \epsilon_4) = \mathbb{E}(X_3) \left[ \mathbb{E}(\epsilon_1 \epsilon_2 \epsilon_4) - \mathbb{E}(\epsilon_1 \epsilon_2) \mathbb{E}(\epsilon_4) \right] = 0,
\]

where the second step follows because \( (\epsilon_1, \epsilon_2, \epsilon_4) \) are independent of \( X_3 \), and the third step follows because \( X_3 \) is mean zero. Suppressing all zero terms and using (E.15) and (E.16), we can write (E.14) as

\[
\frac{2 \text{Cov}(X_1, X_3) \text{Cov}(X_2, X_3) \text{Cov}(X_3, X_4)}{\text{Var}(X_3)} + \text{Cov}(X_1, X_3) \text{Cov}(X_2, \epsilon_4) + \text{Cov}(X_2, X_3) \text{Cov}(X_1, \epsilon_4)
\]

\[= \text{Cov}(X_1, X_3) \text{Cov} \left[ X_2, \frac{\text{Cov}(X_3, X_4)}{\text{Var}(X_3)} X_3 + \epsilon_4 \right] + \text{Cov}(X_2, X_3) \text{Cov} \left[ X_1, \frac{\text{Cov}(X_3, X_4)}{\text{Var}(X_3)} X_3 + \epsilon_4 \right]
\]

\[= \text{Cov}(X_1, X_3) \text{Cov}(X_2, X_4) + \text{Cov}(X_2, X_3) \text{Cov}(X_1, X_4),\]

which implies (E.11).

We next show (E.10) and (E.11) when \( \{X_i\}_{i=1,2,3,4} \) can have a non-zero mean. We set \( \hat{X}_i \equiv X_i - \mathbb{E}(X_i) \) for \( i = 1, 2, 3, 4 \).

We can write the left-hand side of (E.10) as

\[
\text{Cov} \left[ (\mathbb{E}(X_1) + \hat{X}_1)(\mathbb{E}(X_2) + \hat{X}_2), X_3 \right]
\]

\[= \mathbb{E}(X_1) \text{Cov}(\hat{X}_2, X_3) + \mathbb{E}(X_2) \text{Cov}(\hat{X}_1, X_3) + \text{Cov}(\hat{X}_1 \hat{X}_2, X_3)
\]

\[= \mathbb{E}(X_1) \text{Cov}(X_2, X_3) + \mathbb{E}(X_2) \text{Cov}(X_1, X_3) + \text{Cov}(\hat{X}_1 \hat{X}_2, \hat{X}_3). \quad \text{(E.17)}\]

Combining (E.17) with (E.10) applied to \( \{\hat{X}_i\}_{i=1,2,3} \), we find (E.10) applied to \( \{X_i\}_{i=1,2,3} \).

We can write the left-hand side of (E.10) as

\[
\text{Cov} \left[ (\mathbb{E}(X_1) + \hat{X}_1)(\mathbb{E}(X_2) + \hat{X}_2), X_3 X_4 \right]
\]

\[= \mathbb{E}(X_1) \text{Cov}(\hat{X}_2, X_3 X_4) + \mathbb{E}(X_2) \text{Cov}(\hat{X}_1, X_3 X_4) + \text{Cov}(\hat{X}_1 \hat{X}_2, X_3 X_4). \quad \text{(E.18)}\]
Equation (E.10) implies
\[ \text{Cov}(\hat{X}_i, X_3 X_4) = \mathbb{E}(X_3)\text{Cov}(\hat{X}_i, X_4) + \mathbb{E}(X_4)\text{Cov}(\hat{X}_i, X_3) \]
\[ = \mathbb{E}(X_3)\text{Cov}(X_i, X_4) + \mathbb{E}(X_4)\text{Cov}(X_i, X_3) \quad (E.19) \]
for \( i = 1, 2 \). Moreover,
\[ \text{Cov}(\hat{X}_1 \hat{X}_2, X_3 X_4) = \text{Cov} \left[ \hat{X}_1 \hat{X}_2, (\mathbb{E}(X_3) + \hat{X}_3)(\mathbb{E}(X_4) + \hat{X}_4) \right] \]
\[ = \mathbb{E}(X_3)\text{Cov}(\hat{X}_1 \hat{X}_2, X_4) + \mathbb{E}(X_4)\text{Cov}(\hat{X}_1 \hat{X}_2, \hat{X}_3) + \text{Cov}(\hat{X}_1 \hat{X}_2, \hat{X}_3 \hat{X}_4) \]
\[ = \text{Cov}(\hat{X}_1 \hat{X}_2, \hat{X}_3 \hat{X}_4), \quad (E.20) \]
where the second step follows from (E.10) and because \( \hat{X}_i \) is mean zero. Combining (E.18)-(E.20) with (E.11) applied to \( \hat{X}_i \), and noting that
\[ \text{Cov}(\hat{X}_i, \hat{X}_{i'}) = \text{Cov}(X_i, X_{i'}) \]
for \( i = 1, 2 \) and \( i' = 3, 4 \), we find (E.11) applied to \( X_i \).

Lemmas E.4 and E.5 use Lemma E.3 to compute the terms \( \{D_{w,t,T}^{\text{Cov}i}\}_{i=1,2} \) in Lemma E.1 and \( \{D_{w,T}^{\text{Cov}i}\}_{i=1,2} \) in Lemma E.2. To ensure that the normality assumption in Lemma E.3 is met, we restrict trading strategies to be linear, in the sense that strategy weights must be integrals of the Brownian shocks with constant coefficients. The value strategy (4.1), the momentum strategy (4.2), and all strategies of the form \( w_t = (\delta_0 + \delta_1 \hat{C}_t + \delta_2 
abla C_t + \delta_3 \eta_t)p_f \) are linear.

**Lemma E.4.** For linear trading strategies, \( D_{w,t,T}^{\text{Cov}1} = \frac{2}{T} \sum_{i=1}^{6} \int_{u=t}^{t+T} \int_{u'=u}^{t+T} D_{w,t}^{\text{Cov}1,i}(u, u')dudu' \), where
\[ D_{w,t}^{\text{Cov}1,1}(u, u') \equiv \left( f + \frac{k \Delta}{\eta \Sigma f} \right)^2 \mathbb{E}_{I_1}(\lambda_u)\mathbb{E}_{I_1}(\lambda_{u'})\text{Cov}_{I_1}(w_u \Sigma f, w_{u'} \Sigma f), \]
\[ D_{w,t}^{\text{Cov}1,2}(u, u') \equiv \left( f + \frac{k \Delta}{\eta \Sigma f} \right)^2 \mathbb{E}_{I_1}(\lambda_u)\mathbb{E}_{I_1}(w_{u'} \Sigma f')\text{Cov}_{I_1}(w_u \Sigma f', \lambda_{u'}), \]
\[ D_{w,t}^{\text{Cov}1,3}(u, u') \equiv \left( f + \frac{k \Delta}{\eta \Sigma f} \right)^2 \mathbb{E}_{I_1}(w_u \Sigma f')\mathbb{E}_{I_1}(\lambda_{u'})\text{Cov}_{I_1}(\lambda_u, w_{u'} \Sigma f'), \]
\[ D_{w,t}^{\text{Cov}1,4}(u, u') \equiv \left( f + \frac{k \Delta}{\eta \Sigma f} \right)^2 \mathbb{E}_{I_1}(w_u \Sigma f')\mathbb{E}_{I_1}(w_{u'} \Sigma f')\text{Cov}_{I_1}(\lambda_u, \lambda_{u'}), \]
\[ D_{\text{cov1},5}(u, u') \equiv \left( f + \frac{k\Delta}{\eta\Sigma'f} \right)^2 \text{Cov}_{\mathcal{I}_t}(\Lambda_u, \Lambda_{u'}) \text{Cov}_{\mathcal{I}_t}(w_u\Sigma p_f', w_{u'}\Sigma p_f'), \]
\[ D_{\text{cov1},6}(u, u') \equiv \left( f + \frac{k\Delta}{\eta\Sigma'f} \right)^2 \text{Cov}_{\mathcal{I}_t}(\Lambda_u, w_{u'}\Sigma p_f') \text{Cov}_{\mathcal{I}_t}(w_u\Sigma p_f', \Lambda_{u'}), \]

and \[ D_{\text{cov2},2} = \frac{2}{T} \sum_{i=1}^{2} \int_{u=t}^{t+T} \int_{u'=u}^{t+T} D_{\text{cov2},i}(u, u') du du', \]
where
\[ D_{\text{cov2},1}(u, u') \equiv \frac{1}{du} \left( f + \frac{k\Delta}{\eta\Sigma'f} \right) \mathbb{E}_{\mathcal{I}_t} \left[ \hat{w}_u \Lambda_{u'} \text{Cov}_u(dR_u, w_{u'}\Sigma p_f') \right] , \]
\[ D_{\text{cov2},2}(u, u') \equiv \frac{1}{du} \left( f + \frac{k\Delta}{\eta\Sigma'f} \right) \mathbb{E}_{\mathcal{I}_t} \left[ \hat{w}_u w_{u'}\Sigma p_f' \text{Cov}_u(dR_u, \Lambda_{u'}) \right] . \]

**Proof:** Equations (4.3) and (A.12) imply
\[ \hat{w}_t \mathbb{E}_u(dR_t) = \left( w_t - \frac{\text{Cov}_u(w_t dR_t, \eta dR_t)}{\sqrt{\text{Var}_t(\eta dR_t)}} \frac{\eta}{\eta} \right) \left[ \frac{r\alpha f}{\alpha + \alpha} \frac{\eta\Sigma'f}{\eta\Sigma'f} \frac{\Sigma'f}{\Sigma'f} + \left( f + \frac{k\Delta}{\eta\Sigma'f} \right) \Lambda_t \Sigma p_f' \right] dt \]
\[ = \left( w_t - \frac{w_t \Sigma'f}{\eta\Sigma'f} \right) \left[ \frac{r\alpha f}{\alpha + \alpha} \frac{\eta\Sigma'f}{\eta\Sigma'f} \frac{\Sigma'f}{\Sigma'f} + \left( f + \frac{k\Delta}{\eta\Sigma'f} \right) \Lambda_t \Sigma p_f' \right] dt \]
\[ = \left( f + \frac{k\Delta}{\eta\Sigma'f} \right) \Lambda_t w_t \Sigma p_f', \quad \text{(E.21)} \]

where the second step follows from (3.5), (C.19) and \( \eta \Sigma p_f' = 0 \), and the third step follows from \( \eta \Sigma p_f' = 0 \).

Using (E.21), we can write (E.3) as
\[ D_{\text{cov1},T} = \frac{2}{T} \left( f + \frac{k\Delta}{\eta\Sigma'f} \right)^2 \int_{t' = t}^{t + T} \int_{u = u}^{t + T} \text{Cov}_{\mathcal{I}_t}(\Lambda_u w_u \Sigma p_f', \Lambda_{u'} w_{u'} \Sigma p_f') du du'. \quad \text{(E.22)} \]

The equation for \( D_{\text{cov1},T} \) in the lemma follows from (E.22) by using (E.11) and setting \( X_1 = \Lambda_u, X_2 = w_u \Sigma p_f', X_3 = \Lambda_{u'} \) and \( X_4 = w_{u'} \Sigma p_f' \).

Using (E.21), we can write (E.4) as
\[ D_{\text{cov2},T} = \frac{2}{T} \left( f + \frac{k\Delta}{\eta\Sigma'f} \right) \int_{t' = t}^{t + T} \int_{u = u}^{t + T} \mathbb{E}_{\mathcal{I}_t} \left[ \hat{w}_u \text{Cov}_u(dR_u, \Lambda_{u'} w_{u'} \Sigma p_f') \right] du' . \quad \text{(E.23)} \]

The equation for \( D_{\text{cov2},T} \) in the lemma follows from (E.23) by using (E.10) and the Law of Iterative
Expectations and setting \( X_1 = dR_u, X_2 = \Lambda_u' \) and \( X_3 = w_u'\Sigma p'_f \).

**Lemma E.5.** For linear trading strategies, \( \mathcal{D}_{w,T}^{\text{Cov}^1} = 2\sum_{i=1}^6 \int_t^{t+T} (1 - \frac{u-t}{T}) \mathcal{D}_{w}^{\text{Cov}^1,i}(u)du \), where

\[
\mathcal{D}_{w}^{\text{Cov}^1,1}(u) \equiv \left( f + \frac{k\Delta}{\eta \Sigma \eta'} \right)^2 E(\Lambda_t) \text{Cov}(w_t \Sigma p'_f, w_u \Sigma p'_f),
\]

\[
\mathcal{D}_{w}^{\text{Cov}^1,2}(u) \equiv \left( f + \frac{k\Delta}{\eta \Sigma \eta'} \right)^2 \mathbb{E}(\Lambda_t) \mathbb{E}(w_t \Sigma p'_f) \text{Cov}(w_t \Sigma p'_f, \Lambda_u),
\]

\[
\mathcal{D}_{w}^{\text{Cov}^1,3}(u) \equiv \left( f + \frac{k\Delta}{\eta \Sigma \eta'} \right)^2 \mathbb{E}(w_t \Sigma p'_f) \text{Cov}(\Lambda_t, w_u \Sigma p'_f),
\]

\[
\mathcal{D}_{w}^{\text{Cov}^1,4}(u) \equiv \left( f + \frac{k\Delta}{\eta \Sigma \eta'} \right)^2 \text{Cov}(\Lambda_t, \Lambda_u),
\]

\[
\mathcal{D}_{w}^{\text{Cov}^1,5}(u) \equiv \left( f + \frac{k\Delta}{\eta \Sigma \eta'} \right)^2 \text{Cov}(\Lambda_t, w_u \Sigma p'_f) \text{Cov}(w_t \Sigma p'_f, \Lambda_u),
\]

\[
\mathcal{D}_{w}^{\text{Cov}^1,6}(u) \equiv \left( f + \frac{k\Delta}{\eta \Sigma \eta'} \right)^2 \text{Cov}(\Lambda_t, w_u \Sigma p'_f) \text{Cov}(w_t \Sigma p'_f, \Lambda_u),
\]

and \( \mathcal{D}_{w,T}^{\text{Cov}^2} = 2\sum_{i=1}^2 \int_t^{t+T} (1 - \frac{u-t}{T}) \mathcal{D}_{w}^{\text{Cov}^2,i}(u)du \), where

\[
\mathcal{D}_{w}^{\text{Cov}^2,1}(u) \equiv \frac{1}{dt} \left( f + \frac{k\Delta}{\eta \Sigma \eta'} \right) \mathbb{E} \left[ \hat{w}_t \Lambda_u \text{Cov}_t(dR_t, w_u \Sigma p'_f) \right],
\]

\[
\mathcal{D}_{w}^{\text{Cov}^2,2}(u) \equiv \frac{1}{dt} \left( f + \frac{k\Delta}{\eta \Sigma \eta'} \right) \mathbb{E} \left[ \hat{w}_t w_u \Sigma p'_f \text{Cov}_t(dR_t, \Lambda_u) \right].
\]

**Proof:** Using (E.21), we can write (E.8) as

\[
\mathcal{D}_{w,T}^{\text{Cov}^1} = 2 \left( f + \frac{k\Delta}{\eta \Sigma \eta'} \right)^2 \int_t^{t+T} \left( 1 - \frac{u-t}{T} \right) \text{Cov}(\Lambda_t w_t \Sigma p'_f, \Lambda_u w_u \Sigma p'_f)du. \tag{E.24}
\]

The equation for \( \mathcal{D}_{w,T}^{\text{Cov}^1} \) in the lemma follows from (E.24) by using (E.11) and setting \( X_1 = \Lambda_t, X_2 = w_t \Sigma p'_f, X_3 = \Lambda_u \) and \( X_4 = w_u \Sigma p'_f \), and by noting that when expectations are unconditional, \( \mathbb{E}(\Lambda_t) = \mathbb{E}(\Lambda_u) \) and \( \mathbb{E}(w_t \Sigma p'_f) = \mathbb{E}(w_u \Sigma p'_f) \).

Using (E.21), we can write (E.9) as

\[
\mathcal{D}_{w,T}^{\text{Cov}^2} = \frac{2}{dt} \left( f + \frac{k\Delta}{\eta \Sigma \eta'} \right) \int_t^{t+T} \left( 1 - \frac{u-t}{T} \right) \mathbb{E} \left[ \hat{w}_t \text{Cov}_t \left[ dR_t, \Lambda_u w_u \Sigma p'_f \right] \right] du. \tag{E.25}
\]
The equation for $D_{w,T}^{\text{Cov}_2}$ in the lemma follows from (E.25) by using (E.10) and the Law of Iterative Expectations and setting $X_1 = dR_t$, $X_2 = \Lambda_u$ and $X_3 = w_u\Sigma p_f$.

Proposition E.1 computes the unconditional Sharpe ratio over investment horizon $T$ of a general strategy of the form $w_t = (\delta_0 + \delta_1 \hat{C}_t + \delta_2 C_t + \delta_3 y_t) p_f$. The formula for the Sharpe ratio is expressed in terms of integrals. While the integrals can be computed in closed form, we do not present the closed-form solutions because they require introducing additional notation.

**Proposition E.1.** The unconditional Sharpe ratio of a strategy $w_t = (\delta_0 + \delta_1 \hat{C}_t + \delta_2 C_t + \delta_3 y_t) p_f$ over investment horizon $T$ is $SR_{w,T} = \frac{N_w}{\sqrt{D_w + D_{w,T}^{\text{Cov}_1} + D_{w,T}^{\text{Cov}_2}}}$, where

$$N_w = \left[ L_2 \left( \delta_0 + (\delta_1 + \delta_2) \hat{C} + \delta_3 \tilde{y} \right) + H(\gamma_1^R, \gamma_2^R, \gamma_3^R, \delta_1, \delta_2, \delta_3, 0, \nu_0) \right] \frac{\Delta}{\eta \Sigma \eta'},$$

$$D_w = \left( f + \frac{k \Delta}{\eta \Sigma \eta'} \right) \left[ \left( \delta_0 + (\delta_1 + \delta_2) \hat{C} + \delta_3 \tilde{y} \right)^2 + H(\delta_1, \delta_2, \delta_3, \delta_1, \delta_2, \delta_3, 0, \nu_0) \right] \frac{\Delta}{\eta \Sigma \eta'},$$

and $(D_{w,T}^{\text{Cov}_1}, D_{w,T}^{\text{Cov}_2})$ are derived in Lemma E.5, with

$$D_{w,1}^{\text{Cov}_1}(u) = L_2^2 H(\delta_1, \delta_2, \delta_3, \delta_1, \delta_2, \delta_3, u - t, \nu_0) \left( \frac{\Delta}{\eta \Sigma \eta'} \right)^2,$$

$$D_{w,2}^{\text{Cov}_1}(u) = L_2 \left( \delta_0 + (\delta_1 + \delta_2) \hat{C} + \delta_3 \tilde{y} \right) H(\delta_1, \delta_2, \delta_3, \gamma_1^R, \gamma_2^R, \gamma_3^R, \nu_0, u - t, \nu_0) \left( \frac{\Delta}{\eta \Sigma \eta'} \right)^2,$$

$$D_{w,3}^{\text{Cov}_1}(u) = L_2 \left( \delta_0 + (\delta_1 + \delta_2) \hat{C} + \delta_3 \tilde{y} \right) H(\gamma_1^R, \gamma_2^R, \gamma_3^R, \delta_1, \delta_2, \delta_3, u - t, \nu_0) \left( \frac{\Delta}{\eta \Sigma \eta'} \right)^2,$$

$$D_{w,4}^{\text{Cov}_1}(u) = \left( \delta_0 + (\delta_1 + \delta_2) \hat{C} + \delta_3 \tilde{y} \right)^2 H(\gamma_1^R, \gamma_2^R, \gamma_3^R, \gamma_1^R, \gamma_2^R, \gamma_3^R, \nu_0, u - t, \nu_0) \left( \frac{\Delta}{\eta \Sigma \eta'} \right)^2,$$

$$D_{w,5}^{\text{Cov}_1}(u) = H(\gamma_1^R, \gamma_2^R, \gamma_3^R, \gamma_1^R, \gamma_2^R, \gamma_3^R, u - t, \nu_0) H(\delta_1, \delta_2, \delta_3, \delta_1, \delta_2, \delta_3, u - t, \nu_0) \left( \frac{\Delta}{\eta \Sigma \eta'} \right)^2,$$

$$D_{w,6}^{\text{Cov}_1}(u) = H(\gamma_1^R, \gamma_2^R, \gamma_3^R, \delta_1, \delta_2, \delta_3, u - t, \nu_0) H(\delta_1, \delta_2, \delta_3, \gamma_1^R, \gamma_2^R, \gamma_3^R, u - t, \nu_0) \left( \frac{\Delta}{\eta \Sigma \eta'} \right)^2,$$

$$D_{w,1}^{\text{Cov}_2}(u) = \left[ L_2 \left( \delta_0 + (\delta_1 + \delta_2) \hat{C} + \delta_3 \tilde{y} \right) + H(\delta_1, \delta_2, \delta_3, \gamma_1^R, \gamma_2^R, \gamma_3^R, u - t, \nu_0) \right]$$

$$\times G(\delta_1, \delta_2, \delta_3, u - t, \nu_0) \left( \frac{\Delta}{\eta \Sigma \eta'} \right)^2,$$

$$D_{w,2}^{\text{Cov}_2}(u) = \left[ \left( \delta_0 + (\delta_1 + \delta_2) \hat{C} + \delta_3 \tilde{y} \right)^2 + H(\delta_1, \delta_2, \delta_3, \delta_1, \delta_2, \delta_3, u - t, \nu_0) \right]$$
\[ \times G(\gamma_1^R, \gamma_2^R, \gamma_3^R, u - t, \nu_0) \left( \frac{\Delta}{\eta \Sigma \eta'} \right)^2. \]

**Proof:** The Sharpe ratio has the form in Lemma E.2. To compute \( N_w \), we note from Lemma E.2 that

\[
N_w = \left( f + \frac{k \Delta}{\eta \Sigma \eta'} \right) \mathbb{E} \left[ \Lambda_t (\delta_0 + \delta_1 \hat{C}_t + \delta_2 C_t + \delta_3 y_t) \right] p_f \Sigma p'_f \\
= \left( f + \frac{k \Delta}{\eta \Sigma \eta'} \right) \left[ \mathbb{E}(\Lambda_t) \mathbb{E}(\delta_0 + \delta_1 \hat{C}_t + \delta_2 C_t + \delta_3 y_t) + \text{Cov}(\Lambda_t, \delta_0 + \delta_1 \hat{C}_t + \delta_2 C_t + \delta_3 y_t) \right] \frac{\Delta}{\eta \Sigma \eta'} \\
= \left[ L_2 (\delta_0 + (\delta_1 + \delta_2) \hat{C} + \delta_3 \hat{y}) + H(\gamma_1^R, \gamma_2^R, \gamma_3^R, \delta_1, \delta_2, \delta_3, 0, \nu_0) \right] \frac{\Delta}{\eta \Sigma \eta'},
\]

where the first step follows from \( w_t = (\delta_0 + \delta_1 \hat{C}_t + \delta_2 C_t + \delta_3 y_t)p_f \), the second step follows from \( p_f \Sigma p'_f = \frac{\Delta}{\eta \Sigma \eta'} \), and the third step follows from (3.7), (B.10), and (D.8). To compute \( D_w \), we note from Lemma E.2 that

\[
D_w = f \mathbb{E} \left[ (\delta_0 + \delta_1 \hat{C}_t + \delta_2 C_t + \delta_3 y_t)^2 \right] p_f \Sigma p'_f + k \mathbb{E} \left[ (\delta_0 + \delta_1 \hat{C}_t + \delta_2 C_t + \delta_3 y_t)^2 \right] (p_f \Sigma p'_f)^2 \\
= \left( f + \frac{k \Delta}{\eta \Sigma \eta'} \right) \mathbb{E} \left[ (\delta_0 + \delta_1 \hat{C}_t + \delta_2 C_t + \delta_3 y_t)^2 \right] \frac{\Delta}{\eta \Sigma \eta'} \\
= \left( f + \frac{k \Delta}{\eta \Sigma \eta'} \right) \left[ \mathbb{E}(\delta_0 + \delta_1 \hat{C}_t + \delta_2 C_t + \delta_3 y_t)^2 + \text{Var}(\delta_0 + \delta_1 \hat{C}_t + \delta_2 C_t + \delta_3 y_t) \right] \frac{\Delta}{\eta \Sigma \eta'} \\
= \left( f + \frac{k \Delta}{\eta \Sigma \eta'} \right) \left[ (\delta_0 + \delta_1 \hat{C} + \delta_3 \hat{y})^2 + H(\delta_1, \delta_2, \delta_3, \delta_1, \delta_2, \delta_3, 0, \nu_0) \right] \frac{\Delta}{\eta \Sigma \eta'},
\]

where the first step follows from \( w_t = (\delta_0 + \delta_1 \hat{C}_t + \delta_2 C_t + \delta_3 y_t)p_f \) and \( \eta \Sigma p'_f = 0 \), the second step follows from \( p_f \Sigma p'_f = \frac{\Delta}{\eta \Sigma \eta'} \), and the fourth step follows from (B.13). To compute \( \{D_w^\text{cov1}, i(u)\}_{i=1, \ldots, 6} \), we use their definitions in Lemma E.5 together with \( w_t = (\delta_0 + \delta_1 \hat{C}_t + \delta_2 C_t + \delta_3 y_t)p_f \), \( p_f \Sigma p'_f = \frac{\Delta}{\eta \Sigma \eta'} \), (3.7), (B.13) and (D.8). To compute \( \{D_w^\text{cov2}, i(u)\}_{i=1, 2} \), we use their definitions in Lemma E.5 together with \( w_t = (\delta_0 + \delta_1 \hat{C}_t + \delta_2 C_t + \delta_3 y_t)p_f \), \( \eta \Sigma p'_f = 0 \), \( p_f \Sigma p'_f = \frac{\Delta}{\eta \Sigma \eta'} \), (3.7), (B.10), (B.13) and (D.8).

To determine the optimal strategy for a given investment horizon \( T \), we maximize numerically the Sharpe ratio in Proposition E.1 over \( \{\delta_0, \delta_1, \delta_2, \delta_3\} \). Since the Sharpe ratio is the same for \( \{\delta_0, \delta_1, \delta_2, \delta_3\} \) and \( \{\lambda \delta_0, \lambda \delta_1, \lambda \delta_2, \lambda \delta_3\} \) for any \( \lambda > 0 \), we can fix the value of one of the four arguments \( \{\delta_0, \delta_1, \delta_2, \delta_3\} \) to one if the argument is positive at the optimum and to minus one if it is negative.
We do that for $\delta_3$, which we set to minus one because it is negative at the optimum. Proposition E.2 computes the unconditional Sharpe ratio over investment horizon $T$ of the value strategy.

**Proposition E.2.** The unconditional Sharpe ratio of the value strategy (4.1) over investment horizon $T$ is 

$$SR_{w^V,T} = \frac{\mathcal{N}_{w^V}}{\sqrt{\mathcal{D}_{w^V} + \mathcal{D}_{w^V}^{\text{cov1}} + \mathcal{D}_{w^V}^{\text{cov2}}}}$$

where $(\mathcal{N}_{w^V}, \mathcal{D}_{w^V})$ are derived in Proposition D.2 and $(\mathcal{D}_{w^V}^{\text{cov1}}, \mathcal{D}_{w^V}^{\text{cov2}})$ are defined in Lemma E.5, with

$$D_{w^V}^{\text{cov1}, 1}(u) = L_2^2 \left[ \left( \frac{1 - \epsilon}{r + \kappa} [K_1(\gamma_1, \gamma_3, u - t, \nu_0) + K_2(\gamma_1, \gamma_3, u - t, \nu_0)] + H(\gamma_1, \gamma_2, \gamma_3, \gamma_1, \gamma_2, \gamma_3, u - t, \nu_0) \right) (p_f \Sigma^2 p_f')^2 + \frac{(1 - \epsilon)^2 \phi^2}{2(r + \kappa)^2} \nu_0(\kappa, u - t)p_f \Sigma^2 p_f' \right],$$

$$D_{w^V}^{\text{cov1}, 2}(u) = L_2 \left( \frac{L_1}{r} \eta \Sigma^2 p_f' + \frac{L_2}{r} p_f \Sigma^2 p_f' \right) \left( \frac{1 - \epsilon}{r + \kappa} K_2(\gamma_1, \gamma_3, u - t, \nu_0) + H(\gamma_1, \gamma_2, \gamma_3, \gamma_1, \gamma_2, \gamma_3, u - t, \nu_0) \right) p_f \Sigma^2 p_f',$$

$$D_{w^V}^{\text{cov1}, 3}(u) = L_2 \left( \frac{L_1}{r} \eta \Sigma^2 p_f' + \frac{L_2}{r} p_f \Sigma^2 p_f' \right) \left( \frac{1 - \epsilon}{r + \kappa} K_1(\gamma_1, \gamma_3, u - t, \nu_0) + p_f \Sigma^2 p_f' \right),$$

$$D_{w^V}^{\text{cov1}, 4}(u) = \left( \frac{L_1}{r} \eta \Sigma^2 p_f' + \frac{L_2}{r} p_f \Sigma^2 p_f' \right)^2 H(\gamma_1, \gamma_2, \gamma_3, \gamma_1, \gamma_2, \gamma_3, u - t, \nu_0),$$

$$D_{w^V}^{\text{cov1}, 5}(u) = H(\gamma_1, \gamma_2, \gamma_3, \gamma_1, \gamma_2, \gamma_3, u - t, \nu_0) \left[ \left( \frac{1 - \epsilon}{r + \kappa} [K_1(\gamma_1, \gamma_3, u - t, \nu_0) + K_2(\gamma_1, \gamma_3, u - t, \nu_0)] + H(\gamma_1, \gamma_2, \gamma_3, \gamma_1, \gamma_2, \gamma_3, u - t, \nu_0) \right) (p_f \Sigma^2 p_f')^2 + \frac{(1 - \epsilon)^2 \phi^2}{2(r + \kappa)^2} \nu_0(\kappa, u - t)p_f \Sigma^2 p_f' \right],$$

$$D_{w^V}^{\text{cov1}, 6}(u) = \left( \frac{1 - \epsilon}{r + \kappa} K_1(\gamma_1, \gamma_3, u - t, \nu_0) + H(\gamma_1, \gamma_2, \gamma_3, \gamma_1, \gamma_2, \gamma_3, u - t, \nu_0) \right) \left( \frac{1 - \epsilon}{r + \kappa} K_2(\gamma_1, \gamma_3, u - t, \nu_0) + H(\gamma_1, \gamma_2, \gamma_3, \gamma_1, \gamma_2, \gamma_3, u - t, \nu_0) \right) (p_f \Sigma^2 p_f')^2,$$

$$D_{w^V}^{\text{cov2}, 1}(u) = \left[ L_1 L_2 \eta \Sigma^2 p_f' + \left( \frac{L_2}{r} - \frac{1 - \epsilon}{r + \kappa} K_2(\gamma_1, \gamma_3, u - t, \nu_0) \right) + H(\gamma_1, \gamma_2, \gamma_3, \gamma_1, \gamma_2, \gamma_3, u - t, \nu_0) \right) p_f \Sigma^2 p_f'$$

$$\times \left( \frac{(1 - \epsilon)}{(r + \kappa)^2} \delta_2 \gamma_1 \nu_0(\kappa, u - t) + G(\gamma_1, \gamma_2, \gamma_3, u - t, \nu_0) \right) p_f \Sigma^2 p_f'$$

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Combining (E.27) with (3.7), (B.13), (B.15), (D.8), (D.19) and (D.20), we find that 
\( D \) and (3.7) imply

\[
\eta
\]

\( \) Combining (E.26) with (4.3) and 

Proof: The Sharpe ratio has the form in Lemma E.2. To compute \( D_{wV}^{\text{Cov1}}(u) \), we use their definitions in Lemma E.5, together with (3.7), (B.13)-(B.16), (D.8), (D.19), (D.20), and the derivations in the proof of Lemma D.3. To compute \( D_{wV}^{\text{Cov2}}(u) \), we use its definition in Lemma E.5 and note that (B.10), (B.11) and (D.19) imply

\[
\text{Cov}_t(dR_t, w_u V \Sigma p'_f) = \frac{(1 - \epsilon)\phi^2}{(r + \kappa)^2} \left( \Sigma^2 p'_f + \beta_2 \gamma_1 p f \Sigma^2 p'_f \Sigma p'_f \right) \nu_0(\kappa, u - t) dt + G(\gamma_1, \gamma_2, \gamma_3, u - t, \nu_0) p f \Sigma^2 p'_f dt. \tag{E.26}
\]

Combining (E.26) with (4.3) and \( \eta \Sigma p'_f = 0 \), we find

\[
\mathbb{E} \left[ \hat{w}_t^V \Lambda_u \text{Cov}_t(dR_t, w_u V \Sigma p'_f) \right] = \mathbb{E}(w_t^V \Sigma p'_f \Lambda_u) \left( -\frac{(1 - \epsilon)\phi^2}{(r + \kappa)^2} \beta_2 \gamma_1 \nu_0(\kappa, u - t) + G(\gamma_1, \gamma_2, \gamma_3, u - t, \nu_0) \right) p f \Sigma^2 p'_f dt \]

\[
- \mathbb{E} \left( w_t^V \left( \Sigma^2 p'_f - \frac{\Sigma \eta' \eta \Sigma^2 p'_f}{\eta \Sigma \eta'} \right) \Lambda_u \right) \frac{(1 - \epsilon)\phi^2}{(r + \kappa)^2} \nu_0(\kappa, u - t) dt. \tag{E.27}
\]

Combining (E.27) with (3.7), (B.13), (B.15), (D.8), (D.19) and (D.20), we find that \( D_{wV}^{\text{Cov2}}(u) \) is as in the proposition. To compute \( D_{wV}^{\text{Cov2}}(u) \), we use its definition in Lemma E.5 and note that (B.10) and (3.7) imply

\[
\text{Cov}_t(dR_t, \Lambda_u) = \frac{1}{f + \Delta} G(\gamma_1, \gamma_2, \gamma_3, u - t, \nu_0) \Sigma p'_f dt. \tag{E.28}
\]

99
Combining \((E.28)\) with \((4.3)\) and \(\eta \Sigma p'_f = 0\), we find

\[
\mathbb{E} \left[ \tilde{w}^V w^V \Sigma p'_f \text{Cov}_t(dR_t, \Lambda_u) \right] = \frac{1}{f + \frac{k \Delta \rho}{\eta \Sigma p'_f}} \mathbb{E}(w^V \Sigma p'_f w^V \Sigma p'_f) G(\gamma_1^R, \gamma_2^R, \gamma_3^R, u - t, \nu_0) dt. \tag{E.29}
\]

Combining \((E.29)\) with \((D.20)\) and \(\text{Cov}(w^V \Sigma p'_f, w^V \Sigma p'_f)\) from the derivation of \(D^{\text{Cov1,1}}_{w^V}(u)\), we find that \(D^{\text{Cov2,2}}_{w^V}(u)\) is as in the proposition.

Proposition E.3 computes the unconditional Sharpe ratio over investment horizon \(T\) of the momentum strategy.

**Proposition E.3.** The unconditional Sharpe ratio of the momentum strategy \((4.2)\) over investment horizon \(T\) is

\[
\text{SR}_{w^M, T} = \frac{N_{w^M}}{\sqrt{D_{w^M} + D^{\text{Cov1}}_{w^M, T} + D^{\text{Cov2}}_{w^M, T}}},
\]

where \((N_{w^M}, D_{w^M})\) are derived in Proposition D.5 and \((D^{\text{Cov1}}_{w^M, T}, D^{\text{Cov2}}_{w^M, T})\) are defined in Lemma E.5, with

\[
\begin{align*}
D^{\text{Cov1,1}}_{w^M}(u) &= L^2 \left[ (H(\gamma^R, \tau_2^R, \gamma_3^R, \gamma_1^R, \gamma_2^R, \gamma_3^R, \tau', \nu_3) + H(\gamma_1^R, \gamma_2^R, \gamma_3^R, \gamma_1^R, \gamma_2^R, \gamma_3^R, \tau', \nu_4) \\
&+ G(\gamma_1^R, \gamma_2^R, \gamma_3^R, \tau', \nu_3) + G(\gamma_1^R, \gamma_2^R, \gamma_3^R, \tau', \nu_4) + k \max\{\tau + t - u, 0\} \right) (p_f \Sigma^2 p'_f)^2 \\
&+ f \max\{\tau + t - u, 0\} p_f \Sigma^3 p'_f ,
\end{align*}
\]

\[
\begin{align*}
D^{\text{Cov1,2}}_{w^M}(u) &= L_2^2 (L_1 L_1 n \Sigma^2 p'_f + L_2 \tau p_f \Sigma^2 p'_f)(H(\gamma_1^R, \gamma_2^R, \gamma_3^R, \gamma_1^R, \gamma_2^R, \gamma_3^R, \tau', \nu_2) \\
&+ G(\gamma_1^R, \gamma_2^R, \gamma_3^R, \tau', \nu_2)) p_f \Sigma^2 p'_f ,
\end{align*}
\]

\[
\begin{align*}
D^{\text{Cov1,3}}_{w^M}(u) &= L_2^2 (L_1 L_1 n \Sigma^2 p'_f + L_2 \tau p_f \Sigma^2 p'_f)(H(\gamma_1^R, \gamma_2^R, \gamma_3^R, \gamma_1^R, \gamma_2^R, \gamma_3^R, \tau', \nu_1) \\
&+ H(\gamma_1^R, \gamma_2^R, \gamma_3^R, \gamma_1^R, \gamma_2^R, \gamma_3^R, \tau', \nu_2) + G(\gamma_1^R, \gamma_2^R, \gamma_3^R, \tau', \nu_2) p_f \Sigma^2 p'_f ,
\end{align*}
\]

\[
\begin{align*}
D^{\text{Cov1,4}}_{w^M}(u) &= (L_1 L_1 n \Sigma^2 p'_f + L_2 \tau p_f \Sigma^2 p'_f)^2 H(\gamma_1^R, \gamma_2^R, \gamma_3^R, \gamma_1^R, \gamma_2^R, \gamma_3^R, \tau', \nu_0) \\
&+ H(\gamma_1^R, \gamma_2^R, \gamma_3^R, \gamma_1^R, \gamma_2^R, \gamma_3^R, \tau', \nu_0) \right] (H(\gamma_1^R, \gamma_2^R, \gamma_3^R, \gamma_1^R, \gamma_2^R, \gamma_3^R, \tau', \nu_3) \\
&+ H(\gamma_1^R, \gamma_2^R, \gamma_3^R, \gamma_1^R, \gamma_2^R, \gamma_3^R, \tau', \nu_4) + G(\gamma_1^R, \gamma_2^R, \gamma_3^R, \tau', \nu_3) + G(\gamma_1^R, \gamma_2^R, \gamma_3^R, \tau', \nu_4) \\
&+ k \max\{\tau + t - u, 0\} (p_f \Sigma^2 p'_f)^2 + f \max\{\tau + t - u, 0\} p_f \Sigma^3 p'_f ,
\end{align*}
\]

\[
\begin{align*}
D^{\text{Cov1,5}}_{w^M}(u) &= (H(\gamma_1^R, \gamma_2^R, \gamma_3^R, \gamma_1^R, \gamma_2^R, \gamma_3^R, \tau', \nu_2) + G(\gamma_1^R, \gamma_2^R, \gamma_3^R, \tau', \nu_2)) \\
&\times (H(\gamma_1^R, \gamma_2^R, \gamma_3^R, \gamma_1^R, \gamma_2^R, \gamma_3^R, \tau', \nu_1) + H(\gamma_1^R, \gamma_2^R, \gamma_3^R, \gamma_1^R, \gamma_2^R, \gamma_3^R, \tau', \nu_2))
\end{align*}
\]
Lemma E.5 and note that (D.48) implies where the second step follows from (3.5), (3.7) and (B.10). Combining (E.30) with (4.3) and the derivations in the proof of Lemma D.4. To compute \( \Sigma f = 0 \), we find

\[
D_{u,M}^{Cov, 1}(u) = \left[ L_1 L_2 \eta \Sigma^2 p_f' + \left( L_2^2 \tau + H(\gamma_1, \gamma_2, \gamma_3, \gamma_1, \gamma_2, \gamma_3, \tau', \nu_1) + G(\gamma_1, \gamma_2, \gamma_3, \tau', \nu_2) \right) p_f \Sigma^2 p_f' \right]
\]

\[
\times (G(\gamma_1, \gamma_2, \gamma_3, \tau', \nu_1) + k_1 \{ \tau + t - u > 0 \}) p_f \Sigma^2 p_f'
\]

\[
+ \left[ L_1 L_2 \left( \eta \Sigma^3 p_f' - \frac{\eta \Sigma^2 \eta' \eta \Sigma^2 p_f'}{\eta \Sigma \eta'} \right) + \left( L_2^2 \tau + H(\gamma_1, \gamma_2, \gamma_3, \gamma_1, \gamma_2, \gamma_3, \tau', \nu_2) \right) + G(\gamma_1, \gamma_2, \gamma_3, \tau', \nu_2) \right) \left( p_f \Sigma^2 p_f' - \frac{(\eta \Sigma^2 p_f')^2}{\eta \Sigma \eta'} \right) \right] f_1(\tau + t - u > 0),
\]

\[
D_{u,M}^{Cov, 2}(u) = \left[ \left( L_1 \eta \Sigma^2 p_f' + L_2 \eta \Sigma^2 p_f' \right)^2 + \left( H(\gamma_1, \gamma_2, \gamma_3, \gamma_1, \gamma_2, \gamma_3, \tau', \nu_3) \right) + G(\gamma_1, \gamma_2, \gamma_3, \tau', \nu_4) + G(\gamma_1, \gamma_2, \gamma_3, \tau', \nu_3) + G(\gamma_1, \gamma_2, \gamma_3, \tau', \nu_4) + k \max\{ \tau + t - u, 0 \} \left( p_f \Sigma^2 p_f' \right)^2 + f \max\{ \tau + t - u, 0 \} p_f \Sigma^2 p_f' \right] \eta \Sigma \eta' f_1(\tau + t - u > 0),
\]

where \( \tau' = (u - t, \tau) \) and \( \tau' = (t - u, \tau) \).

**Proof:** The Sharpe ratio has the form in Lemma E.2. To compute \( \{ D_{u,M}^{Cov, i}(u) \}_{i=1,..,6} \), we use their definitions in Lemma E.5 together with (3.5), (3.7), (B.10), (B.13), (D.8), (D.48), (D.49), (D.51) and the derivations in the proof of Lemma D.4. To compute \( D_{u,M}^{Cov, 1}(u) \), we use its definition in Lemma E.5 and note that (D.48) implies

\[
Cov_t(\Delta R_t, u \Sigma p_f') = \left( f + \frac{k \Delta}{\eta \Sigma^2} \right) \left( \int_{u - \tau}^u \text{Cov}_t(\Delta R_t, \Lambda_u) dt' \right) p_f \Sigma^2 p_f' + \text{Cov}_t(\Delta R_t, \Delta R_t') 1_{\{ \tau + t - u > 0 \}} \Sigma p_f'
\]

\[
= G(\gamma_1, \gamma_2, \gamma_3, \tau', \nu_1) p_f \Sigma^2 p_f' \Sigma p_f' + \left( f \Sigma^2 p_f' + kp_f \Sigma^2 p_f' \Sigma p_f' \right) 1_{\{ \tau + t - u > 0 \}},
\]

(E.30)

where the second step follows from (3.5), (3.7) and (B.10). Combining (E.30) with (4.3) and \( \eta \Sigma p_f' = 0 \), we find

\[
\text{E} \left[ \tilde{w}_t^M \Lambda_u \text{Cov}_t(\Delta R_t, u \Sigma p_f') \right] = \text{E}(w_t^M \Sigma p_f' \Lambda_u) \left( G(\gamma_1, \gamma_2, \gamma_3, \tau', \nu_1) + k_1 \{ \tau + t - u > 0 \} \right) p_f \Sigma^2 p_f' dt
\]

\[
- \text{E} \left( w_t^M \left( \Sigma^2 p_f' - \frac{\Sigma \eta'}{\eta \Sigma \eta'} \eta \Sigma^2 p_f' \right) \Lambda_u \right) f_1(\tau + t - u > 0) dt.
\]

(E.31)
Combining (E.31) with (D.8), (D.49) and $\text{Cov}(w_t^M \Sigma p_f', \Lambda_u)$ from the derivation of $D_{w^M}^{\text{Cov},1}(u)$, we find that $D_{w^M}^{\text{Cov},1}(u)$ is as in the proposition. To compute $D_{w^M}^{\text{Cov},2}(u)$, we use its definition in Lemma E.5 and combine

$$
\mathbb{E} \left[ \hat{w}_t^M w_u^M \Sigma p_f' \text{Cov}_t(dR_t, \Lambda_u) \right] = \frac{1}{f + \frac{k_{\mathfrak{m}}}{\eta \Sigma y'}} \mathbb{E}(w_t^M \Sigma p_f' w_u^M \Sigma p_f') G(\gamma_1^R, \gamma_2^R, \gamma_3^R, u - t, \nu_0) dt,
$$

which is the counterpart of (E.29) for momentum, with (D.49) and $\text{Cov}(w_t^M \Sigma p_f', w_u^M \Sigma p_f')$ from the derivation of $D_{w^M}^{\text{Cov},1}(u)$.

Proposition E.4 computes the Sharpe ratios of the value strategy and the momentum strategy conditional on $(\hat{C}_t, y_t)$ and over investment horizon $T$.

**Proposition E.4.** The Sharpe ratios of the value strategy (4.1) and the momentum strategy (4.2) conditional on $(\hat{C}_t, y_t)$ and over investment horizon $T$ are

$$
SR_{w^V, t, T} = \frac{N_{w^V, t, T}}{\sqrt{D_{w^V, t, T} + D_{w^M, t, T} + D_{w^V, t, T}^\text{Cov}}},
$$

where $j = V$ for value and $j = M$ for momentum,

$$
N_{w^j, t, T} = \frac{1}{T} \int_t^{t+T} \left[ \left( L_2 + \delta_{12, u-t}(\hat{C}_t - \bar{C}) + \delta_{3, u-t}(y_t - \bar{y}) \right) \right.
\times \left( L_1 z^j \Sigma p_f' + \left( L_2 z^j + \delta_{12, u-t}(\hat{C}_t - \bar{C}) + \delta_{3, u-t}(y_t - \bar{y}) \right) p_f \Sigma^2 p_f' \right] du,
$$

$$
D_{w^j, t, T} = \frac{1}{T} \int_t^{t+T} \left[ \left( L_1 z^j \Sigma p_f' + \left( L_2 z^j + \delta_{12, u-t}(\hat{C}_t - \bar{C}) + \delta_{3, u-t}(y_t - \bar{y}) \right) p_f \Sigma^2 p_f' \right) \right.
\times \left( \left( L_2 z^j + \delta_{12, u-t}(\hat{C}_t - \bar{C}) + \delta_{3, u-t}(y_t - \bar{y}) \right)^2 + C_{u-t, u-t}^j \right) \Delta_3
\left. + C_{u-t, u-t}^{j, \Sigma} \Delta_4 \right] du,
$$

and $(D_{w^j, t, T}^{\text{Cov}}, D_{w^j, t, T}^{\text{Cov}})$ are defined in Lemma E.4, with

$$
D_{w^V, t, T}^{\text{Cov},1}(u, u') = \left( L_2 + \delta_{12, u-t}(\hat{C}_t - \bar{C}) + \delta_{3, u-t}(y_t - \bar{y}) \right)
\times \left( L_1 z^j \Sigma p_f' + \left( L_2 z^j + \delta_{12, u-t}(\hat{C}_t - \bar{C}) + \delta_{3, u-t}(y_t - \bar{y}) \right) p_f \Sigma^2 p_f' \right) C_{u-t, u-t}^{j, \Sigma} p_f',
$$

$$
D_{w^V, t, T}^{\text{Cov},2}(u, u') = \left( L_2 + \delta_{12, u-t}(\hat{C}_t - \bar{C}) + \delta_{3, u-t}(y_t - \bar{y}) \right)
\times \left( L_1 z^j \Sigma p_f' + \left( L_2 z^j + \delta_{12, u-t}(\hat{C}_t - \bar{C}) + \delta_{3, u-t}(y_t - \bar{y}) \right) p_f \Sigma^2 p_f' \right) C_{u-t, u-t}^{j, \Sigma} p_f',
$$

$$
D_{w^V, t, T}^{\text{Cov},3}(u, u') = \left( L_1 z^j \Sigma p_f' + \left( L_2 z^j + \delta_{12, u-t}(\hat{C}_t - \bar{C}) + \delta_{3, u-t}(y_t - \bar{y}) \right) p_f \Sigma^2 p_f' \right).
$$
\[ D_{u,t}^{\text{Cov},1} (u, u') = \left( L_1 z^j \eta \Sigma_p' p_f + \left( L_2 z^j + \delta_{12, u-t} (\mathcal{C}_t - \bar{C}) + \delta_{3, u-t} (y_t - y_i) \right) p_f \Sigma^2 p_f' \right) \times \left( L_1 z^j \eta \Sigma_p' p_f + \left( L_2 z^j + \delta_{12, u-t} (\mathcal{C}_t - \bar{C}) + \delta_{3, u-t} (y_t - y_i) \right) p_f \Sigma^2 p_f' \right) C_{u-t, u-t}^{\Lambda}, \]

Proof: The Sharpe ratio has the form in Lemma E.1. To compute \( N_{u, t, T} \), we note that (D.1) and (E.1) imply

\[ N_{u, t, T} = \frac{1}{T} \int_t^{t+T} \left( f + \frac{k \Delta}{\eta \Sigma' \eta} \right) \mathbb{E}_{\mathcal{I}_t} \left( \Lambda_u w'^i \Sigma p_f' \right) du, \]
and we compute the integrand using the decomposition in (D.17) together with (D.24)-(D.26), (D.56) and (D.57). To compute \( D_{w^j,t,T} \), we note that (D.2) and (E.2) imply

\[
D_{w^j,t,T} \equiv \frac{1}{T} \int_t^{t+T} \left\{ f \left[ \mathbb{E}_{\mathcal{I}_t} \left( w^j_u \Sigma (w^j_u)' \right) - \frac{\mathbb{E}_{\mathcal{I}_t} \left[ (w^j_u \Sigma \eta')^2 \right]}{\eta \Sigma \eta'} \right] + k \mathbb{E}_{\mathcal{I}_t} \left[ (w^j_u \Sigma p_f')^2 \right] \right\} du,
\]

and we compute the integrand using the decomposition in (D.18) together with (D.24), (D.28) (D.56) and (D.59). To compute \( \{D_{\text{Cov}^1, i}(u)\}_{i=1,...,6} \), we use their definitions in Lemma E.4 together with (D.24)-(D.29) and (D.56)-(D.59). To compute \( \{D_{\text{Cov}^2, i}(u)\}_{i=1,2} \), we use their definitions in Lemma E.4 and proceed as in the proofs of Propositions E.2 and E.3 replacing unconditional expectations \( \mathbb{E}(w^j_t \Sigma p_f' \Lambda_t) \) and \( \mathbb{E}(w^j_t \Sigma p_f' w^j_u \Sigma p_f') \) by conditional expectations \( \mathbb{E}_{\mathcal{I}_t}(w^j_t \Sigma p_f' \Lambda_t) \) and \( \mathbb{E}_{\mathcal{I}_t}(w^j_t \Sigma p_f' w^j_u \Sigma p_f') \).

Lemma E.6 computes the unconditional covariance between the returns of (the index-adjusted versions of) two strategies \((w^A_t, w^B_t)\) over investment horizon \(T\).

**Lemma E.6.** The unconditional covariance between the returns of \((w^A_t, w^B_t)\) over investment horizon \(T\) is given by

\[
\frac{1}{T} \text{Cov} \left( \int_0^T \hat{w}^A_t dR_t, \int_0^T \hat{w}^B_t dR_t \right) = G_{w^A,w^B} + G_{w^A,w^B,T}^{\text{Cov}^1} + G_{w^A,w^B,T}^{\text{Cov}^2},
\]

where

\[
G_{w^A,w^B} = \frac{1}{dt} \text{Cov} \left( \hat{w}^A_t dR_t, \hat{w}^B_t dR_t \right)
\]

\[
= f \left[ \mathbb{E} \left( w^A_t \Sigma (w^B_t)' \right) - \frac{\mathbb{E} \left( w^A_t \Sigma \eta' w^B_t \Sigma \eta' \right)}{\eta \Sigma \eta'} \right] + k \mathbb{E} \left( w^A_t \Sigma p_f' w^B_t \Sigma p_f' \right),
\]

is the unconditional version of \( G_{w^A,w^B,t} \) defined in Lemma D.6, and

\[
G_{w^A,w^B,T}^{\text{Cov}^1} = \frac{1}{dt} \int_t^{t+T} \left( 1 - \frac{u-t}{T} \right) \left\{ \text{Cov} \left[ \hat{w}^A_t \mathbb{E}_t(dR_t), \hat{w}^B_u \mathbb{E}_u(dR_u) \right] + \text{Cov} \left[ \hat{w}^B_t \mathbb{E}_t(dR_t), \hat{w}^A_u \mathbb{E}_u(dR_u) \right] \right\},
\]

(E.33)
\[
\mathcal{G}_{w^A, w^B, T}^{\text{Cov}2} = \frac{1}{dt} \int_t^{t+T} \left( 1 - \frac{u - t}{T} \right) \left( \mathbb{E} \left\{ \hat{w}^A_t \text{Cov}_t \left[ dR_t, \hat{w}^B_u \mathbb{E}_u (dR_u) \right] \right\} + \mathbb{E} \left\{ \hat{w}^B_t \text{Cov}_t \left[ dR_t, \hat{w}^A_u \mathbb{E}_u (dR_u) \right] \right\} \right).
\]

(E.34)

When the strategies \((w^A_t, w^B_t)\) are linear, \(\mathcal{G}_{w^A, w^B, T}^{\text{Cov}1} = \sum_{i=1}^6 \int_t^{t+T} \left( 1 - \frac{u - t}{T} \right) \left[ \mathcal{G}_{w^A, w^B}^{\text{Cov}1,i} (u) + \mathcal{G}_{w^B, w^A}^{\text{Cov}1,i} (u) \right] du,
\]

where

\[
\mathcal{G}_{w^A, w^B}^{\text{Cov}1,i} (u) \equiv \left( f + \frac{k \Delta}{\eta \Sigma \eta'} \right)^2 \mathbb{E} \left( \Lambda_t \right)^2 \text{Cov} (w^A_t \Sigma \mathbf{p}'_f, w^B_t \Sigma \mathbf{p}'_f),
\]

and \(\mathcal{G}_{w^A, w^B, T}^{\text{Cov}2} = \sum_{i=1}^2 \int_t^{t+T} \left( 1 - \frac{u - t}{T} \right) \left[ \mathcal{G}_{w^A, w^B}^{\text{Cov}2,i} (u) + \mathcal{G}_{w^B, w^A}^{\text{Cov}2,i} (u) \right] du,
\]

where

\[
\mathcal{G}_{w^A, w^B}^{\text{Cov}2,i} (u) \equiv \frac{1}{dt} \left( f + \frac{k \Delta}{\eta \Sigma \eta'} \right) \mathbb{E} \left[ \hat{w}^A_t \Lambda_u \text{Cov}_t \left( dR_t, w^B_u \Sigma \mathbf{p}'_f \right) \right],
\]

for \(j, k \in \{A, B\} \) and \(j \neq k\).

**Proof:** The covariance between the returns of \((w^A_t, w^B_t)\) conditional on \(\mathcal{I}_t\) and over investment horizon \(T\) is given by

\[
\frac{1}{T} \text{Cov}_{\mathcal{I}_t} \left( \int_t^{t+T} \hat{w}^A_u dR_u, \int_t^{t+T} \hat{w}^B_u dR_u \right) = \frac{1}{T} \int_t^{t+T} \text{Cov}_{\mathcal{I}_t} \left( \hat{w}^A_u dR_u, \hat{w}^B_u dR_u \right) + \frac{1}{T} \int_{u'=t}^{t+T} \int_{u'=t}^{t+T} \text{Cov}_{\mathcal{I}_t} \left( \hat{w}^A_u dR_u, \hat{w}^B_u dR_{u'} \right)
\]

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\[ \frac{1}{T} \int_{t}^{t+T} \text{Cov}_{\mathcal{I}_t} (\hat{w}_u^A dR_u, \hat{w}_u^B dR_u) + \frac{1}{T} \int_{u=t}^{t+T} \int_{u'=u}^{t+T} \text{Cov}_{\mathcal{I}_t} (\hat{w}_u^A dR_u, \hat{w}_{u'}^B dR_{u'}) \]

\[ + \frac{1}{T} \int_{u=t}^{t+T} \int_{u'=u}^{t+T} \text{Cov}_{\mathcal{I}_t} (\hat{w}_u^B dR_u, \hat{w}_{u'}^A dR_{u'}) \]  \hspace{1cm} (E.35)

where the first step follows by separating the covariance between contemporaneous returns and the covariance between lagged returns. Proceeding as in the derivation of (E.7), we can write the second term in (E.35) as

\[ \frac{1}{T} \int_{u=t}^{t+T} \int_{u'=u}^{t+T} \{ \text{Cov}_{\mathcal{I}_t} (\hat{w}_u^A E_u(dR_u), \hat{w}_{u'}^B E_{u'}(dR_{u'})) + \mathbb{E}_{\mathcal{I}_t} [\hat{w}_u^B \text{Cov}_u(dR_u, \hat{w}_{u'}^A E_{u'}(dR_{u'}))] \} \]  \hspace{1cm} (E.36)

and the third term as

\[ \frac{1}{T} \int_{u=t}^{t+T} \int_{u'=u}^{t+T} \{ \text{Cov}_{\mathcal{I}_t} (\hat{w}_u^B E_u(dR_u), \hat{w}_{u'}^A E_{u'}(dR_{u'})) + \mathbb{E}_{\mathcal{I}_t} [\hat{w}_u^B \text{Cov}_u(dR_u, \hat{w}_{u'}^A E_{u'}(dR_{u'}))] \} \]  \hspace{1cm} (E.37)

When the covariance is unconditional (\( \mathcal{I}_t = \emptyset \)), we can proceed as in the proof of Lemma E.2 to show that the first term in (E.35) becomes \( \mathcal{G}_{w^A,w^B} \), and (E.36) and (E.37) become \( \mathcal{G}_{w^A,w^B}^{\text{cov1}} \) and \( \mathcal{G}_{w^A,w^B}^{\text{cov2}} \), respectively. When the strategies \( (w^A_t, w^B_t) \) are linear, we can proceed as in the proof of Lemma E.5 to show

\[ \mathcal{G}_{w^A,w^B}^{\text{cov1}} = \sum_{i=1}^{6} \int_{t}^{t+T} (1 - \frac{u-t}{T}) \left[ \mathcal{G}_{w^A,w^B}^{\text{cov1},i}(u) + \mathcal{G}_{w^A,w^A}^{\text{cov1},i}(u) \right] du \]  \hspace{1cm} (E.36)

\[ \mathcal{G}_{w^A,w^B}^{\text{cov2}} = \sum_{i=1}^{2} \int_{t}^{t+T} (1 - \frac{u-t}{T}) \left[ \mathcal{G}_{w^A,w^B}^{\text{cov2},i}(u) + \mathcal{G}_{w^A,w^A}^{\text{cov2},i}(u) \right] du \]  \hspace{1cm} (E.37)

as in the proposition.

Proposition E.5 computes the unconditional instantaneous correlation of a general strategy of the form \( w_t = (\delta_0 + \delta_1 \hat{C}_t + \delta_2 C_t + \delta_3 \hat{y}_t) p_f \) with value and momentum.

Proposition E.5. The unconditional instantaneous correlation of a strategy \( w_t = (\delta_0 + \delta_1 \hat{C}_t + \delta_2 C_t + \delta_3 \hat{y}_t) p_f \) with value and momentum is

\[ \text{Corr}(\hat{w}_t^j dR_t, \hat{w}_t dR_t) = \frac{\mathcal{G}_{w^j,w}^{\text{cov}}}{\sqrt{\mathcal{D}_{w^j} \mathcal{D}_w}}, \]  \hspace{1cm} where \( j = V \) for value and \( j = M \) for momentum, \( \mathcal{D}_w \) is derived in Proposition E.1, \( \{\mathcal{D}_{w^j}\}_{j=V,M} \) are derived in Propositions D.2 and D.5,

\[ \mathcal{G}_{w^V,w} = \left( f + \frac{k \Delta}{\eta \Sigma f} \right) \left[ \left( \frac{L_1}{r} \eta \Sigma^2 f' + \frac{L_2}{r} p_f \Sigma^2 p_f' \right) (\delta_0 + (\delta_1 + \delta_2) \hat{C} + \delta_3 \hat{y}_t) + \left( -\frac{1 - \epsilon}{r + \kappa} K_1(\delta_1, \delta_3, 0, \nu_0) + H(\gamma_1, \gamma_2, \gamma_3, \delta_1, \delta_2, \delta_3, 0, \nu_0) \right) p_f \Sigma^2 p_f' \right] , \]
Proposition E.6. The unconditional correlation of a strategy \( w_t = (\delta_0 + \delta_1 \hat{C}_t + \delta_2 C_t + \delta_3 y_t) p_f \) with value and momentum over investment horizon \( T \) is

\[
\text{Corr} \left( \int_t^{t+T} \hat{w}_u^y dR_u, \int_t^{t+T} \hat{w}_u dR_u \right) = \frac{\mathcal{G}_{w^y,w} + \mathcal{G}_{w^y,w,T} + \mathcal{G}_{w^y,w,T}^\text{Cov}}{\sqrt{\mathcal{D}_{w^y} + \mathcal{D}_{w^y,T} + \mathcal{D}_{w^y,T}^\text{Cov}} \left( \mathcal{D}_{w} + \mathcal{D}_{w,T}^\text{Cov} + \mathcal{D}_{w,T}^\text{Cov} \right)}.
\]

Proof: Since \( \mathcal{D}_{w^y} = \Var(\hat{w}_t^y dR_t) \), \( \mathcal{D}_{w^M} = \Var(\hat{w}_t^M dR_t) \) and \( \mathcal{D}_w = \Var(\hat{w}_t dR_t) \), the results on the unconditional instantaneous correlation will follow if we show \( \Cov(\hat{w}_t^y dR_t, \hat{w}_t dR_t) = \mathcal{G}_{w^y,w} dt \) and \( \Cov(\hat{w}_t^M dR_t, \hat{w}_t dR_t) = \mathcal{G}_{w^M,w} dt \). Using Lemma D.6 and noting that \( \eta \Sigma p_f = 0 \) implies \( w_t \Sigma \eta' = 0 \), we find

\[
\Cov \left( \hat{w}_t^y dR_t, \hat{w}_t dR_t \right) = f \mathbb{E} \left( \hat{w}_t^y \Sigma w_t' \right) + k \mathbb{E} \left( \hat{w}_t^y \Sigma p_f w_t \Sigma p_f' \right)
\]

\[
= f \mathbb{E}(w_t^y) \mathbb{E}(w_t') + k \mathbb{E} \left( w_t^y \Sigma p_f \mathbb{E}(w_t) \Sigma p_f' + f \Cov \left( w_t^y, \Sigma w_t' \right) + k \Cov \left( w_t^y \Sigma p_f', w_t \Sigma p_f' \right) \right)
\]

\[
= \left( f + \frac{k \Delta}{\eta \Sigma p_f} \right) \left[ \mathbb{E}(w_t^y) \Sigma p_f' \left( \delta_0 + (\delta_1 + \delta_2) \hat{C} + \delta_3 \hat{y} \right) + \Cov \left( w_t^y, \delta_0 + \delta_1 \hat{C}_t + \delta_2 C_t + \delta_3 y_t \right) \Sigma p_f' \right]
\]

(E.38)

for \( j = V, M \), where the third step follows from \( w_t = (\delta_0 + \delta_1 \hat{C}_t + \delta_2 C_t + \delta_3 y_t) p_f \). Combining (E.38) with (D.20), and computing \( \Cov \left( w_t^y, \delta_0 + \delta_1 \hat{C}_t + \delta_2 C_t + \delta_3 y_t \right) \) as in the derivation of (D.22), we find \( \Cov(\hat{w}_t^y dR_t, \hat{w}_t dR_t) = \mathcal{G}_{w^y,w} dt \). Combining (E.38) with (D.49), and computing \( \Cov \left( w_t^M, \delta_0 + \delta_1 \hat{C}_t + \delta_2 C_t + \delta_3 y_t \right) \) as in the derivation of (D.52), we find \( \Cov(\hat{w}_t^M dR_t, \hat{w}_t dR_t) = \mathcal{G}_{w^M,w} dt \).

Proposition E.6 computes the unconditional correlation of a general strategy of the form \( w_t = (\delta_0 + \delta_1 \hat{C}_t + \delta_2 C_t + \delta_3 y_t) p_f \) with value and momentum over investment horizon \( T \).
where \( j = V \) for value and \( j = M \) for momentum, \( \{ D_{\omega}, D^{C_{V1}}_{\omega}, D^{C_{V3}}_{\omega} \} \) are derived in Proposition E.1, \( \{ D_{\omega} \}_{j=V,M} \) are derived in Propositions D.2 and D.5, \( \left\{ \left( T^{C_{V1}}_{\omega,T}, T^{C_{V3}}_{\omega,T} \right) \right\}_{j=V,M} \) are derived in Propositions E.2 and E.3, \( \{ G_{\omega,j} \}_{j=V,M} \) are derived in Proposition E.5, and \( \{ \left( G^{C_{V1}}_{\omega,j,T}, G^{C_{V3}}_{\omega,j,T} \right) \}_{j=V,M} \) are defined in Lemma E.6, with

\[
G^{C_{V1},1}_{\omega,V,w} (u) = L^2 \left( -\frac{1 - \epsilon}{r + \kappa} K_2(\delta_1, \delta_3, u - t, \nu_0) + H(\gamma_1, \gamma_2, \gamma_3, \delta_1, \delta_2, \delta_3, u - t, \nu_0) \right) \frac{p_J \Sigma^2 p_J^f \Delta}{\eta \Sigma \eta'},
\]

\[
G^{C_{V1},1}_{\omega,u,V} (u) = L^2 \left( -\frac{1 - \epsilon}{r + \kappa} K_1(\delta_1, \delta_3, u - t, \nu_0) + H(\delta_1, \delta_2, \delta_3, \gamma_1, \gamma_2, \gamma_3, u - t, \nu_0) \right) \frac{p_J \Sigma^2 p_J^f \Delta}{\eta \Sigma \eta'},
\]

\[
G^{C_{V1},1}_{\omega,M,w} (u) = L^2 \left( H(\gamma_1^R, \gamma_2^R, \gamma_3^R, \delta_1, \delta_2, \delta_3, T^-, \nu_2) + G(\delta_1, \delta_2, \delta_3, T^-, \nu_2) \right) \frac{p_J \Sigma^2 p_J^f \Delta}{\eta \Sigma \eta'},
\]

\[
G^{C_{V1},1}_{\omega,M,M} (u) = L^2 \left( H(\delta_1, \delta_2, \delta_3, \gamma_1^R, \gamma_2^R, \gamma_3^R, T^-, \nu_1) + H(\gamma_1^R, \gamma_2^R, \gamma_3^R, \delta_1, \delta_2, \delta_3, T^-, \nu_2) + G(\delta_1, \delta_2, \delta_3, T^-, \nu_2) \right) \frac{p_J \Sigma^2 p_J^f \Delta}{\eta \Sigma \eta'},
\]

\[
G^{C_{V1},2}_{\omega,V,w} (u) = L^2 \left( \left( -\frac{1 - \epsilon}{r + \kappa} K_2(\gamma_1^R, \gamma_3^R, u - t, \nu_0) + H(\gamma_1, \gamma_2, \gamma_3, \gamma_1^R, \gamma_2^R, \gamma_3^R, u - t, \nu_0) \right) \frac{p_J \Sigma^2 p_J^f \Delta}{\eta \Sigma \eta'}, \right)
\]

\[
G^{C_{V1},2}_{\omega,u,V} (u) = L^2 \left( \left( \frac{L_1}{r} \eta \Sigma \Sigma^2 p_f, \frac{L_2}{r} p_f \Sigma^2 p_f, \right) H(\delta_1, \delta_2, \delta_3, \gamma_1^R, \gamma_2^R, \gamma_3^R, u - t, \nu_0) \frac{\Delta}{\eta \Sigma \eta'}, \right)
\]

\[
G^{C_{V1},2}_{\omega,M,w} (u) = L^2 \left( \left( -\frac{1 - \epsilon}{r + \kappa} K_2(\gamma_1^R, \gamma_3^R, \gamma_1^R, \gamma_2^R, \gamma_3^R, \gamma_3^R, u - t, \nu_0) \frac{p_J \Sigma^2 p_J^f \Delta}{\eta \Sigma \eta'}, \right)
\]

\[
G^{C_{V1},2}_{\omega,M,M} (u) = L^2 \left( \left( -\frac{1 - \epsilon}{r + \kappa} K_2(\gamma_1^R, \gamma_3^R, \gamma_1^R, \gamma_2^R, \gamma_3^R, \gamma_3^R, u - t, \nu_0) \frac{p_J \Sigma^2 p_J^f \Delta}{\eta \Sigma \eta'}, \right)
\]

\[
G^{C_{V1},3}_{\omega,V,w} (u) = L^2 \left( \left( -\frac{1 - \epsilon}{r + \kappa} K_2(\gamma_1^R, \gamma_3^R, u - t, \nu_0) + H(\gamma_1^R, \gamma_2^R, \gamma_3^R, \gamma_1^R, \gamma_2^R, \gamma_3^R, u - t, \nu_0) \frac{p_J \Sigma^2 p_J^f \Delta}{\eta \Sigma \eta'}, \right)
\]

\[
G^{C_{V1},3}_{\omega,u,V} (u) = L^2 \left( \left( \frac{L_1}{r} \eta \Sigma \Sigma^2 p_f, \frac{L_2}{r} p_f \Sigma^2 p_f, \right) H(\gamma_1^R, \gamma_2^R, \gamma_3^R, \gamma_1^R, \gamma_2^R, \gamma_3^R, \gamma_3^R, u - t, \nu_0) \frac{\Delta}{\eta \Sigma \eta'}, \right)
\]

\[
G^{C_{V1},3}_{\omega,M,w} (u) = L^2 \left( \left( -\frac{1 - \epsilon}{r + \kappa} K_2(\gamma_1^R, \gamma_3^R, \gamma_1^R, \gamma_2^R, \gamma_3^R, \gamma_3^R, u - t, \nu_0) \frac{p_J \Sigma^2 p_J^f \Delta}{\eta \Sigma \eta'}, \right)
\]

\[
G^{C_{V1},3}_{\omega,M,M} (u) = L^2 \left( \left( -\frac{1 - \epsilon}{r + \kappa} K_2(\gamma_1^R, \gamma_3^R, \gamma_1^R, \gamma_2^R, \gamma_3^R, \gamma_3^R, u - t, \nu_0) \frac{p_J \Sigma^2 p_J^f \Delta}{\eta \Sigma \eta'}, \right)
\]
\[ G_{w, w}^{\text{Cov}, 3}(u) = L_2 \left( \delta_0 + (\delta_1 + \delta_2)C + \delta_3 \bar{y} \right) \left( H(\gamma_1^R, \gamma_2^R, \gamma_3^R, \gamma_1^R, \gamma_2^R, \gamma_3^R, T', \nu_1) + H(\gamma_1^R, \gamma_2^R, \gamma_3^R, \gamma_1^R, \gamma_2^R, \gamma_3^R, T', \nu_2) + G(\gamma_1^R, \gamma_2^R, \gamma_3^R, T', \nu_2) \right) \frac{\eta \Sigma^2 \bar{p}_f \Delta}{\eta \Sigma \eta'}, \]

\[ G_{w, w}^{\text{Cov}, 4}(u) = G_{w, w}^{\text{Cov}, 4}(u) = \left( L_1 \eta \Sigma^2 \bar{p}_f + L_2 \eta \Sigma^2 \bar{p}_f \right) \left( \delta_0 + (\delta_1 + \delta_2)C + \delta_3 \bar{y} \right) \]

\[ \times H(\gamma_1^R, \gamma_2^R, \gamma_3^R, \gamma_1^R, \gamma_2^R, \gamma_3^R, u - t, \nu_0) \frac{\Delta}{\eta \Sigma \eta'}, \]

\[ G_{w, w}^{\text{Cov}, 4}(u) = G_{w, w}^{\text{Cov}, 4}(u) = \left( L_1 \tau \eta \Sigma^2 \bar{p}_f + L_2 \tau \eta \Sigma^2 \bar{p}_f \right) \left( \delta_0 + (\delta_1 + \delta_2)C + \delta_3 \bar{y} \right) \]

\[ \times H(\gamma_1^R, \gamma_2^R, \gamma_3^R, \gamma_1^R, \gamma_2^R, \gamma_3^R, u - t, \nu_0) \frac{\Delta}{\eta \Sigma \eta'}, \]

\[ G_{w, w}^{\text{Cov}, 5}(u) = H(\gamma_1^R, \gamma_2^R, \gamma_3^R, \gamma_1^R, \gamma_2^R, \gamma_3^R, u - t, \nu_0) \]

\[ \times \left( \frac{1 - \epsilon}{r + \kappa} K_2(\delta_1, \delta_3, u - t, \nu_0) + H(\gamma_1^R, \gamma_2^R, \gamma_3^R, \gamma_1^R, \gamma_2^R, \gamma_3^R, u - t, \nu_0) \right) \frac{\eta \Sigma^2 \bar{p}_f \Delta}{\eta \Sigma \eta'}, \]

\[ G_{w, w}^{\text{Cov}, 5}(u) = H(\gamma_1^R, \gamma_2^R, \gamma_3^R, \gamma_1^R, \gamma_2^R, \gamma_3^R, u - t, \nu_0) \]

\[ \times \left( \frac{1 - \epsilon}{r + \kappa} K_1(\delta_1, \delta_3, u - t, \nu_0) + H(\delta_2, \delta_3, \gamma_1^R, \gamma_2^R, \gamma_3^R, u - t, \nu_0) \right) \frac{\eta \Sigma^2 \bar{p}_f \Delta}{\eta \Sigma \eta'}, \]

\[ G_{w, w}^{\text{Cov}, 5}(u) = H(\gamma_1^R, \gamma_2^R, \gamma_3^R, \gamma_1^R, \gamma_2^R, \gamma_3^R, u - t, \nu_0) \]

\[ \times \left( H(\gamma_1^R, \gamma_2^R, \gamma_3^R, \gamma_1^R, \gamma_2^R, \gamma_3^R, u - t, \nu_0) \right) \frac{\eta \Sigma^2 \bar{p}_f \Delta}{\eta \Sigma \eta'}, \]

\[ G_{w, w}^{\text{Cov}, 5}(u) = H(\gamma_1^R, \gamma_2^R, \gamma_3^R, \gamma_1^R, \gamma_2^R, \gamma_3^R, u - t, \nu_0) \]

\[ \times \left( H(\gamma_1^R, \gamma_2^R, \gamma_3^R, \gamma_1^R, \gamma_2^R, \gamma_3^R, u - t, \nu_0) \right) \frac{\eta \Sigma^2 \bar{p}_f \Delta}{\eta \Sigma \eta'}, \]

\[ G_{w, w}^{\text{Cov}, 6}(u) = H(\gamma_1^R, \gamma_2^R, \gamma_3^R, \gamma_1^R, \gamma_2^R, \gamma_3^R, u - t, \nu_0) \]

\[ \times \left( \frac{1 - \epsilon}{r + \kappa} K_2(\gamma_1^R, \gamma_3^R, u - t, \nu_0) + H(\gamma_1^R, \gamma_2^R, \gamma_3^R, \gamma_1^R, \gamma_2^R, \gamma_3^R, u - t, \nu_0) \right) \frac{\eta \Sigma^2 \bar{p}_f \Delta}{\eta \Sigma \eta'}, \]

\[ G_{w, w}^{\text{Cov}, 6}(u) = H(\gamma_1^R, \gamma_2^R, \gamma_3^R, \gamma_1^R, \gamma_2^R, \gamma_3^R, u - t, \nu_0) \]

\[ \times \left( \frac{1 - \epsilon}{r + \kappa} K_1(\gamma_1^R, \gamma_3^R, u - t, \nu_0) + H(\gamma_1^R, \gamma_2^R, \gamma_3^R, \gamma_1^R, \gamma_2^R, \gamma_3^R, u - t, \nu_0) \right) \frac{\eta \Sigma^2 \bar{p}_f \Delta}{\eta \Sigma \eta'}, \]

\[ G_{w, w}^{\text{Cov}, 6}(u) = H(\gamma_1^R, \gamma_2^R, \gamma_3^R, \gamma_1^R, \gamma_2^R, \gamma_3^R, u - t, \nu_0) \]

\[ \times \left( \frac{1 - \epsilon}{r + \kappa} K_1(\gamma_1^R, \gamma_3^R, u - t, \nu_0) + H(\gamma_1^R, \gamma_2^R, \gamma_3^R, \gamma_1^R, \gamma_2^R, \gamma_3^R, u - t, \nu_0) \right) \frac{\eta \Sigma^2 \bar{p}_f \Delta}{\eta \Sigma \eta'}, \]

\[ G_{w, w}^{\text{Cov}, 6}(u) = (H(\gamma_1^R, \gamma_2^R, \gamma_3^R, \gamma_1^R, \gamma_2^R, \gamma_3^R, T', \nu_2) + G(\gamma_1^R, \gamma_2^R, \gamma_3^R, T', \nu_2)) \]
\[
G_{\omega,\omega,M}^{\text{Cov},1}(u) = H(\delta_1, \delta_2, \delta_3, \gamma_1, \gamma_2, \gamma_3, u - t, \nu_0) \left( H(\gamma_1, \gamma_2, \gamma_3, \gamma_1, \gamma_2, \gamma_3, T', \nu_1) + H(\gamma_1, \gamma_2, \gamma_3, \gamma_1, \gamma_2, \gamma_3, T', \nu_2) + G(\gamma_1, \gamma_2, \gamma_3, T', \nu_2) \right) \frac{p_f \Sigma^2 \sigma_f' \Delta}{\eta \Sigma' f},
\]

\[
G_{\omega,\omega,V}^{\text{Cov},2}(u) = \left[ \frac{L_1 L_2}{r} \eta \Sigma^2 \sigma_f' \left( \frac{L_2}{r} - \frac{1 - \epsilon}{r + \kappa} K_2(\gamma_1, \gamma_2, \gamma_3, u - t, \nu_0) + H(\gamma_1, \gamma_2, \gamma_3, \gamma_1, \gamma_2, \gamma_3, u - t, \nu_0) \right) \right] \frac{p_f \Sigma^2 \sigma_f' \Delta}{\eta \Sigma' f},
\]

\[
G_{\omega,\omega,M}^{\text{Cov},1}(u) = \left[ L_2 \left( \delta_0 + (\delta_1 + \delta_2) \tilde{C} + \delta_3 \tilde{y} \right) + H(\delta_1, \delta_2, \delta_3, \gamma_1, \gamma_2, \gamma_3, u - t, \nu_0) \right] \frac{p_f \Sigma^2 \sigma_f' \Delta}{\eta \Sigma' f},
\]

\[
G_{\omega,\omega,V}^{\text{Cov},1}(u) = \left[ L_2 \left( \delta_0 + (\delta_1 + \delta_2) \tilde{C} + \delta_3 \tilde{y} \right) + H(\delta_1, \delta_2, \delta_3, \gamma_1, \gamma_2, \gamma_3, u - t, \nu_0) \right] \frac{p_f \Sigma^2 \sigma_f' \Delta}{\eta \Sigma' f},
\]

\[
G_{\omega,\omega,M}^{\text{Cov},2}(u) = \left[ \frac{L_1 L_2}{r} \eta \Sigma^2 \sigma_f' \left( \delta_0 + (\delta_1 + \delta_2) \tilde{C} + \delta_3 \tilde{y} \right) + \left( \frac{1 - \epsilon}{r + \kappa} K_2(\delta_1, \delta_3, u - t, \nu_0) \right) \right] \frac{p_f \Sigma^2 \sigma_f' \Delta}{\eta \Sigma' f},
\]

\[
G_{\omega,\omega,V}^{\text{Cov},2}(u) = \left[ \frac{L_1 L_2}{r} \eta \Sigma^2 \sigma_f' \left( \delta_0 + (\delta_1 + \delta_2) \tilde{C} + \delta_3 \tilde{y} \right) + \left( \frac{1 - \epsilon}{r + \kappa} K_1(\delta_1, \delta_3, u - t, \nu_0) \right) \right] \frac{p_f \Sigma^2 \sigma_f' \Delta}{\eta \Sigma' f},
\]

\[
G_{\omega,\omega,M}^{\text{Cov},2}(u) = \left[ L_1 \tau \eta \Sigma^2 \sigma_f' + L_2 \eta \Sigma^2 \sigma_f' \right] \left( \delta_0 + (\delta_1 + \delta_2) \tilde{C} + \delta_3 \tilde{y} \right) \frac{p_f \Sigma^2 \sigma_f' \Delta}{\eta \Sigma' f},
\]

\[
G_{\omega,\omega,V}^{\text{Cov},2}(u) = \left[ L_1 \tau \eta \Sigma^2 \sigma_f' + L_2 \eta \Sigma^2 \sigma_f' \right] \left( \delta_0 + (\delta_1 + \delta_2) \tilde{C} + \delta_3 \tilde{y} \right) \frac{p_f \Sigma^2 \sigma_f' \Delta}{\eta \Sigma' f},
\]

\[
G_{\omega,\omega,M}^{\text{Cov},2}(u) = \left[ L_1 \tau \eta \Sigma^2 \sigma_f' + L_2 \eta \Sigma^2 \sigma_f' \right] \left( \delta_0 + (\delta_1 + \delta_2) \tilde{C} + \delta_3 \tilde{y} \right) \frac{p_f \Sigma^2 \sigma_f' \Delta}{\eta \Sigma' f},
\]

\[
G_{\omega,\omega,M}^{\text{Cov},2}(u) = \left[ L_1 \tau \eta \Sigma^2 \sigma_f' + L_2 \eta \Sigma^2 \sigma_f' \right] \left( \delta_0 + (\delta_1 + \delta_2) \tilde{C} + \delta_3 \tilde{y} \right) \frac{p_f \Sigma^2 \sigma_f' \Delta}{\eta \Sigma' f},
\]

\[
G_{\omega,\omega,M}^{\text{Cov},2}(u) = \left[ L_1 \tau \eta \Sigma^2 \sigma_f' + L_2 \eta \Sigma^2 \sigma_f' \right] \left( \delta_0 + (\delta_1 + \delta_2) \tilde{C} + \delta_3 \tilde{y} \right) \frac{p_f \Sigma^2 \sigma_f' \Delta}{\eta \Sigma' f},
\]

\[
G_{\omega,\omega,M}^{\text{Cov},2}(u) = \left[ L_1 \tau \eta \Sigma^2 \sigma_f' + L_2 \eta \Sigma^2 \sigma_f' \right] \left( \delta_0 + (\delta_1 + \delta_2) \tilde{C} + \delta_3 \tilde{y} \right) \frac{p_f \Sigma^2 \sigma_f' \Delta}{\eta \Sigma' f},
\]

\[
G_{\omega,\omega,M}^{\text{Cov},2}(u) = \left[ L_1 \tau \eta \Sigma^2 \sigma_f' + L_2 \eta \Sigma^2 \sigma_f' \right] \left( \delta_0 + (\delta_1 + \delta_2) \tilde{C} + \delta_3 \tilde{y} \right) \frac{p_f \Sigma^2 \sigma_f' \Delta}{\eta \Sigma' f},
\]

\[
G_{\omega,\omega,M}^{\text{Cov},2}(u) = \left[ L_1 \tau \eta \Sigma^2 \sigma_f' + L_2 \eta \Sigma^2 \sigma_f' \right] \left( \delta_0 + (\delta_1 + \delta_2) \tilde{C} + \delta_3 \tilde{y} \right) \frac{p_f \Sigma^2 \sigma_f' \Delta}{\eta \Sigma' f},
\]

\[
G_{\omega,\omega,M}^{\text{Cov},2}(u) = \left[ L_1 \tau \eta \Sigma^2 \sigma_f' + L_2 \eta \Sigma^2 \sigma_f' \right] \left( \delta_0 + (\delta_1 + \delta_2) \tilde{C} + \delta_3 \tilde{y} \right) \frac{p_f \Sigma^2 \sigma_f' \Delta}{\eta \Sigma' f},
\]

\[
G_{\omega,\omega,M}^{\text{Cov},2}(u) = \left[ L_1 \tau \eta \Sigma^2 \sigma_f' + L_2 \eta \Sigma^2 \sigma_f' \right] \left( \delta_0 + (\delta_1 + \delta_2) \tilde{C} + \delta_3 \tilde{y} \right) \frac{p_f \Sigma^2 \sigma_f' \Delta}{\eta \Sigma' f},
\]

\[
G_{\omega,\omega,M}^{\text{Cov},2}(u) = \left[ L_1 \tau \eta \Sigma^2 \sigma_f' + L_2 \eta \Sigma^2 \sigma_f' \right] \left( \delta_0 + (\delta_1 + \delta_2) \tilde{C} + \delta_3 \tilde{y} \right) \frac{p_f \Sigma^2 \sigma_f' \Delta}{\eta \Sigma' f},
\]

\[
G_{\omega,\omega,M}^{\text{Cov},2}(u) = \left[ L_1 \tau \eta \Sigma^2 \sigma_f' + L_2 \eta \Sigma^2 \sigma_f' \right] \left( \delta_0 + (\delta_1 + \delta_2) \tilde{C} + \delta_3 \tilde{y} \right) \frac{p_f \Sigma^2 \sigma_f' \Delta}{\eta \Sigma' f},
\]

\[
G_{\omega,\omega,M}^{\text{Cov},2}(u) = \left[ L_1 \tau \eta \Sigma^2 \sigma_f' + L_2 \eta \Sigma^2 \sigma_f' \right] \left( \delta_0 + (\delta_1 + \delta_2) \tilde{C} + \delta_3 \tilde{y} \right) \frac{p_f \Sigma^2 \sigma_f' \Delta}{\eta \Sigma' f},
\]

\[
G_{\omega,\omega,M}^{\text{Cov},2}(u) = \left[ L_1 \tau \eta \Sigma^2 \sigma_f' + L_2 \eta \Sigma^2 \sigma_f' \right] \left( \delta_0 + (\delta_1 + \delta_2) \tilde{C} + \delta_3 \tilde{y} \right) \frac{p_f \Sigma^2 \sigma_f' \Delta}{\eta \Sigma' f},
\]

\[
G_{\omega,\omega,M}^{\text{Cov},2}(u) = \left[ L_1 \tau \eta \Sigma^2 \sigma_f' + L_2 \eta \Sigma^2 \sigma_f' \right] \left( \delta_0 + (\delta_1 + \delta_2) \tilde{C} + \delta_3 \tilde{y} \right) \frac{p_f \Sigma^2 \sigma_f' \Delta}{\eta \Sigma' f},
\]
where $\mathcal{T}' = (u - t, \tau)$ and $\mathcal{T}'^- = (t - u, \tau)$.

**Proof:** To show the proposition, we need to show that the definitions of \( \{ G_{w_i,w,T}^{\text{Cov},i} (u) \}_{i=1,\ldots,6} \), \( \{ G_{w^{\dagger}_i,w,T}^{\text{Cov},i} (u) \}_{i=1,2} \) in Lemma E.6 yield the equations in the proposition. The equations follow from the derivations in Propositions E.2 and E.3. (These derivations determine the covariances \( \{ \text{Cov}(w^i_j \Sigma p_j, (\delta_0 + \delta_1 \hat{C}_u + \delta_2 C_u + \delta_3 y_u)) \}_{j=V,M} \) and \( \{ \text{Cov}(\delta_0 + \delta_1 \hat{C}_t + \delta_2 C_t + \delta_3 y_t), w^i_j \Sigma p_j' ) \}_{j=V,M} \) by replacing \((\gamma_1^R, \gamma_2^R, \gamma_3^R)\) by \((\delta_1, \delta_2, \delta_3)\).

Proposition E.7 computes the unconditional correlation between the returns of value and momentum strategies over investment horizon $T$.

**Proposition E.7.** The unconditional correlation between the returns of the value strategy (4.1) and the momentum strategy (4.2) over investment horizon $T$ is

\[
\text{Corr} \left( \int_t^{t+T} \hat{w}^{V}_u dR_u, \int_t^{t+T} \hat{w}^{M}_u dR_u \right) = \frac{G_{w^{V},w^{M}} + G_{w^{V},w^{M},T}^{\text{Cov}1} + G_{w^{V},w^{M},T}^{\text{Cov}2}}{\sqrt{(D_{w^{V}} + D_{w^{V},T}^{\text{Cov1}} + D_{w^{V},T}^{\text{Cov2}}) (D_{w^{M}} + D_{w^{M},T}^{\text{Cov1}} + D_{w^{M},T}^{\text{Cov2}})}},
\]

(E.40)

where $D_{w^{V}}$, $D_{w^{M}}$ and $G_{w^{V},w^{M}}$ are derived in Propositions D.2, D.5 and D.7, respectively, \((D_{w^{V},T}^{\text{Cov1}}, D_{w^{V},T}^{\text{Cov2}})\) and \((D_{w^{M},T}^{\text{Cov1}}, D_{w^{M},T}^{\text{Cov2}})\) are derived in Propositions E.2 and E.3, respectively, and \((G_{w^{V},w^{M},T}^{\text{Cov}1}, G_{w^{V},w^{M},T}^{\text{Cov}2})\) are defined in Lemma E.6, with

\[
G_{w^{V},w^{M},T}^{\text{Cov}1,1}(u) = L_2 \left[ \left( -\frac{1 - \epsilon}{r + \kappa} [K_2(\gamma_1^R, \gamma_3^R, \mathcal{T}', \nu_1) + K_1(\gamma_1^R, \gamma_3^R, \mathcal{T}', \nu_2)] - \frac{(1 - \epsilon) \phi^2}{(r + \kappa)^2} \beta_2 \gamma_1 \nu_2(\kappa, \mathcal{T}') \right]
\]

\[
+ H(\gamma_1, \gamma_2, \gamma_3, \gamma_1^R, \gamma_3^R, \mathcal{T}', \nu_1) + H(\gamma_1, \gamma_2, \gamma_3, \gamma_1^R, \gamma_3^R, \mathcal{T}', \nu_2)
\]

\[
+ G(\gamma_1, \gamma_2, \gamma_3, \mathcal{T}', \nu_2) \right] (p_f \Sigma p_f')^2 - \frac{(1 - \epsilon) \phi^2}{(r + \kappa)^2} \nu_2(\kappa, \mathcal{T}') p_f \Sigma^3 p_f',
\]

\[
G_{w^{M},w^{V},T}^{\text{Cov}1,1}(u) = L_2 \left[ \left( -\frac{1 - \epsilon}{r + \kappa} [K_2(\gamma_1^R, \gamma_3^R, \mathcal{T}', -\nu_1) - \frac{(1 - \epsilon) \phi^2}{(r + \kappa)^2} \beta_2 \gamma_1 \nu_2(\kappa, -\mathcal{T}') + H(\gamma_1, \gamma_2, \gamma_3, \gamma_1^R, \gamma_3^R, \mathcal{T}', -\nu_2)
\]

\[
+ G(\gamma_1, \gamma_2, \gamma_3, \mathcal{T}', -\nu_2) \right] (p_f \Sigma p_f')^2 - \frac{(1 - \epsilon) \phi^2}{(r + \kappa)^2} \nu_2(\kappa, -\mathcal{T}') p_f \Sigma^3 p_f'
\]

\[
G_{w^{V},w^{M},T}^{\text{Cov}2}(u) = L_2 \left( \gamma_1 \tau \Sigma^2 p_f + L_2 \tau p_f \Sigma^2 p_f' \right)
\]

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\begin{align*}
&\times \left( -\frac{1-\epsilon}{r + \kappa} R^2(\gamma_1^R, \gamma_3^R, u - t, \nu_0) + H(\gamma_1, \gamma_2, \gamma_3, \gamma_1^R, \gamma_2^R, \gamma_3^R, u - t, \nu_0) \right) p_f \Sigma^2 p_f', \\
G_{w,M}^{Cov,1,2}(u) &= L_2 \left( \frac{L_1}{r} \eta \Sigma^2 p_f' + \frac{L_2}{r} p_f \Sigma^2 p_f' \right) \\
&\times \left( H(\gamma_1^R, \gamma_2^R, \gamma_3^R, \gamma_1^R, \gamma_2^R, \gamma_3^R, T', \nu_2) + G(\gamma_1^R, \gamma_2^R, \gamma_3^R, T', \nu_2) \right) p_f \Sigma^2 p_f', \\
G_{w,V,w,M}^{Cov,1,3}(u) &= L_2 \left( \frac{L_1}{r} \eta \Sigma^2 p_f' + \frac{L_2}{r} p_f \Sigma^2 p_f' \right) \\
&\times \left( H(\gamma_1^R, \gamma_2^R, \gamma_3^R, \gamma_1^R, \gamma_2^R, \gamma_3^R, T', \nu_1) \\
&+ H(\gamma_1^R, \gamma_2^R, \gamma_3^R, \gamma_1^R, \gamma_2^R, \gamma_3^R, T', \nu_2) + G(\gamma_1^R, \gamma_2^R, \gamma_3^R, T', \nu_2) \right) p_f \Sigma^2 p_f', \\
G_{w,V,w,M}^{Cov,1,3}(u) &= L_2 \left( L_1 \eta \Sigma^2 p_f' + L_2 p_f \Sigma^2 p_f' \right) \\
&\times \left( -\frac{1-\epsilon}{r + \kappa} R^2(\gamma_1^R, \gamma_3^R, u - t, \nu_0) + H(\gamma_1, \gamma_2, \gamma_3, \gamma_1^R, \gamma_2^R, \gamma_3^R, u - t, \nu_0) \right) p_f \Sigma^2 p_f', \\
G_{w,V,w,M}^{Cov,1,4}(u) &= G_{w,V,w,M}^{Cov,1,4}(u) = \left( L_1 \eta \Sigma^2 p_f' + L_2 p_f \Sigma^2 p_f' \right)^2 \frac{\tau}{r} H(\gamma_1^R, \gamma_2^R, \gamma_3^R, \gamma_1^R, \gamma_2^R, \gamma_3^R, u - t, \nu_0), \\
G_{w,V,w,M}^{Cov,1,5}(u) &= H(\gamma_1^R, \gamma_2^R, \gamma_3^R, \gamma_1^R, \gamma_2^R, \gamma_3^R, u - t, \nu_0) \left[ \left( -\frac{1-\epsilon}{r + \kappa} K_1(\gamma_1^R, \gamma_3^R, T', \nu_1) + K_1(\gamma_1^R, \gamma_3^R, T', \nu_2) \right) \\
&- \frac{(1-\epsilon)\phi^2}{(r + \kappa)^2} \beta_2 \gamma_1^R \nu_2(\kappa, T') + H(\gamma_1, \gamma_2, \gamma_3, \gamma_1^R, \gamma_2^R, \gamma_3^R, T', \nu_1) + H(\gamma_1, \gamma_2, \gamma_3, \gamma_1^R, \gamma_2^R, \gamma_3^R, T', \nu_2) \\
&+ G(\gamma_1, \gamma_2, \gamma_3, T', \nu_2) \right] \left( p_f \Sigma p_f' \right)^2 - \frac{(1-\epsilon)\phi^2}{(r + \kappa)^2} \nu_2(\kappa, T') p_f \Sigma^2 p_f', \\
G_{w,V,w,M}^{Cov,1,6}(u) &= \left( -\frac{1-\epsilon}{r + \kappa} K_2(\gamma_1^R, \gamma_3^R, u - t, \nu_0) + H(\gamma_1, \gamma_2, \gamma_3, \gamma_1^R, \gamma_2^R, \gamma_3^R, u - t, \nu_0) \right) \\
&\times \left( H(\gamma_1^R, \gamma_2^R, \gamma_3^R, \gamma_1^R, \gamma_2^R, \gamma_3^R, T', \nu_1) + H(\gamma_1^R, \gamma_2^R, \gamma_3^R, \gamma_1^R, \gamma_2^R, \gamma_3^R, T', \nu_2) \\
&+ G(\gamma_1^R, \gamma_2^R, \gamma_3^R, T', \nu_2) \right) \left( p_f \Sigma^2 p_f' \right)^2, \\
G_{w,V,w,M}^{Cov,1,6}(u) &= \left( H(\gamma_1^R, \gamma_2^R, \gamma_3^R, \gamma_1^R, \gamma_2^R, \gamma_3^R, T', \nu_2) + G(\gamma_1^R, \gamma_2^R, \gamma_3^R, T', \nu_2) \right) \\
&\times \left( -\frac{1-\epsilon}{r + \kappa} K_1(\gamma_1^R, \gamma_3^R, u - t, \nu_0) + H(\gamma_1^R, \gamma_2^R, \gamma_3^R, \gamma_1^R, \gamma_2^R, \gamma_3^R, u - t, \nu_0) \right) \left( p_f \Sigma^2 p_f' \right)^2, \\
\end{align*}
\[ G_{\text{Cov,1}}^{uV,\text{w}^H}(u) = \left[ \frac{L_1L_2}{r} \eta \Sigma^2 p_f' + \left( \frac{L_2^2}{r} - \frac{1 - \epsilon}{r + \kappa} K_2(\gamma_1 R, \gamma_3 R, u, t, \nu_0) + H(\gamma_1, \gamma_2, \gamma_3, \gamma_1^R, \gamma_2^R, \gamma_3^R, u - t, \nu_0) \right) \right] \times p_f \Sigma^2 p_f' \\
\times p_f \Sigma^2 p_f' \left( G(\gamma_1 R, \gamma_2 R, \gamma_3 R, T', \nu_1) + k1_{\{\tau + t - u > 0\}} \right) p_f \Sigma^2 p_f' \\
+ \left[ \frac{L_1L_2}{r} \left( \eta \Sigma^2 p_f' - \frac{\eta \Sigma^2 p_f' \eta \Sigma^2 p_f'}{\eta \Sigma t'} \right) \right] \left( \frac{L_2^2}{r} - \frac{1 - \epsilon}{r + \kappa} K_2(\gamma_1^R, \gamma_3^R, u, t, \nu_0) \\
+ H(\gamma_1, \gamma_2, \gamma_3, \gamma_1^R, \gamma_2^R, \gamma_3^R, u - t, \nu_0) \right) \left( p_f \Sigma^2 p_f' - \frac{(\eta \Sigma^2 p_f')^2}{\eta \Sigma t'} \right) \right] f1_{\{\tau + t - u > 0\}}, \]

\[ G_{\text{Cov,1}}^{uV,\text{wV}}(u) = \left[ L_1L_2 \eta \Sigma^2 p_f' + \left( L_2^2 \tau + H(\gamma_1 R, \gamma_2 R, \gamma_3 R, \gamma_1, \gamma_2, \gamma_3, \gamma_1^R, \gamma_2^R, \gamma_3^R, T' - \nu_2) + G(\gamma_1 R, \gamma_2 R, \gamma_3 R, T' - \nu_2) \right) \right] \times p_f \Sigma^2 p_f' \\
\times p_f \Sigma^2 p_f' \left( - \frac{(1 - \epsilon) \phi^2}{(r + \kappa)^2} \right) \beta_2 \gamma_1 \nu_0(\kappa, u - t) + G(\gamma_1 R, \gamma_2 R, \gamma_3 R, u - t, \nu_0) p_f \Sigma^2 p_f' \\
- \left[ \left( \eta \Sigma^2 p_f' - \frac{\eta \Sigma^2 p_f' \eta \Sigma^2 p_f'}{\eta \Sigma t'} \right) + \left( \frac{L_2^2}{r} + H(\gamma_1 R, \gamma_2 R, \gamma_3 R, \gamma_1, \gamma_2, \gamma_3, \gamma_1^R, \gamma_2^R, \gamma_3^R, T' - \nu_2) \\
+ G(\gamma_1 R, \gamma_2 R, \gamma_3 R, T' - \nu_2) \right) \left( p_f \Sigma^2 p_f' - \frac{(\eta \Sigma^2 p_f')^2}{\eta \Sigma t'} \right) \right] (1 - \epsilon) \phi^2 \left( \frac{(r + \kappa)^2}{(r + \kappa)^2} \right) \nu_0(\kappa, u - t), \]

\[ G_{\text{Cov,2}}^{uV,\text{w}^H}(u) = \left[ \left( \frac{L_1 \eta \Sigma^2 p_f' + L_2 p_f \Sigma^2 p_f' \right)^2 \frac{\tau}{r} + \left( \frac{1 - \epsilon}{r + \kappa} \left[ K_2(\gamma_1 R, \gamma_3 R, T', \nu_1) + K_1(\gamma_1 R, \gamma_3 R, T', \nu_2) \right] \\
\frac{(1 - \epsilon) \phi^2}{(r + \kappa)^2} \beta_2 \gamma_1 \nu_2(\kappa, T') + H(\gamma_1, \gamma_2, \gamma_3, \gamma_1 ^R, \gamma_2 ^R, \gamma_3 ^R, T', \nu_1) + H(\gamma_1 R, \gamma_2 R, \gamma_3 R, \gamma_1, \gamma_2, \gamma_3, \gamma_1 ^R, \gamma_2 ^R, \gamma_3 ^R, T', \nu_2) \\
+ G(\gamma_1 R, \gamma_2 R, \gamma_3 R, T' - \nu_2) \right) \left( p_f \Sigma^2 p_f' \right)^2 - \frac{(1 - \epsilon) \phi^2}{(r + \kappa)^2} \nu_0(\kappa, T', \nu_0) \right] G(\gamma_1 R, \gamma_2 R, \gamma_3 R, u - t, \nu_0), \]

\[ G_{\text{Cov,2}}^{uV,\text{wV}}(u) = \left[ \left( \frac{L_1 \eta \Sigma^2 p_f' + L_2 p_f \Sigma^2 p_f' \right)^2 \frac{\tau}{r} + \left( \frac{1 - \epsilon}{r + \kappa} K_1(\gamma_1 R, \gamma_3 R, T' - \nu_1) \right. \\
\frac{(1 - \epsilon) \phi^2}{(r + \kappa)^2} \beta_2 \gamma_1 \nu_2(\kappa, T' - \nu_2) + H(\gamma_1 R, \gamma_2 R, \gamma_3 R, \gamma_1, \gamma_2, \gamma_3, T' - \nu_2) \\
+ G(\gamma_1, \gamma_2, \gamma_3, T' - \nu_2) \right) \left( p_f \Sigma^2 p_f' \right)^2 - \frac{(1 - \epsilon) \phi^2}{(r + \kappa)^2} \nu_0(\kappa, T' - \nu_2) \right] G(\gamma_1 R, \gamma_2 R, \gamma_3 R, u - t, \nu_0), \]

where \( T' = (u - t, \tau) \) and \( T' - = (t - u, \tau) \).

**Proof:** To show the proposition, we need to show that the definitions of \( \{ G_{\text{Cov,1}}^{uV,\text{w}^H}(u) \}_{i=1,..,6} \) and \( \{ G_{\text{Cov,2}}^{uV,\text{wV,b}}(u) \}_{i=1,2} \) in Lemma E.6 yield the equations in the proposition. The equations follow from the derivations in Propositions E.2 and E.3, and the derivations of \( \text{Cov}(w_i^V \Sigma p_f', w_i^M \Sigma p_f') \) and
\[ \text{Proposition E.8} \text{ computes the unconditional correlation between the return of strategy } w^j_t \text{ over an interval } [t, t + T] \text{ and the return of strategy } w^k_t \text{ over a subsequent interval } [t', t' + T'], \text{ with } j, k \in \{V, M\} \text{ and } t' \geq t + T. \text{ The autocorrelation of value returns follows by setting } j = k = V, \text{ the autocorrelation of momentum returns follows by setting } j = k = M, \text{ and the cross-autocorrelations between the two strategies’ returns follow by setting } (j, k) = (V, M) \text{ and } (j, k) = (M, V). \\

\textbf{Proposition E.8.} \text{ Consider intervals } [t, t + T] \text{ and } [t', t' + T'], \text{ with } (T, T') \text{ positive and } t' \geq t + T. \text{ The autocorrelation between the return of strategy } w^j_t \text{ over the interval } [t, t + T] \text{ and the return of strategy } w^k_t \text{ over the interval } [t', t' + T'], \text{ with } j, k \in \{V, M\}, \text{ is} \\

\[ \text{Corr} \left( \int_t^{t+T} \hat{w}^j_u dR_u, \int_{t'}^{t'+T'} \hat{w}^k_u dR_u \right) = \frac{A_{w^j, w^k, T, t', T'}^{\text{Cov1}} + A_{w^j, w^k, T, t', T'}^{\text{Cov2}}}{\sqrt{\left(D_{w^j} + D_{w^j, T}^{\text{Cov1}} + D_{w^j, T}^{\text{Cov2}} \right) \left(D_{w^k} + D_{w^k, T}^{\text{Cov1}} + D_{w^k, T}^{\text{Cov2}} \right)}}, \]

\text{where } \{D_{w^j}\}_{j=V,M} \text{ are derived in Propositions D.2 and D.5, } \left\{\left(\text{Cov1}_{w^j, w^k, T, t', T'}, \text{Cov2}_{w^j, w^k, T, t', T'} \right)\right\}_{j=V,M} \text{ are derived in Propositions E.2 and E.3, and } (A_{w^j, w^k, T, t', T'}, A_{w^j, w^k, T, t', T'}) \text{ are defined as} \\

\[ A_{w^j, w^k, T, t', T'}^{\text{Cov1}} = \frac{1}{\sqrt{TT'}} \sum_{i=1}^{6} \int_{t'-T}^{t'+T'} F(u) D_{w^j, i}^{\text{Cov1}}(u) du, \]

\[ A_{w^j, w^k, T, t', T'}^{\text{Cov2}} = \frac{1}{\sqrt{TT'}} \sum_{i=1}^{2} \int_{t'-T}^{t'+T'} F(u) D_{w^j, i}^{\text{Cov2}}(u) du, \]

\text{for } j = k \text{ with } \left\{D_{w^j, i}^{\text{Cov1}}(u)\right\}_{i=1, \ldots, 6}, \left\{D_{w^j, i}^{\text{Cov2}}(u)\right\}_{i=1, 2} \text{ derived in Propositions E.2 and E.3 and } F(u) \equiv \min\{T, t' + T' - u\} - \max\{0, t' - u\}, \text{ and as} \\

\[ A_{w^j, w^k, T, t', T'}^{\text{Cov1}} = \frac{1}{\sqrt{TT'}} \sum_{i=1}^{6} \int_{t'-T}^{t'+T'} F(u) G_{w^j, w^k, i}^{\text{Cov1}}(u) du, \]

\[ A_{w^j, w^k, T, t', T'}^{\text{Cov2}} = \frac{1}{\sqrt{TT'}} \sum_{i=1}^{2} \int_{t'-T}^{t'+T'} F(u) G_{w^j, w^k, i}^{\text{Cov2}}(u) du, \]

\text{for } j \neq k \text{ with } \left\{G_{w^j, w^k, i}^{\text{Cov1}}(u)\right\}_{i=1, \ldots, 6}, \left\{G_{w^j, w^k, i}^{\text{Cov2}}(u)\right\}_{i=1, 2} \text{ derived in Proposition E.7.} \]
Proof: To show the proposition, we need to show

\[
\text{Cov} \left( \int_t^{t+T} \hat{w}_u^j dR_u, \int_{t'}^{t'+T'} \hat{w}_u^k dR_u \right) = \sqrt{TT'} \left( A_{\text{Cov}_1}^{\text{cov}_{1},w,k,T,t',T'} + A_{\text{Cov}_2}^{\text{cov}_{2},w,k,T,t',t'} \right). \tag{E.41}
\]

Proceeding as in the proof of Lemma E.1 and using \( t' \geq t + T \), we find

\[
\text{Cov} \left( \int_t^{t+T} \hat{w}_u^j dR_u, \int_{t'}^{t'+T'} \hat{w}_u^k dR_u \right)
\]

\[
= \int_u^{t+T} \int_{u'=t'}^{t'+T'} \text{Cov} \left( \hat{w}_u^j dR_u, \hat{w}_{u'}^k dR_{u'} \right)
\]

\[
= \int_u^{t+T} \int_{u'=t'}^{t'+T'} \left\{ \text{Cov} \left[ \hat{w}_u^j \mathbb{E}_u(dR_u), \hat{w}_{u'}^k \mathbb{E}_{u'}(dR_{u'}) \right] + \mathbb{E} \left[ \hat{w}_u^j \text{Cov}_u(dR_u, \hat{w}_{u'}^k \mathbb{E}_{u'}(dR_{u'})) \right] \right\}
\]

\[
= \text{Equation (E.42)}
\]

To compute (E.42), we proceed as in the proof of Lemma E.2. Equation (E.42) becomes

\[
\int_{u=t}^{t+T} \int_{s=t'-u}^{t'+T'} \left\{ \text{Cov} \left[ \hat{w}_u^j \mathbb{E}_u(dR_u), \hat{w}_{u+s}^k \mathbb{E}_{u+s}(dR_{u+s}) \right] + \mathbb{E} \left[ \hat{w}_u^j \text{Cov}_u(dR_u, \hat{w}_{u+s}^k \mathbb{E}_{u+s}(dR_{u+s})) \right] \right\} du
\]

\[
= \int_{s=t'-u}^{t'-s} \int_{u=\text{max}(t,s)}^{t+T} \frac{1}{du} \left\{ \text{Cov} \left[ \hat{w}_u^j \mathbb{E}_u(dR_u), \hat{w}_{u+s}^k \mathbb{E}_{u+s}(dR_{u+s}) \right] + \mathbb{E} \left[ \hat{w}_u^j \text{Cov}_u(dR_u, \hat{w}_{u+s}^k \mathbb{E}_{u+s}(dR_{u+s})) \right] \right\} du
\]

\[
= \int_{s=t'-u}^{t'-s} \int_{u=\text{max}(t,s)}^{t+T} \frac{1}{dt} \left\{ \text{Cov} \left[ \hat{w}_u^j \mathbb{E}_t(dR_t), \hat{w}_{t+s}^k \mathbb{E}_{t+s}(dR_{t+s}) \right] + \mathbb{E} \left[ \hat{w}_u^j \text{Cov}_t(dR_t, \hat{w}_{t+s}^k \mathbb{E}_{t+s}(dR_{t+s})) \right] \right\} du
\]

\[
= \int_{s=t'-u}^{t'-s} \int_{u=\text{max}(t,s)}^{t+T} \frac{1}{dt} \left\{ \text{Cov} \left[ \hat{w}_u^j \mathbb{E}_t(dR_t), \hat{w}_{t+s}^k \mathbb{E}_{t+s}(dR_{t+s}) \right] + \mathbb{E} \left[ \hat{w}_u^j \text{Cov}_t(dR_t, \hat{w}_{t+s}^k \mathbb{E}_{t+s}(dR_{t+s})) \right] \right\} du
\]

\[
= \int_{u=t'-u}^{t'-u} \left\{ \text{Cov} \left[ \hat{w}_u^j \mathbb{E}_u(dR_u), \hat{w}_{u+t}^k \mathbb{E}_{u+t}(dR_{u+t}) \right] + \mathbb{E} \left[ \hat{w}_u^j \text{Cov}_u(dR_u, \hat{w}_{u+t}^k \mathbb{E}_{u+t}(dR_{u+t})) \right] \right\}
\]

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except that the budget constraint (C.10) is replaced by

\[ \eta \]

Proposition E.9.

Suppose \( \text{Symmetric assets.} \)

Consider an investor at time \( t \). The proposition assumes symmetric assets.

**Proposition E.9.** Suppose \( \eta = 1 \) and \( \Sigma = \sigma^2(I + \omega 11') \). The weights of value and momentum in their combination that maximizes an unconditional mean-variance objective over investment horizon \( T \) are

\[
\hat{y}^V = \frac{1}{a} \frac{SR_{w^V, T} - SR_{w^M, T} \text{Corr}\left( \int_t^{t+T} \hat{w}^V_u dR_u, \int_t^{t+T} \hat{w}^M_u dR_u \right)}{1 - \text{Corr}\left( \int_t^{t+T} \hat{w}^V_u dR_u, \int_t^{t+T} \hat{w}^M_u dR_u \right)^2} \sqrt{\frac{T}{\text{Var}\left( \int_t^{t+T} \hat{w}^V_u dR_u \right)}}
\]

\[ (E.44) \]

\[
\hat{y}^M = \frac{1}{a} \frac{SR_{w^M, T} - SR_{w^V, T} \text{Corr}\left( \int_t^{t+T} \hat{w}^V_u dR_u, \int_t^{t+T} \hat{w}^M_u dR_u \right)}{1 - \text{Corr}\left( \int_t^{t+T} \hat{w}^V_u dR_u, \int_t^{t+T} \hat{w}^M_u dR_u \right)^2} \sqrt{\frac{T}{\text{Var}\left( \int_t^{t+T} \hat{w}^M_u dR_u \right)}}.
\]

\[ (E.45) \]

**Proof:** Consider an investor at time \( t \) with horizon \( T \), who can invest in the riskless asset, the index \( \eta \) and the strategies \( (w^V_t, w^M_t) \). The investor’s optimization problem is as in Lemma C.2, except that the budget constraint (C.10) is replaced by

\[
\Delta W_{t+T} = \hat{x}_t \int_t^{t+T} \eta dR_u + \hat{y}_t^V \int_t^{t+T} \hat{w}^V_u dR_u + \hat{y}_t^M \int_t^{t+T} \hat{w}^M_u dR_u.
\]

\[ (E.46) \]
Substituting $\Delta W_{t+T}$ from (E.46) and setting $\mathcal{L}_t = 0$, we can write the investor’s objective (C.7) as

$$
\hat{x} \mathbb{E} \left( \int_t^{t+T} \eta dR_u \right) + \hat{y} \mathbb{E} \left( \int_t^{t+T} \hat{w}^V dR_u \right) + \hat{y} M \mathbb{E} \left( \int_t^{t+T} \hat{w}^M dR_u \right) \\
- \frac{a}{2} \left[ \hat{x}^2 \text{var} \left( \int_t^{t+T} \eta dR_u \right) + (\hat{y} V)^2 \text{var} \left( \int_t^{t+T} \hat{w}^V dR_u \right) + (\hat{y} M)^2 \text{var} \left( \int_t^{t+T} \hat{w}^M dR_u \right) \right] \\
+ 2 \hat{x} \hat{y} V \text{Cov} \left( \int_t^{t+T} \eta dR_u, \int_t^{t+T} \hat{w}^V dR_u \right) + 2 \hat{x} \hat{y} M \text{Cov} \left( \int_t^{t+T} \eta dR_u, \int_t^{t+T} \hat{w}^M dR_u \right) \\
+ 2 \hat{y} \hat{y} M \text{Cov} \left( \int_t^{t+T} \hat{w}^V dR_u, \int_t^{t+T} \hat{w}^M dR_u \right) \right]. 
$$

(E.47)

The proof of Lemma C.2 implies that the first and second covariances in (E.47) are zero if $\text{Cov}(\eta dR_u, \hat{w}^j u dR_u) = 0$ for $u < u'$ and $j = V, M$. The proof of Lemma E.1 implies

$$
\text{Cov}(\eta dR_u, \hat{w}^j u dR_u) = \text{Cov} \left[ \eta \mathbb{E}_u (dR_u), \hat{w}^j u \mathbb{E}_{u'} (dR_u') \right] + \mathbb{E} \left[ \eta \text{Cov}_{u'} (dR_u, \hat{w}^j u \mathbb{E}_{u'} (dR_u')) \right] \\
= \mathbb{E} \left[ \eta \text{Cov}_{u'} (dR_u, \hat{w}^j u \mathbb{E}_{u'} (dR_u')) \right] \\
= \frac{1}{du} \left( f + \frac{k \Delta}{\eta \Sigma_{u'}} \right) \left\{ \mathbb{E} \left[ \eta \Lambda_{u'} \text{Cov}_{u'} (dR_u, u' M \Sigma p_f') \right] + \mathbb{E} \left[ \eta \Lambda_{u'} \Sigma p_f' \text{Cov}_{u'} (dR_u, \Lambda_{u'}) \right] \right\}, 
$$

(E.48)

where the second step follows because (C.6) implies $\eta \mathbb{E}_u (dR_u) = \frac{ra_f}{a+\alpha} \eta \Sigma \theta dt$, which is constant over time, and the third step follows from the proof of Lemma E.4. Since (3.7) and (B.10) imply that $\text{Cov}_{u'} (dR_u, \Lambda_{u'})$ is collinear to $\Sigma p_f'$, the second term in (E.48) is zero because $\eta \Sigma p_f' = 0$. Since (B.10), (B.11) and (D.19) imply that $\text{Cov}_{u'} (dR_u, u' M \Sigma p_f')$ is a linear combination of $\Sigma p_f'$ and $\Sigma^2 p_f'$, the first term in (E.48) is zero for $j = V$ because $\eta \Sigma p_f' = 0$ and because for symmetric assets, Lemma C.6 implies $\eta \Sigma^2 p_f' = 0$. Since (3.5), (B.10) and (D.48) imply that $\text{Cov}_{u'} (dR_u, u' M \Sigma p_f')$ is a linear combination of $\Sigma p_f'$ and $\Sigma^2 p_f'$, the first term in (E.48) is zero for $j = M$ because $\eta \Sigma p_f' = \eta \Sigma^2 p_f' = 0$.

Setting the first and second covariances in (E.47) to zero, we can simplify (E.47) to

$$
\hat{x} \mathbb{E} \left( \int_t^{t+T} \eta dR_u \right) + \hat{y} V \mathbb{E} \left( \int_t^{t+T} \hat{w}^V dR_u \right) + \hat{y} M \mathbb{E} \left( \int_t^{t+T} \hat{w}^M dR_u \right) \\
- \frac{a}{2} \left[ \hat{x}^2 \text{var} \left( \int_t^{t+T} \eta dR_u \right) + (\hat{y} V)^2 \text{var} \left( \int_t^{t+T} \hat{w}^V dR_u \right) + (\hat{y} M)^2 \text{var} \left( \int_t^{t+T} \hat{w}^M dR_u \right) \right] \right]. 
$$

(E.47)
The first-order conditions over \( \hat{y}_t^V \) and \( \hat{y}_t^M \) are

\[
E \left( \int_t^{t+T} \hat{w}_u^V dR_u \right) = a \left[ \hat{y}_t^V \text{Var} \left( \int_t^{t+T} \hat{w}_u^V dR_u \right) + \hat{y}_t^M \text{Cov} \left( \int_t^{t+T} \hat{w}_u^V dR_u, \int_t^{t+T} \hat{w}_u^M dR_u \right) \right],
\]

(E.49)

\[
E \left( \int_t^{t+T} \hat{w}_u^M dR_u \right) = a \left[ \hat{y}_t^V \text{Cov} \left( \int_t^{t+T} \hat{w}_u^V dR_u, \int_t^{t+T} \hat{w}_u^M dR_u \right) + \hat{y}_t^M \text{Var} \left( \int_t^{t+T} \hat{w}_u^M dR_u \right) \right],
\]

(E.50)

respectively. Solving the linear system of (E.49) and (E.50), we find (E.44) and (E.45). We normalize \( \hat{y}_t^V \) and \( \hat{y}_t^M \) by setting \( a \sqrt{\text{Var} \left( \int_t^{t+T} \hat{w}_u^V dR_u \right)} = a \sqrt{\text{Var} \left( \int_t^{t+T} \hat{w}_u^M dR_u \right)} = 1 \).

Proposition E.10 computes the weights of value and momentum in the combination that best approximates the strategy that is optimal over investment horizon \( T \). We construct the approximating combination by minimizing the unconditional variance of the difference in returns over horizon \( T \) between that combination and the optimal strategy. The proposition assumes symmetric assets.

**Proposition E.10.** Suppose \( \eta = 1' \) and \( \Sigma = \hat{\sigma}^2 (I + \omega 11') \). The weights \( (\lambda^V, \lambda^M) \) of value and momentum in the combination that minimizes

\[
\text{Var} \left[ \int_t^{t+T} w_u dR_u - \left( \lambda^V \int_t^{t+T} \eta dR_u + \lambda^M \int_t^{t+T} \hat{w}_u^V dR_u + \lambda^M \int_t^{t+T} \hat{w}_u^M dR_u \right) \right],
\]

(E.51)

where \( w_t \) is the optimal strategy over investment horizon \( T \) derived in Section 6.1, are

\[
\lambda^V = \left[ \text{Corr} \left( \int_t^{t+T} \hat{w}_u^V dR_u, \int_t^{t+T} w_u dR_u \right) - \text{Corr} \left( \int_t^{t+T} \hat{w}_u^M dR_u, \int_t^{t+T} w_u dR_u \right) \right] \times \text{Corr} \left( \int_t^{t+T} \hat{w}_u^V dR_u, \int_t^{t+T} \hat{w}_u^M dR_u \right) \frac{1}{1 - \text{Corr} \left( \int_t^{t+T} \hat{w}_u^V dR_u, \int_t^{t+T} \hat{w}_u^M dR_u \right)^2} \left[ \text{Var} \left( \int_t^{t+T} \hat{w}_u^V dR_u \right) \right] \left[ \text{Var} \left( \int_t^{t+T} \hat{w}_u^M dR_u \right) \right]^{-1/2},
\]

(E.52)

\[
\lambda^M = \left[ \text{Corr} \left( \int_t^{t+T} \hat{w}_u^M dR_u, \int_t^{t+T} w_u dR_u \right) - \text{Corr} \left( \int_t^{t+T} \hat{w}_u^V dR_u, \int_t^{t+T} w_u dR_u \right) \right] \times \text{Corr} \left( \int_t^{t+T} \hat{w}_u^V dR_u, \int_t^{t+T} \hat{w}_u^M dR_u \right) \frac{1}{1 - \text{Corr} \left( \int_t^{t+T} \hat{w}_u^V dR_u, \int_t^{t+T} \hat{w}_u^M dR_u \right)^2} \left[ \text{Var} \left( \int_t^{t+T} \hat{w}_u^M dR_u \right) \right] \left[ \text{Var} \left( \int_t^{t+T} \hat{w}_u^V dR_u \right) \right]^{-1/2}.
\]
\[ \times \text{Corr} \left( \int_t^{t+T} \hat{w}_u^V dR_u, \int_t^{t+T} \hat{w}_u^M dR_u \right) \frac{1}{1 - \text{Corr} \left( \int_t^{t+T} \hat{w}_u^V dR_u, \int_t^{t+T} \hat{w}_u^M dR_u \right)} \left[ \frac{\text{Var} \left( \int_t^{t+T} w_u dR_u \right)}{\text{Var} \left( \int_t^{t+T} \hat{w}_u^M dR_u \right)} \right]. \] 

(E.53)

**Proof:** The first-order conditions from minimizing (E.51) over \( \lambda^V \) and \( \lambda^M \) are

\[
\text{Cov} \left( \int_t^{t+T} \hat{w}_u^V dR_u, \int_t^{t+T} w_u dR_u \right) = \lambda^V \text{Cov} \left( \int_t^{t+T} \eta dR_u, \int_t^{t+T} \hat{w}_u^V dR_u \right) + \lambda^V \text{Var} \left( \int_t^{t+T} \hat{w}_u^V dR_u \right) + \lambda^M \text{Cov} \left( \int_t^{t+T} \hat{w}_u^M dR_u, \int_t^{t+T} w_u dR_u \right) = 0,
\]

(E.54)

\[
\text{Cov} \left( \int_t^{t+T} \hat{w}_u^M dR_u, \int_t^{t+T} \hat{w}_u^M dR_u \right) = \lambda^M \text{Cov} \left( \int_t^{t+T} \eta dR_u, \int_t^{t+T} \hat{w}_u^M dR_u \right) + \lambda^M \text{Var} \left( \int_t^{t+T} \hat{w}_u^M dR_u \right) = 0,
\]

(E.55)

respectively. Since with symmetric assets Proposition E.9 implies

\[
\text{Cov} \left( \int_t^{t+T} \eta dR_u, \int_t^{t+T} \hat{w}_u^V dR_u \right) = \text{Cov} \left( \int_t^{t+T} \eta dR_u, \int_t^{t+T} \hat{w}_u^M dR_u \right) = 0,
\]

(E.54) and (E.55) imply (E.52) and (E.53).

To translate the weights \( \lambda^V \) and \( \lambda^M \) to weights \( \hat{y}^V \) and \( \hat{y}^M \) as in Proposition E.9, we multiply them by the weight \( \hat{y} \) given to the optimal strategy \( w_t \). Proceeding as in Proposition E.9, we find

\[
\text{Cov} \left( \int_t^{t+T} \eta dR_u, \int_t^{t+T} w_u dR_u \right) = 0
\]

because \( \text{Cov}_u(dR_u, w_u \Sigma p'_f) \) is collinear to \( \Sigma p'_f \). Therefore, maximization of (C.7) yields

\[
\hat{y} = \frac{1}{a} \frac{\text{SR}_{w,T}}{\sqrt{\text{Var} \left( \int_t^{t+T} w_u dR_u \right)}}.
\]

The resulting weights \( \hat{y}^V \) and \( \hat{y}^M \) are

\[
\hat{y}^V = \hat{y} \lambda^V = \frac{1}{a} \left[ \text{Corr} \left( \int_t^{t+T} \hat{w}_u^V dR_u, \int_t^{t+T} w_u dR_u \right) - \text{Corr} \left( \int_t^{t+T} \hat{w}_u^M dR_u, \int_t^{t+T} w_u dR_u \right) \right]
\]

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\[ \times \text{Corr} \left( \int_t^{t+T} \hat{w}_u^V dR_u, \int_t^{t+T} \hat{w}_u^M dR_u \right) \frac{SR_{w,T}}{1 - \text{Corr} \left( \int_t^{t+T} \hat{w}_u^V dR_u, \int_t^{t+T} \hat{w}_u^M dR_u \right)} \sqrt{\text{Var} \left( \int_t^{t+T} \hat{w}_u^V dR_u \right)}, \]  

(E.56)

\[ \hat{y}^M = \hat{y}^M = \frac{1}{a} \left[ \text{Corr} \left( \int_t^{t+T} \hat{w}_u^M dR_u, \int_t^{t+T} w_u dR_u \right) - \text{Corr} \left( \int_t^{t+T} \hat{w}_u^V dR_u, \int_t^{t+T} w_u dR_u \right) \right] \times \text{Corr} \left( \int_t^{t+T} \hat{w}_u^V dR_u, \int_t^{t+T} \hat{w}_u^M dR_u \right) \frac{SR_{w,T}}{1 - \text{Corr} \left( \int_t^{t+T} \hat{w}_u^V dR_u, \int_t^{t+T} \hat{w}_u^M dR_u \right)} \sqrt{\text{Var} \left( \int_t^{t+T} \hat{w}_u^M dR_u \right)}. \]  

(E.57)

Equations (E.56) and (E.57) are analogous to (E.44) and (E.45) in Proposition E.9. We normalize \( \hat{y}^V \) and \( \hat{y}^M \) by setting \( a \sqrt{\text{Var} \left( \int_t^{t+T} \hat{w}_u^V dR_u \right)} = a \sqrt{\text{Var} \left( \int_t^{t+T} \hat{w}_u^M dR_u \right)} = 1. \)