Optimal Asset Management Contracts with Hidden Savings

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November 2016

Abstract

We characterize optimal asset management contracts in a classic portfolio-investment setting. When the agent has access to hidden savings, his incentives to misbehave depend on his precautionary saving motive. The contract distorts his access to capital to manipulate his precautionary saving motive and reduce incentives for misbehavior. As a result, implementing the optimal contract requires history-dependent equity and leverage constraints. We extend our results to incorporate market risk, hidden investment, and renegotiation. We provide a sufficient condition for the validity of the first-order approach: if the agent’s precautionary saving motive weakens after bad outcomes, the contract is globally incentive compatible.

1 Introduction

Delegated asset management is ubiquitous in modern economies, from fund managers investing in financial assets to CEOs or entrepreneurs managing real capital assets. These intermediaries facilitate the flow of funds to the most productive uses, but their activity is hindered by financial frictions. To align incentives, asset managers must retain a stake in their investment activity. However, hidden savings pose a significant challenge to incentive
schemes. The asset manager can undo incentives by saving to self insure against bad outcomes. We characterize the optimal dynamic asset management contract when the agent has access to hidden savings.

The problem of hidden savings in a dynamic principal-agent framework is an old, hard problem because of the possibility of double deviations. As a result, a large part of the literature on agency problems assumes that the agent doesn’t have access to hidden savings. How much hidden savings matter for the attainable surplus and for the shape of the optimal contract is still an open question. Our results show that hidden savings are far from innocuous and have important implications for the kind of financial frictions generated by the agency problem and the dynamic behavior of the optimal contract.

This paper has two main contributions. First, we build a model of delegated asset management in a classic portfolio-investment setting that can be embedded in macroeconomic or financial frameworks: an agent with CRRA preferences continuously invests in risky assets, but can secretly divert returns and has access to hidden savings. Hidden savings lead to dynamic distortions in access to capital. As a result, the optimal contract is implemented with history-dependent retained equity and leverage constraints. In contrast, the optimal contract without hidden savings requires only a constant retained equity constraint. Second, on the methodological side, we provide a general verification theorem for global incentive compatibility that is valid for a wide class of contracts. Global incentive compatibility is ensured as long as the agent’s precautionary saving motive weakens after bad outcomes.

The agent’s precautionary saving motive plays a prominent role in the analysis. When the agent expects a risky consumption stream in the future he places a high value on hidden savings that he can use to self insure, which makes fund diversion more attractive. The optimal contract must therefore manage the agent’s precautionary saving motive by committing to limit his future risk exposure, especially after bad outcomes. This is accomplished by controlling the agent’s access to capital. Giving the agent capital to manage requires exposing him to risk in order to align incentives. By promising an inefficiently low amount of capital (and therefore risk) in the future, especially after bad outcomes, the principal makes fund diversion less attractive today and reduces the cost of giving capital to the agent up front. This dynamic tradeoff leads to history-dependent distortions in the agent’s access to capital. After good outcomes the agent’s access to capital improves, which allows him to keep growing rapidly; after bad outcomes he gets starved for capital and stagnates. The flip side is that the agent’s consumption is somewhat insured on the downside, and
he is punished instead with lower consumption growth. This hurts the agent, but hidden savings can’t help him get around the incentive scheme.

The presence of hidden savings has important implications for the types of financial frictions facing the agent. The optimal contract can be implemented with a simple capital structure subject to *equity constraints* and *leverage constraints*. The agent can issue equity and debt to buy assets, but he must retain an equity stake and the fund/firm leverage (assets over total equity) is capped. The equity constraint is related to the *hidden action* component of the agency problem; i.e. the agent must keep some “skin in the game” to deter fund diversion. The leverage constraint is related to *hidden savings*; the optimal contract without hidden savings only requires a constant retained equity constraint. A future leverage constraint, which tightens after bad outcomes, restricts the agent’s future access to capital and allows the principal to relax the equity constraint today. Intuitively, a binding leverage constraint makes the marginal value of inside equity larger than the marginal value of hidden savings, because an extra dollar in inside equity allows the agent to also get more capital. In addition, after bad outcomes the agent is punished with tighter financial frictions. As a result of both effects, a smaller retained equity stake is enough to deter misbehavior. Thus, the leverage constraint in our model arises from the presence of hidden savings and reflects a completely different logic than that of models with limited commitment such as Hart and Moore (1994) and Kiyotaki and Moore (1997). The leverage constraint exists not because the agent can walk away, but because it weakens the agent’s precautionary saving motive and allows the principal to relax the equity constraint.

Since the principal uses the agent’s access to capital to provide incentives, it is natural to ask how *hidden investment* affects results. If the agent can secretly invest his hidden savings in risky capital (for example by secretly investing more than indicated by the contract), the principal finds it harder to provide incentives. However, there are still gains from trade because the principal can provide some risk sharing for the capital invested through the contract, while the agent must bear all the risk on the capital he invests on his own. As a result, the optimal contract with hidden investment has the same qualitative features, and can also be implemented with retained equity and leverage constraints. We also add observable *market risk* that commands a premium into our setting, and allow the agent to also invest his hidden savings in the market. Our agency model can therefore be fully embedded within the standard setting of continuous-time dynamic asset pricing theory and macro-finance.

A potential concern is that the optimal contract requires commitment. The principal
relaxes the agent’s precautionary saving motive by promising an inefficiently low amount of capital in the future, especially after bad outcomes. It is therefore tempting at that point to renegotiate and start over. To address this issue, we also characterize the optimal renegotiation-proof contract. This leads to a stationary leverage constraint that restricts the agent’s access to capital to reduce his precautionary saving motive in a uniform way.

One of the main methodological contributions of this paper is to provide an analytical verification of the validity of the first order approach. Contractual environments where the agent has access to hidden savings are often difficult to analyze because we need to ensure incentive compatibility with respect to double deviations. Dealing with single deviations is relatively straightforward using the first-order approach, introduced to the problem of hidden savings by Werning (2001). We can deter the agent from diverting funds and immediately consuming them by giving him some skin in the game. Likewise, we can ensure that he will not secretly save his recommended consumption for later by incorporating his Euler equation as a constraint on the contract design. But what if the agent diverts returns and saves them for later? Since diverting funds makes bad outcomes more likely, the agent expects to be punished with lower consumption (high marginal utility) in the future, so this could potentially be an attractive double deviation. Establishing global incentive compatibility is a difficult problem, as the characterization of the agent’s deviation incentives has one more dimension than the recursive structure of the contract on path. Kocherlakota (2004) provides a well-known example in which double deviations are profitable (and the first-order approach fails) when cost of effort is linear. To establish the validity of the first-order approach, the existing literature has often pursued the numerical option (e.g. see Farhi and Werning (2013)).

We prove the validity of the first order approach analytically by establishing an upper bound on the agent’s continuation utility for any deviation, after any history. Double deviations are not profitable under the sufficient condition that agent’s precautionary saving motive becomes weaker after bad outcomes. Intuitively, if the contract becomes less risky and the precautionary saving motive gets weaker after bad outcomes (produced by the fund diversion), then hidden savings become less valuable to the agent. We show that this is sufficient to rule out double deviations. It is important to note that this verification argument applies beyond the optimal contract, to any contract in which the precautionary saving motive weakens after bad outcomes. If the agent has access to hidden investment, verifying global incentive compatibility is potentially more difficult, as the agent has even more valid deviations. However, the same sufficient condition is valid for the case with
hidden investment.

**Literature Review.** This paper fits within the literature on dynamic agency problems. It builds upon the standard recursive techniques, but adds the problem of persistent private information. In our case, about the agent’s hidden savings. There is extensive literature that uses recursive methods to characterize optimal contracts, including Spear and Srivastava (1987), Phelan and Townsend (1991), Sannikov (2008), He (2011), Biais et al. (2007), and Hopenhayn and Clementi (2006). The agency problem we study is one of cash flow diversion, as in DeMarzo and Sannikov (2006) and DeMarzo et al. (2012), but unlike their models we have CRRA rather than risk neutral preferences. With risk-neutral preferences, the optimal contracts with and without hidden savings are the same. Once concave preferences are introduced, the principal has incentives to front load consumption in order to reduce the private benefit of cash diversion and allow better risk sharing. This opens the door to potential distortions to control the agent’s incentives to save, but also presents the problem of double deviations. In some settings distortions do not arise, e.g. the CARA settings, such as He (2011), and Williams (2013), where the ratio of the agent’s current utility to continuation utility is invariant to contract design.\(^1\) However, when distortions do arise, it is difficult to characterize the specific form they take, and the first-order approach may fail. We are able to provide a sharp characterization of the optimal contract and the distortions generated by the presence of hidden savings, and provide a verification theorem for global incentive compatibility which is valid for a wide class of contracts.

Our paper is related to the literature on persistent private information, since the agent has private information about savings. The growing literature in this area includes the fundamental approach of Fernandes and Phelan (2000), who propose to keep track of the agent’s entire off-equilibrium value function, and the first-order approach, such as He et al. (2015) and DeMarzo and Sannikov (2016), who use a recursive structure that includes the agent’s “information rent,” i.e. the derivative of the agent’s payoff with respect to private information. In our case, information rent is the marginal utility of consumption (of an extra unit of savings). While the connection to our work may appear subtle, it is actually quite direct - key issues are (1) the way that information rents enter the incentive constraint, (2) distortions that arise from this interaction and (3) forces that affect the validity of the first-order approach. In our case, we can verify the validity of the first-order approach.

\(^1\)Likewise, the dynamic incentive accounts of Edmans et al. (2011) exhibit no distortions either, as hidden action enters multiplicatively and project size is fixed.
approach by characterizing an analytic upper bound, related to the CRRA utility function, on the agent’s payoff after deviations. The bound coincides with the agent’s utility on path (hence the first-order approach is valid), and its derivative with respect to hidden savings is the agent’s “information rent.” It is also possible to find the agent’s value function after deviations explicitly, and hence verify the first-order approach, numerically. Farhi and Werning (2013) do this in the context of insurance with unobservable skill shocks. Other papers that study the problems of persistent private information via a recursive structure that includes information rents include Garrett and Pavan (2015), Cisternas (2014), Kapička (2013), and Williams (2011).

We use a classic portfolio-investment environment widely used in macroeconomic and financial applications. Our model provides a unified account of equity and leverage constraints, which are two of the most commonly used financial frictions in the macro-finance literature in the tradition of Bernanke and Gertler (1989) and Kiyotaki and Moore (1997).\(^2\) Di Tella (2014) adopts a version of our setting without hidden savings to study optimal financial regulation policy in a general equilibrium environment. Our paper is also related to models of incomplete idiosyncratic risk sharing, such as Aiyagari (1994) and Krusell and Smith (1998). Here the focus is on risky capital income, as in Angeletos (2006) or Christiano et al. (2014) (rather than risky labor income). This affords us a degree of scale invariance that allows us to provide a sharp characterization of the optimal contract. In our setting, after good performance, the agent does not need to be retired nor outgrow moral hazard as in Sannikov (2008) or Hopenhayn and Clementi (2006) respectively. Likewise, since the project can be scaled down, neither will the agent retire after sufficiently bad outcomes as in DeMarzo and Sannikov (2006). Rather, the optimal contract dynamically scales the size of the agent’s fund with performance, taking into account his precautionary saving motive. Access to capital provides the principal with an important incentive tool. He is able to relax the incentive constraints and improve risk sharing by committing to distort project size below optimum over time and after bad performance. This result stands in contrast to Cole and Kocherlakota (2001), where project scale is fixed and the optimal contract is risk-free debt. We recover the result of Cole and Kocherlakota (2001) only in the special case when the agent can secretly invest on his own just as efficiently as through the principal, so the

principal cannot control the scale of investment at all.

This paper is organized as follows. Section 2 presents the model in the absence of the stock market. Section 3 solves for the optimal contract and provides a suitable sufficient condition to verify the validity of the first-order approach. Section 4 discusses the implementation of the optimal contract as a portfolio problem subject to equity and leverage constraints. Section 5 compares the optimal contract with several simple benchmark contracts, in order to better understand dynamics and financial frictions. Section 6 incorporates both aggregate risk and hidden investment into the setting, and Section 7 introduces renegotiation. Section 8 concludes.

2 The model

Let \((\Omega, P, \mathcal{F})\) be a complete probability space equipped with filtration \(\mathcal{F}\) generated by a Brownian motion \(Z\), with the usual conditions. Throughout, all stochastic processes are adapted to \(\mathcal{F}\). There is a complete financial market with equivalent martingale measure \(Q\). The risk-free interest rate is \(r > 0\) and \(Z\) is idiosyncratic risk and therefore not priced by the market. In the baseline setting there is no aggregate risk so \(Q = P\), but later we will allow them to differ.

The agent can manage capital to obtain a risky return that exceeds the required return of \(r\), but he may also get a private monetary benefit by diverting returns. If the diversion rate is \(a_t\), the observed return per dollar invested in capital is

\[ dR_t = (r + \alpha - a_t) \, dt + \sigma \, dZ_t \]

where \(\alpha > 0\) is the excess return and \(\sigma > 0\) is the volatility. \(Z\) is agent-specific idiosyncratic risk. If we think of the agent as a fund manager, it represents the outcome of his particular investment/trading activity.\(^3\) If we take the agent to be an entrepreneur it represents the outcome of his particular project.

If capital is \(k_t \geq 0\), diversion of \(a_t \geq 0\) gives the agent a flow of \(\phi a_t k_t\). For each stolen dollar, the agent keeps only fraction \(\phi \in (0, 1)\). If the agent also receives payments \(c_t \geq 0\)

\(^3\)If we give $1 to invest to two fund managers, they will obtain different returns depending on exactly which assets they buy or sell, and the exact timing and price of their trades.
from the principal and consumes $\tilde{c}_t \geq 0$ then his hidden savings $h_t \geq 0$ evolve according to\(^4\)

$$dh_t = (rh_t + c_t - \tilde{c}_t + \phi k_t a_t)\, dt.$$ 

The agent invests his hidden savings at the risk-free rate $r$. Later we will introduce hidden investment, and allow the agent to also invest his hidden savings in risky capital.

The agent wants to raise funds and share risk with the market, which we refer to as the principal. The principal observes returns $R$ but not the agent’s diversion $a$, consumption $\tilde{c}$, or hidden savings $h$. The principal can commit to a fully history-dependent contract $C = (c, k)$ that specifies payments to the agent $c_t$ and capital $k_t$ as a function of the history of realized returns $R$ up to time $t$. After signing the contract $C$ the agent can choose a strategy $(\tilde{c}, a)$ that specifies $\tilde{c}_t$ and $a_t$, also as a history of returns up to time $t$. 

The agent has CRRA preferences. Given contract $C$, under strategy $(\tilde{c}, a)$ the agent gets utility

$$U_{0}^{\tilde{c}, a} = E \left[ \int_{0}^{\infty} e^{-\rho t} \frac{\tilde{c}_t^{1-\gamma}}{1-\gamma} dt \right]$$ (1)

Given contract $C$, we say a strategy $(\tilde{c}, a)$ is feasible if 1) there is a finite utility $U_{0}^{\tilde{c}, a}$, and 2) $h_t \geq 0$ always. Let $S(C)$ be the set of feasible strategies $(\tilde{c}, a)$ given contract $C$.

The principal pays for the agent’s consumption, but keeps the excess return $\alpha$ on the capital that the agent manages. He tries to minimize the cost of delivering utility $u_0$ to the agent

$$J_0 = E^{Q} \left[ \int_{0}^{\infty} e^{-rt} (c_t - k_t \alpha) dt \right]$$ (2)

A standard argument in this setting implies that the optimal contract must implement no stealing, i.e. $a = 0$. In addition, without loss of generality and for analytic convenience, we can restrict attention to contracts in which $h = 0$ and $\tilde{c} = c$, i.e. the principal saves for the agent.\(^5\) Of course, the optimal contract has many equivalent and more natural forms, in which the agent maintains savings, but all these forms can be deduced easily from the optimal contract with $\tilde{c} = c$.

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\(^4\)We don’t allow negative hidden savings, $h_t \geq 0$. This is without loss of generality if the contract can exhaust the agent’s credit capacity.

\(^5\)Lemma 19 establishes this in the more general setting of Section 6, with both aggregate risk and hidden investment.
We say a contract $C = (c, k)$ is admissible if 1) there is a finite utility $U_{c,0}$, and 2)\(^6\)

$$\mathbb{E}^Q \left[ \int_0^\infty e^{-rt} |c_t + k_t\alpha| dt \right] < \infty$$  \hspace{1cm} (3)

We say an admissible contract $C$ is incentive compatible if

$$(c, 0) \in \arg \max_{(c,a) \in S(C)} U_{c,a}$$

Let $\mathcal{IC}$ be the set of incentive compatible contracts. For an initial utility $u_0$ for the agent, an incentive compatible contract is optimal if it minimizes the cost of delivering initial utility $u_0$ to the agent

$$v_0 = \min_{(c,k)} J_0$$

$$st: \quad U_{c,0} \geq u_0$$

$$(c, k) \in \mathcal{IC}$$

By changing $u_0$ we can trace the Pareto frontier for this problem. To make the problem well defined and avoid infinite profits/utility, we assume throughout that $\rho > r(1 - \gamma)$, and

$$\alpha \leq \bar{\alpha} \equiv \frac{\phi \sigma \gamma \sqrt{2}}{\sqrt{1 + \gamma}} \sqrt{\frac{\rho - r(1 - \gamma)}{\gamma}}$$

### 3 Solving the model

We solve the model as follows. We first derive necessary first-order incentive-compatibility conditions for the agent’s effort and savings choice, using two appropriate state variables: the agent’s continuation utility and consumption level. This allows us to formulate the principal’s relaxed problem, minimizing the cost subject to only first-order conditions, as a control problem. We use the HJB equation to solve this problem.

We then derive a sufficient condition for global incentive compatibility (against all deviations, not just local), which uses the same two state variables. The condition is on-path, i.e. for a particular recommended strategy of the agent, but it is sufficient because it al-

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\(^6\)This assumption plays the role of a no-Ponzi condition, making sure the principal’s objective function is well defined. It rules out exploding strategies where the present value of both consumption and capital is infinity.
allows us to bound the agent’s payoff off-path after arbitrary deviations. We show that the solution to the relaxed problem satisfies the sufficient condition, thereby proving it is the optimal contract. More generally, the sufficient condition identifies a whole class of globally incentive compatible contracts, and is useful in a broader context as we show in the next section.

**Incentive compatibility**

We use the continuation utility of the agent as a state variable for the contract

\[
U_t^{c,0} = \mathbb{E}_t \left[ \int_t^{\infty} e^{-\rho(s-t)} \frac{c_t^{1-\gamma}}{1-\gamma} ds \right]
\]

First we obtain the law of motion for the agent’s continuation utility.

**Lemma 1.** For any admissible contract \( C = (c, k) \), the agent’s continuation utility \( U_t^{c,0} \) satisfies

\[
dU_t^{c,0} = \left( \rho U_t^{c,0} - \frac{c_t^{1-\gamma}}{1-\gamma} \right) dt + \Delta_t \left( dR_t - (\alpha + r) dt \right)
\]

for some stochastic process \( \Delta \).

Faced with this contract, the agent might consider stealing and immediately consuming the proceeds, i.e. following a strategy \( (c + \phi k_\alpha, a) \) for some \( a \), which results in savings \( h = 0 \). The agent adds \( \phi k_t a_t \) to his consumption, but reduces the observed returns \( dR_t \), and therefore his continuation utility \( U_t^{c,0} \) by \( \Delta_t a_t \). Incentive compatibility therefore requires

\[
0 \in \arg \max_{a \geq 0} \frac{(c_t + \phi k_t a_t)^{1-\gamma}}{1-\gamma} - \Delta_t a
\]

Taking FOC yields

\[
\Delta_t \geq c_t^{-\gamma} \phi k_t
\]

which is positive. We need to give the agent some “skin in game”, which exposes him to risk. This is costly because the principal is risk-neutral with respect to \( Z \) so he would like to provide full insurance to the agent.

Notice how the private benefit of the hidden action depends on the marginal utility of consumption \( c_t^{-\gamma} \), so the principal would like to front load the agent’s consumption to relax
the risk-sharing constraint. With hidden savings this is not possible. If the principal tries to front load the agent’s consumption, he will secretly save and consume when his marginal utility is higher. The optimal contract must therefore respect the agent’s Euler equation: the discounted marginal utility \( e^{(r-\rho)\mathcal{U}_t} c_t^{-\gamma} \) must be a supermartingale.\(^7\)

The following lemma summarizes all the necessary conditions for incentive compatibility and their implication on the path of the agent’s consumption. We present sufficient conditions in the next subsection.

**Lemma 2.** If \( \mathcal{C} = (c,k) \) is an incentive compatible contract, then (6) must hold and the agent’s consumption must satisfy

\[
\frac{dc_t}{c_t} = \left( \frac{r - \rho}{\gamma} + \frac{1 + \gamma}{2} (\sigma^c_t)^2 \right) dt + \sigma^c_t \left( \frac{1}{\sigma} (dR_t - (\alpha + r) dt) + dL_t \right) + dZ_t \text{ if } a_t = 0
\]

for some \( \sigma^c \) and a weakly increasing process \( L \).

Equation (7) imposes a lower bound on the growth rate of the agent’s consumption. The first term \( \frac{r - \rho}{\gamma} \) captures the benefit of postponing consumption without risk, given by the risk-free rate \( r \), the discount rate \( \rho \), and the elasticity of intertemporal substitution \( 1/\gamma \). The second term \( \frac{1 + \gamma}{2} (\sigma^c_t)^2 \) captures the agent’s precautionary saving motive. A risky consumption profile induces the agent to postpone consumption to self-insure, resulting in a steeper consumption profile.

**State space**

It is convenient to work with the following transformation of the state variables

\[
x_t = \left( (1 - \gamma) \mathcal{U}_t^{0.0} \right)^{\frac{1}{1-\gamma}} > 0
\]

\[
\hat{c}_t = \frac{c_t}{x_t} \geq 0
\]

Variable \( x \) is just a monotone transformation of continuation utility, but it is measured in consumption units (up to a constant). As a result, \( \hat{c} \) measures how front loaded the agent’s consumption is. The state \( \hat{c} \) is related to the agent’s precautionary saving motive

\(^7\)The agent can expect lower marginal utility in the future, because he can’t borrow. If the agent could have hidden debt, then \( e^{(r-\rho)\mathcal{U}_t} c_t^{-\gamma} \) would have to be a proper martingale. As it turns out, this is the case in the optimal contract.
for saving. If the agent faces risk looking forward, he will want to postpone consumption in an attempt to self insure (low \( \hat{c} \)). As a result, while \( x_t \) can take any positive value, \( \hat{c}_t \) has an upper bound.

**Lemma 3.** For any incentive compatible contract \( C \), at all times \( t, \hat{c}_t \in (0, \hat{c}_h] \), where

\[
\hat{c}_h \equiv \left( \frac{\rho - r(1 - \gamma)}{\gamma} \right)^{\frac{1}{1-\gamma}} > 0
\]  

If ever \( \hat{c}_t = \hat{c}_h \), then the continuation contract satisfies \( k_{t+s} = 0 \) and \( \hat{c}_{t+s} = \hat{c}_h \) at all future times \( t + s \) and gives the agent a unique deterministic consumption path with growth \( (r - \rho)/\gamma \). The contract has cost \( \hat{v}_h x_t \) to the principal, where \( \hat{v}_h \equiv \hat{c}_h^\gamma \).

This upper bound has a simple interpretation. The contract that minimizes the agent’s precautionary saving motive and maximizes \( \hat{c} \) is the fully safe contract, which gives the agent no capital to manage and lets consumption grow at the deterministic growth rate of \( \frac{r - \rho}{\gamma} \). This corresponds to \( \hat{c}_h \). A lower \( \hat{c} \) means the agent expects to manage capital and be exposed to risk in the future. The safe contract minimizes the cost of the agent’s consumption, but is very costly because it doesn’t give any capital to the agent and so doesn’t take advantage of the excess return \( \alpha > 0 \).

**The relaxed problem**

Using Ito’s lemma we can obtain laws of motion for \( x_t \) and \( \hat{c}_t \) from (4) and (7). Using the normalization \( \Delta_t \sigma/U^c_{t,0} = (1 - \gamma)\sigma_t^x \), we obtain

\[
\frac{dx_t}{x_t} = \left( \frac{\rho - \hat{c}_t^{1-\gamma}}{1 - \gamma} + \frac{\gamma (\sigma_t^x)^2}{2} \right) dt + \sigma_t^x dZ_t
\]  

and

\[
\frac{d\hat{c}_t}{\hat{c}_t} = \left( \frac{r - \rho}{\gamma} + \frac{\hat{c}_t^{1-\gamma} - \rho}{1 - \gamma} + \frac{(\sigma_t^x)^2}{2} + \frac{1 + \gamma}{2} (\sigma_t^\hat{c})^2 \right) dt + \sigma_t^\hat{c} dZ_t + dL_t
\]  

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for some $\sigma_i^0 = \sigma_i - \sigma_i'$. The constraint (6) can be rewritten as

$$\sigma_i^x \geq \hat{c}_i - \gamma \hat{k}_t \phi \sigma,$$

where $\hat{k}_t = \frac{b_t}{x_t}$. It will always be binding because conditional on $\sigma_i^x$ it is always better to give the agent more capital to manage. The cost flow for the principal is

$$c_t - k_t \alpha = x_t \left( \hat{c}_t - \hat{c}_t^0 \frac{\alpha}{\phi \sigma} \sigma^x_t \right)$$

(12)

We obtain a stochastic control problem with laws of motion of the state variables (9) and (10) with absorption at $\hat{c}_h$, controls $\sigma_i^x$ and $\sigma_i^\hat{c}$, and cost flow (12). This is the relaxed problem.

The principal would like to give capital to the agent because it yields an excess return $\alpha$. However, he must expose the agent to risk to provide incentives. This is costly for two reasons. First, since the agent is risk averse he must be compensated for the exposure to risk with higher utility in the future. This is reflected in the drift of $x_t$ in equation (9). Second, because of hidden savings, exposing the agent to risk induces a precautionary saving motive - a low $\hat{c}$ - that distorts the optimal intertemporal consumption profile. In fact, these two costs interact because a low $\hat{c}$ further increases the incentives to steal through $\hat{c}^{-\gamma}$ in (11).

Here’s how this works in the recursive formulation. While $x_0$ is determined by the initial utility $u_0$, the principal must choose an initial $\hat{c}_0 \leq \hat{c}_h$. A lower $\hat{c}_t$ means that the agent expects to be given capital and therefore be exposed to risk in the future, so we can interpret $\hat{c}_h - \hat{c}_t$ as the principal’s “budget” for risk exposure. If the principal exposes the agent to risk today with $\sigma_i^x$, he must expose him to less risk in the future. This is reflected in equation (10) where the drift of $\hat{c}$ is increasing in $\sigma_i^x$: exposing the agent to risk today eats up the budget for risk exposure in the future. A higher $\hat{c}_t$ creates distortions because it reduces the principal’s ability to give capital to the agent. Indeed, if $\hat{c}_t$ ever reached the upper bound $\hat{c}_h$, the principal would have to give him the perfectly safe contract without any capital. The principal must therefore choose the initial $\hat{c}_0$ weighting the benefit of giving capital to the agent against the distortions in intertemporal consumption smoothing and risk sharing. After that, he manages his budget for risk exposure using $\sigma^x$ and $\sigma^\hat{c}$.

In addition to $\sigma_i^x$ the principal can use a negative $\sigma_i^\hat{c}$ to relax the agent’s precautionary saving motive. The agent is concerned the most about risk exposure after bad outcomes, when his utility is low. Thus, the principal can reduce the agent’s precautionary saving
motive by giving him less risk - raising \( \hat{c}_t \) - in the event that \( x_t \) goes down. This is captured by the drift of \( \hat{c}_t \) in equation (10), which can be reduced with a negative \( \sigma^c \). As it turns out the principal will also prefer to raise \( \hat{c}_t \) after bad outcomes for dynamic hedging reasons: he would rather restrict his ability to give capital to the agent (with a high \( \hat{c}_t \)) when he must deliver less utility \( x_t \) to the agent, and relax it (with a low \( \hat{c}_t \)) when he must deliver more utility \( x_t \).

It turns out that relaxing the agent’s precautionary saving motive after bad outcomes, \( \sigma^c \leq 0 \), is also a sufficient condition for the contract to be globally incentive compatible if the necessary conditions of Lemma 2 hold. While our paper is the first to our knowledge to prove this form of a result analytically, the intuition is simple. The marginal value of hidden savings rises in the agent’s precautionary saving motive, i.e. savings become more valuable when the agent is exposed to risk going forward and therefore \( \hat{c}_t \) is lower. Therefore, whenever a contract reduces the precautionary saving motive after bad outcomes - these outcomes become more likely when the agent steals - stealing and saving is an unattractive deviation. The agent expects to have hidden savings when they are less valuable to him. This intuition is formalized below in Theorem 3.

**The HJB equation**

Because preferences are homothetic and the principal’s objective is linear, we know the principal’s cost function takes the form \( v(x, \hat{c}) = \hat{v}(\hat{c}) x \) with \( \hat{v}(\hat{c}_h) = \hat{v}_h > 0 \). Notice that since we could always raise \( \hat{c}_t \) using \( dL_t \), we know that \( \hat{v}(\hat{c}) \) must be weakly increasing. In fact, we show below that \( \hat{v}(\hat{c}) \) increases strictly over the interval in which \( \hat{c}_t \) stays over the course of the optimal contract, so we can drop the term \( dL_t \) from what follows. We will also sometimes write \( \hat{v}_t = \hat{v}(\hat{c}_t) \), and \( \hat{v} \) instead of \( \hat{v}(\hat{c}) \).

The HJB equation associated with this problem is

\[
r \hat{v} x = \min_{\sigma^x, \sigma^c} \left( \hat{c} - \hat{k} \alpha \right) x + \mathbb{E}^Q_t \left[ d(\hat{v}_t x_t) \right]
\]

subject to (9), (10), and (11), and \( \hat{k} \geq 0 \). Using Ito’s lemma and canceling the \( x \) on both sides, we get

\[
r \hat{v} = \min_{\sigma^x, \sigma^c} \hat{c} - \sigma^x \sigma^c \frac{\alpha}{\hat{c}} + \hat{v} \left( \frac{\rho - \hat{c}^{1-\gamma}}{1 - \gamma} + \frac{\gamma}{2} \left( \sigma^x \right)^2 \right) + \hat{v}' \hat{c} \left( \frac{\sigma^2}{\hat{c}} + (1 + \gamma) \sigma^x \sigma^c + \frac{1 + \gamma}{2} \left( \sigma^c \right)^2 - \frac{\rho - \hat{c}^{1-\gamma}}{1 - \gamma} \right) + \frac{\hat{v}''}{2} \hat{c}^2 \left( \sigma^c \right)^2
\]

14
Even though we have two state variables, \( \hat{c} \) and \( x \), the HJB equation boils down to a second order ODE in \( \hat{c} \). This is a feature of homothetic preferences and linear technology that makes the problem more tractable. The following lemma characterizes the shape of the principal’s cost function and the range of \( \hat{c}_t \) under the optimal contract.

**Theorem 1.** The principal’s cost function \( \hat{v}(\hat{c}) \) has a flat portion on \([0, \hat{c}_l]\) and a strictly increasing portion on \([\hat{c}_l, \hat{c}_h]\), for some \( \hat{c}_l \in (0, \hat{c}_h) \). The HJB equation (13) holds with equality for \( \hat{c} \geq \hat{c}_l \). For \( \hat{c} < \hat{c}_l \), we have \( \hat{v}(\hat{c}) = \hat{v}(\hat{c}_l) \equiv \hat{v}_l \) and the HJB holds as an inequality

\[
A(\hat{c}, \hat{v}_l) \equiv \min_{\sigma^2} \hat{c} - \sigma^2 \hat{c}^2 - r\hat{v}_l + \hat{v}_l \left( \frac{\rho - \hat{c}_l - \gamma}{1 - \gamma} + \frac{\gamma}{2} (\sigma^2)^2 \right) > 0 \quad \forall \hat{c} < \hat{c}_l. \tag{14}
\]

At \( \hat{c}_l \) the cost function \( \hat{v}(\hat{c}) \) satisfies \( \hat{v}'(\hat{c}_l) = 0 \), \( \hat{v}''(\hat{c}_l) > 0 \), and \( A(\hat{c}_l, \hat{v}_l) = 0 \).

The optimal contract starts at \( \hat{c}_0 = \hat{c}_l \), where \( \sigma_0^2 \) is chosen without taking into account its effect on the agent’s precautionary saving motive, to maximize

\[
\sigma^2 \hat{c}_l^\gamma \frac{\alpha}{\phi \sigma} - \hat{v}(\hat{c}_l) \left( \frac{\gamma}{2} (\sigma^2)^2 \right) \tag{15}
\]

At \( \hat{c}_l \) we have \( \mu^\hat{c}(\hat{c}_l) > 0 \) and \( \sigma^\hat{c}(\hat{c}_l) = 0 \). For all \( t > 0 \), \( \hat{c}_t \in [\hat{c}_l, \hat{c}_h] \), \( \sigma^\hat{c}_t \leq 0 \) and \( \sigma^x_t \geq 0 \).

Figures 1 and 2 show the cost function and the drift and volatility of state variables \( x \) and \( \hat{c} \) for a numerical solution. To understand why the cost function has a flat portion and an increasing portion, consider the problem of the optimal choice of \( \hat{c}_0 \). Choosing \( \hat{c}_0 = \hat{c}_h \) is suboptimal, because then the principal cannot give the agent any capital. It is beneficial to give the agent capital and expose him to risk because capital generates an excess return of \( \alpha \). However, risk exposure is costly because the agent is risk averse. In addition, with hidden savings risk exposure also generates a precautionary saving motive which lowers \( \hat{c} \), distorting the intertemporal consumption profile and further tightening the incentive constraint. Eventually, the costs of risk exposure outweigh the benefits. Under these trade-offs, we denote by \( \hat{c}_l \) the value that minimizes the cost of delivering utility to the agent, indicated with a blue dot in Figure 1. For \( \hat{c} < \hat{c}_l \), the principal has the option to raise \( \hat{c} \) to \( \hat{c}_l \) immediately using the process \( dL_t \). Hence, the cost function is flat over \([0, \hat{c}_l]\). For \( \hat{c} > \hat{c}_l \), he must give the agent an inefficiently low risk exposure to reduce his precautionary saving motive. As a result the cost function \( \hat{v}(\hat{c}) \) is increasing over \([\hat{c}_l, \hat{c}_h]\), and the optimal contract starts at \( \hat{c}_0 = \hat{c}_l \).
Figure 1: The cost function $\hat{v}(\hat{c})$ solid in blue for the optimal contract, and dashed in red for stationary contracts. The starting point of the optimal contract is indicated by the blue dot, the optimal stationary contract by the red dot, the optimal portfolio plan with $\tilde{\phi} = \phi$ is indicated the black dot, and the optimal contract without hidden savings by the green dot. The dashed black curve is the locus $A(\hat{c}, \hat{v}) = 0$. Parameters: $\rho = r = 5\%$, $\alpha = 1.7\%$, $\gamma = 1/3$, $\phi \sigma = 0.2$.

Figure 2: The drift, $\mu^{\hat{c}}$ and $\mu^{x}$, and volatility, $\sigma^{\hat{c}}$ and $\sigma^{x}$, of the state variables $\hat{c}$ and $x$. The starting point of the optimal contract is indicated by the blue dot, the optimal stationary contract by the red dot, the optimal portfolio plan with $\tilde{\phi} = \phi$ is indicated the black dot, and the optimal contract without hidden savings by the green dot. Parameters: $\rho = r = 5\%$, $\alpha = 1.7\%$, $\gamma = 1/3$, $\phi \sigma = 0.2$. 
The optimal contract starts at the optimal \( \hat{c}_l \) but becomes less risky over time - \( \hat{c} \) drifts up to a steady state, as shown in Figure 2. Why does \( \hat{c}_t \) not stay at the cost-minimizing level \( \hat{c}_l \) forever? By promising a less risky contract in the future (higher \( \hat{c} \)), the principal can weaken the agent’s precautionary saving motive and relax his incentives to steal today.

To see this in more detail, consider the first-order condition\(^8\) for \( \sigma^x \)

\[
\alpha = \gamma (\hat{v} \hat{c}^{-\gamma} \phi \sigma^x) + \phi \sigma \hat{v}' \hat{c}^{1-\gamma} \left( (1 + \gamma) \sigma \hat{c} + \sigma^x \right)
\]

(16)

The left hand side and the first term on the right capture the tradeoff between excess return \( \alpha \) and exposure to risk \( \sigma^x \). The second term captures an intertemporal tradeoff.

The principal can give more capital to the agent today (exposing him to more risk \( \sigma^x \) today) without a larger precautionary saving motive (lower \( \hat{c} \)) if he promises a less risky contract in the future (the drift of \( \hat{c} \) is increasing in \( \sigma^x \)). Since less risky contracts require an inefficiently low level of capital - \( \hat{v}(\hat{c}) \) is strictly increasing in \( \hat{c} \) as described above - this is a potentially costly tradeoff for the principal. At the optimal point \( \hat{c}_l \), however, the principal is indifferent about small changes in \( \hat{c} \) because \( \hat{v}'(\hat{c}_l) = 0 \), so the intertemporal trade-off vanishes. The principal picks \( \sigma^x \) to myopically maximize (15), and doesn’t care that he is promising less risk in the future. Intuitively, distorting future \( \sigma^x_t \) relaxes the incentive problem at \( t = 0 \), but distorting \( \sigma^x_0 \) does not help relax the incentive problem in the future. As a result, the drift of \( \hat{c} \) is initially positive. As \( \hat{c} \) moves up from \( \hat{c}_l \), the cost of the contract goes up. The principal would benefit from reducing \( \hat{c} \) - giving the agent a more risky contract - but he cannot do so because he must keep his promise to the agent.

The principal then chooses \( \sigma^x \) taking into account this intertemporal tradeoff.

The principal can reduce the cost by setting \( \sigma^x < 0 \), as shown in Figure 2, i.e. after bad outcomes the contract becomes less risky for the agent. This has two benefits, as shown by the FOC for \( \sigma^x \)

\[
\hat{v}' \left( \gamma \sigma^x + (1 + \gamma) \sigma^x \right) + \hat{v}' \sigma^x + \hat{v}'' \sigma \hat{c}^x = 0
\]

(17)

The first term says that by setting \( \sigma^x < 0 \) the principal can reduce the agent’s precautionary

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\(^8\)This first-order condition characterizes a minimum (rather than a maximum) because \( \hat{v} \gamma + \hat{v}' \hat{c} > 0 \), since \( \hat{v} > 0 \) and \( \hat{v}' \geq 0 \).
saving motive. The principal promises the agent less risk - a higher $\hat{c}$ - after bad outcomes, when it matters most to him. What is going on is that the agent’s consumption $c_t = \hat{c}_t x_t$ is somewhat insured because $\hat{c}$ and $x$ move in opposite directions. As a result, the agent’s precautionary saving motive is weaker and the drift of $\hat{c}$ lower, which is attractive to the principal because $\hat{v}(\hat{c})$ is increasing. The second term in the FOC captures a hedging motive for the principal: he also prefers to use the relatively costly contracts with high $\hat{c}$ when he must deliver less utility $x$, because then he can use the relatively less costly contracts with low $\hat{c}$ when he must deliver more utility $x$. Both motives induce a negative $\sigma_{\hat{c}}$.

If the agent’s consumption is somewhat insured and he faces less risk after bad outcomes, how exactly is he being punished? The answer is that he faces a lower growth rate in his consumption. After bad outcomes the agent’s access to capital is restricted, and he is given an inefficiently safe contract with low consumption growth. This hurts the agent, but he can’t use hidden savings to get around it. The principal could punish the agent by proportionally scaling down his capital and consumption (keeping $\hat{c}$ constant), but this would be too costly: it would make the agent’s consumption too risky and give him a precautionary saving motive for savings that would distort intertemporal consumption smoothing and increase his incentives to steal. Instead, the principal uses the agent’s access to capital to manipulate his precautionary saving motive.

**Long-run behavior.** The contract is scale invariant with respect to $x$. After a very good history, the agent’s continuation utility $x$ will be large, and he will get a large amount of
capital and consumption. Conversely, after a bad history his continuation utility \( x \) will be small and he will get a small amount of capital and consumption.

The behavior of \( \hat{c} \), however, creates non-trivial dynamics. While the contract starts at the lower end of the domain \( \hat{c}_0 = \hat{c}_l \), with high growth and risk, in the absence of shocks the drift of \( \hat{c} \) takes the contract to a “steady state” \( \hat{c}_{ss} \in (\hat{c}_l, \hat{c}_h) \), as shown in Figure 2. But this steady state can be a misleading guide to the long-run behavior of the contract. If the volatility \( \sigma \hat{c} \) is high near the steady state, the contract might spend very little time there. Figure 3 shows the stationary distribution of \( \hat{c} \). For this numerical solution, the contract spends most of the time near the upper bound \( \hat{c}_h \), with low growth and risk, where both the drift and volatility of \( \hat{c} \) are small.

### Verification theorems

We know that the principal’s cost function in the relaxed problem satisfies the HJB equation, but how do we know that the equation has no other solutions? And how do we know that we have identified the true (non-relaxed) optimal contract? The following theorem shows that if an appropriate solution has been found, e.g. numerically, then it must be the true cost function, and we can use it to build the optimal contract.

**Theorem 2** (Verification Theorem). Let \( \hat{v}(\hat{c}) : [\hat{c}_l, \hat{c}_h] \to [\hat{v}_l, \hat{v}_h] \) be a strictly increasing \( C^2 \) solution to the HJB equation (13) for some \( \hat{c}_l \in (0, \hat{c}_h) \), such that \( \hat{v}_l \equiv \hat{v}(\hat{c}_l) \in (0, \hat{v}_h] \), \( \hat{v}'(\hat{c}_l) = 0 \), \( \hat{v}''(\hat{c}_l) > 0 \) and \( \hat{v}(\hat{c}_h) = \hat{v}_h \). If \( \gamma < \frac{1}{2} \), we also need to check that

\[
1 - \hat{v}_l (\hat{c}_l^{\gamma} + \hat{c}_l^{2\gamma-1} \alpha^2 (\phi \sigma)^{-2} \hat{v}_l^{-2}) \leq 0
\] (18)

Then,

1) For any incentive compatible contract \( \mathcal{C} = (c, k) \) that delivers at least utility \( u_0 \) to the agent, we have \( \hat{v}(\hat{c}_l) ((1 - \gamma)u_0)^{\frac{1}{1-\gamma}} \leq J_0(\mathcal{C}) \).

2) Let \( \mathcal{C}^* \) be a contract generated by the policy functions of the HJB. Specifically, the state variables \( x^* \) and \( \hat{c}^* \) are solutions to (9) and (10) (with potential absorption at \( \hat{c}_h \)), with initial values \( x_0^* = ((1 - \gamma)u_0)^{\frac{1}{1-\gamma}} \) and \( \hat{c}_0^* = \hat{c}_l \). If \( \mathcal{C}^* \) is admissible, and \( \sigma \hat{c}^* \) is bounded, then \( \mathcal{C}^* \) is an optimal contract, with cost \( J_0(\mathcal{C}^*) = \hat{v}(\hat{c}_l) ((1 - \gamma)u_0)^{\frac{1}{1-\gamma}} \).

The HJB equation can be solved as an ODE by plugging in the FOCs. We only need
to verify condition (18) in case $\gamma < \frac{1}{2}$, and that the contract generated by the HJB $C^*$ is admissible. The following sufficient condition can be useful.

Lemma 4. If the candidate contract $C^*$ constructed in Theorem 2 has $\mu^{C^*} < r$, then $C^*$ is admissible and delivers utility $u_0$ to the agent.

Global incentive compatibility. To finish the section, we provide sufficient conditions for global incentive compatibility of any contract that satisfies the local constraints on savings (10) and effort (11). This result is used in Theorem 2 to verify that the candidate optimal contract $C^*$ is incentive compatible. However, it is more general than that and can be used to check incentive compatibility of many other contracts of interest (see Section 4).

While (10) and (11) ensure that neither stealing and immediately consuming, nor secretly saving without stealing are attractive on their own, they leave open the possibility that a double deviation (stealing and saving the proceeds for later) could be attractive to the agent. To see how this can happen, notice that since stealing makes bad outcomes more likely, it increases the expected marginal utility of consumption in the future $E_t^0 \left[ e^{(r-\rho)u} c^{-\gamma}_t \right]$. Saving the stolen funds for consumption later could therefore be very attractive. However, hidden savings have decreasing marginal value (the first dollar yields $c^{-\gamma}_t$, the second one less than that), which depends on the agent’s precautionary saving motive. This observation allows us to derive a sufficient condition to rule out profitable double deviations.

Theorem 3. Let $C = (c,k)$ be an admissible contract with associated processes $x$ and $\hat{c}$ satisfying (9) and (10) and (11), with bounded $\mu^x$, $\mu^{\hat{c}}$, and $\sigma^{\hat{c}}$, and with $\hat{c}$ uniformly bounded away from zero and bounded above by $\hat{c}_h$. Suppose that the contract satisfies the following property

$$\sigma^{\hat{c}}_t \leq 0$$

Then for any feasible strategy $(\hat{c},a)$, with associated hidden savings $h$, we have the following upper bound on the agent’s utility, after any history

$$U^c_{t+\hat{c},a} \leq \left( 1 + \frac{h_t}{x_t} c^{-\gamma}_t \right)^{1-\gamma} U^{c,0}_t$$

In particular, since $h_0 = 0$, for any feasible strategy $U^c_{0,0} \leq U^{c,0}_0$, and the contract $C$ is

\[In other words, even if $e^{(r-\rho)u} c^{-\gamma}_t$ is a martingale under $P$, it might be a submartingale under $P^a$.\]
therefore incentive compatible.

Theorem 3 shows that condition (19) is sufficient by providing a closed-form explicit upper bound (20) on the agent’s off-equilibrium payoff for any savings level \( h_t \geq 0 \). According to (20), if the agent does not have any hidden savings, \( h_t = 0 \), the most utility he could get is \( U_t^{c,0} \), i.e. the utility level he obtains from “good behavior” \((c,0)\). Hence, good behavior is incentive compatible. However, if the agent had somehow accumulated hidden savings in the past, he would want to deviate from \((c,0)\) in the future, at the very least to increase his consumption, and attain a greater utility. Inequality (20) bounds the utility the agent can get, and the bound tightens as \( \hat{c}_t \) rises and the agent’s precautionary saving motive decreases. The bound is consistent with the intuition that the marginal value of hidden savings becomes lower as the agent’s precautionary saving motive decreases (however, remember that this is just an upper bound on achievable utility).

The sufficient condition \( \sigma_t^{\hat{c}} \leq 0 \) can be understood as follows. Hidden savings become more valuable when the agent faces more risk, i.e. has a higher precautionary saving motive. With \( \sigma_t^{\hat{c}} \leq 0 \) the contract becomes less risky for the agent after bad outcomes (hidden savings becomes less valuable). Since stealing makes bad outcomes more likely, if the agent steals and saves for later, he expects to have a hidden dollar when it is least valuable to him. This makes double deviations unprofitable.

As shown above, the optimal contract has the property that \( \sigma_t^{\hat{c}} \leq 0 \) because the principal wants to contain the agent’s precautionary saving motive and the most efficient way to do this is by reducing the agent’s risk exposure after bad outcomes. As it happens it is the same property that is sufficient for global incentive compatibility.

We would like emphasize once more that Theorem 3 identifies \( \sigma_t^{\hat{c}} \leq 0 \) as a general sufficient condition for incentive compatibility, without even assuming that the contract is recursive in variables \( x \) and \( \hat{c} \). These variables are well-defined for an arbitrary contract, and their laws of motion do not need to be Markov for Theorem 3 to apply. Condition \( \sigma_t^{\hat{c}} \leq 0 \) can be used to verify global incentive compatibility of other contracts. For example, it implies that stationary contracts that we discuss below are also all globally incentive compatible.
4 Implementation of the Optimal Contract

In this section we derive an implementation of the optimal contract. A capital structure implements a contract if it leads to exactly the same map from histories of portfolio returns $R$ to the agent’s compensation $c$ and funds invested $k$ as the contract.

Consider a family of capital structures, in which the fund’s assets $k_t$ are financed by debt $d_t$ and outside equity $e_t$ held by the investors, as well as inside equity $n_t$, the agent’s legitimate wealth, so that

$$k_t = n_t + e_t + d_t. \quad (21)$$

Suppose that the stakes held by the principal, debt, and outside equity earn the required return of $r$ in expectation. This restriction on capital structure ensures that investors are always willing to provide the required fund inflows and outflows. Suppose that inside and outside equity absorb risks proportional to $e_t$ and $n_t$, respectively. Subject to these rules, any capital structure within this family can specify the evolution of the stakes $n_t$, $e_t$ and $d_t$ as functions of the history of returns. We call capital structures of this form standard.

Let us introduce some terminology about this family of capital structures. Denote by $\tilde{\phi}_t = \frac{n_t}{n_t + e_t}$ the fraction of equity held by the agent and by $\lambda_t = \frac{k}{n_t + e_t}$ the fund/firm leverage ratio. These two quantities determine the inflows and outflows of the fund, given the requirements about the division of risk and return. Those requirements imply that the returns of debt and outside equity are then $r dt$ and $r dt + \lambda_t \sigma dZ_t$, respectively. The return of inside equity $n_t$ has to be, therefore, $(r + \lambda_t \alpha \tilde{\phi}_t) dt + \lambda_t \sigma dZ_t$. Given the payout rate of the agent’s equity of $c_t/n_t = \theta_t$, the agent’s inside equity $n_t$ follows the budget constraint of

$$\frac{dn_t}{n_t} = \left( r + \lambda_t \alpha \tilde{\phi}_t - \theta_t \right) dt + \lambda_t \sigma dZ_t. \quad (22)$$

It is convenient to specify standard capital structures through the triplet of processes $S = (\lambda, \tilde{\phi}, \theta)$, all contingent on the history of returns $R$, with an associated process for agent’s equity given by (22). A standard capital structure $S = (\lambda, \tilde{\phi}, \theta)$ implements contract $C = (c, k)$ if $c_t = \theta_t \times n_t$ and $k_t = \lambda_t (n_t + e_t) = \lambda_t / \tilde{\phi}_t \times n_t$.

By matching appropriate terms, we obtain the unique implementation of the optimal

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10Notice that leverage $\lambda_t$ gets the agent an excess return $\alpha \tilde{\phi}_t$ because while he retains only a fraction $\tilde{\phi}_t$ of the equity, he appropriates all the excess returns $\alpha k_t = \alpha \lambda_t (n_t + e_t) = \alpha (\lambda_t / \tilde{\phi}_t) n_t$. 

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contract with a standard capital structure. In order to proceed, notice first that the value of the stake of outside investors has to be 

\[ dt + et = k_t - x_t \hat{\nu}(\hat{c}_t), \]

since the cost of compensating the agent under the optimal contract is 

\[ n_t = x_t \hat{\nu}(\hat{c}_t). \]

In order to obtain \( \tilde{\phi}_t \), we measure the risk of the agent’s equity of 

\[ x_t \hat{\nu}(\hat{c}_t), \]

which is \( \left( \sigma_t^2 x_t \hat{\nu}(\hat{c}_t) + x_t \hat{\nu}'(\hat{c}_t) \sigma_t^2 \hat{c}_t \right) dZ_t, \) against total capital risk of \( \sigma k_t dZ_t. \) Since 

\[ k_t = \frac{x_t \sigma \tilde{c}_t^{\gamma}}{\tilde{\phi}_t}, \]

it follows that 

\[ \tilde{\phi}_t = \frac{\sigma_t^2 x_t \hat{\nu}_t + x_t \hat{\nu}'(\hat{c}_t) \sigma_t^2 \hat{c}_t}{\sigma k_t} = \phi \left( \frac{\hat{\nu}_t}{\tilde{c}_t} + \frac{\hat{\nu}'(\hat{c}_t)}{\sigma} \right). \] (23)

We obtain the fund leverage from 

\[ \lambda_t = \frac{\sigma_t^2 \hat{c}_t^{\gamma}}{\phi \sigma \hat{\nu}_t} \tilde{\phi}_t = \frac{\sigma_t^2}{\sigma} + \frac{\hat{\nu}'(\hat{c}_t)}{\hat{\nu}_t} \] (24)

and the payout policy is simply 

\[ \theta_t = \hat{c}_t / \hat{\nu}_t. \] (25)

This spells out the triplet of processes \( S = (\lambda, \tilde{\phi}, \theta) \) as functions of \( \hat{c}_t \), which itself is determined by the history of observed returns. The following Lemma summarizes the implementation.

**Lemma 5.** The optimal contract has a unique implementation with a standard capital structure, with inside equity share \( \tilde{\phi}_t = \frac{n_t}{n_t + e_t} \) given by (23), leverage \( \lambda_t = \frac{k_t}{n_t + e_t} \) given by (24), a payout policy \( \theta_t = \frac{\hat{c}_t}{\hat{\nu}_t} \) given by (25) and assets given by (21).

The agent obtains utility in consumption units \( x_t = \hat{\nu}(\hat{c}_t)^{-1} n_t \), so the marginal value of equity \( n_t \) is \( \hat{\nu}(\hat{c}_t)^{-1} \). The marginal value of hidden savings (also measured in consumption units \( x_t \)) is \( \hat{c}_t^{\gamma} \), and is always less than the marginal value of inside equity, \( \hat{\nu}(\hat{c}_t)^{-1} > \hat{c}_t^{\gamma} \). This wedge plays an important role for the agent’s incentives. In particular, (23) implies that \( \tilde{\phi}_t < \phi \) always. Figure 4 shows the implementation for the optimal contract, and a set of relevant benchmarks which will be discussed below.

**Lemma 6.** In the implementation of the optimal contract with a standard capital structure, for all \( \hat{c} \in (\hat{c}_l, \hat{c}_h) \) the marginal value of inside equity is larger than the marginal value of hidden savings, \( \hat{\nu}(\hat{c})^{-1} > \hat{c}^{\gamma} \), and the inside equity share \( \tilde{\phi}(\hat{c}) < \phi \).

The optimal contract with hidden savings has two important features. First, there is a leverage constraint that restricts the agent’s access to capital. Second, the capital structure
Figure 4: The implementation of the optimal contract. The starting point of the optimal contract is indicated by the blue dot, the optimal stationary contract by the red dot, the optimal portfolio plan with $\tilde{\phi} = \phi$ is indicated the black dot, and the optimal contract without hidden savings by the green dot. Parameters: $\rho = r = 5\%$, $\alpha = 1.7\%$, $\gamma = 1/3$, $\phi\sigma = 0.2$. 

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depends on the history of returns $R$. As Figure 4 shows, leverage $\lambda$ decreases with $\hat{c}$, so it goes down over time (up to a steady state) and especially after bad outcomes. Both features allow the contract to relax the agent’s equity share, $\tilde{\phi}_t < \phi$, which also depends on the history of returns $R$. Figure 4 shows that the agent’s inside equity share $\tilde{\phi}$ is non-monotonic in $\hat{c}$.

To understand how it is possible to provide incentives against fund diversion with an equity share $\tilde{\phi}_t < \phi$, notice that because of the leverage constraint, the marginal value of inside equity $\hat{v}(\hat{c}_t)^{-1}$ is larger than the marginal value of hidden savings $\hat{c}_t^{-\gamma}$ (both measured in consumption units $x$). Intuitively, an extra dollar in inside equity gives the agent not only more consumption, but also allows him to access more capital. Hidden savings in contrast can only be used to consume. Because of the leverage constraint the agent has too little capital, so he values inside equity more than hidden savings,\footnote{If the contract leverage was too high, this would reverse: the value of hidden savings would be larger than the value of inside equity.} $\hat{v}(\hat{c}_t)^{-1} > \hat{c}_t^{-\gamma}$ – the agent is nonetheless willing to consume as indicated because with low leverage his exposure to risk and precautionary saving motive are small, so the marginal value of hidden savings is low. As a result, the losses in inside equity after stealing are worth more than the gains in hidden savings, so incentive compatibility is satisfied with a smaller equity share. This is captured by the first term of (23), $\phi\hat{v}_t\hat{c}_t^{-\gamma} < \phi$.

The dynamic incentive scheme helps further reduce the inside equity share $\tilde{\phi}_t$. First, because the marginal value of inside equity $\hat{v}(\hat{c}_t)^{-1}$ is forward-looking, we can relax the equity share today by reducing leverage in the future, but not the other way. This asymmetry means we want to back-load distortions. Leverage starts high and then falls as $\hat{c}_t$ increases (up to its steady state). Second, leverage falls after bad outcomes ($\sigma \hat{c} \leq 0$), which reduces the marginal value of inside equity $\hat{v}(\hat{c}_t)^{-1}$. This allows us to further relax the equity share $\tilde{\phi}_t$. To the extent that we punish the agent with less access to capital after bad outcomes, we don’t need to punish him with less equity. This is captured by the second term of (23), $\phi\hat{v}_t\sigma^2\hat{c}_t\hat{c}_t\left(\sigma_x^2\hat{c}_t^{-\gamma}\right)^{-1} < 0$.

5 Discussion: The Dynamics of Financial Frictions.

In this section, we discuss the implications of hidden savings on financial frictions. The optimal contract without hidden savings has a stationary capital structure where the agent must retain an equity stake for incentive reasons, but leverage is chosen optimally condi-
tional on the equity constraint. In contrast, the optimal contract with hidden savings has a capital structure with history-dependent leverage constraints and retained equity stake. Hidden savings therefore link leverage and equity constraints, and introduce a dynamic behavior into an otherwise stationary environment. The agent starts with high leverage, large exposure to risk and high growth expectations. Over time, and especially after bad outcomes, his leverage is reduced, he faces less risk, but gives up growth ($\hat{c}$ rises). This is reversed after good outcomes, as leverage, risk and growth rise ($\hat{c}$ falls). As a result, agents who are initially lucky and have good outcomes will have access to capital and high growth rates, while those that are initially less lucky get starved for capital and languish.

To understand the role of history-dependent financial frictions, it is useful to consider a simple benchmark where the agent keeps a fraction $\phi$ of the equity but is otherwise free to choose leverage and payouts. Starting from this benchmark, the cost of reducing leverage is second-order because it was being chosen optimally. However, lower leverage reduces the agent’s exposure to risk and his precautionary saving motive, driving a wedge between the marginal value of inside equity and hidden savings. This allows us to relax the agent’s inside equity share $\tilde{\phi} < \phi$. Since the equity constraint was binding the benefit is first-order, so we improve the contract. Thus, even stationary contracts that use a restriction on leverage, together with an appropriate incentive-compatible inside equity share $\tilde{\phi} < \phi$ improve upon this simple incentive scheme.

Non-stationary elements can improve the contract further. Distortions to the agent’s risk exposure further in the future reduce the agent’s precautionary saving motive, relaxing the incentive constraint at all times leading to those future histories. Thus, it makes sense that under the optimal contract leverage declines over time. Furthermore, distortions to the agent’s risk exposure have the most impact on the precautionary saving motive, and are least costly, if imposed after bad performance. Thus, in the optimal contract leverage declines after bad performance and may rise after good performance.

Of course, if the agent didn’t have access to hidden savings, we could do even better. The principal could simply front-load the agents’ consumption to reduce the private benefit of stealing, which allows him to reduce the agent’s retained equity without distorting leverage. Without the need to manage precautionary saving motive dynamically, the optimal contract without hidden savings is stationary.

We illustrate this logic by designing several benchmark contracts in sequence. We start by presenting the contract that solves the agent’s optimal portfolio choice problem conditional on retaining a fraction $\phi$ of equity, without any leverage constraint. Leverage
constraints reduce the agent’s precautionary saving motive, so we embed the solution of this portfolio choice problem within the family of stationary contracts. We compare all these contracts to the optimal contract, which manages leverage and the agent’s share dynamically. We finish this section by considering optimal contracting without hidden savings.

**Benchmark #1: Optimal Portfolio with an Equity Constraint**

Consider the reasonable incentive scheme where the agent must retain a fraction \( \phi \) of equity, but can otherwise choose the leverage and consumption ratio freely at any moment of time. His budget constraint is then

\[
\frac{dn_t}{n_t} = \left( r + \lambda_t \alpha / \phi - \theta_t \right) dt + \lambda_t \sigma dZ_t
\]  

(26)

This is a classic optimal portfolio choice problem, in which the risky asset has the Sharpe ratio of \( \alpha / (\phi \sigma) \). The solution is well-known and has leverage, consumption ratio and value function given by

\[
\lambda_p = \frac{\alpha}{\gamma \phi \sigma^2}, \quad \theta_p = \frac{\rho - r(1 - \gamma)}{\gamma} - \frac{1 - \gamma}{2} \left( \frac{\alpha}{\gamma \phi \sigma} \right)^2, \quad \text{and} \quad \hat{v}_p = \theta_p \frac{\gamma^2}{1 - \gamma}.
\]  

(27)

The resulting contract \( C_p = (c_p, k_p) \) corresponds to the black dot in Figures 1, 2, and 4. It has a constant \( \hat{c}_p = \theta_p \hat{v}_p = \theta_p \frac{\gamma^2}{1 - \gamma}, \) and is therefore incentive compatible.

**Lemma 7.** The contract implemented with a constant equity constraint \( C_p \) is an incentive compatible contract.

To understand the intuition, suppose the agent diverts a dollar. The agent absorbs fraction \( \phi \) of the loss, i.e. his inside equity \( n_t \) drops by \( \phi \). At the same time, the agent gets extra \( \phi \) dollars in hidden savings. Because the consumption rate \( \theta_p \) is chosen optimally by the agent, the marginal value of inside equity \( \hat{v}_p^{-1} = \theta_p \frac{\gamma^2}{1 - \gamma} \) and hidden savings \( \hat{c}_p^{-1} = \theta_p \frac{\gamma^2}{1 - \gamma} \) is the same. In addition, since the contract is stationary, the marginal value of equity \( \hat{v}_p^{-1} \) is not affected by bad performance, so the inside equity share \( \tilde{\phi} = \phi \) given by (23) ensures incentive compatibility.

Contract \( C_p \) is suboptimal. It is possible to improve upon \( C_p \) even through stationary contracts that combine a higher payout ratio \( \theta \), lower leverage \( \lambda \), and a lower incentive-compatible equity share \( \tilde{\phi} < \phi \). The very fact that an equity share \( \tilde{\phi} < \phi \) can be incentive
compatible may appear surprising. Simple stationary contracts illustrate how this is possible.

**Benchmark #2: Stationary Contracts**

Consider now the wider family of stationary contracts with a constant \( \hat{c} \). We can chose any \( \hat{c} \), and set \( \sigma_{\hat{c}} = 0 \) and 

\[
\sigma_x = \sigma^x_r(\hat{c}) \equiv \sqrt{2} \sqrt{\frac{\rho - r}{\gamma} + \frac{\rho - \hat{c}^{1-\gamma}}{1 - \gamma}}
\]  

(28)

so that \( \mu^x = 0 \) in (10). If we initiate the contract with \( x_0 = ((1 - \gamma)u_0)^{\frac{1}{1-\gamma}} \) we get a stationary contract that gives the agent utility \( u_0 \). Stationary contracts are indicated by the red dashed line in Figures 1, 2, and 4.

Equation (28) highlights the trade-off between the risk exposure and precautionary saving motive, since \( \sigma_x \) increases from 0 while \( \hat{c} \) decreases from \( \hat{c}_h \). Higher permanent risk exposure \( \sigma_x \) corresponds to a greater precautionary saving motive and a lower \( \hat{c} \). The expected growth rate \( \mu^x \) given by (9) is increasing in the risk exposure \( \sigma_x \) of the stationary contract. To guarantee admissibility, we have to restrict \( \hat{c} \) to be in the range \( (\hat{c}^*, \hat{c}_h) \), where \( \hat{c}_s \in (0, \hat{c}_h) \) is given by

\[
\hat{c}_s = \left( \frac{2\gamma}{1 + \gamma} \frac{\rho - r(1 - \gamma)}{\gamma} \right)^{\frac{1}{1-\gamma}} \triangleq \hat{c}_h
\]  

(29)

so that \( \mu^x < r \) and the No-Ponzi condition (3) is satisfied. Other than that, Theorem 3 is general enough to ensure that stationary contracts are globally incentive compatible. The HJB equation (13) yields the cost of the stationary contract. Thus, we obtain the following characterization.

**Lemma 8.** For any \( \hat{c} \in (\hat{c}_s, \hat{c}_h) \), the corresponding stationary contract with \( \sigma^x = 0 \) and \( \sigma^x = \sigma^x_r(\hat{c}) \) given by (28) is globally incentive compatible and has cost \( \hat{v}_r(\hat{c})x_0 \), where

\[
\hat{v}_r(\hat{c}) = \frac{\hat{c} - \frac{\sigma^x}{\sigma^x_r(\hat{c})}}{2r - \rho - (1 + \gamma)\frac{\rho - \hat{c}^{1-\gamma}}{1-\gamma}}.
\]

(30)

Since stationary contracts are incentive compatible, we have \( \hat{v}(\hat{c}) \leq \hat{v}_r(\hat{c}) \).\(^{12}\)

Figures 1 shows that the cost function \( \hat{v}_r(\hat{c}) \) is non-monotonic in \( \hat{c} \) (or \( \sigma^x \)). Starting from \( \sigma^x = 0 \) and \( \hat{c}_h \), raising \( \sigma^x \) and increasing capital is attractive at first because capital

\(^{12}\)Notice that \( \hat{v}_r(\hat{c}_h) = \hat{v}_h = \hat{c}_h^\gamma \), which makes sense because at \( \hat{c}_h \) the stationary contract has \( \sigma^x = 0 \).
yields an excess return $\alpha$, and the distortions to risk-sharing and intertemporal consumption are second order. However, at some point these distortions become too costly. The best stationary contract with minimum cost $C_{r}^{\text{min}}$ is indicated by the red point at $(\hat{c}_{r}^{\text{min}}, \hat{v}_{r}^{\text{min}})$, where $c_{r}^{\text{min}} \equiv \arg \max_{\hat{c} \in (\hat{c}_{p}, \hat{c}_{h})} \hat{v}_{r}(\hat{c})$ and $\hat{v}_{r}^{\text{min}} \equiv \hat{v}_{r}(c_{r}^{\text{min}})$.

The best stationary contract $C_{r}^{\text{min}}$ is better than the optimal portfolio contract with an equity constraint $C_{p}$, which is also an incentive compatible stationary contract indicated by the black dot. To understand the improvement, consider a small reduction in leverage starting from $C_{p}$. Since leverage was being chosen optimally, the effect on the agent’s utility is second-order. However, the effect on the precautionary saving motive is first-order. Incentive-compatible payouts go up, and a first-order drop in the agent’s marginal utility of consumption leads it to drop below the marginal value of net worth (on which the leverage constraint has a second-order effect). The wedge leads to a lower incentive-compatible equity share $\tilde{\phi}$ and a first-order positive effect on the agent’s utility.\footnote{Of course, lower $\tilde{\phi}$ implies even lower precautionary saving motive, so $\tilde{\phi}$ can be lowered even further, until a fixed point.}

**Lemma 9.** Any stationary contract can be implemented with a constant capital structure with

\[
\tilde{\phi}_{r}(\hat{c}) = \frac{\hat{v}_{r}(\hat{c})}{\hat{c}^{\gamma}}, \quad \lambda_{r}(\hat{c}) = \frac{\sigma^{2}(\hat{c})}{\sigma} \quad \text{and} \quad \theta_{r}(\hat{c}) = \frac{\hat{c}}{\hat{v}_{r}(\hat{c})}
\]  

(31)

The contract implemented with a constant equity constraint $C_{p}$ is the stationary contract corresponding to $\hat{c}_{p}$, but is not optimal. The best stationary contract $C_{r}^{\text{min}}$ is less risky for the agent, i.e. we have $\hat{c}_{r} < \hat{c}_{p} < \hat{c}_{r}^{\text{min}}$. For all $\hat{c} \in (\hat{c}_{p}, \hat{c}_{h})$ the marginal value of equity is larger than the marginal value of hidden savings, $\hat{v}_{r}^{-1}(\hat{c}) > \hat{c}^{-\gamma}$, and the inside equity share $\tilde{\phi}_{r}(\hat{c}) < \phi$.

For stationary contracts with $\hat{c} \in (\hat{c}_{p}, \hat{c}_{h})$ it is worth making several observations. First, since there is a wedge between the marginal utility of inside equity and the marginal utility of consumption, $\hat{v}_{r}(\hat{c})^{-1} > \hat{c}^{-\gamma}$, we can relax the inside equity share, $\tilde{\phi}_{r}(\hat{c}) < \phi$. Second, the fund must mandate the payout rate $\theta_{r}(\hat{c})$ since the agent would prefer to postpone consumption. However, the agent is willing to consume as indicated, rather than save, because the payout rate corresponds to his precautionary saving motive. Third, hidden savings create a leverage constraint for a completely different reason from models of limited commitment, as in Hart and Moore (1994) and Kiyotaki and Moore (1997), where the agent can “run away” with some resources. Here the leverage constraint reduces the agent’s precautionary saving motive, leading to a lower incentive compatible risk exposure.
Benchmark #3: The Optimal Contract

The best stationary contract illustrates how restricting the agent’s access to capital through a leverage constraint can help provide incentives. The optimal contract does even better by adding a dynamic dimension to the financial frictions. The principal front loads the agent’s access to capital and punishes him after bad outcomes by further restricting his access to capital, rather than taking away equity or consumption (he gives the agent more capital after good outcomes). This improves risk sharing and relaxes the agent’s precautionary saving motive, making stealing less attractive, and allows the principal to use the more efficient continuation contracts when he must deliver more utility to the agent (after good outcomes).

Lemma 10. For any \( \hat{c} \in [\hat{c}_l, \hat{c}_h] \), the optimal contract has a lower equity constraint than the corresponding stationary contract, i.e.

\[
\tilde{\phi}(\hat{c}) \equiv \phi \left( \frac{\hat{v}(\hat{c}_t)}{\hat{c}_t^\gamma} + \frac{\hat{v}'(\hat{c}_t)\sigma_t^2 \hat{c}_t}{\sigma_t^2 \hat{c}_t^\gamma} \right) \leq \tilde{\phi}_r(\hat{c}).
\]  

Of course, if the agent did not have access to hidden savings, we could do even better. Let us now turn to the optimal contract without hidden savings.

Benchmark #4: No hidden savings

If the agent had no access to hidden savings, the principal can control the agent’s consumption directly. Hence, the principal specifies the agent’s consumption and capital to maximize profit and subject to incentive compatibility with respect to diversion of returns, without regard to the agent’s precautionary saving motive. While we present it as a benchmark here, optimal contracting without hidden savings is a classic problem that is interesting in its own right. The solution we derive here, in particular, is of special interest because it has a closed form and a stationary structure. The convenient form of the solution should prove helpful in applications. In fact, Di Tella (2014) adopts a generalized version of this model in the context of financial regulation, and provides important characterizations, some of which we cite here.

Without hidden savings, (6) is still the incentive compatibility constraint for return diversion and (9) is still the law of motion of \( x_t \), but the agent’s consumption \( c_t \) and ratio \( \hat{c}_t = c_t/x_t \) are no longer bound by the Euler equation (7) or (10). Rather, \( \hat{c}_t \) is now the
principal's control rather than a state variable. We can write the HJB equation for the problem without hidden savings by taking (13), removing terms that contain \( \hat{v}' \) and \( \hat{v}'' \), and replacing control \( \sigma \hat{c} \) with \( \hat{c} \). We obtain the HJB equation

\[
r\hat{v} = \min_{\sigma^x, \hat{c}} \hat{c} - \sigma^x \hat{c}^\gamma \frac{\alpha}{\phi \sigma} + \hat{v} \left( \frac{\rho - \hat{c}^{1-\gamma}}{1 - \gamma} + \frac{\gamma}{2} (\sigma^x)^2 \right),
\]

(33)

where \( v(x) = \hat{v} x \) is the principal's cost function.

The optimal contract takes a simple stationary form, with constant volatility \( \sigma^x \) and ratio \( \hat{c} \) and hence constant growth \( \mu^x \). The solution exists only if \( \gamma \leq 1/2 \) and only if \( \alpha \) is sufficiently low; otherwise the principal's value function becomes infinite. More generally, Di Tella (2014) shows that with Epstein-Zin preferences, the elasticity of intertemporal substitution must be greater than 2 in order for the solution to exist. Intuitively, if the EIS is low a principal who can control the agent's consumption has a lot of power over him and can effectively eliminate the moral hazard problem and obtain an arbitrage for any \( \alpha > 0 \).

An example of a solution for \( \gamma = 1/3 \) is indicated with a green dot in Figures 1, 2, and 4.

The agent's consumption in the optimal contract is determined without regard to the agent's precautionary saving motive. The first-order condition for optimal consumption is

\[
1 = \hat{v} \hat{c}^{-\gamma} \hat{c} + \hat{v} \gamma (\sigma^x)^2 \hat{c}^{-1}
\]

(34)

The left-hand side is the marginal cost of paying the agent, one. The two terms on the right-hand side capture the benefits of paying the agent. First, the principal reduces the agent’s continuation utility by paying today. Second, according to (6), higher consumption today reduces the agent’s incentives to divert returns, and hence reduces the required risk exposure given any level of capital under management. Hence, the agent's consumption in the optimal contract becomes front-loaded. This can be expressed in several ways. The Euler equation does not hold, i.e. \( e^{-(r-\rho)t} c_t^{-\gamma} \) is a strict submartingale and so, if the agent had access to hidden savings, he would want to save.\(^{14}\) Equivalently, given choice of \( \sigma^x_n \) in the optimal contract, the consumption rate that reflects the precautionary saving

\(^{14}\) It is interesting to note that the standard inverse Euler equation of agency problems without hidden savings does not hold in our setting either. The inverse Euler equation requires that \( e^{-(r-\rho)t} c_t^{-\gamma} \) is a martingale, where \( c_t^{-\gamma} \) is the marginal cost of delivering utility to the agent, or inverse marginal utility. If the inverse Euler equation holds, then the agent wants to save if he could, i.e. consumption is more front-loaded relative to the hidden-savings case. In our problem, because consumption enters the agent’s incentive constraint, it is even more front-loaded than the profile of the inverse Euler equation.
motive, given by (28), is lower than the consumption rate \( \hat{c}_n \) of the optimal contract. To repeat the main intuition, front-loaded consumption is beneficial because it reduces the agent’s marginal utility from consumption, and therefore makes stealing and immediately consuming less attractive. This relaxes the incentive constraint.

We can implement the contract by following the derivations of Section 4. By stationarity, the optimal contract has fixed leverage, equity shares held by the agent and principal, and payout rate. We have analogous formulas as (31),

\[
\tilde{\phi}_n = \phi \frac{\hat{v}_n}{\hat{c}_n}, \quad \lambda_n = \frac{\sigma_x}{\sigma} \quad \text{and} \quad \theta_n = \frac{\hat{c}_n}{\hat{v}_n}.
\]  

We summarize important properties of the implementation in the following lemma.

**Lemma 11.** The optimal contract without hidden savings is implemented with a constant capital structure given by (35). The marginal value of equity is larger than the marginal value of consumption, \( \hat{v}_n^{-1} > \hat{c}_n^{-\gamma} \), and the inside equity share \( \tilde{\phi}_n < \phi \). There is no leverage constraint; leverage is chosen optimally conditional on \( \tilde{\phi}_n \), \( \lambda_n = \frac{\alpha}{\gamma \tilde{\phi}_n \sigma} \).

For a long time, researchers have been wondering to what extent the possibility of hidden savings matters. The view that the savings constraint matters very little leads to a justification of models without savings, especially given the technical difficulties that hidden savings pose. Without solving for the optimal contract, this question is difficult to address, although Farhi and Werning (2008) provide important assessments based on ingenious explorations of the Euler and inverse Euler equations. In our setting, given full characterization of optimal contracts, we can answer the question of how much hidden savings matter more definitively. The answer depends on parameters, but potentially the difference between the optimal contracts with and without hidden savings can be very large. In fact, for some parameters the principal’s value function without savings becomes infinite, while it is finite with hidden savings.

### 6 Aggregate risk and hidden investment

The previous results show how the principal can use the agent’s access to capital to manipulate his precautionary saving motive. This suggests that hidden investment could be an important constraint in this setting. To study the role of hidden investment, it is useful to also introduce aggregate risk, and allow the agent to invest his hidden savings in it.
Fortunately, we can easily incorporate hidden investment into our setting and extend our tools to study what role they play. Appendix B has all the formal results. Here we will focus on the main economic insights.

The return on capital is now

$$dR_t = (r + \pi \tilde{\sigma} + \alpha - a_t) dt + \sigma dZ_t + \tilde{\sigma} d\tilde{Z}_t$$

Where $\tilde{Z}$ is an independent Brownian motion that represents aggregate risk, with market price $\pi$. Capital has a loading $\tilde{\sigma}$ on aggregate risk, so the excess return on capital for the agent is $\alpha$, as in the baseline. Let $Q$ be the associated martingale measure. Now the agent can invest his hidden savings (besides the risk-free rate). We always allow the agent to invest in aggregate risk, in the same way the principal would be able to. This is really an extension of the notion of hidden savings to an environment with aggregate risk. In addition, the agent may be able to invest his hidden savings in his own private technology. His hidden savings therefore follow the law of motion

$$dh_t = (rh_t + z_t h_t(\alpha + \pi \tilde{\sigma}) + \tilde{z}_t h_t \pi + c_t - \tilde{c}_t + \phi k_t a_t) dt + z_t h_t \left( \sigma dZ_t + \tilde{\sigma} d\tilde{Z}_t \right) + \tilde{z}_t h_t d\tilde{Z}_t$$

where $z$ is the portfolio weight on his own private technology, and $\tilde{z}$ the weight on aggregate risk. While the agent can choose any position on aggregate risk, $\tilde{z}_t \in \mathbb{R}$, for his hidden private investment we consider two cases: 1) no hidden private investment, $z_t \in H = \{0\}$, and 2) hidden private investment, $z_t \in H = \mathbb{R}_+$. A valid interpretation for hidden investment in the private technology $z_t$ is that the principal can give the agent an amount of capital $k_t$ that he monitors, but the agent can secretly invest more.

A contract $C = (c, k)$ specifies the contractible payments $c$ and capital $k$, contingent not only on returns $R$ but also on the observable aggregate shock $\tilde{Z}$. After signing the contract the agent can choose a strategy $(\tilde{c}, a, z, \tilde{z})$ to maximize his utility. The agent’s utility and the principal’s objective function are still given by (1) and (2). As in the baseline setting, it is without loss of generality to look for a contract where the agent does not steal, has no hidden savings, and no hidden investment. A contract is therefore incentive compatible if the agent’s optimal strategy is $(c, 0, 0, 0)$, or $(c, 0)$ for short.

\footnote{We can also study other cases where the agent may not be able to invest in aggregate risk, or only take a positive position, which requires small modifications to the relevant incentive compatibility constraints. We focus on the economically most relevant case, where the agent can always invest his hidden savings in the market in the same way the principal would.}
Since there is now aggregate risk that pays a premium, we need to slightly modify the parameter restrictions

\[
\left( \frac{\rho - r(1 - \gamma) - \frac{1 - \gamma}{2} \left( \frac{\pi}{\gamma} \right)^2}{\gamma} \right)^{\frac{1}{1 - \gamma}} > 0
\]  

and

\[
\alpha < \bar{\alpha} \equiv \frac{\phi \sigma \gamma \sqrt{2}}{\sqrt{1 + \gamma}} \sqrt{\frac{\rho - r(1 - \gamma)}{\gamma} - \frac{1}{2} \frac{(1 - \gamma)(\frac{\pi}{\gamma})^2}{\gamma^2}}
\]

We can check that with \( \pi = 0 \) we recover the formulas without aggregate risk.

Since the contract can depend on the history of aggregate shocks \( \tilde{Z} \), so can his continuation utility \( U^{c,0} \) and his consumption \( c \). However, because the agent is not responsible for aggregate shocks, incentive compatibility does not place any constraints on his exposure to aggregate risk. On the other hand, since the agent can invest his hidden savings in aggregate risk, his Euler equation needs to be modified appropriately. His discounted marginal utility must be a supermartingale under any valid trading strategy. Otherwise, the agent could save a dollar, invest it in aggregate risk or his private technology, and consume it later. As a result, the laws of motion for the state variables \( x \) and \( \dot{c} \) need to be modified to

\[
\frac{dx_t}{x_t} = \left( \frac{\rho - \dot{c}_t^{1-\gamma}}{1 - \gamma} + \frac{\gamma}{2} (\sigma^x_t)^2 + \frac{\gamma}{2} (\tilde{\sigma}^x_t)^2 \right) dt + \sigma^x_t dZ_t + \tilde{\sigma}^x_t d\tilde{Z}_t \tag{38}
\]

\[
\frac{d\dot{c}_t}{\dot{c}_t} = \left( \frac{r - \rho}{\gamma} - \frac{\rho - \dot{c}_t^{1-\gamma}}{1 - \gamma} + \frac{(\sigma^x_t)^2}{2} + \gamma \sigma^x_t \sigma^{\dot{c}}_t + \frac{1 + \gamma (\sigma^{\dot{c}}_t)^2}{2} \right) dt + \sigma^{\dot{c}}_t dZ_t + \tilde{\sigma}^{\dot{c}}_t d\tilde{Z}_t + dL_t \tag{39}
\]

The “skin in the game” IC constraint is unchanged

\[
\sigma^x_t = \dot{c}_t^{-\gamma} \phi k_t \sigma
\]

Since the agent can invest his hidden savings in aggregate risk (both long and short) we have a new IC constraint

\[
\tilde{\sigma}^{\dot{c}}_t + \tilde{\sigma}^x_t = \frac{\pi}{\gamma} \tag{41}
\]

If the agent can also invest his hidden savings in his private technology, \( H = \mathbb{R}_+ \), we also
have another IC constraint
\[ \sigma^x_t + \sigma^\hat{c}_t \geq \frac{\alpha}{\sigma \gamma} \] (42)

The interpretation for (41) and (42) is simple: for the agent, aggregate risk has a premium \( \pi \) and his idiosyncratic risk a premium \( \frac{\alpha}{\sigma \gamma} \). The principal must allow him to invest in these sources of risk through the contract, or he will do it on his own. This restricts the principal’s ability to provide incentives. In particular, the principal cannot promise to give the agent a perfectly deterministic consumption at some point in the future. If he tried to do this, the agent would just take risk on his own. This is reflected in a lower upper bound \( \hat{c}_h \), which is costly because the principal would like to use a promise of future safety to relax the agent’s precautionary saving motive. If the agent cannot invest in his private technology, we have
\[ \hat{c}_h = \left( \frac{\rho - r (1 - \gamma)}{\gamma} - \frac{1 - \gamma}{2} \left( \frac{\pi}{\gamma} \right)^2 \right)^{1/(1 - \gamma)} \]
This is the \( \hat{c} \) corresponding to the optimal portfolio strategy when the agent can invest only in aggregate risk. Notice that if we take \( \pi = 0 \) we recover expression (8) in the baseline setting, corresponding to the optimal portfolio if the agent can only invest in a risk-free asset. If the agent can also invest in his private technology, then \( \hat{c}_h = \left( \frac{\rho - r (1 - \gamma)}{\gamma} - \frac{1 - \gamma}{2} \left( \frac{\alpha}{\sigma \gamma} \right)^2 - \frac{1 - \gamma}{2} \left( \frac{\pi}{\gamma} \right)^2 \right)^{1/(1 - \gamma)} \), which is even lower and corresponds to the \( \hat{c} \) in an optimal portfolio strategy when the agent can invest in both aggregate risk and in his private technology on his own (so he can’t get any risk sharing).\(^{16}\)

We can now characterize the relaxed problem with an HJB equation and appropriate constraints

\[
0 = \min_{\sigma^x, \sigma^\hat{c}, \tilde{\sigma}^x, \tilde{\sigma}^\hat{c}} \hat{c} - r \hat{v} - \sigma^x \hat{c}^2 \frac{\alpha}{\phi \sigma} + \hat{v} \left( \frac{\rho - \hat{c}^{1 - \gamma}}{1 - \gamma} + \frac{\gamma}{2} (\sigma^x)^2 + \frac{\gamma}{2} (\tilde{\sigma}^{\hat{c}})^2 - \pi (\sigma^x) \right) + \hat{v}' \hat{c} \left( \frac{r - \rho}{\gamma} - \frac{\rho - \hat{c}^{1 - \gamma}}{1 - \gamma} + \frac{(\sigma^x)^2}{2} + (1 + \gamma) \sigma^x \sigma^{\hat{c}} + \frac{1 + \gamma}{2} (\sigma^{\hat{c}})^2 + \frac{(\tilde{\sigma}^x)^2}{2} \right) + (1 + \gamma) \tilde{\sigma}^x \tilde{\sigma}^\hat{c} + \frac{1 + \gamma}{2} (\tilde{\sigma}^{\hat{c}})^2 - 2 \tilde{\sigma}^x \pi + \frac{\hat{v}''}{2} \hat{c}^2 \left( (\sigma^{\hat{c}})^2 + (\tilde{\sigma}^{\hat{c}})^2 \right) \]

subject to \( \sigma^x \geq 0 \) and (41) and (42).

Figure 5 shows the optimal contract with hidden investment. The optimal contract has the same features as in the baseline setting. Hidden investment, however, restricts the principal’s ability to manipulate the agent’s precautionary saving motive by promising

\(^{16}\)The parameter restrictions (36) and (37) ensure \( \hat{c}_h \) is positive in both cases.
Figure 5: The cost function $\hat{v}(\hat{c})$ solid in blue without hidden investment, green with hidden investment, and dashed in red for stationary contracts. The the starting point without hidden investment is indicated by the blue dot, with hidden investment by the green dot, the optimal stationary contract by the red dot, the optimal portfolio plan with $\hat{\phi} = \phi$ by the black dot. The black dashed line is the locus $A(\hat{c}, \hat{v}) = 0$. Parameters: $\rho = r = 5\%$, $\alpha = 6\%$, $\gamma = 3$, $\phi \sigma = 0.2$, $\pi = 4\%$.

Figure 6: The drift, $\mu^{\hat{c}}$ and $\mu^{x}$, and volatility, $\sigma^{\hat{c}}$ and $\sigma^{x}$, of the state variables $\hat{c}$ and $x$, without hidden investment (blue) and with hidden investment (green). The black dashed lines indicate $\hat{c}_l$ and $\hat{c}_h$ with hidden investment, and $\hat{c}_l$ without hidden investment. Parameters: $\rho = r = 5\%$, $\alpha = 6\%$, $\gamma = 3$, $\phi \sigma = 0.2$, $\pi = 4\%$. 

less risk in the future and after bad outcomes, and thus leads to a higher cost. The contract starts at some \( \hat{c}_t \) and then moves immediately into the interior of the domain \([\hat{c}_t, \hat{c}_h]\). Because the principal cannot promise a completely deterministic contract in the future (\( \hat{c}_h \) is lower with hidden investment), the agent’s precautionary saving motive is stronger, and the contract starts at a lower \( \hat{c}_t \).

Turn now to the contract’s dynamic behavior. First, using (41) to eliminate \( \tilde{\sigma} \hat{c} \), and taking FOC for \( \tilde{\sigma} x \), we obtain

\[
\tilde{\sigma} x = \pi \gamma \tilde{\sigma} \hat{c} \quad \tilde{\sigma} = 0
\]

This is the first best exposure to aggregate risk. The principal and the agent don’t have any conflict about aggregate risk, and the principal cannot use it to relax the moral hazard problem, so they implement the first best aggregate risk sharing.\(^{17}\) The presence of this investment opportunity still affects the optimal contract, however, by limiting the principal’s ability to manipulate the agent’s precautionary saving motive (it lowers \( \hat{c}_h \)).

In contrast, the FOC for \( \sigma^x \) and \( \sigma^\hat{c} \) depend on whether the agent can invest his hidden savings in his private technology, as shown in Figure 6. Without hidden investment, the FOCs are the same as in the baseline, (16) and (17), and capture the intertemporal and hedging motives described in Section 3. With hidden investment, the IC constraint (42) can be binding in some region of the state space. This is the case in Figure 6 near the upper bound of the hidden investment contract \( \hat{c}_h \). It is still true, however, that \( \mu \hat{c} (\hat{c}_l) > 0 \) and \( \sigma^\hat{c} \leq 0 \) always, so the contract dynamics are the same as in the baseline: the agent gets less capital and less risk over time (up to a “steady state”) and after bad outcomes.

The optimal contract can be implemented with a capital structure \( S = (\lambda, \tilde{\phi}, \theta, \tilde{\sigma}^n) \) given by (23), (24), (25), and \( \tilde{\sigma}^n = \pi / \gamma \), with the law of motion for the agent’s inside equity

\[
\frac{dn_t}{n_t} = \left( r + \lambda_t \alpha / \tilde{\phi}_t + \tilde{\sigma}^n_t \pi - \theta_t \right) dt + \lambda_t \sigma dZ_t + \tilde{\sigma}^n_t d\tilde{Z}
\]

There is still a leverage constraint that allows the principal to relax the agent’s inside equity stake \( \tilde{\phi} < \phi \). How come the principal is able to impose a leverage constraint if the agent can invest on his own with his hidden savings? If the agent invests on his own he is the residual claimant and must bear the whole risk, while if he invests through the principal he gets to share some risk and keep only a fraction \( \tilde{\phi}_t < 1 \). The leverage constraint really limits the

\(^{17}\) If the agent didn’t have access to hidden investment in aggregate risk, and the agent’s private technology is exposed to aggregate risk \( \tilde{\sigma} \neq 0 \), then the principal could potentially use the agent’s exposure to aggregate risk to relax the moral hazard problem.
amount of capital the principal will agree to share risk on. As in the setting without hidden investment, the optimal contract starts with a non-binding leverage constraint. It is worth pointing out that although \( \hat{c}_h \) corresponds to the consumption profile under autarky, there are still gains from trade if the agent is able to invest in his private technology, so that the cost for the principal at this point is lower \( \hat{v}(\hat{c}_h) < \hat{c}_h^\gamma \). To see why, notice that the principal can give the agent the same consumption process that he would get in autarky, but more capital. Since \( \hat{\phi} < \phi < 1 \), the agent can manage more capital than under autarky and yet have the same exposure to risk. Since he keeps the total exposure to risk corresponding to autarky, which is relatively low given that the agent must bear the whole risk, there is a binding leverage constraint here. In other words, if the agent could invest more capital and keep only a fraction \( \hat{\phi} < 1 \) he would like to invest more, but not if he has to keep all the risk.

It is useful to ask under what conditions the gains from trade are completely exhausted, and the optimal contract corresponds to letting the agent invest on his own without any risk sharing. This is true in the special (but salient) case with \( \phi = 1 \), where we have a pure misreporting problem. Without hidden investment, the are still gains from trade, and the optimal contract has the same qualitative features described in Section 3. While the agent can save on his own, the principal can still control his access to capital to provide incentives, and can therefore provide some risk sharing. However, with hidden investment, the optimal contract coincides with autarky. Intuitively, the agent can both save and invest on his own, so the principal cannot provide any risk sharing in an incentive compatible way. We can see this case as a limit in Figure 5. If we let \( \phi \to 1 \), while also adjusting \( \sigma \) so that \( \phi \sigma \) is constant, all the curves corresponding to case with no hidden investment remain unchanged. The optimal contract with hidden investment, however, becomes progressively worse, because the agent finds investing on his own more attractive (the green curve shifts up). This is reflected in a falling \( \hat{c}_h \) \( \downarrow \hat{c}_p = \left( \frac{\rho - r(1 - \gamma)}{\gamma} - \frac{1 - \gamma}{2} \left( \frac{\alpha}{\gamma \phi \sigma} \right)^2 - \frac{1 - \gamma}{2} \left( \frac{\bar{z}}{\gamma} \right)^2 \right) \), where \( \hat{c}_p \) corresponds to the optimal portfolio with an equity constraint \( \hat{\phi} = \phi \) (which in the limit is 1).

**Verifying global incentive compatibility with hidden investment**

The agent’s ability to invest his hidden savings makes the verification of global incentive compatibility potentially more difficult. The agent could find it attractive to steal and save the proceeds for later, while investing them in aggregate risk or even his own pri-
vate technology. However, we can extend the results of Theorem 3 to deal with hidden investment.

**Theorem 4.** Let $C = (c, k)$ be an admissible contract with associated processes $x$ and $\hat{c}$ satisfying (62) and (63), and (64), (65), and (66), with bounded $\mu^x$, $\mu^\hat{c}$, $\sigma^\hat{c}$, and $\tilde{\sigma}^\hat{c}$, and with $\hat{c}$ uniformly bounded away for zero and bounded above by $\hat{c}_h$. Suppose that the contract satisfies the following property

$$\sigma^\hat{c}_t \leq 0$$

Then for any feasible strategy $(\hat{c}, a, z, \tilde{z})$, with associated hidden savings $h$, we have the following upper bound on the agent’s utility, after any history

$$U^{\hat{c}, a, z, \tilde{z}}_t \leq \left(1 + \frac{h_t}{x_t} \hat{c} - \gamma \right)^{1-\gamma} U^{c, 0}_t$$

In particular, since $h_0 = 0$, for any feasible strategy $U^{\hat{c}, a, z, \tilde{z}}_0 \leq U^{c, 0}_0$, and the contract $C$ is therefore incentive compatible.

## 7 Renegotiation

The optimal contract requires commitment. The principal can relax the agent’s precautionary saving motive by promising an inefficiently low amount of capital and risk over time and after bad outcomes. This suggests that the agent and principal could be tempted to renegotiate the contract and “start over”, undoing the whole incentive scheme. Here we formalize a notion of renegotiation, and characterize the optimal renegotiation-proof contract. As it turns out, the optimal renegotiation-proof contract is the best stationary contract, so even without commitment we are able to do better than letting the agent invest while keeping a fraction $\phi$ of his risk, and a leverage constraint can be used to relax the equity constraint. This section is consistent with the presence of aggregate risk and hidden investment introduce in Section 6.

After signing an incentive compatible contract $C = (c, k)$, the principal can at any time offer a new continuation contract that leaves the agent at least as well off (the offer is “take it or leave it”). The question is what kind of contracts can he offer - what is a valid “challenger” to the original contract? Here we explore a notion of internal consistency. If $C$ is renegotiation proof, then surely appropriately scaled parts of it should be a valid challenger. This means that at any stopping time $\tau$, the principal can renegotiate and get
a continuation cost $x_r \times \inf \hat{v}(\omega, t)$. With this in mind, we say that an incentive compatible contract is renegotiation-proof (RP) if

$$
\infty = \arg \min_{\tau} \mathbb{E}^Q \left[ \int_0^\tau e^{-rt} (c - k_t \alpha) dt + e^{-r\tau} x_r \inf \hat{v} \right]
$$

The optimal contract with hidden savings is not renegotiation proof, because after any history $\hat{v}_t > \hat{v}_l = \inf \hat{v}(\omega, s)$, so the principal is always tempted to “start over”. In fact, it is easy to see that RP contracts must have a constant $\hat{v}_t$. The converse it also true.

**Lemma 12.** An incentive compatible contract $C$ is renegotiation proof if and only if the continuation cost $\hat{v}$ is constant.

Stationary contracts have a constant $\hat{v}$, because $\hat{c}$ is constant. However, those contracts were built using $dL_t = 0$. There are other contracts with a constant $\hat{c}$ that use $dL_t > 0$, i.e. the drift of $\hat{c}$ would be negative without $dL_t$. In addition, there could be non-stationary contracts with a constant cost $\hat{v}(\hat{c})$ for all $\hat{c}$ in the domain. The next Lemma shows they are all worse than the best stationary contract $C_{r, min}$, with cost $\hat{v}_{r, min} = \min_{\hat{c} \in (\hat{c}_r, \hat{c}_h]} \hat{v}_r(\hat{c})$.

**Theorem 5.** The optimal renegotiation-proof contract is the optimal stationary contract $C_{r, min}$ with cost $\hat{v}_{r, min}$.

**Remark.** It is possible that $\hat{v}_{r, min} = \hat{c}_h$ if the agent can invest his hidden savings and $\phi$ is close enough to 1. In the special case with hidden investment and $\phi = 1$, we have $\hat{c}_{r, min} = \hat{c}_p = \hat{c}_h$, as show in Lemma 28.

This result shows that even without full commitment, hidden savings generate a leverage constraint. The principal can do better than just giving the agent an equity constraint $\tilde{\phi} = \phi$ by restricting leverage $\lambda = \frac{k}{n+\epsilon}$.

### 8 Conclusions

We study the role of hidden savings in a classic portfolio-investment problem with fund diversion, which can be embedded in macroeconomic and financial environments. Without hidden savings, the principal front-loads the agent’s consumption to reduce the private benefit of fund diversion. The agent must keep an equity stake to provide incentives, but he is otherwise free to choose his leverage. With hidden savings, the principal cannot control the agent’s consumption, and must use the agent’s access to capital to manipulate his
precautionary saving motive. By reducing access to capital over time, and especially after bad outcomes, the principal reduces incentives to divert funds and secretly save, reducing the cost of providing incentives. As a result, the optimal contract requires a leverage constraint in addition to the equity constraint. Hidden savings generate a leverage constraint for completely different reason than models of limited commitment such as Hart and Moore (1994) and Kiyotaki and Moore (1997) – a binding leverage constraint reduces the agent’s precautionary saving motive and allows the principal to relax the equity constraint. Our results are robust to introducing market risk, hidden investment, and renegotiation.

An important methodological contribution is that we provide a sufficient analytical condition for the validity of the first-order approach. If the agent’s precautionary saving motive is weaker after bad outcomes, the contract is globally incentive compatible. This condition holds in the optimal contract and in a wider class of contracts beyond the optimal one. In fact, the sufficient condition does not even require a recursive structure, and it is valid even with aggregate risk and hidden investment.
References


Appendix A - Omitted Proofs

Lemma 1

Consider
\[ Y_t = \mathbb{E}_t \left[ \int_0^\infty e^{-\rho s} \frac{c_s^{1-\gamma}}{1-\gamma} ds \right] = \int_0^t e^{-\rho s} \frac{c_s^{1-\gamma}}{1-\gamma} ds + e^{-\rho t} \bar{U}^{c,0} \]
Since \( Y \) is an \( \mathbb{F} \)-adapted \( P \)-martingale, and \( \mathbb{F} \) is generated by \( Z \), we can apply a martingale representation theorem to obtain
\[ dY_t = e^{-\rho t} \frac{c_t^{1-\gamma}}{1-\gamma} dt + e^{-\rho t} dU^{c,0}_t - \rho e^{-\rho t} U^{c,0}_t dt = e^{-\rho t} \Delta_t \sigma \ dZ_t \]
for some stochastic process \( \Delta \) also adapted to \( \mathbb{F} \). Dividing by \( e^{-\rho t} \) and rearranging we get (4).

Lemma 2

First, take the strategy \((c + \phi k, a)\). Here \( a \) is zero where (5) is satisfied and \( a = a^*_\gamma \wedge \bar{a} \) where \( a^*_\gamma \) achieves the maximum in (5) and \( \bar{a} > 0 \) is an arbitrary bound. Because the objective in (5) is concave and it’s \( \frac{1-\gamma}{1-\gamma} \) for \( a = 0 \), then for \( a^*_\gamma \wedge \bar{a} \) it is strictly greater when (5) fails. Notice \( h_t = 0 \) by construction, and we can follow strategy \((\hat{c}^n, a^n)\) until some stopping time \( \tau^n \) and then revert to good behavior, where the stopping time \( \tau^n \to \infty \) a.s., ensures it’s feasible and reduces the stochastic integral. We can compare the utility from this strategy \( U^{c^n,a^n} \) with the utility from good behavior \( U^{c,0} \):
\[ U^{c^n,a^n} - U^{c,0} = \mathbb{E} \left[ \int_0^{\tau^n} e^{-\rho t} \left( \frac{(c_t + \phi k_t a_t)^{1-\gamma}}{1-\gamma} - \frac{c_t^{1-\gamma}}{1-\gamma} - \Delta_t a_t \right) dt \right] \]
(44)
where we have used \( Z^n_t = Z_t + \int_0^t a_s ds \) to express \( U^{c,0} \) as an expected integral under \( P^n \), and also the fact that \( U^{c^n,a^n} = U^{c,0} \). Taking \( n \to \infty \) and using the monotone convergence theorem (the integrand is always positive), if (5) fails we get \( U^{c^n,a^n} > U^{c,0} \) for some large \( n \), and \( \bar{c} \) is not incentive compatible. (6) is simply the FOC necessary condition for (5) (and sufficient because of concavity).

For (7) this is the result of \( e^{-(\rho-r)t} c_t e^{-\gamma} \) being a supermartingale, which is a standard necessary condition in a savings/consumption problem. Note this doesn’t involve stealing \( a_t \), just hidden consumption \( c \). We can then write it \( e^{-(\rho-r)t} c_t = M_t - A_t \), where \( M_t = \int_0^t \sigma_M dZ_t \) is a local martingale, and \( A \) a weakly increasing process. Using Ito’s lemma, we get the desired expression.

Lemma 3

For the bound, since both \( c_t \geq 0 \) and \( x_t \geq 0 \), we only need to show that \( \hat{c}_t \leq \hat{c}_0 \). Marginal utility of consumption is \( m_t = c_t e^{-\gamma} \) and the utility flow \( \frac{c_t^{1-\gamma}}{1-\gamma} = \frac{1}{1-\gamma} \frac{c_t^{-1} e^{\rho t}}{m_t} \). This is a convex and decreasing function of \( m_t \). From Lemma 2, we have by (7) that \( \mathbb{E} [m_{t+u}] \leq e^{(\rho-r)u} m_t \). Given any \( m_0 \) we have by Jensen’s inequality
\[ \mathbb{E} \left[ \frac{c_t^{1-\gamma}}{1-\gamma} \right] \geq \frac{1}{1-\gamma} \mathbb{E} [m_t] \geq \frac{1}{1-\gamma} \frac{m_0}{e^{(\rho-r)\frac{2}{\gamma} t}} \geq \frac{c_0^{1-\gamma}}{1-\gamma} e^{(\rho-r)\frac{2}{\gamma} t} \]
with equality only if $c$ is deterministic. So

$$U_i^{c, 0} = \mathbb{E}_t \left[ \int_t^\infty e^{-\rho(u-t)} \frac{\epsilon^{1-\gamma} u}{1-\gamma} du \right]$$

$$\geq \int_t^\infty e^{-\rho(u-t)} \frac{\epsilon^{1-\gamma} u}{1-\gamma} e^{(\rho-r)\frac{\gamma}{1-\gamma}(u-t)} ds = \frac{c_t^{1-\gamma}}{1-\gamma} \rho - r(1-\gamma)$$

where the second equality uses the fact that the termination contract must deliver $\hat{U}$ to the agent. This bound implies

$$x_t = ((1-\gamma)U_i^{c, 0})^{1/(1-\gamma)} \geq c_t \left( \frac{\gamma}{\rho - r(1-\gamma)} \right)^{1/(1-\gamma)}$$

So $\hat{c}_t = c_t/x_t$ has an upper bound

$$\hat{c}_t \leq \left( \frac{\rho - r(1-\gamma)}{\gamma} \right)^{1/(1-\gamma)} = \hat{c}_h$$

In addition, since the upper bound can only be achieved with a deterministic consumption, $U_i^{c, 0}$ is deterministic too. This implies that both $c_t$ and $x_t$ grow at rate $\frac{\rho - r}{1-\gamma}$, so $\hat{c}_h$ is an absorbing state. In light of (6), we must have $k_{t+u} = 0$ in the continuation contract, so we have the “retirement” contract with cost $\hat{v}_h x_t$. This completes the proof.

**Theorem 1**

The proof is split into parts.

1) The cost function must be bounded above by $\hat{v}_h$ since we can always just give consumption to the agent without any capital, and obtain cost $\hat{v}_h$. It must be strictly positive because if $\hat{v}(\hat{c}) = 0$ for any $\hat{c} \in [0, \hat{c}_h]$, then we can scale up the contract and give infinite utility to the agent at zero cost, or else achieve infinite profits.

2) Because we can always move $\hat{c}$ up using $dL_t$, we know that $\hat{v}$ must be weakly increasing. It is useful to write the function $A(\hat{c}; \hat{v})$ from (14) as

$$A(\hat{c}, \hat{v}) \equiv \hat{c} - r\hat{v} = \frac{1}{2} \left( \frac{\hat{c}_h \hat{v}^{1-\gamma}}{\hat{v}_h} \right)^2 + \hat{v}^\rho - \epsilon^{1-\gamma}$$

which is the HJB equation when $\hat{v}(\hat{c})$ is flat. The region where the HJB equation holds cannot have any flat parts, because this would mean that $A(\hat{c}, \hat{v}(\hat{c})) = 0$. We know however from Lemma 13 (with $\pi = 0$) that this function can have at most two roots, so $\hat{v}$ must be strictly increasing in the region where the HJB holds.

The only way the HJB could not hold is if the optimal contract never spends any time there, i.e. if we ever found ourselves there, it would be optimal to immediately jump out using $dL_t$. The value in that region then must be constant and equal to the value at the destination point (the upper end of the flat region). The HJB should hold as an inequality with $A(\hat{c}; \hat{v}(\hat{c})) \geq 0$, since otherwise we could improve by lingering in the flat region for a while before jumping. Since $\hat{v}(\hat{c})$ is strictly increasing when the HJB equation holds,
the contract will start with $c_0$ at the upper end of a flat region. It cannot be that $c_0 = 0$ because of Inada conditions, so we must have at least one flat interval $[0, c_1]$.

3) We would like to show that this is the only flat interval, and the HJB equation holds as an equality in the strictly increasing region $[c_1, c_2]$, and both regions are connected with smooth pasting, i.e. $\hat{v}'(\hat{c}_1) = 0$. Suppose then that there is a region $[c_1, \hat{c}_2] \subset (0, c_2)$ where the cost function is flat, and it’s strictly increasing immediately above it (possibly, $\hat{c}_1 = 0$). Let’s show that as $\hat{c} \searrow \hat{c}_2$, $A(\hat{c}, \hat{v}(\hat{c})) \to 0$ and $\hat{v}'(\hat{c}) \to 0$ (i.e. we have smooth pasting). Towards contradiction, imagine there is a kink at $\hat{c}_2$, i.e. the right-derivative of $\hat{v}(\hat{c})$ is strictly positive. This can only happen if $\sigma^x(\hat{c} + \epsilon) \to 0$ and $\mu^x(\hat{c}_2 + \epsilon) \geq 0$ as $\epsilon \to 0$, since otherwise we would cross into the flat region where the HJB doesn’t hold (with $\hat{v}'(\hat{c}) > 0$ we must have $dL = 0$). First consider the case with $\lim inf \mu^x(\hat{c}_2 + \epsilon) > 0$. We can contemplate the following deviation: start at $\hat{c}_2 - \delta$, with the same $\sigma^x$ and $\sigma^e = 0$: by continuity $\mu^x > 0$ along this plan, so after some time we will end up at $\hat{c}_2$ and we can go back to the optimal contract and obtain continuation cost $\hat{v}(\hat{c}_2)$. The value of this strategy for $\hat{c} < \hat{c}_2$ extends the cost function $\hat{v}(\hat{c})$ below $\hat{c}_2$ and satisfies the HJB equation (with $\sigma^e = 0$, so it’s a first order ODE with boundary condition given by $\hat{v}(\hat{c}_2)$ at $\hat{c}_2$). However, because $\hat{v}'(\hat{c}_2) > 0$, we obtain a lower cost at $\hat{c}_2 - \delta$. This is therefore an attractive deviation, which is a contradiction, so $\mu^x(\hat{c}_2 + \epsilon) \to 0$ is the only option left. In this case, since we also have $\sigma^x(\hat{c} + \epsilon) \to 0$, it means we have a stationary contract, and because we are at a local minimum of the cost function, we must also be at a local minimum of the cost of stationary contracts $\hat{v}_r(\hat{c})$ given by (30), implying $\hat{v}_r'(\hat{c}_2) = 0$. However, because $\hat{v}'(\hat{c}_2) > 0$, we get $\hat{v}(\hat{c} + \epsilon) > \hat{v}_r(\hat{c} + \epsilon)$, which cannot be. We conclude that $\hat{v}'(\hat{c}_2) = 0$.

We still need to show that $A(\hat{c}, \hat{v}(\hat{c})) \to 0$ as $\hat{c} \searrow \hat{c}_2$. To see this it’s useful to use the FOC for $\sigma^x$ conditional on $\sigma^e$ to obtain

$$\sigma^x = \frac{\sigma^x \gamma - \hat{v}'(1 + \gamma)\sigma^e}{\hat{v}'}$$

and plug it into the HJB equation. We can then re-write the HJB

$$0 = \min_{\sigma^e} A + B\sigma^x + \frac{1}{2} C(\sigma^e)^2$$

with

$$A = \hat{c} - r \hat{v} - \frac{(\sigma^x \gamma)^2}{2 \hat{v}'} + \hat{v} \left( \frac{\sigma^x \gamma}{1 - \gamma} \right) + \hat{v}'(1 + \gamma) \left( \frac{\sigma^x \gamma}{1 - \gamma} \right)$$

$$B = \hat{v}'(1 + \gamma) \frac{\sigma^x \gamma}{\hat{v}'} + \frac{\sigma^x \gamma}{\hat{v}'} \geq 0$$

$$C = \gamma \hat{v}'(1 + \gamma) \frac{\hat{v} - \hat{v}' \gamma}{\hat{v}'} + \hat{v}'' \gamma$$

For the HJB to have a minimum, it must be that $C \geq 0$. If $C = 0$ we can only have a minimum if $B = 0$ as well, in which case $A \equiv 0$, and with $\hat{v}'(\hat{c}_2) = 0$ we get $A(\hat{c}, \hat{v}(\hat{c})) \to 0$ as $\hat{c} \searrow \hat{c}_2$ as desired. If instead $C > 0$, then

$$\sigma^e = -\frac{B}{C} \leq 0$$

With $\hat{v}'(\hat{c}_2) = 0$ we must have $\hat{v}''(\hat{c} + \epsilon) \geq 0$ for small $\epsilon$ (or else $\hat{v}'(\hat{c} + \epsilon) < 0$). We can then show that $\frac{\hat{c}}{\hat{c}_2} \to \infty$ as $\hat{c} \searrow \hat{c}_2$, which implies $\frac{B^2}{C} \to 0$ and therefore $A \to 0$, and therefore $A(\hat{c}, \hat{v}(\hat{c})) \to 0$ as $\hat{c} \searrow \hat{c}_2$. 47
as desired. Since this is true in particular when \( \hat{c}_1 = 0 \) and \( \hat{c}_2 = \hat{c}_t \), we have proven the smooth pasting condition.

Now since at \( \hat{c}_2 \) we have \( A(\hat{c}, \hat{v}(\hat{c})) = 0 \), this is a root of \( A \). Below that we have \( A(\hat{c}, \hat{v}(\hat{c})) \geq 0 \). From Lemma 13 we know that this can only be the case if \( \hat{c}_2 \) is the first root of \( A(\hat{c}, \hat{v}(\hat{c}_2)) \). We would like to show that it cannot be the case that for \( \hat{c} < \hat{c}_1 < \hat{c}_2 \) the cost function is lower, i.e. \( \hat{v}(\hat{c}_1 + \delta) < \hat{v}(\hat{c}_2) = \hat{v}(\hat{c}_2) \).

To see this, imagine the same problem with a smaller \( \alpha' < \alpha \), and cost function \( \hat{v}_{\alpha'}(\hat{c}) \geq \hat{v}(\hat{c}) \). We can pick \( \alpha' \) small enough that \( \hat{v}_{\alpha'}(\hat{c}') = \hat{v}(\hat{c}_2) \), where \( \hat{c}' \) is the upper end of the first flat region for the contract with \( \alpha' \) (and it minimizes \( v_{\alpha'}(\hat{c}) \)). We can do this because \( \hat{v}_{\alpha'}(\hat{c}) \) is continuously increasing in \( \alpha' \) for any \( \hat{c} \), and has \( \hat{v}_{\alpha'}(\hat{c}) = \hat{v}_b \) for all \( \hat{c} \leq \hat{c}_b \) if \( \alpha' = 0 \). It must be that \( \hat{c}' \leq \hat{c}_2 \), because to the right of \( \hat{c}_2 \) \( \hat{v}_{\alpha'}(\hat{c}) \geq \hat{v}(\hat{c}_2) \) for all \( \hat{c} > \hat{c}_2 \). However, looking at \( A(\hat{c}, \hat{v}(\hat{c}_2)) \) we notice it is decreasing in \( \alpha \), so \( A_{\alpha'}(\hat{c}', \hat{v}(\hat{c}_2)) = A(\hat{c}', \hat{v}(\hat{c}_2)) \geq 0 \). But the previous argument shows that \( A_{\alpha'}(\hat{c}'', \hat{v}(\hat{c}_2)) = A_{\alpha'}(\hat{c}', \hat{v}(\hat{c}_2)) = 0 \) because \( \hat{c}' \) is the upper end of the first flat region of the optimal contract for \( \alpha' \). This is a contradiction, so we cannot have \( \hat{v}(\hat{c}_1 - \delta) < \hat{v}(\hat{c}_1) = \hat{v}(\hat{c}_2) \).

Putting all of this together, we only have one flat region, \((0, \hat{c}_1)\), where the HJB equation holds as an inequality \( A(\hat{c}, \hat{v}(\hat{c})) > 0 \) (the inequality is strict because of Lemma 13 and the fact that \( \hat{c}_1 \) must be the first root), and a strictly increasing region \([\hat{c}_1, \hat{c}_t]\) where the HJB equation holds with equality. In the flat region we have \( \hat{v}(\hat{c}) = \hat{v}(\hat{c}_1) \equiv \hat{v}_t \) for all \( \hat{c} \leq \hat{c}_1 \). At \( \hat{c}_1 \) we have \( \hat{v}'(\hat{c}_1) = 0 \) and \( A(\hat{c}_1, \hat{v}(\hat{c}_1)) = 0 \).

4) Now we want to show that \( \hat{v}'(\hat{c}) > 0 \). First suppose the \( A(\hat{c}, \hat{v}(\hat{c}_1)) \) is strictly positive below \( \hat{c}_1 \) and strictly negative above it, so that \( A(\hat{c}_1, \hat{v}_t) < 0 \) (we will show that this must indeed be the case below).

Consider the first order ODE that results from fixing \( \sigma'' = 0 \) and \( \sigma^x = \frac{\alpha}{\sigma_0} \) in the HJB equation:

\[
\hat{c} - \alpha \frac{\sigma'}{\sigma} + \hat{v}_{\sigma_0} \left( \frac{r - \sigma}{1 - \gamma} + \frac{\gamma}{2}(\sigma^x)^2 \right) + \hat{v}_{\sigma_0}^\prime \left( \frac{r - \sigma}{\gamma} + \frac{(\sigma^x)^2}{2} - \frac{\rho - \sigma}{1 - \gamma} \right) = 0
\]

and consider the solution with boundary condition \( \hat{v}_{\sigma_0}(\hat{c}_1) = \hat{v}(\hat{c}_1) \). Since we already know that \( A(\hat{c}_1, \hat{v}(\hat{c}_1)) = 0 \), we must have \( \hat{v}'_{\sigma_0}(\hat{c}_1) = 0 \) because the term in parenthesis is \( \mu^c = 0 \), and Lemma 15 shows this is strictly positive under these conditions. Furthermore, we must have \( \hat{v}'_{\sigma_0}(\hat{c}_1) = \hat{v}'(\hat{c}_1) \). To see this, if \( \hat{v}'_{\sigma_0}(\hat{c}_1) > \hat{v}'(\hat{c}_1) \geq 0 \), then \( \hat{v}_{\sigma_0}(\hat{c}_1 + \epsilon) > \hat{v}'(\hat{c}_1 + \epsilon) \), while \( \hat{v}_{\sigma_0}(\hat{c}_1 + \epsilon) = \hat{v}(\hat{c}_1 + \epsilon) + o(\epsilon) \) (because both have first derivatives equal to zero) for some small \( \epsilon \). By continuity, it will still be the case that \( \mu^c > 0 \) for \( \hat{c}_1 + \epsilon \); so starting at \( \hat{c}_1 + \epsilon - \delta \) we will eventually get to \( \hat{c}_1 + \epsilon \). We can then solve the first order ODE backwards with boundary condition \( \hat{v}_{\sigma_0}(\hat{c}_1 + \epsilon) = \hat{v}(\hat{c}_1 + \epsilon) \) and we will obtain a lower cost for \( \hat{c}_1 + \epsilon - \delta \), i.e. \( \hat{v}_{\sigma_0}(\hat{c}_1 + \epsilon - \delta) < \hat{v}''(\hat{c}_1 + \epsilon - \delta) \), because by continuity \( \hat{v}_{\sigma_0}(\hat{c}_1 + \epsilon) > \hat{v}'(\hat{c}_1 + \epsilon) \) for this solution as well. This cannot be, so we must have \( \hat{v}'_{\sigma_0}(\hat{c}_1) \leq \hat{v}'(\hat{c}_1) \).

Differentiating the first order ODE with respect to \( \hat{c} \) we obtain:

\[
0 = \hat{v}_{\sigma_0}(\hat{c}_1) \frac{\partial A}{\partial \hat{c}} (\hat{c}_1, \hat{v}_t) + \hat{v}_{\sigma_0}'(\hat{c}_1) \hat{c}_t \frac{r - \rho}{\gamma} + \frac{(\sigma^x)^2}{2} - \frac{\rho - \sigma}{1 - \gamma}
\]

where we have used \( \hat{v}_{\sigma_0}'(\hat{c}_1) = 0 \) and the envelope theorem to compute the derivative \( \hat{v}_{\sigma_0}(\hat{c}_1) \). It follows that since \( A_2(\hat{c}_1, \hat{v}_t) < 0 \), then either \( \hat{v}_{\sigma_0}'(\hat{c}_1) > 0 \) and \( \frac{r - \rho}{\gamma} + \frac{(\sigma^x)^2}{2} - \frac{\rho - \sigma}{1 - \gamma} > 0 \), or both are strictly negative. But we already know that \( \frac{r - \rho}{\gamma} + \frac{(\sigma^x)^2}{2} - \frac{\rho - \sigma}{1 - \gamma} > 0 \), so this leaves only \( \hat{v}'(\hat{c}_1) \geq \hat{v}_{\sigma_0}'(\hat{c}_1) > 0 \).
Now we would like to take the limit. The stopped stochastic integrals are therefore martingales, so take expectations under $Q$ to obtain

$$
\mathbb{E}_Q^0 \left[ e^{-\gamma \tau^n} \phi(\xi_{\tau^n} \alpha) \right] \geq \mathbb{E}_Q^0 \left[ \int_0^{\tau^n} e^{-r t} \left( c_t - k_t \alpha \right) dt \right]
$$

for the localizing sequence of stopping times $\{\tau^n\}_{n \in \mathbb{N}}$:

$$
\tau^n = \inf \left\{ T \geq 0 : \int_0^T \left| e^{-rt} \phi(\xi_t \alpha) x_t \right| \left( \frac{\phi'(\xi_t \alpha) x_t}{\phi(\xi_t \alpha)} \right) \left( \hat{c}_t \right) + \sigma_t \right| dt \geq n \right\}
$$

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$$
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$$

Now we would like to take the limit $n \to \infty$, but we need to use the dominated convergence theorem. First,

$$
\left| \int_0^{\tau^n} e^{-rt} (c_t - k_t \alpha) dt \right| \leq \int_0^\infty e^{-rt} |c_t - k_t \alpha| dt \leq \int_0^\infty e^{-rt} (|c_t| + k_t \alpha) dt
$$

5) $\hat{v}''(\hat{c}_t) > 0$ implies $\sigma^\hat{c}(\hat{c}_t) = 0$ and $\mu^\hat{c}(\hat{c}_t) > 0$. This in turn proves that $\hat{c}_t > \hat{c}_0 = \hat{c}_t$. From (50) we get $\sigma^\hat{c}(\hat{c}) \leq 0$, and (46) implies $\sigma^\hat{c}(\hat{c}) \geq 0$. It also implies that at $\hat{c}_t$, since $\hat{v}'(\hat{c}_t) = 0$, the choice of $\sigma^\hat{c}$ maximizes (15).

**Theorem 2**

First let’s extend the function $\hat{v}(\hat{c})$ as described above, with $\hat{v}(\hat{c}) = \hat{v}_1 \equiv \hat{v}(\hat{c}_t)$ for all $\hat{c} < \hat{c}_t$ (we always have $\hat{c} \in [0, \hat{c}_t]$). The HJB holds as an equality for $\hat{c} \geq \hat{c}_t$, but we need to check that it holds as an inequality for $\hat{c} < \hat{c}_t$. Using the version of $A(\hat{c}, \hat{v})$ in (45), notice $A(\hat{c}_t, \hat{v}_1) = 0$, so $\hat{c}_t$ is a root of $A(\hat{c}; \hat{v}_1)$. If $\gamma > \frac{1}{2}$, Lemma 13 (with $\tau = 0$) says that it can have at most one root, and it’s positive for small $\hat{c}$, so $A(\hat{c}; \hat{v}_1) \geq 0$ for all $\hat{c} < \hat{c}_t$. For $\gamma < \frac{1}{2}$, it’s convex and can have at most two roots. Condition (18) guarantees that the derivative is negative, so $\hat{c}_t$ is the smaller root, and we also have $A(\hat{c}; \hat{v}_1) \geq 0$ for all $\hat{c} < \hat{c}_t$. Notice that we only need to check (18) if $\gamma < \frac{1}{2}$.

Consider any incentive compatible contract $C = (c, k)$ that delivers utility of at least $u_0$ to the agent, with associated state variables $x$ and $c$. Because $\hat{v}'(\hat{c}_t) = 0$ we can use Ito’s lemma\(^{18}\) and the HJB equation to obtain

\[ e^{-r \tau^n} \phi(\xi_{\tau^n} \alpha) \geq \mathbb{E}_Q^0 \left[ \int_0^{\tau^n} e^{-r t} \left( \frac{\hat{v}'(\hat{c}_t \alpha) x_t}{\hat{v}(\hat{c}_t \alpha) x_t} \right) \left( \hat{c}_t \hat{c} + \sigma_t \right) dt \right] \]

for the localizing sequence of stopping times $\{\tau^n\}_{n \in \mathbb{N}}$:

\[ \tau^n = \inf \left\{ T \geq 0 : \int_0^T \left| e^{-rt} \phi(\xi_t \alpha) x_t \right| \left( \frac{\phi'(\xi_t \alpha) x_t}{\phi(\xi_t \alpha)} \right) \left( \hat{c}_t \hat{c} + \sigma_t \right) dt \geq n \right\} \]

The stopped stochastic integrals are therefore martingales, so take expectations under $Q$ to obtain

\[ \mathbb{E}_Q^0 \left[ e^{-\gamma \tau^n} \phi(\xi_{\tau^n} \alpha) \right] \geq \mathbb{E}_Q^0 \left[ \int_0^{\tau^n} e^{-r t} \left( c_t - k_t \alpha \right) dt \right] \]

\[ (51) \]

Now we would like to take the limit $n \to \infty$, but we need to use the dominated convergence theorem. First,

\[ \left| \int_0^{\tau^n} e^{-rt} (c_t - k_t \alpha) dt \right| \leq \int_0^\infty e^{-rt} |c_t - k_t \alpha| dt \leq \int_0^\infty e^{-rt} (|c_t| + k_t \alpha) dt \]

\(^{18}\) Notice $\hat{v}''$ is discontinuous at $\hat{c}_t$, but this doesn’t change Ito’s formula.
which is integrable because the contract is admissible. Second, for an admissible contract

\[ 0 \leq \lim_{n \to \infty} \mathbb{E}_0^Q \left[ e^{-r^n} \tilde{v}(c_n)x_n \right] \leq \lim_{n \to \infty} \mathbb{E}_0^Q \left[ e^{-r^n} \hat{v}_x x_n \right] = 0 \]

To see why the last equality holds, notice that since \( \hat{v}_x x \) is the cheapest way of delivering utility to the agent without capital, the cost of consumption on the contract is

\[ \mathbb{E}_0^Q \left[ \int_0^\infty e^{-rt} c_t dt \right] \geq \mathbb{E}_0^Q \left[ \int_0^\infty e^{-rt} c_t dt + e^{-r^n} \hat{v}_x x_n \right] \]

Taking the limit \( n \to \infty \) and using the monotone convergence theorem, we obtain \( 0 \leq \lim_{n \to \infty} \mathbb{E}_0^Q \left[ e^{-r^n} \hat{v}_x x_n \right] \leq 0 \).

Upon taking the limit \( n \to \infty \) in (51), we obtain \( \tau^* \to \infty \) a.s. and therefore

\[ \mathbb{E}_0^Q \left[ \int_0^\infty e^{-rt} (c_t - k_t \alpha) dt \right] \geq \hat{v}(\tilde{c}_0)x_0 \]

Using \( \hat{v}(\tilde{c}_0) \leq \hat{v}(\bar{c}_0) \) and \( x_0 \geq ((1 - \gamma)u_0)^\frac{1}{1-\gamma} \) we obtain the first result.

For the second part, first let's show that \( C^* \) is incentive compatible. We already know that for the HJB to have a solution with well defined policy functions it must be the case that \( \bar{B} \geq 0 \) and \( \bar{C} > 0 \), defined in (48) and (49), and therefore \( \sigma^{x*} \leq 0 \) and \( \sigma^{x*} \geq 0 \). If \( \bar{C}_t \in [\bar{c}_l, \bar{c}_h] \), because \( \sigma^{x*} \) is bounded, so is \( \sigma^{x*} \) and therefore so is \( \mu^{x*} \) and \( \mu^{x*} \). We can then use Lemma 3 and the fact that \( \bar{h}_0 = 0 \) to show that

\[ U_{0}^{x,0} \leq U_{0}^{x,0} = u_0 \]

for any feasible strategy \((\bar{c}, \bar{a})\), and therefore \( C^* \) is indeed incentive compatible. To show that the cost of the contract is \( \hat{v}_x x_0 \), we can use the HJB. If \( \bar{c} \in [\bar{c}_l, \bar{c}_h] \) always, where the HJB holds, then the same argument as in the first part shows the cost is \( \hat{v}_x x_0 \). Notice that with \( \hat{v}'(\bar{c}_l) = 0 \) and \( \hat{v}''(\bar{c}_l) > 0 \) we get that as \( \bar{c} \to \bar{c}_l \), \( \bar{B} \to 0 \) and \( \bar{C} \to \bar{C}_l \), so \( \sigma^{x*} \to 0 \), \( \sigma^{x*} \to \frac{\bar{c}_l}{\bar{c}_l} \), and \( A(\bar{c}_l; \bar{v}) = 0 \). From the first part we know \( \hat{v}(\bar{c}) \) is weakly below the true cost function, which is also bounded above by \( \hat{v}_x \), so there is a finite cost function and \( \hat{v}_x \) is weakly below it, and therefore \( \hat{v}_x \leq \hat{v}_p \) from equation (27). Lemma 15 shows that under these conditions \( \mu^{x*}(\bar{c}_l) > 0 \) and therefore \( \bar{c} \in [\bar{c}_l, \bar{c}_h] \). Notice that the candidate contract does indeed deliver utility \( u_0 \) to the agent. To see this let \( U^* = \frac{\mu^{x*}}{\sigma^{x*}} \), so using the law of motion of \( x^* \), (9), we get

\[ U_{0}^{x,0} = \mathbb{E} \left[ \int_0^\infty e^{-\rho t} \frac{\mu^{x*}}{1-\gamma} dt + e^{-\rho^n} U_{x^n}^{x,0} \right] \]

with some sequence \( \tau^n \to \infty \) a.s. Use the monotone convergence theorem and notice that

\[ \lim_{n \to \infty} \mathbb{E} \left[ e^{-\rho^n} U_{x^n}^{x,0} \right] = 0 \]

because \( \rho - (1 - \gamma)(\mu^{x*} - \frac{1}{2}(\sigma^{x*})^2) = \bar{c}^{1-\gamma} \geq \min\{\bar{c}_l, \bar{c}_h\} > 0 \). We then get that \( U_{0}^{x,0} = U_{0}^{x,0} = u_0 \).

This completes the proof.
Lemma 4

We know from Lemma 15 that \( \hat{c}_t \in [\hat{c}_l, \hat{c}_h] \) and recall that \( \hat{c}_l > 0 \). Then an upper bounded \( \mu \tilde{s}^* < r \) implies a bounded \( 0 \leq \sigma \tilde{s} \leq \sigma \). Then

\[
\mathbb{E}^Q \left[ \int_0^\infty e^{-r t} (|c_t^*| + |k_t^*|^2 \alpha) \right] \leq 2 \max \left\{ \hat{c}_h, \frac{\sigma x \hat{c}_h}{\phi \sigma} \right\} \mathbb{E}^Q \left[ \int_0^\infty e^{-r t} x_t^* dt \right] < \infty
\]

where the last inequality follows from \( \mu \tilde{s}^* < r \). Let \( U^* = \frac{\omega(x^*)_{1-\gamma}}{1-\gamma} \), so using the law of motion of \( x^* \), (9), we get

\[
U_{0}^{*} = \mathbb{E} \left[ \int_0^{\tau} e^{-\mu t} \frac{c_{1-\gamma}^*}{1-\gamma} dt + e^{-\rho \tau} U_{\tau}^{*} \right]
\]

with \( \tau \to \infty \) a.s. Use the monotone convergence theorem and notice that

\[
\lim_{n \to \infty} \mathbb{E} \left[ e^{-\rho \tau} U_{\tau}^{*} \right] = 0
\]

because \( \rho - (1-\gamma)(\mu \tilde{s}^* - \frac{\gamma}{2} \sigma \tilde{s}^2) = \tilde{c}^{1-\gamma} \geq \min \{ \tilde{c}_l^{1-\gamma}, \tilde{c}_h^{1-\gamma} \} > 0 \). We then get that \( U_{0}^{*} = U_{0} = u_0 \).

We conclude that the contract is indeed admissible.

Theorem 3

We will focus on the case with \( dL_t = 0 \), but the proof can be extended quite naturally. Let \( F(\hat{h}, \hat{c}) = (1 + \hat{h} \tilde{c}^{-\gamma})^{1-\gamma} \) where \( \hat{h} = h/x \), and let \( F_t = F(\hat{h}_t, \hat{c}_t) \), and likewise for derivatives, e.g. \( F_{c,t} = \partial_t F(\hat{h}_t, \hat{c}_t) \).

Also, define \( \hat{\hat{c}} = \tilde{c}/x \). Write

\[
e^{-\mu t} \left( U_t^{c,a} - F(\hat{h}_t, \hat{c}_t)U_t^{c,0} \right) = \mathbb{E}_t^a \left[ \int_t^{\tau} e^{-\rho u} \frac{c_{1-\gamma}^*}{1-\gamma} du + \int_t^{\tau} d \left( e^{-\rho u} F_u U_u^{c,0} \right) + e^{-\rho \tau} \left( U_{\tau}^{c,a} - F_{\tau} U_{\tau}^{c,0} \right) \right]
\]

for a localizing sequence \( \{\tau^n\}_{n \in \mathbb{N}} \) with \( \tau^n \to \infty \) a.s. We will show that the rhs is non-positive. First write the integral part

\[
\mathbb{E}_t^a \left[ \int_t^{\tau} e^{-\rho u} \frac{c_{1-\gamma}^*}{1-\gamma} du + \int_t^{\tau} d \left( e^{-\rho u} F_u U_u^{c,0} \right) \right] = \mathbb{E}_t^a \left[ \int_t^{\tau} e^{-\rho u} (1-\gamma)U_u^{c,0} Y_u du \right]
\]  \( (52) \)

with

\[
Y_t = \frac{\tilde{c}_t^{1-\gamma}}{1-\gamma} - \frac{\rho F_t + \rho F_t - \tilde{c}_t^{1-\gamma} F_t}{1-\gamma}
\]

\[
+ \frac{F_{\hat{c},t}}{(1-\gamma)} \hat{c}_t \left( \frac{r - \rho}{\gamma} + \frac{1 + \gamma}{2} (\sigma_t^2 + \sigma_t^2) - \frac{\rho - \tilde{c}_t^{1-\gamma} - \frac{\gamma}{2} (\sigma_t^2)^2 - \sigma_t^2 \sigma_t^2} \right) + \frac{F_{\hat{h},t}}{(1-\gamma)} (1-\gamma) \sigma_t^2 \sigma_t \sigma_t
\]

\[
+ \frac{F_{\hat{h},t}}{(1-\gamma)} \hat{h}_t \left( r + \frac{\tilde{c}_t - \tilde{c}_h}{\hat{h}_t} - \frac{\rho - \tilde{c}_h^{1-\gamma}}{1-\gamma} - \frac{\gamma}{2} (\sigma_t^2)^2 + (\sigma_t^2)^2 \right) - \frac{F_{\hat{h},t}}{(1-\gamma)} (1-\gamma)(\sigma_t^2)^2 \hat{h}_t
\]
\[
\frac{(\sigma_t^x)^2 \hat{c}_t^2 F_{\hat{c}_t,t}^2 - 2\sigma_t^x \sigma_t^e \hat{c}_t \hat{h}_t F_{\hat{c}_t \hat{h}_t,t} + (\sigma_t^x)^2 \hat{h}_t^2 F_{\hat{h}_t,t}}{2(1 - \gamma)} + \frac{1}{(1 - \gamma)} \left( (1 - \gamma) \sigma_t^e \hat{F}_t - \sigma_t^e \sigma_t^e \hat{c}_t + (\sigma_t^e \hat{h}_t + \hat{k}_t \phi \sigma) F_{\hat{h}_t,t} \right) \frac{a_t}{\sigma_t^e}
\]

where

\[
F_{\hat{c}_t,t} = \gamma (\gamma - 1) \hat{h}_t \hat{c}_t^{-1} \left( 1 + \hat{h}_t \hat{c}_t^{-1} \right)^{-\gamma}
\]

\[
F_{\hat{h}_t,t} = (1 - \gamma) \hat{c}_t^{-1} \left( 1 + \hat{h}_t \hat{c}_t^{-1} \right)^{-\gamma}
\]

\[
F_{\hat{c}_t} = -\gamma (\gamma - 1) \hat{h}_t \hat{c}_t^{-2\gamma} \left( 1 + \hat{h}_t \hat{c}_t^{-1} \right)^{-\gamma} - \gamma^2 (\gamma - 1) \hat{h}_t \hat{c}_t^{-\gamma - 2} \left( 1 + \hat{h}_t \hat{c}_t^{-1} \right)^{-\gamma - 1} \left( 1 - \hat{h}_t (\gamma - 1) \hat{c}_t^{-1} \right)
\]

\[
F_{\hat{h}_t} = \gamma (\gamma - 1) \hat{c}_t^{-\gamma - 1} \left( 1 + \hat{h}_t \hat{c}_t^{-1} \right)^{-\gamma} - \gamma^2 (\gamma - 1) \hat{h}_t \hat{c}_t^{-\gamma - 1} \left( 1 + \hat{h}_t \hat{c}_t^{-1} \right)^{-1\gamma - 1}
\]

\[
F_{\hat{h}_t \hat{c}_t,t} = -\gamma \hat{c}_t^{-2\gamma} (1 - \gamma) \left( 1 + \hat{h}_t \hat{c}_t^{-1} \right)^{-\gamma - 1}
\]

We know \( e^{-\gamma t} (1 - \gamma) U_{t,0} > 0 \), and we will show that \( Y_t \leq 0 \). To do this, we will split the expression into three parts, \( Y_t = A_t + B_t + C_t \), and show each one is non-positive. First take the terms multiplying \( s_t \)

\[
A_t = \frac{1}{(1 - \gamma)} \left( (1 - \gamma) \sigma_t^e \left( 1 + \hat{h}_t \hat{c}_t^{-1} \right)^{-\gamma} \right)
\]

\[
- \gamma (\gamma - 1) \hat{h}_t \hat{c}_t^{-\gamma - 1} \left( 1 + \hat{h}_t \hat{c}_t^{-1} \right)^{-\gamma} \sigma_t^e \hat{c}_t + (\sigma_t^e \hat{h}_t + \hat{k}_t \phi \sigma) (1 - \gamma) \hat{c}_t^{-1} \left( 1 + \hat{h}_t \hat{c}_t^{-1} \right)^{-\gamma}
\]

\[
A_t = \left( 1 + \hat{h}_t \hat{c}_t^{-1} \right)^{-\gamma} \left( -\sigma_t^e \left( 1 + \hat{h}_t \hat{c}_t^{-1} \right) + \gamma \hat{h}_t \hat{c}_t^{-\gamma - 1} \sigma_t^e \hat{c}_t + (\sigma_t^e \hat{h}_t + \hat{k}_t \phi \sigma) \hat{c}_t \right)
\]

\[
A_t \leq \left( 1 + \hat{h}_t \hat{c}_t^{-1} \right)^{-\gamma} \gamma \hat{h}_t \hat{c}_t^{-\gamma - 1} \sigma_t^e \hat{c}_t \leq 0
\]

where we have used \( \sigma_t^e \geq \hat{k}_t \phi \sigma \hat{c}_t^{-\gamma} \) in the first inequality, and \( \hat{h}_t \geq 0 \) and \( \sigma_t^e \leq 0 \) in the second. Here is the only place where \( \sigma_t^e \leq 0 \) is used.

Second, all the remaining terms that have \( \sigma_t^x \) or \( \sigma_t^e \) are

\[
B_t = \frac{F_{\hat{c}_t,t}}{(1 - \gamma)} \hat{c}_t \left( \frac{1 + \gamma (\sigma_t^x + \sigma_t^e)^2 - \gamma^2 (\sigma_t^x)^2 - \sigma_t^x \sigma_t^e}{2(1 - \gamma)} \right) + \frac{F_{\hat{h}_t,t}}{(1 - \gamma)} (1 - \gamma) \sigma_t^x \sigma_t^e \hat{c}_t
\]

\[
+ \frac{F_{\hat{c}_t,t} - F_{\hat{h}_t,t} (1 - \gamma) \gamma}{(1 - \gamma)} \hat{c}_t \left( \frac{1 + \gamma (\sigma_t^x + \sigma_t^e)^2 - \gamma^2 (\sigma_t^e)^2 - \gamma \sigma_t^x \sigma_t^e}{2(1 - \gamma)} \right) + \frac{F_{\hat{h}_t,t}}{(1 - \gamma)} \hat{h}_t \left( \frac{\gamma}{2} \right) \gamma \sigma_t^e \sigma_t^e \hat{c}_t + (\sigma_t^e)^2 \hat{c}_t^2 \left( \frac{F_{\hat{c}_t,t}^2 - 2\sigma_t^x \sigma_t^e \hat{c}_t \hat{h}_t F_{\hat{c}_t \hat{h}_t,t} + (\sigma_t^e)^2 \hat{h}_t^2 F_{\hat{h}_h,t}}{2(1 - \gamma)} \right)
\]

\[
B_t = \frac{(\sigma_t^x)^2}{2} \left( \frac{F_{\hat{c}_t,t}^2 - F_{\hat{h}_t,t} (1 - \gamma) \gamma + F_{\hat{h}_t,t} \hat{h}_t^2}{2(1 - \gamma)} \right) + \frac{(\sigma_t^e)^2}{2} \left( \frac{F_{\hat{c}_t,t} (1 + \gamma) \hat{c}_t + F_{\hat{c}_t \hat{h}_t,t}}{2(1 - \gamma)} \right)
\]

\[52\]
\[-\sigma_t^\xi\sigma_t^\delta \left( -\frac{F_{\xi t}}{1-\gamma} \hat{c}_t + \frac{F_{\delta t}}{1-\gamma} \hat{c}_t \right) \]

which plugging in the $F$’s gets us

\[
B_t = \frac{(\sigma_t^\xi)^2}{2} \left( \frac{\gamma(\gamma - 1)\hat{h}_t \hat{c}_t^{-\gamma - 1} \left( 1 + \hat{h}_t \hat{c}_t^{-\gamma} \right)^{-\gamma}}{1-\gamma} \hat{c}_t + \frac{1-\gamma}{1-\gamma} \frac{\gamma^2 (\gamma - 1)\hat{h}_t(1 + \hat{h}_t \hat{c}_t^{-\gamma - 2})(1 + \hat{h}_t \hat{c}_t^{-\gamma})^{-\gamma - 1}}{1-\gamma}(1 + \gamma)\hat{c}_t + \frac{(\sigma_t^\xi)^2}{2} \left( \frac{\gamma(\gamma - 1)\hat{h}_t \hat{c}_t^{-\gamma - 1} \left( 1 + \hat{h}_t \hat{c}_t^{-\gamma} \right)^{-\gamma}}{1-\gamma} - \gamma^2 (\gamma - 1)\hat{h}_t \hat{c}_t^{-\gamma - 2} \left( 1 + \hat{h}_t \hat{c}_t^{-\gamma} \right)^{-\gamma - 1} \hat{c}_t \right) \right) 
\]

\[-\sigma_t^\xi \sigma_t^\delta \left( \frac{-\gamma(\gamma - 1)\hat{h}_t \hat{c}_t^{-\gamma - 1} \left( 1 + \hat{h}_t \hat{c}_t^{-\gamma} \right)^{-\gamma}}{1-\gamma} - \gamma^2 (\gamma - 1)\hat{h}_t \hat{c}_t^{-\gamma - 1} \left( 1 + \hat{h}_t \hat{c}_t^{-\gamma} \right)^{-\gamma - 1} \hat{c}_t \right) \]

which simplifies to:

\[
B_t = \frac{(\sigma_t^\xi)^2}{2} \left( -\gamma\hat{h}_t \hat{c}_t^{-\gamma - 1} \left( 1 + \hat{h}_t \hat{c}_t^{-\gamma} \right)^{-\gamma} \hat{c}_t + \hat{c}_t^{-\gamma} \left( 1 + \hat{h}_t \hat{c}_t^{-\gamma} \right)^{-\gamma} \hat{h}_t \gamma - \gamma^2 \hat{c}_t^{-\gamma - 1} \left( 1 + \hat{h}_t \hat{c}_t^{-\gamma} \right)^{-\gamma - 1} \hat{h}_t \right) 
\]

\[
+ \frac{(\sigma_t^\xi)^2}{2} \left( \gamma\hat{h}_t \hat{c}_t^{-\gamma - 1} \left( 1 + \hat{h}_t \hat{c}_t^{-\gamma} \right)^{-\gamma} \hat{c}_t + \gamma\hat{h}_t \hat{c}_t^{-\gamma - 2} \left( 1 + \hat{h}_t \hat{c}_t^{-\gamma} \right)^{-\gamma} \hat{c}_t^2 \right) 
\]

\[-\gamma^2 \hat{h}_t \hat{c}_t^{-\gamma - 2} \left( 1 + \hat{h}_t \hat{c}_t^{-\gamma} \right)^{-\gamma - 1} \left( 1 - \hat{h}_t(\gamma - 1)\hat{c}_t^{-\gamma} \right) \hat{c}_t^2 \]

\[-\sigma_t^\xi \sigma_t^\delta \left( \gamma\hat{h}_t \hat{c}_t^{-\gamma - 1} \left( 1 + \hat{h}_t \hat{c}_t^{-\gamma} \right)^{-\gamma} \hat{c}_t - \gamma\hat{c}_t^{-\gamma - 1} \left( 1 + \hat{h}_t \hat{c}_t^{-\gamma} \right)^{-\gamma} \hat{c}_t \hat{h}_t + \gamma^2 \hat{h}_t \hat{c}_t^{-\gamma - 1} \left( 1 + \hat{h}_t \hat{c}_t^{-\gamma} \right)^{-\gamma - 1} \hat{c}_t \hat{h}_t \right) \]

and

\[
B_t = \frac{-(\sigma_t^\xi)^2}{2} \left( 1 + \hat{h}_t \hat{c}_t^{-\gamma} \right)^{-\gamma - 1} \hat{c}_t^{-\gamma - 1} \gamma\hat{c}_t^{1-\gamma} \hat{h}_t^2 
\]

\[-\frac{(\sigma_t^\xi)^2}{2} \left( 1 + \hat{h}_t \hat{c}_t^{-\gamma} \right)^{-\gamma - 1} \hat{c}_t^{-1}\hat{h}_t^2 \gamma^2 \hat{c}_t^{1-\gamma} 
\]

\[-\sigma_t^\xi \sigma_t^\delta \left( 1 + \hat{h}_t \hat{c}_t^{-\gamma} \right)^{-\gamma - 1} \hat{c}_t^{-1} \hat{h}_t^2 \gamma^2 \hat{c}_t^{1-\gamma} \hat{h}_t^2 \]

\[
B_t = \frac{(\sigma_t^\xi + \sigma_t^\delta)^2}{2} \gamma \hat{h}_t \hat{c}_t^{-\gamma} \left( 1 + \hat{h}_t \hat{c}_t^{-\gamma} \right)^{-\gamma - 1} \hat{c}_t^{-\gamma - 1} \leq 0 
\]
And finally all the remaining terms that don’t involve $\sigma_t^k$ or $\sigma_t^l$ are

\[
C_t = \frac{\tilde{c}_t^{1-\gamma}}{1-\gamma} \frac{\tilde{c}_t^{1-\gamma} F_t}{(1-\gamma)} + \frac{\gamma(\gamma - 1) \hat{h}_t \tilde{c}_t^{1-\gamma-1} (1 + \hat{h}_t \tilde{c}_t^{\gamma})^{-\gamma}}{(1-\gamma)} \hat{c}_t \left( \frac{r - \rho}{\gamma} - \frac{\rho - \tilde{c}_t^{1-\gamma}}{1-\gamma} \right) \\
+ \frac{(1 - \gamma) \tilde{c}_t^{1-\gamma} (1 + \hat{h}_t \tilde{c}_t^{\gamma})^{-\gamma}}{(1-\gamma)} \frac{\hat{h}_t}{r + \frac{\hat{c}_t - (1 + \hat{h}_t \tilde{c}_t^{\gamma}) \hat{c}_t}{\hat{h}_t}} \frac{\rho - \tilde{c}_t^{1-\gamma}}{1-\gamma}
\]

which is maximized for $\hat{c}_t = \hat{c}_t + \hat{h}_t \tilde{c}_t^{1-\gamma}$. Plugging this in yields

\[
C_t \leq \frac{\tilde{c}_t^{1-\gamma}}{1-\gamma} \frac{\tilde{c}_t^{1-\gamma} F_t}{(1-\gamma)} + \frac{\gamma(\gamma - 1) \hat{h}_t \tilde{c}_t^{1-\gamma-1} (1 + \hat{h}_t \tilde{c}_t^{\gamma})^{-\gamma}}{(1-\gamma)} \hat{c}_t \left( \frac{r - \rho}{\gamma} - \frac{\rho - \tilde{c}_t^{1-\gamma}}{1-\gamma} \right) \\
+ \frac{(1 - \gamma) \tilde{c}_t^{1-\gamma} (1 + \hat{h}_t \tilde{c}_t^{\gamma})^{-\gamma}}{(1-\gamma)} \frac{\hat{h}_t}{r + \frac{\hat{c}_t - (1 + \hat{h}_t \tilde{c}_t^{\gamma}) \hat{c}_t}{\hat{h}_t}} \frac{\rho - \tilde{c}_t^{1-\gamma}}{1-\gamma}
\]

\[
C_t \leq -\gamma \hat{h}_t \tilde{c}_t^{1-\gamma} \left( 1 + \hat{h}_t \tilde{c}_t^{\gamma} \right) \left( \frac{r - \rho}{\gamma} - \frac{\rho - \tilde{c}_t^{1-\gamma}}{1-\gamma} \right) \\
+ \tilde{c}_t^{1-\gamma} \left( 1 + \hat{h}_t \tilde{c}_t^{\gamma} \right) \hat{h}_t \left( \rho - r + \frac{\rho - \tilde{c}_t^{1-\gamma}}{1-\gamma} + r - \tilde{c}_t^{1-\gamma} - \frac{\rho - \tilde{c}_t^{1-\gamma}}{1-\gamma} \right)
\]

\[
C_t \leq \left( 1 + \hat{h}_t \tilde{c}_t^{\gamma} \right) \left( 1 + \hat{h}_t \tilde{c}_t^{\gamma} \right) \hat{h}_t \left( \rho - r + \frac{\rho - \tilde{c}_t^{1-\gamma}}{1-\gamma} + r - \tilde{c}_t^{1-\gamma} - \frac{\rho - \tilde{c}_t^{1-\gamma}}{1-\gamma} \right) = 0
\]

Since $A_a, B_a, C_u \leq 0$ we conclude that $Y_t \leq 0$.

Now for the last term, since the agent’s strategy $(\hat{c}, a)$ is feasible, we have that

\[
\lim_{n \to \infty} E_t^a \left[ e^{-\rho^a_U} U^\tau_{r+} \right] = 0
\]

For $\gamma < 1$ we use Fatou’s lemma to show

\[
\lim_{n \to \infty} E_t^a \left[ e^{-\rho^a_F} F_{r+} U^c_{r+} \right] \geq 0
\]

As a result, when we take $n \to \infty$ we get $E_t^a \left[ e^{-\rho^a_U} \left( U^\tau_{r+} - F(\hat{h}_t, \hat{c}_t) U^c_{r+} \right) \right] \leq 0$.

For $\gamma > 1$, if $\lim_{n \to \infty} E_t^a \left[ e^{-\rho^a_F} F_{r+} U^c_{r+} \right] = 0$ for any feasible strategy $(\hat{c}, a)$, then we are done. To show this, let $N_t = x_t + \hat{h}_t \tilde{c}_t^{1-\gamma}$, so that $F_t U^c_{r+} = \frac{N_t^{1-\gamma}}{1-\gamma}$. The law of motion of $N_t$ satisfies

\[
dN_t \leq (\lambda_1 N_t - \lambda_2 \tilde{c}_t) dt + \sigma_t^N N_t dZ_t^a
\]

where $\lambda_2 = \tilde{c}_t^{1-\gamma} > 0$, and $\lambda_1 = \tilde{\mu}^x + r + \tilde{c}_t^{1-\gamma} + \gamma |\tilde{\mu}^s| + \frac{1}{2} \gamma (1 + \gamma) (\tilde{\sigma}^s)^2$. Here’s where we use the assumption that $\mu^x$, $\mu^s$, and $\sigma^s$ are bounded and $\tilde{c}_t \leq \tilde{c}_h$ is uniformly bounded away from 0; $\tilde{\mu}^x$, $\tilde{\mu}^s$, $\tilde{\sigma}$, and $\tilde{\sigma}^s$ are appropriate bounds on each process. Notice that stealing only reduces the drift of $N_t$, since the change in

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the drift of \( x \) and \( h \) cancel out, and it increases the drift of \( \hat{c} \). Since \( N_t > 0 \) always, Lemma 14 and it's corollary ensure the desired limit.

**Lemma 5**

Start with the optimal contract \( C = (c,k) \), with associated processes \( x, \hat{c}, \) and \( \hat{k} \). Define \( n_t = \hat{v}_t x_t \). We have from Ito's lemma the geometric volatility

\[
\sigma_i^n = \sigma_i^t + \frac{\hat{v}_t^t}{\hat{v}_t} \hat{c}_t \sigma_i^t = \hat{c}_t^{-\gamma} k_t \phi \sigma + \frac{\hat{v}_t^t}{\hat{v}_t} \hat{c}_t \sigma_i^t
\]

\[
= \left( \hat{c}_t^{-\gamma} \phi + \frac{\hat{v}_t^t}{\hat{v}_t} \hat{c}_t (\sigma k_t)^{-1} \sigma_i^t \right) \hat{k}_t \sigma
\]

Use \( \hat{k}_t = \frac{\sigma_i^t \hat{c}_t^2}{\phi \alpha} \) and multiply and divide by \( \hat{v}_t \) to obtain:

\[
\sigma_i^n = \left( \hat{v}_t \hat{c}_t^{-\gamma} \phi + \frac{\hat{v}_t^t}{\hat{v}_t} \hat{c}_t^{1-\gamma} (\sigma k_t)^{-1} \sigma_i^t \phi \right) \frac{k_t}{n_t} \frac{1}{\sigma}
\]

Using the definition of \( \tilde{\phi}_t \) in (23) and of of \( \lambda_t \) in (24), we obtain

\[
\sigma_i^n = \tilde{\phi}_t \frac{k_t}{n_t} = \lambda_t \sigma
\]

The net worth \( n_t \) is the principal’s continuation cost, so it satisfies the HJB equation (13). This immediately yields geometric drift

\[
\mu_i^n = r + \frac{k_t}{n_t} \alpha - \frac{c_t}{n_t}
\]

using the definition of \( \theta_t \) in (25) and \( \lambda_t \) in (24) we obtain the budget constraint (22). Since this is a linear SDE with locally bounded stochastic coefficients, if we fix the capital structure \( S = (\lambda, \tilde{\phi}, \theta) \), it has a unique solution \( n_t = x_t \hat{v}_t \). We get \( c_t = \theta_t n_t \) and \( k_t = \lambda_t / \tilde{\phi}_t \times n_t \), so \( S \) implements the optimal contract \( C \).

**Lemma 6**

Lemma 18 establishes \( \hat{v}(\hat{c}) < \hat{c}^{\gamma} \) for all \( \hat{c} \in (\hat{c}_l, \hat{c}_h) \), and together with \( \hat{v}' \geq 0, \sigma^x \geq 0 \), and \( \sigma^\delta \leq 0 \) from Theorem 1, and the definition of \( \tilde{\phi} \) in (23), we obtain \( \tilde{\phi}(\hat{c}) < \phi \).

**Lemma 7**

The optimal portfolio plan generates a stationary contract with \( \hat{c}_p = \theta_{\hat{c}_p}^{\frac{1}{\gamma}} \). Lemma 17 ensures that \( \hat{c}_p \in (\hat{c}_*, \hat{c}_h) \) and therefore from Lemma 8 we know we have an incentive compatible contract.

**Lemma 8**

We use the HJB equation (13) with \( \mu^p = \sigma^p = 0 \). Lemma 16 ensures that \( \hat{v}(\hat{c}) > 0 \) for all \( \hat{c} \in (\hat{c}_*, \hat{c}_h) \). The same argument as in Theorem 2 shows that \( \hat{v}_r(\hat{c}) \) from (30) is the cost corresponding to the stationary
contract with $\hat{c}$ and $\sigma^x$ given by (28), as long as the contract is indeed admissible and delivers utility $u_0$ to the agent. We can check that $\mu^x < r$ for the stationary contract if and only if $\hat{c} > \hat{c}_*$, where $\hat{c}_*$ is given by (29). In this case, since $\mu^x < r$ arguing as in the proof of Lemma 4 we can show that the stationary contract is admissible and delivers utility $u_0$ to the agent if and only if $\hat{c} > \hat{c}_*$. Since the contract satisfies (9), (10), and (11) by construction, Theorem 3 then ensures that it is incentive compatible.

**Lemma 9**

The capital structure in (31) is a special case of the implementation in (23), (24), and (25). The same argument as in Lemma 5 works here. We already know that the contract implemented with a constant equity stake $\phi$, $C_p$ is stationary. Lemma 17 ensures that $\hat{c}_p \in (\hat{c}_*, \hat{c}_h)$ and Lemma 8 that it is incentive compatible.

The best stationary contract $C_{\min}^r$ has $\hat{c}_{\min}^r > \hat{c}_p > \hat{c}_*$ from part 1) of Lemma 17. So $C_p$ is not the best stationary contract. Part 2) of Lemma 17 shows that $\hs_{\gamma}^{-1}(\hat{c}) > \hat{c}_r^{-\gamma}$ for all $\hat{c} \in (\hat{c}_p, \hat{c}_h)$, with equality at $\hat{c}_p$ and $\hat{c}_h$. The expression for $\tilde{\phi}_r(\hat{c})$ in (31) then shows that $\tilde{\phi}_r(\hat{c}) < \phi$ in this region.

**Lemma 10**

For any given $\hat{c} \in [\hat{c}_l, \hat{c}_h]$, we have $\tilde{v}(\hat{c}) \leq \tilde{v}_r(\hat{c})$, because stationary contracts are incentive compatible and always available as continuation contracts. Furthermore, $\tilde{v}'(\hat{c}) \geq 0$ and $\sigma^x(\hat{c}) \leq 0$. Using the definition of $\tilde{\phi}(\hat{c})$ in (23) and $\tilde{\phi}_r(\hat{c})$ in (31), we obtain $\tilde{\phi}(\hat{c}) \leq \phi_r(\hat{c})$.

**Lemma 11**

The same argument as in Lemma 5 shows that the capital structure in (35) implements the optimal contract without hidden savings. From the FOC for $\hat{c}$ (34) we obtain $\hat{v}_n < \hat{c}_n^\gamma$, and therefore $\tilde{\phi}_n < \phi$. Finally, the FOC for $k$ in the HJB can be written

$$\sigma_n^x = \frac{1}{\gamma} \frac{\alpha}{\gamma \tilde{\phi}_n} \hat{v}_n$$

$$\Rightarrow \sigma_n^x = \frac{\alpha}{\gamma \phi_n \sigma} \Rightarrow \lambda_n = \frac{\alpha}{\gamma \phi_n \sigma^2}$$

which is precisely the leverage the agent would choose on his own. This completes the proof.

**Lemma 12**

*Proof.* If ever $\hat{v}_t > \inf \hat{v}(\omega, s)$, then renegotiating at that point is better than never renegotiating and obtaining $\hat{v}_0$. In the other direction, if $\hat{v}$ is constant, any stopping time $\tau$ yields the same value to the principal, so $\tau = \infty$ is an optimal choice.
Theorem 5
Proof. Since the optimal stationary contract is incentive compatible and has a constant \( \hat{v} \), we only need to show that any incentive compatible contract with constant \( \hat{v} \) has \( \hat{v} \geq \hat{v}_{r}^{\min} \). This is clearly true for all stationary contracts as defined in Lemma 8, or Lemma 25 with aggregate risk.

There could also be stationary contracts with a constant \( \hat{c} \) but \( dL_t > 0 \). For these contracts the drift \( \mu^\hat{c} < 0 \) in the absence of \( dL_t \). Consider the optimization problem

\[
0 = \min_{\hat{c}} \hat{c} - r\hat{v} - \sigma^\hat{c} \gamma \frac{\alpha}{\phi \sigma} + \hat{v} \left( \frac{\rho - \hat{c}^{1-\gamma}}{1-\gamma} + \frac{\gamma}{2} (\sigma^\gamma)^2 - \frac{\gamma}{2} \left( \frac{\pi}{\gamma} \right)^2 \right)
\]

\[
st: \quad \frac{r - \rho - \hat{c}^{1-\gamma}}{1-\gamma} + \frac{1}{2} (\sigma^\gamma)^2 + \frac{1}{2} \left( \frac{\pi}{\gamma} \right)^2 \leq 0
\]

If the constraint is binding, we get the stationary contracts with \( dL_t = 0 \), so \( \hat{v} = \hat{v}_r \). We want to show that it must be binding. Towards contradiction, if the constraint is not binding we have \( \sigma^\hat{c} = \frac{\phi \sigma}{\hat{c}^{1-\gamma}} \) and therefore we have \( A(\hat{c}, \hat{v}) = 0 \), where \( A \) is defined as in Lemma 13. If \( \hat{v} \leq \hat{v}_{r}^{\min} \) then \( \hat{v} \leq \hat{v}_p \), because the portfolio plan with \( \hat{\phi} = \phi \) is always one of the incentive compatible stationary contracts (\( \hat{c}_p \leq \hat{c}_h \) for any valid hidden investment). Then Lemma 30 ensures that \( \frac{\tau}{\gamma} - \frac{\rho - \hat{c}^{1-\gamma}}{1-\gamma} + \frac{1}{2} (\sigma^\gamma)^2 + \frac{1}{2} \left( \frac{\pi}{\gamma} \right)^2 > 0 \), which violates the constraint. This means that \( \hat{v} \geq \hat{v}_{r}^{\min} \).

Finally, if we have a non-stationary contract with a constant \( \hat{v} < \hat{v}_{r}^{\min} \), the domain of \( \hat{c} \) must have an upper bound \( \hat{c}^* \leq \hat{c}_h \) because otherwise they would have a lower cost than the optimal contract near \( \hat{c}_h \), and this cannot be for an IC contract. For the upper bound \( \hat{c}^* \) we must have \( \sigma^\hat{c} = 0 \) and \( \mu^\hat{c} \leq 0 \). But this is the same situation with stationary contracts with \( dL_t > 0 \), and we know their cost is above \( \hat{v}_{r}^{\min} \).

\[
\square
\]

Intermediate lemmas

Lemma 13. Define the function

\[
A(\hat{c}, \hat{v}) \equiv \hat{c} - r\hat{v} - \frac{1}{2} \left( \frac{\hat{c}^{1-\gamma}}{\phi \sigma} \right)^2 + \hat{v} \left( \frac{\rho - \hat{c}^{1-\gamma}}{1-\gamma} - \frac{1}{2} \left( \frac{\pi}{\gamma} \right)^2 \right)
\]

For any \( \hat{v} \in (0, (\hat{c}^*)^\gamma) \), we have \( A(\hat{c}; \hat{v}) > 0 \) for \( \hat{c} \) near 0, where \( \hat{c}^* = \left( \frac{\rho - r(1-\gamma)}{\gamma} - \frac{1}{2} \left( \frac{\pi}{\gamma} \right)^2 \right)^{1/\gamma} \).

In addition, if \( \gamma \geq \frac{1}{2} \) then \( A(\hat{c}; \hat{v}) \) has at most one root in \([0, \hat{c}^*] \). If instead \( \gamma < \frac{1}{2} \), \( A(\hat{c}; \hat{v}) \) is convex and has at most two roots.

Proof. First, for \( \gamma < 1 \lim_{\hat{c} \to 0} A(\hat{c}; \hat{v}) = \hat{v} \left( \frac{\rho - r(1-\gamma)}{\gamma} - \frac{1}{2} \left( \frac{\pi}{\gamma} \right)^2 \right) > 0 \). For \( \gamma > 1 \), \( \lim_{\hat{c} \to 0} A(\hat{c}; \hat{v}) = \infty \).

For \( \gamma \geq 1/2 \), to show that \( A(\hat{c}; \hat{v}) \) has at most one root in \([0, \hat{c}^*] \) for any \( \hat{v} \in (0, \hat{v}_h) \), we will show that \( A'(\hat{c}; \hat{v}) = 0 \implies A(\hat{c}; \hat{v}) > 0 \) for all \( \hat{c} < \hat{c}^* \). Compute the derivative (dropping the arguments to avoid
(clutter) \[ A'_c = 1 - \hat{\beta} e^{-\gamma} - \hat{e}^{2\gamma - 1} \left( \frac{\alpha}{\phi \sigma} \right)^2 \frac{1}{\hat{v}} \]

So

\[ A'_c = 0 \implies \hat{\beta} = \hat{e}^{2\gamma} \left( \frac{\alpha}{\phi \sigma} \right)^2 \frac{1}{\hat{v}} \]

Plug this into the formula for \( A \) to get

\[ A = \hat{\beta} - r\hat{v} + \hat{e}^2 + \frac{\rho - \hat{e}^{1-\gamma}}{1 - \gamma} = \frac{1}{2\gamma} (\hat{\beta} - \hat{e}^{1-\gamma}) - \frac{\hat{v} \pi^2}{2} \gamma \]

\[ = \frac{2\gamma - 1}{2\gamma} \hat{c} + \frac{1 - 3\gamma}{2\gamma} \hat{e}^{1-\gamma} + \frac{\rho - r(1 - \gamma)}{1 - \gamma} - \frac{\hat{v} \pi^2}{2} \gamma = B(\hat{c}, \hat{v}) \]

\( B(\hat{c}, \hat{v}) \) is convex in \( \hat{c} \) because \( 1 - 3\gamma < 0 \) for \( \gamma \geq \frac{1}{2} \), so it’s minimized in \( \hat{c} \) when \( B'_c = 0 \):

\[ \frac{2\gamma - 1}{3\gamma - 1} = \hat{e}^{2\gamma} \]

and it is strictly decreasing before this point. Now we have two possible cases:

**CASE 1:** The minimum of \( B \) is achieved for \( \hat{c} \geq \hat{c}^* \), so in the relevant range, it is minimized at \( \hat{c}_h \). So let’s plug in \( \hat{c}^* \) into \( B(\hat{c}, \hat{v}) \):

\[ 2\gamma B(\hat{c}^*, \hat{v}) = (2\gamma - 1) \hat{c}^* + \frac{\hat{v}}{1 - \gamma} \left( (\rho - r(1 - \gamma)) 2\gamma + (1 - 3\gamma) (\hat{c}^*)^{1-\gamma} \right) - \frac{\hat{v} \pi^2}{2} \gamma \]

\[ = (2\gamma - 1) \hat{c}^* + \frac{\hat{v}}{1 - \gamma} \left( \frac{\rho - r(1 - \gamma)}{\gamma} \right) (2\gamma^2 + (1 - 3\gamma)) - \frac{\hat{v}}{1 - \gamma} (1 - 3\gamma) \frac{1}{2} (1 - \gamma) \left( \frac{\pi^2}{\gamma} \right)^2 - \frac{\hat{v} \pi^2}{2} \gamma \]

\[ = (2\gamma - 1) \hat{c}^* + \hat{v} \left( \frac{\rho - r(1 - \gamma)}{\gamma} \right) (1 - 2\gamma) - \frac{1}{2} (1 - \gamma) \left( \frac{\pi^2}{\gamma} \right)^2 \hat{v} (1 - 2\gamma) \]

\[ = (2\gamma - 1) \left( \hat{c}^* - \hat{v} \left( \frac{\rho - r(1 - \gamma)}{\gamma} \right) - \frac{1}{2} (1 - \gamma) \left( \frac{\pi^2}{\gamma} \right)^2 \right) \geq 0 \]

and the inequality is strict if \( \hat{v} < (\hat{c}^*)^\gamma \). So \( A(\hat{c}, \hat{v}) = B(\hat{c}, \hat{v}) + B(\hat{c}^*, \hat{v}) \geq 0 \) for any \( \hat{c} < \hat{c}^* \).

**CASE 2:** If the minimum is achieved for \( \hat{c}_m \in [0, \hat{c}^*] \) it must be that \( \gamma > 1/2 \). Then plugging in \( \hat{c}^* \) into \( B \):

\[ B(\hat{c}, \hat{v}) \geq \frac{2\gamma - 1}{2\gamma} \hat{c}_m - \frac{\hat{e}^{2\gamma - 1}}{2\gamma} \hat{v}^{1 - \gamma} + \hat{e}^2 + \frac{\rho - r(1 - \gamma)}{1 - \gamma} - \frac{\hat{v} \pi^2}{2} \gamma \]

\[ = \frac{1 - 2\gamma}{2} \hat{c}_m + \hat{e}^2 + \frac{\rho - r(1 - \gamma)}{1 - \gamma} - \frac{\hat{v} \pi^2}{2} \gamma \]

\[ = \frac{1 - 2\gamma}{2} \hat{c}_m + \frac{2\gamma - 1}{3\gamma - 1} \hat{c}_m \left( \frac{\rho - r(1 - \gamma)}{1 - \gamma} - \frac{\pi^2}{2} \gamma \right) \]
and dividing throughout by $2\gamma - 1 > 0$

$$= -\frac{1}{2} \overline{c}_m + \frac{\overline{c}_m^\gamma}{3\gamma - 1} \left( \frac{\rho - r(1-\gamma)}{1-\gamma} - 1 \frac{\pi^2}{2 \gamma} \right)$$

and multiplying by $\frac{\overline{c}_m^{1-\gamma}}{1-\gamma}$ and using $\frac{\overline{c}_m^{1-\gamma}}{1-\gamma} < \frac{(\overline{c}_m^*)^{1-\gamma}}{1-\gamma}$:

$$> -\frac{1}{2} (\overline{c}_m^*)^{1-\gamma} + \frac{1}{3\gamma - 1} \left( \frac{\rho - r(1-\gamma)}{1-\gamma} - 1 \frac{\pi^2}{2 \gamma} \right)$$

$$= \left( \frac{\rho - r(1-\gamma)}{\gamma} - 1 \frac{(1-\gamma)(\pi^2)}{2} \right) \left( -\frac{1}{2} \frac{1}{1-\gamma} + \frac{\gamma}{(3\gamma - 1)(1-\gamma)} \right) \left( \frac{\rho - r(1-\gamma)}{\gamma} - 1 \frac{(1-\gamma)(\pi^2)}{2} \right) \frac{1}{(3\gamma - 1)^2} > 0$$

So $A(\overline{c}; \overline{v}) \geq B(\overline{c}, \overline{v}) > 0$ for all $\overline{c} \in [0, \overline{c}^*]$.

For the case with $\gamma < \frac{1}{2}$, the second derivative of $A$ is

$$A''_v = \gamma \overline{v} e^{-\gamma - 1} - (2\gamma - 1) \overline{v}^{2\gamma - 2} \left( \frac{\alpha}{\phi \sigma} \right)^2 \frac{1}{\overline{v}} > 0$$

So $A(\overline{c}; \overline{v})$ is strictly convex and so can have at most two roots.

\[\square\]

**Lemma 14.** Assume there are some constants $\lambda_1, \lambda_2, \lambda_3 \in \mathbb{R}$, and $\lambda_4 > 0$ such that for any feasible strategy $(\overline{c}, a, z, \overline{z})$ there is a non-negative process $N$ with

$$dN_t \leq ((\lambda_1 + \lambda_2 \sigma_t^N + \lambda_3 \hat{\sigma}_t^N) N_t - \lambda_4 \overline{c}_t) dt + \sigma_t^N N_t dZ_t + \hat{\sigma}_t^N N_t d\tilde{Z}_t$$

for some processes $\sigma^N$ and $\hat{\sigma}^N$, which can depend on the strategy. Then there is a constant $\lambda_5 > 0$ such that for any $T > 0$ and any feasible strategy $(\overline{c}, a, z, \overline{z})$

$$\mathbb{E}^a \left[ \int_0^T e^{-\mu t} \frac{\overline{c}_t^{1-\gamma}}{1-\gamma} dt \right] \leq \lambda_5 \frac{N_0^{1-\gamma}}{1-\gamma}$$

**Proof.** First define $n_t$ as the solution to the SDE

$$dn_t = \left( (\lambda_1 + \lambda_2 \sigma_t^N + \lambda_3 \hat{\sigma}_t^N) n_t - \overline{c}_t \right) dt + \sigma_t^n n_t dZ_t^a + \hat{\sigma}_t^n n_t d\tilde{Z}_t$$

and $n_0 = \frac{N_0}{\lambda_4}$. It follows that $n_t \geq \frac{N_0}{\lambda_4} \geq 0$. Now define $\zeta$ as

$$\frac{d\zeta_t}{\zeta_t} = -\lambda_1 dt - \lambda_2 dZ_t^a - \lambda_3 d\tilde{Z}_t, \quad \zeta_0 = 1$$

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and
\[ \tilde{n}_t = \int_0^t \zeta_a \tilde{c}_a ds + \zeta_t \]

We can check that \( \tilde{n}_t \) is a local martingale under \( P^a \). Since \( \zeta_t > 0 \) and \( n_t \geq 0 \) it follows that

\[
\mathbb{E}^a \left[ \int_0^{\tau^m \land T} \zeta_a \tilde{c}_a ds \right] \leq \mathbb{E}^a \left[ \int_0^{\tau^m \land T} \zeta_a \tilde{c}_a ds + \zeta_{\tau^m \land T} n_{\tau^m \land T} \right] = n_0
\]

where \( \{\tau^m\} \) reduces the stochastic integral and has \( \lim_{m \to \infty} \tau^m = \infty \) a.s. Taking \( m \to \infty \) and using the monotone convergence theorem we obtain

\[
\mathbb{E}^a \left[ \int_0^T \zeta_a \tilde{c}_a ds \right] \leq n_0
\]

Now we want to maximize \( \mathbb{E}^a \left[ \int_0^T e^{-\rho t} \frac{c_t^{1-\gamma}}{1-\gamma} dt \right] \) subject to this budget constraint. Notice that \( \gamma \) appears both in the budget constraint and objective function, but does not affect the law of motion of \( \zeta \) under \( P^a \), so we can ignore it since we are choosing \( \tilde{c} \). The candidate solution \( c \) has

\[
e^{-\rho t} \tilde{c}_t^{1-\gamma} = \zeta_t \mu
\]

where \( \mu > 0 \) is the Lagrange multiplier and is chosen so that the budget constraint holds with equality. For any \( \tilde{c} \) that satisfies the budget constraint we have

\[
\mathbb{E}^a \left[ \int_0^T e^{-\rho t} \frac{\tilde{c}_t^{1-\gamma}}{1-\gamma} dt \right] \leq \mathbb{E}^a \left[ \int_0^T e^{-\rho t} \left( \frac{\tilde{c}_t^{1-\gamma}}{1-\gamma} + \tilde{c}_t^{1-\gamma} (\tilde{c}_t - c_t) \right) dt \right] = \mathbb{E}^a \left[ \int_0^T e^{-\rho t} \frac{\tilde{c}_t^{1-\gamma}}{1-\gamma} dt \right] + \mu \mathbb{E}^a \left[ \int_0^T \zeta_t (\tilde{c}_t - c_t) dt \right] \leq \mathbb{E}^a \left[ \int_0^T e^{-\rho t} \frac{\tilde{c}_t^{1-\gamma}}{1-\gamma} dt \right]
\]

Now since \( c_t = (\zeta_t \mu)^{-\frac{1}{\gamma}} e^{-\frac{\tilde{c}_t}{c_t}} \) it follows a geometric Brownian motion so \( \mathbb{E}^a \left[ \int_0^T e^{-\rho t} \frac{\tilde{c}_t^{1-\gamma}}{1-\gamma} dt \right] \) is finite.

Because of homothetic preferences, we know that \( \mathbb{E}^a \left[ \int_0^T e^{-\rho t} \frac{\tilde{c}_t^{1-\gamma}}{1-\gamma} dt \right] = \lambda_5 \frac{N^{1-\gamma}}{1-\gamma} = \lambda_5 \frac{N^{1-\gamma}}{1-\gamma} \) for some \( \lambda_5 > 0 \).

\[ \square \]

**Corollary.** For \( \gamma > 1 \), \( \lim_{n \to \infty} \mathbb{E}_t^a \left[ e^{-\rho N_{1-\gamma} \frac{N^{1-\gamma}}{1-\gamma}} \right] = 0 \) for any feasible strategy \((\tilde{c}, a, z, \tilde{z})\).

**Proof.** The continuation utility at any stopping time \( \tau^n < \infty \) has

\[
U^\tilde{c}_n = \mathbb{E}_t^a \left[ \int_{\tau^n}^{\tau^{n+T}} e^{-\rho(t-\tau^n)} \frac{\tilde{c}_t^{1-\gamma}}{1-\gamma} dt + e^{-\rho(T-\tau^n)} U^\tilde{c}_n (\tau^{n+T}) \right]
\]

\[
\leq \mathbb{E}_t^a \left[ \int_{\tau^n}^{\tau^{n+T}} e^{-\rho(t-\tau^n)} \frac{\tilde{c}_t^{1-\gamma}}{1-\gamma} dt \right] \leq \lambda_5 \frac{N^{1-\gamma}}{1-\gamma}
\]

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So at \( t = 0 \) we get
\[
U_{0}^{c,a} = \mathbb{E}^{a} \left[ \int_{0}^{\tau_{n}} e^{-\rho t} \frac{\hat{c}_{1}^{1-\gamma}}{1-\gamma} dt + e^{-\rho \tau_{n}} U_{\tau_{n}}^{c,a} \right] \leq \mathbb{E}^{a} \left[ \int_{0}^{\tau_{n}} e^{-\rho t} \frac{\hat{c}_{1}^{1-\gamma}}{1-\gamma} dt + e^{-\rho \tau_{n}} \lambda_{0} \frac{N_{1}^{1-\gamma}}{1-\gamma} \right]
\]

Take limits \( n \to \infty \) and use the monotone convergence theorem on the first term on the right hand side to get
\[
0 \geq \lim_{n \to \infty} \mathbb{E}^{a} \left[ e^{-\rho \tau_{n}} \frac{N_{1}^{1-\gamma}}{1-\gamma} \right] \geq 0.
\]

\[\square\]

**Lemma 15.** Let \( \hat{c}_{i} \in (0, \hat{c}_{h}) \) and \( \hat{v}_{i} \leq \hat{v}_{p} \) for any \( \hat{c} \in (0, \hat{c}_{h}) \). If \( \sigma^{\hat{c}} = 0 \) and \( \sigma^{x} = \frac{\alpha}{\sigma} \hat{c}_{i}^{\gamma} \) and \( A(\hat{c}_{i}, \hat{v}_{i}) = 0 \), then
\[
\mu^{\hat{c}} = \frac{r - \rho}{\gamma} - \frac{\rho - \hat{c}_{1}^{1-\gamma}}{1-\gamma} + \frac{(\sigma^{x})^{2}}{2} > 0
\]

**Proof.** Looking at (10), with \( \sigma^{\hat{c}} = 0 \) we get for the drift
\[
\mu^{\hat{c}} = \frac{r - \rho}{\gamma} + \frac{1}{2} (\sigma^{x})^{2} - \frac{\rho - \hat{c}_{1}^{1-\gamma}}{1-\gamma}
\]

So for any \( \hat{c}, \mu^{\hat{c}} > 0 \) implies
\[
\frac{1}{2} (\sigma^{x})^{2} > \frac{\rho - r}{\gamma} + \frac{\rho - \hat{c}_{1}^{1-\gamma}}{1-\gamma}
\]

Since we also want \( A(\hat{c}; \hat{v}) = 0 \), we get
\[
0 = \hat{c} - r \hat{v} + \hat{v} \left( \frac{\rho - \hat{c}_{1}^{1-\gamma}}{1-\gamma} - \frac{\gamma (\sigma^{x})^{2}}{2} \right) > \hat{c} - \hat{v} \hat{c}_{1}^{1-\gamma} \equiv M
\]

Notice that if \( \hat{v} = \hat{c}_{1}^{\gamma} \) we have \( M = 0 \). If \( \hat{v} > \hat{c}_{1}^{\gamma} \) we have \( M < 0 \) and if \( \hat{v} < \hat{c}_{1}^{\gamma} \) we have \( M > 0 \). So for \( A(\hat{c}; \hat{v}) = 0 \) and \( \mu^{\hat{c}} > 0 \) we need \( \hat{v} < \hat{c}_{1}^{\gamma} \). In fact, if \( \hat{v} = \hat{c}_{1}^{\gamma} \) and in addition
\[
\frac{1}{2} \left( \frac{\alpha}{\sigma} \right)^{2} = \frac{\rho - \hat{c}_{1}^{1-\gamma}}{1-\gamma} + \frac{\rho - r}{\gamma}
\]

then we have \( A = 0 \) and \( \mu^{\hat{c}} = 0 \). In this case, because we have \( \mu^{\hat{c}} = 0 \) we therefore have the value of a stationary contract, i.e. \( \hat{v} = \hat{v}_{r}(\hat{c}) \). This point corresponds to the optimal portfolio with \( \hat{\phi} = \phi \), \( (\hat{c}_{p}, \hat{v}_{p}) \).

We know from Lemma 17 that \( \hat{c}_{p} \in [\hat{c}_{r}, \hat{c}_{h}] \). By assumption, \( \hat{v}_{i} \leq \hat{v}_{p} \).

First we will show that \( \mu^{\hat{c}} \geq 0 \), and then make the inequality strict. Towards contradiction, suppose \( \mu^{\hat{c}} < 0 \) at \( \hat{c}_{i} \). Then it must be the case that \( \hat{v}_{i} > \hat{c}_{1}^{\gamma} \) because we have \( A(\hat{c}_{i}, \hat{v}_{i}) = 0 \). We will show that \( A(\hat{c}_{i}, \hat{v}_{i}) > 0 \) and get a contradiction. First take the derivative of \( A \):
\[
A_{\hat{c}}(\hat{c}_{i}, \hat{v}_{i}) = 1 - \hat{v}_{i} \left( \hat{c}_{1}^{\gamma} + \hat{c}_{i}^{2\gamma - 1} \left( \frac{\alpha}{\sigma} \right)^{2} \frac{1}{\hat{v}_{i}^{2}} \right) < 0
\]
where the inequality holds for all \( \dot{c} < \hat{v}_l^\frac{1}{\gamma} \). So \( A(\hat{c}_l, \hat{v}_l) > A(\hat{v}_l^\frac{1}{\gamma}, \hat{v}_l) \). Letting \( \hat{c}_m = \hat{v}_l^\frac{1}{\gamma} \) we get

\[
A(\hat{c}_l, \hat{v}_l) > \hat{c}_m - r\hat{v}_l + \hat{v}_l \left( \frac{\rho - \hat{c}_m^{1-\gamma}}{1-\gamma} - \frac{1}{2} \left( \frac{\alpha}{\phi \sigma} \right)^2 \frac{1}{\gamma} \right)
\]

\[
= \hat{c}_m - r\hat{v}_l + \hat{v}_l \left( \frac{\rho - \hat{c}_m^{1-\gamma}}{1-\gamma} - \frac{\rho - \hat{c}_l^{1-\gamma}}{1-\gamma} - (\rho - r) \right)
\]

\[
= \hat{c}_m - r\hat{v}_l + \hat{v}_l \frac{\rho - \hat{c}_m^{1-\gamma} - \hat{c}_l^{1-\gamma}}{1-\gamma} \geq 0
\]

where the last equality uses \( \hat{v}_l = \hat{c}_m^\gamma \) and the last inequality uses \( \hat{c}_m = \hat{v}_l^\frac{1}{\gamma} \leq \hat{v}_p^\frac{1}{\gamma} = \hat{c}_p \). This is a contradiction, and therefore it must be the case that \( \mu^\beta \geq 0 \) at \( \hat{c}_l \).

It’s clear from the previous argument that \( \mu^\beta(\hat{c}_l) = 0 \) only if \( (\hat{c}_l, \hat{v}_l) = (\hat{c}_p, \hat{v}_p) \). We will show this cannot be the case because \( \alpha > 0 \). First, note that \( (\hat{c}_p, \hat{v}_p) \) is a tangency point where \( \hat{v}_r(\hat{c}) \) touches the locus \( \hat{v}_b(\hat{c}) \) defined by \( A(\hat{c}; \hat{v}_b(\hat{c})) = 0 \). If \( (\hat{c}_l, \hat{v}_l) = (\hat{c}_p, \hat{v}_p) \) then this must be the minimum point for \( \hat{v}_r(\hat{c}) \), so the derivative of both \( \hat{v}_r(\hat{c}) \) and \( \hat{v}_b(\hat{c}) \) must be zero. This means that \( A'_r(\hat{c}_l, \hat{v}_l) = 0 \). However,

\[
1 - \dot{v}_l \left( \hat{c}_l^{-\gamma} + \hat{c}_l^{2\gamma-1} \left( \frac{\alpha}{\phi \sigma} \right)^2 \frac{1}{\hat{v}_l^2} \right) < 0
\]

where the inequality follows from \( \dot{v}_l = \hat{v}_p = \hat{c}_l^\gamma \) (note that \( \hat{c}_l > 0 \) because as Lemma 13 shows \( A(\hat{c}, \hat{v}_l) \) is strictly positive for \( \hat{c} \) near 0). This can’t be a minimum of \( \hat{v}_r(\hat{c}) \). Therefore \( (\hat{c}_l, \hat{v}_l) \neq (\hat{c}_p, \hat{v}_p) \) and \( \mu^\beta(\hat{c}_l) > 0 \). This completes the proof.

\[
\square
\]

**Lemma 16.** The cost function of stationary contracts \( \hat{v}_r(\hat{c}) \) defined by (30) is strictly positive for all \( \hat{c} \in (\hat{c}_*, \hat{c}_h) \) if and only if

\[
\alpha < \bar{\alpha} \equiv \phi \sigma \gamma \sqrt{\frac{2}{1+\gamma}} \sqrt{\frac{\rho - r(1-\gamma)}{\gamma}}
\]

**Proof.** We need to check the numerator in (30), since the denominator is positive for all \( \hat{c} \geq \hat{c}_* \):

\[
\hat{c} - \frac{\alpha}{\phi \sigma} \hat{c}_l^\gamma \sqrt{2 \sqrt{\frac{\rho - r(1-\gamma)}{\gamma}} - \hat{c}_l^{1-\gamma}}
\]

The rest of the proof consists of evaluating this expression at \( \hat{c} = \hat{c}_* \) and showing it is non-positive iff the bound is violated. We get \( \hat{c} \) times

\[
1 - \frac{\alpha}{\phi \sigma} \sqrt{2 \sqrt{\frac{\rho - r(1-\gamma)}{\gamma}} \frac{1}{2\gamma} \left( \frac{\rho - r(1-\gamma)}{\gamma} \right)^{-1}}
\]

So if \( \alpha \geq \bar{\alpha} \) the numerator is non-positive, and if \( \alpha < \bar{\alpha} \) then it’s strictly positive. This completes the proof.

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Lemma 17. Let 

\[ \hat{c}_p \equiv \left( \frac{\rho - r(1 - \gamma)}{\gamma} - \frac{1 - \gamma}{2} \left( \frac{\alpha}{\phi \sigma^2} \right)^2 - \frac{1 - \gamma}{2} \left( \frac{\pi}{\gamma} \right)^2 \right)^{\frac{1}{1 - \gamma}} \]

\[ \hat{v}_p \equiv \hat{c}_p^\gamma \]

be the \( \hat{c} \) and \( \hat{v} \) corresponding to the optimal portfolio plan with \( \hat{\phi} = \phi \). We have the following properties

1) \( \hat{c}_s < \hat{c}_p < \hat{c}_p^{\min} \leq \hat{c}_h \), for any valid hidden investment setting 

2) \( \hat{c}_s \) intersects \( \hat{v}_r(\hat{c}) \) only at \( \hat{c}_p \) and \( \hat{c}^* = \left( \frac{\rho - r(1 - \gamma)}{\gamma} - \frac{1 - \gamma}{2} \left( \frac{\pi}{\gamma} \right)^2 \right)^{1/(1 - \gamma)} \) in \([0, \hat{c}^*]\). Furthermore, \( \hat{c}_s \geq \hat{v}_r(\hat{c}) \) for all \( \hat{c} \in [\hat{c}_p, \hat{c}^*] \), and \( \hat{c}_s \leq \hat{v}_r(\hat{c}) \) for all \( \hat{c} \in [\hat{c}_s, \hat{c}_p] \), with strict inequality in the interior of each region.

3) \( A(\hat{c}, \hat{c}^\gamma) = 0 \) only at \( \hat{c} = 0 \) and \( \hat{c}_p \). Furthermore, \( A(\hat{c}, \hat{c}^\gamma) \leq 0 \) for all \( \hat{c} \in [\hat{c}_p, \hat{c}_h] \) and \( A(\hat{c}, \hat{c}^\gamma) \geq 0 \) for all \( \hat{c} \in [0, \hat{c}_p] \), and \( \partial_1 A(\hat{c}, \hat{c}^\gamma) < 0 \) for all \( \hat{c} \in (0, \hat{c}_h] \).

Proof. First let’s show that \( \hat{c}_p \in (\hat{c}_s, \hat{c}_h) \). Clearly, \( \hat{c}_p \leq \hat{c}_h \) for any type of valid hidden investment, because \( \phi < 1 \). Now write \( \hat{c}_p \)

\[ \hat{c}_p = \left( \frac{\rho - r(1 - \gamma)}{\gamma} - \frac{1 - \gamma}{2} \left( \frac{\alpha}{\phi \sigma^2} \right)^2 - \frac{1 - \gamma}{2} \left( \frac{\pi}{\gamma} \right)^2 \right)^{\frac{1}{1 - \gamma}} \]

where the inequality comes from \( \alpha < \bar{\alpha} = \frac{\phi \sigma^2 \sqrt{\pi}}{\sqrt{1 + \gamma}} \sqrt{\frac{\rho - r(1 - \gamma) - \frac{1 - \gamma}{2} \left( \frac{\pi}{\gamma} \right)^2}{1 + \gamma}} \). Notice \( 1 - \frac{1 - \gamma}{1 + \gamma} = \frac{2 - \gamma}{1 + \gamma} \) and use the definition of \( \hat{c}_s \):

\[ \hat{c}_s = \left( \frac{2\gamma}{1 + \gamma} \right)^{1/(1 - \gamma)} \left( \frac{\rho - r(1 - \gamma)}{\gamma} - \frac{1 - \gamma}{2} \left( \frac{\pi}{\gamma} \right)^2 \right)^{1/(1 - \gamma)} \]

to conclude that \( \hat{c}_s < \hat{c}_p \). The cost of this portfolio contract is \( \hat{v}_p = \hat{c}_p^\gamma \).

Now go to 2). We are looking for roots of \( \hat{v}_r(\hat{c}) = \hat{c}_s \):

\[ \hat{c} - \frac{\alpha}{\phi \sigma} \hat{c} \sqrt{2} \left( \frac{\rho - r(1 - \gamma)}{\gamma} - \frac{1 - \gamma}{2} \left( \frac{\pi}{\gamma} \right)^2 - \hat{c} \right) = \hat{c}^\gamma \left( 2\gamma - \rho - \frac{1 + \gamma}{1 - \gamma} \rho + \gamma \left( \frac{\pi}{\gamma} \right)^2 + \frac{\hat{c}^{1 - \gamma}}{1 - \gamma} (1 + \gamma) \right) \]

Divide throughout by \( \hat{c}^{1 - \gamma} > 0 \) and reorganize the right hand side

\[ \frac{\hat{c}^{1 - \gamma}}{1 - \gamma} (1 + \gamma) - \frac{\alpha}{\phi \sigma} \sqrt{2} \left( \frac{\rho - r(1 - \gamma)}{\gamma} - \frac{1 - \gamma}{2} \left( \frac{\pi}{\gamma} \right)^2 - \hat{c} \right) = -2\gamma \left( \frac{\rho - r(1 - \gamma) - \frac{1 - \gamma}{2} \left( \frac{\pi}{\gamma} \right)^2}{1 - \gamma} + \frac{\hat{c}^{1 - \gamma}}{1 - \gamma} (1 + \gamma) \right) + \frac{\hat{c}^{1 - \gamma}}{1 - \gamma} (1 + \gamma) \]
\[-\frac{\alpha}{\phi \sigma} \sqrt{2} \left( \frac{\rho - r(1 - \gamma)}{\gamma} - \frac{1 - \gamma}{2} \left( \frac{\pi}{\gamma} \right)^2 \right) - \hat{c}^{1 - \gamma} \right] = -2 \gamma \left( \frac{\rho - r(1 - \gamma)}{\gamma} - \frac{1 - \gamma}{2} \left( \frac{\pi}{\gamma} \right)^2 \right) + \hat{c}^{1 - \gamma}
\]

\[\frac{\alpha}{\phi \sigma} \sqrt{2} \left( \frac{\rho - r(1 - \gamma)}{\gamma} - \frac{1 - \gamma}{2} \left( \frac{\pi}{\gamma} \right)^2 \right) - \hat{c}^{1 - \gamma} \right] = 2 \gamma \left( \frac{\rho - r(1 - \gamma)}{\gamma} - \frac{1 - \gamma}{2} \left( \frac{\pi}{\gamma} \right)^2 \right) - \hat{c}^{1 - \gamma}
\]

If \( \hat{c} = \left( \frac{\rho - r(1 - \gamma)}{\gamma} - \frac{1 - \gamma}{2} \left( \frac{\pi}{\gamma} \right)^2 \right)^{\frac{1}{1 - \gamma}} \) we have a root. If not, then we can write

\[\frac{\alpha}{\phi \sigma} = \sqrt{2} \left( \frac{\rho - r(1 - \gamma)}{\gamma} - \frac{1 - \gamma}{2} \left( \frac{\pi}{\gamma} \right)^2 \right) - \hat{c}^{1 - \gamma}
\]

\[\hat{c} = \left( \frac{\rho - r(1 - \gamma)}{\gamma} - \frac{1 - \gamma}{2} \left( \frac{\alpha}{\phi \sigma} \right) \right)^{\frac{1}{1 - \gamma}} = \hat{c}_p < \left( \frac{\rho - r(1 - \gamma)}{\gamma} - \frac{1 - \gamma}{2} \left( \frac{\pi}{\gamma} \right)^2 \right)^{\frac{1}{1 - \gamma}}
\]

We know that at \( \hat{c} = 0, \hat{c}^2 = 0 \), while \( \hat{v}_r(\hat{c}) \) is always positive above \( \hat{c}_s \) and diverges to infinity as \( \hat{c} \searrow \hat{c}_s \).

So we know that \( \hat{c}_p \) is the first time they intersect and therefore \( \hat{c}^2 \) intersects \( \hat{v}_r(\hat{c}) \) from below. Since they won’t intersect again until \( \hat{c}^* \), we get the other inequality.

Back to 1), consider the locus \( \hat{v}_b(\hat{c}) \) defined by \( A(\hat{c}, \hat{v}_b(\hat{c})) = 0 \). Since \( A(\hat{c}, \hat{v}) \) minimizes over \( \sigma_r \), it is always below \( \hat{v}_r(\hat{c}) \). At \( (\hat{c}_p, \hat{v}_b) \), we have \( \hat{v}_b(\hat{c}) = \hat{v}_r(\hat{c}) \) by part 3) below, which means this is a tangency point of \( \hat{v}_b \) and \( \hat{v}_r \). We can now show that \( A'_u(\hat{c}_p, \hat{v}_p) < 0 \) and \( A'_v(\hat{c}_p, \hat{v}_p) < 0 \), so that \( \hat{v}_b(\hat{c}_p) = \hat{v}_r(\hat{c}_p) < 0 \) which means that the \( \hat{c}_p \) is not the optimal stationary contract, since \( \hat{c}_p < \hat{c}_b \).

Write

\[A'_u(\hat{c}_p, \hat{v}_p) = 1 - \hat{v}_p \left( \hat{c}^{\gamma - 1} + \hat{c}^{2 \gamma - 1} \left( \frac{\alpha}{\phi \sigma} \right) \right)^{\frac{1}{\gamma}} = -\hat{c}^{\gamma - 1} \left( \frac{\alpha}{\phi \sigma} \right) < 0
\]

\[A'_v(\hat{c}_p, \hat{v}_p) = \frac{1}{1 - \gamma} \left( \gamma \left( \frac{\rho - r(1 - \gamma)}{\gamma} - \frac{1 - \gamma}{2} \left( \frac{\pi}{\gamma} \right)^2 \right) - \hat{c}^{1 - \gamma} \right) \frac{1}{\hat{v}_p^2} \left( \frac{\alpha}{\phi \sigma} \right)^2
\]

\[= \frac{1}{1 - \gamma} \left( \gamma \left( \frac{\rho - r(1 - \gamma)}{\gamma} - \frac{1 - \gamma}{2} \left( \frac{\pi}{\gamma} \right)^2 \right) - \hat{c}^{1 - \gamma} \right) \frac{1}{\hat{v}_p^2} \left( \frac{\alpha}{\phi \sigma} \right)^2 + \frac{1}{2} \left( \frac{\alpha}{\phi \sigma} \right)^2 < 0
\]

where the last inequalities follows from the bound on \( \alpha < \alpha \equiv \frac{\phi \sigma \sqrt{\gamma}}{\sqrt{1 + \gamma}} \sqrt{\frac{\rho - r(1 - \gamma)}{\gamma} - \frac{1 - \gamma}{2} \left( \frac{\pi}{\gamma} \right)^2} \). From Lemma 9 or Lemma 74, the minimum stationary contract can be implemented with a constant capital structure \( S = (\lambda, \phi, \theta, \sigma) \). We know it has a lower cost than \( \hat{v}_p \), which is optimal subject to the equity share \( \phi \), so this means the agent must keep a smaller equity share \( \hat{\phi}_r \). Since \( (\sigma_r)^{\gamma} = \phi \hat{\phi}_r(\hat{c}_r^{\gamma}) \gamma \sigma \), we get

\[\hat{\phi}_r = \hat{\phi}_r(\hat{c}_r^{\gamma}) \gamma \phi < \phi
\]

which implies \( \hat{c}_r^{\gamma} \times (\hat{c}_r^{\gamma})^{-\gamma} < 1 \) and therefore \( \hat{c}_r^{\gamma} \gamma \phi < \phi \). From 2) this means \( \hat{c}_r^{\gamma} > \hat{c}_p \). We know
\( \hat{c}_{min} \leq \hat{c}_h \) from the definition of \( \hat{c}_{min} \).

For 3), we are looking for roots of

\[
\hat{c} - r\hat{c}^\gamma - \frac{1}{2} \left( \frac{\alpha \hat{c}^\gamma}{\varphi} \right)^2 + \hat{c}^\gamma \left( \frac{\rho - \hat{c}^{1-\gamma}}{1 - \gamma} - \frac{1}{2} \frac{\pi^2}{\gamma} \right) = 0
\]

This works for \( \hat{c} = 0 \). Otherwise, divide by \( \hat{c}^\gamma \)

\[
\frac{\hat{c}^{1-\gamma}}{1 - \gamma} - r - \frac{1}{2} \left( \frac{\alpha}{\varphi \gamma} \right)^2 + \frac{\rho - \hat{c}^{1-\gamma}}{1 - \gamma} - \frac{1}{2} \frac{\pi^2}{\gamma} = 0
\]

\[
\frac{\rho - r(1 - \gamma)}{1 - \gamma} - \gamma \left( \frac{\alpha}{\varphi \gamma} \right)^2 - \frac{\gamma}{2} \left( \frac{\pi}{\gamma} \right)^2 = \hat{c}^{1-\gamma}
\]

\[
\frac{\rho - r(1 - \gamma)}{\gamma} - \frac{1 - \gamma}{2} \left( \frac{\alpha}{\varphi \gamma} \right)^2 - \frac{1 - \gamma}{2} \left( \frac{\pi}{\gamma} \right)^2 = \hat{c}^{1-\gamma}
\]

\[
\hat{c} = \left( \frac{\rho - r(1 - \gamma)}{\gamma} - \frac{1 - \gamma}{2} \left( \frac{\alpha}{\varphi \gamma} \right)^2 - \frac{1 - \gamma}{2} \left( \frac{\pi}{\gamma} \right)^2 \right)^{\frac{1}{1-\gamma}} = \hat{c}_p
\]

So we have only \( \hat{c}_p \) and \( \hat{c} = 0 \) as roots. This argument also shows that \( A(\hat{c}, \hat{c}^\gamma) \leq 0 \) for \( \hat{c} \in [\hat{c}_p, \hat{c}_h] \), and \( A(\hat{c}, \hat{c}^\gamma) \geq 0 \) for \( \hat{c} \in [0, \hat{c}_p] \). Also, evaluating the derivative \( \partial_1 A(\hat{c}, \hat{c}^\gamma) \)

\[
\partial_1 A(\hat{c}, \hat{c}^\gamma) = 1 - \hat{c}^\gamma \hat{c}^{-\gamma} - \hat{c}^{\gamma-1} \left( \frac{\alpha}{\varphi \gamma} \right)^2 \frac{1}{\hat{c}^\gamma}
\]

\[
\partial_1 A(\hat{c}, \hat{c}^\gamma) = 1 - \hat{c}^{\gamma-1} \left( \frac{\alpha}{\varphi \gamma} \right)^2 = -\hat{c}^{\gamma-1} \left( \frac{\alpha}{\varphi \gamma} \right)^2 < 0
\]

for all \( \hat{c} \in (0, \hat{c}_h] \).

\[
\text{Lemma 18. For all } \hat{c} \in [\hat{c}_l, \hat{c}_h]
\]

\[
\hat{v}(\hat{c}) < \hat{c}^\gamma
\]

If \( \hat{c}_h = \left( \frac{\rho - r(1 - \gamma)}{\gamma} - \frac{1 - \gamma}{2} \left( \frac{\pi}{\gamma} \right)^2 \right)^{\frac{1}{1-\gamma}} \) then \( \hat{v}(\hat{c}_h) = \hat{c}_h^\gamma \). If \( \hat{c}_h < \left( \frac{\rho - r(1 - \gamma)}{\gamma} - \frac{1 - \gamma}{2} \left( \frac{\pi}{\gamma} \right)^2 \right)^{\frac{1}{1-\gamma}} \) then \( \hat{v}(\hat{c}_h) < \hat{c}^\gamma \).

\[
\text{Proof. We already know from the proof of Lemma 30 that at } \hat{c}_l \text{ we have } \hat{v}(\hat{c}_l) < \hat{c}_l^\gamma. \text{ We also know that } \hat{v}(\hat{c}) \leq \hat{v}(\hat{c}), \text{ so from Lemma 17 if ever } \hat{v}(\hat{c}) = \hat{c}^\gamma \text{ for some } \hat{c} > \hat{c}_l, \text{ it must be either that } \hat{c} = \hat{c}^* = \left( \frac{\rho - r(1 - \gamma)}{\gamma} - \frac{1 - \gamma}{2} \left( \frac{\pi}{\gamma} \right)^2 \right)^{\frac{1}{1-\gamma}} \geq \hat{c}_h; \text{ or that } \hat{c} \leq \hat{c}_p \text{ and therefore } A(\hat{c}, \hat{c}^\gamma) = A(\hat{c}, \hat{v}(\hat{c})) \geq 0 \text{ and } \partial_1 A(\hat{c}, \hat{v}(\hat{c})) < 0 \text{ (because } \hat{c} \geq \hat{c}_l > 0). \text{ From Lemma 13 we know that } A(\hat{c}, \hat{v}) \text{ is positive near 0 and either has one root in } \hat{c} \text{ if } \gamma \geq 1/2, \text{ or is convex with at most two roots if } \hat{c} \leq 1/2. \text{ This means that } A(\hat{c} - \delta, \hat{c}^\gamma) > 0 \text{ for all } \delta \in (0, \hat{c}).
\]

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Now we’ll use the same reasoning as in Theorem 1. We can pick an $\alpha' < \alpha$ so that $\hat{v}_{\alpha'}(\hat{c}^{\gamma'}) = \hat{v}(\hat{c})$, and $\hat{c}^{\gamma'} < \hat{c}$, because $\hat{v}_{\alpha'}(\hat{c})$ is decreasing in $\alpha'$. However, $A_{\alpha'}(\hat{c}, \hat{c}^{\gamma})$ is decreasing in $\alpha$, so we get $A_{\alpha'}(\hat{c}^{\gamma'}, \hat{c}^{\gamma}) > A(\hat{c}^{\gamma'}, \hat{c}^{\gamma}) > 0$, which contradicts $A_{\alpha'}(\hat{c}^{\gamma'}, \hat{v}_{\alpha'}(\hat{c}^{\gamma'})) = 0$.

So we conclude that $\hat{v}(\hat{c}) < \hat{c}^{\gamma}$ for all $\hat{c} \in [\hat{c}_l, \hat{c}_h]$. If $\hat{c}_h = \hat{c}^*$ then we have $\hat{v}(\hat{c}_h) = \hat{c}_h^1$, but if $\hat{c}_h < \hat{c}^*$ then we must also have $\hat{v}(\hat{c}_h) < \hat{c}_h^1$.

\[ \square \]

**Corollary.** For all $\hat{c} \in [\hat{c}_l, \hat{c}_h]$

\[ \hat{\phi}(\hat{c}) = \left(\hat{v}\hat{c}^\gamma\right) + \hat{v}'\hat{c}k^{-1}\sigma^{-1}\hat{c} \leq \phi \]

\[ \in (0, 1) \leq 0 \]
Appendix B: aggregate risk and hidden investment

This appendix formally introduces aggregate risk and hidden investment into the baseline environment. The return on capital is now

\[
    dR_t = (r + \pi \tilde{\sigma} + \alpha - a_t) \, dt + \sigma dZ_t + \tilde{\sigma} d\tilde{Z}_t
\]

Where \( \tilde{Z} \) is an independent Brownian motion that represents aggregate risk, with market price \( \pi \). Capital has a loading \( \tilde{\sigma} \) on aggregate risk, so the excess return on capital for the agent is \( \alpha \), as in the baseline. Let \( Q \) be the associated martingale measure.

The agent receives cumulative payments \( I \) from the principal and manages capital \( k \) for him. Payments \( I \) can be any semimartingale (it could be decreasing if the agent must pay the principal). This nests the relevant case where the contract gives the agent only what he will consume, i.e. \( dI_t = \gamma dt \). As in the baseline setting, the agent can steal from the principal at rate \( a \geq 0 \) and decide when to consume \( \tilde{c} \geq 0 \). He can invest his hidden savings in the same way the principal would, not only in a risk-free asset, but also in aggregate risk \( \tilde{Z} \). In addition, the agent may be able to invest his hidden savings in his private technology. His hidden savings follow the law of motion

\[
    dh_t = dI_t + (rh_t + z_t h_t (\alpha + \pi \tilde{\sigma}) + \tilde{z}_t h_t \pi - \tilde{c}_t + \phi k_t a_t) \, dt + z_t h_t \left( \sigma dZ_t + \tilde{\sigma} d\tilde{Z}_t \right) + \tilde{z}_t h_t d\tilde{Z}_t
\]

where \( z \) is the portfolio weight on his own private technology, and \( \tilde{z} \) the weight on aggregate risk. While the agent can chose any position on his own private technology, and \( \tilde{z} \) the weight on aggregate risk. While the agent can chose any position on aggregate risk, \( \tilde{z}_t \in \mathbb{R} \), for his hidden private investment we consider two cases: 1) no hidden private investment, \( z_t \in H = \{0\} \), and 2) hidden private investment, \( z_t \in H = \mathbb{R}_+ \).

The cost to the principal is

\[
    J_0 = \mathbb{E}^Q \left[ \int_0^\infty e^{-rt} (dI_t(R^a) - (\alpha - a_t)k_t(R^a)) \, dt \right]
\]

where \( R^a \) the observed return that results from the agent’s stealing activity \( a \).

A contract \( \mathcal{C} = (I,k,\tilde{c},a,z,\tilde{z}) \) specifies the contractible payments \( I \) and capital \( k \), and recommends the hidden action \( (\tilde{c},a,z,\tilde{z}) \), all contingent not only on returns \( R \) but also on

\[ \text{We can also study other cases where the agent may not be able to invest in aggregate risk, or only take a positive position, which requires small modifications to the relevant incentive compatibility constraints. We focus on the economically most relevant case, where the agent can always invest his hidden savings in the market in the same way the principal would.} \]
the observable aggregate shock $\tilde{Z}$. After signing the contract the agent can choose a strategy $(\tilde{c}, a, z, \tilde{z})$ to maximize his utility (potentially different from the one recommended by the principal). Given contract $C$, a strategy is feasible if 1) there is a finite utility $U_{\tilde{c},a,z,\tilde{z}}$, and 2) hidden savings $h_t \geq 0$ always. Since the agent can secretly invest in his private technology, we also impose the No-Ponzi condition on him 3) $E^Q \left[ \int_0^{\infty} e^{-rt} (\tilde{c}_t + \alpha z_t h_t) \, dt \right] < \infty$.

A contract $C = (I, k, \tilde{c}, a, z, \tilde{z})$ is admissible if 1) $(\tilde{c}, a, z, \tilde{z})$ is feasible given $C$, and 2) $E^Q \left[ \int_0^{\infty} e^{-rt} (dI_t(R^a) + k_t(R^a)adt + a_t k_t(R^a)dt) \right] < \infty$.

An admissible contract $C = (I, k, \tilde{c}, a, z, \tilde{z})$ is incentive compatible if the agent’s optimal strategy is $(\tilde{c}, a, z, \tilde{z})$ as recommended by the principal. An incentive compatible contract is optimal if it minimizes the principal’s cost

$$v_0 = \min_C J_0(C)$$

$$st : \quad U_{\tilde{c},a,z,\tilde{z}}^0 \geq u_0$$

$$C \in I^C$$

To incorporate aggregate risk into the setting, we need to slightly modify the parameter restrictions. We assume throughout that

$$\frac{\rho - r(1 - \gamma)}{\gamma} - \frac{1 - \gamma}{2} \left( \frac{\pi}{\gamma} \right)^2 > 0$$

$$\alpha < \bar{\alpha} \equiv \frac{\phi \pi \gamma \sqrt{2}}{\sqrt{1 + \gamma}} \sqrt{\frac{\rho - r(1 - \gamma)}{\gamma} - \frac{1 - \gamma}{2} \left( \frac{\pi}{\gamma} \right)^2}$$

No stealing or hidden savings in the optimal contract

Just like in the baseline setting, it is without loss of generality to look only at contracts which induce no stealing, no hidden savings, and no hidden investment.

**Lemma 19.** It is without loss of generality to look only at contracts that induce no stealing $a = 0$, no hidden savings, $h = 0$, and no hidden investment, $z = \tilde{z} = 0$.

**Remark.** This lemma is also valid for the baseline setting without aggregate risk or hidden investment.
Proof. Imagine the principal is offering contract \( C = (I, k, c, a, z, \tilde{z}) \) with associated hidden savings \( h \). Let \( k^h = zh \) and \( \tilde{k}^h = \tilde{z}h \) be the agent’s absolute hidden positions in his private technology and aggregate risk respectively. We will show that we can offer a new contract \( C' = (I', k', dI', 0, 0, 0) \) under which it is optimal for the agent to choose not to steal, no hidden savings, and no hidden investment, i.e. \( \tilde{c}' = dI' \), \( a = z = \tilde{z} = 0 \). The new contract has \( I' = \int_0^t \tilde{c}_i dt \) and \( k' = k(R^a) + k^h \).

If the agent now chooses \( \tilde{c}' = dI' \), \( a = z = \tilde{z} = 0 \), he gets hidden savings \( h' = 0 \) and consumption \( \hat{c} \), so he gets the same utility as under the original contract and this strategy is therefore feasible under the new contract. If instead he chooses a different feasible strategy \( (\tilde{c}', a', z', \tilde{z}') \), he gets the utility associated with \( \tilde{c}' \). We will show that he could achieve this utility under the original contract by picking consumption \( \tilde{c}' \), stealing \( dR - dR^a(R^a) \), hidden investment in private technology \( k^h(R^a) + (\tilde{k}^h)' \), and hidden investment in aggregate risk \( \tilde{k}^h(R^a) + (\tilde{k}^h)' \). Since the strategy \( (\tilde{c}', a', z', \tilde{z}') \) is feasible under the new contract \( C' \), and \( (\tilde{c}, a, z, \tilde{z}) \) feasible under the old original contract, then in order to ensure the new strategy is feasible under the original contract we only need to show that hidden savings remain non-negative always

\[
h'_t = \int_0^t e^{(t-s)} \left( dI_t(R^a(R^a)) - \tilde{c}'_t dt + \phi k_t(R^a(R^a))(dR_t - dR^a_t(R^a)) + (k^h(R^a) + (\tilde{k}^h)'(R^a))(\pi dt + d\tilde{Z}_t) \right)
\]

To show this is always non-negative, we will show it’s greater or equal to the sum of two non-negative terms. First, the hidden savings under the original contract, following the original feasible strategy, had \( R^a' \) been the true return

\[
A_t = \int_0^t e^{(t-s)} \left( dI_t(R^a(R^a)) - \tilde{c}'_t dt + \phi k_t(R^a(R^a))(dR_t - dR^a_t(R^a)) + k^h(R^a)dt + (\tilde{k}^h)'(R^a)(\pi dt + d\tilde{Z}_t) \right) \geq 0
\]

Second, hidden savings under the new contract, following the feasible new strategy

\[
B_t = \int_0^t e^{(t-s)} \left( \tilde{c}_t(R^a)dt - \tilde{c}'_t dt + \phi(k_t(R^a(R^a)) + k^h(R^a))(dR_t - dR^a_t) + (\tilde{k}^h)'(R^a)(\pi dt + d\tilde{Z}_t) \right) \geq 0
\]

If \( \phi = 1 \) then \( h'_t = A_t + B_t \geq 0 \). With \( \phi < 1 \), we have \( h'_t \geq A_t + B_t \geq 0 \), because \( dR_t - dR^a_t = a' dt \geq 0 \) and \( k^h(R^a) \geq 0 \). This means that \( \tilde{c}' = c' \), \( a = z = \tilde{z} = 0 \) is the agent’s optimal choice under the new contract \( C' \), since any other choice delivers an utility that he could have obtained - but chose not to - under the original contract \( C \).

We can now compute the principal’s cost under the new contract

\[
J_0 = E^Q \left[ \int_0^\infty e^{-rt} \left( \tilde{c}_t - \alpha k_t(R^a) + k^h \right) dt + e^{-r\infty} J^t_\infty \right] =
\]

\[
E^Q \left[ \int_0^\infty e^{-rt} (dI_t(R^a) - (\alpha - a_t)k_t(R^a)dt) + e^{-r\infty} J^t_\infty \right] - E^Q \left[ \int_0^\infty e^{-rt} a_t k_t(R^a)(1 - \phi)dt \right]
\]

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\[
-E^Q \left[ \int_0^{\tau_n} e^{-rt} \left( dI_t(R^n) - \tilde{c}_t dt + \phi k_t(R^n) a_t dt + k^h_t \alpha dt \right) \right] + E^Q \left[ e^{-r\tau_n} (J^*_n - J_{\tau_n}) \right]
\]

On the rhs, the first term is the cost under the original contract; the second term the destruction produced by stealing under the original contract, which is non-negative; and the third term is \( E^Q \left[ e^{-r\tau_n} h_{\tau_n} \right] \geq 0 \), where \( h \) is the agent’s hidden savings under the original contract. To see this, write

\[
dh_t = h_t r dt + dI_t(R^n) - \tilde{c}_t dt + \phi k_t(R^n) a_t dt + k^h_t ((\alpha + \pi \tilde{\sigma}) dt + \sigma dZ_t + \tilde{\sigma} d\tilde{Z}_t) + \tilde{k}^h_t (\pi dt + d\tilde{Z}_t)
\]

So

\[
d(e^{-rt} h_t) = e^{-rt} dh_t - re^{-rt} h_t dt
\]

\[
= e^{-rt} \left( dI_t(R^n) - \tilde{c}_t dt + \phi k_t(R^n) a_t dt + k^h_t ((\alpha + \pi \tilde{\sigma}) dt + \sigma dZ_t + \tilde{\sigma} d\tilde{Z}_t) + \tilde{k}^h_t (\pi dt + d\tilde{Z}_t) \right)
\]

Now take expectations under \( Q \), choosing the localizing process appropriately to get

\[
E^Q \left[ \int_0^{\tau_n} d(e^{-rt} h_t) \right] = E^Q \left[ \int_0^{\tau_n} e^{-rt} \left( dI_t(R^n) - \tilde{c}_t dt + \phi k_t(R^n) a_t dt + k^h_t \alpha dt \right) \right] = E^Q \left[ e^{-r\tau_n} h_{\tau_n} - h_0 \right] \geq 0
\]

Given these inequalities, we can write:

\[
J^*_0 - J_0 \leq E^Q \left[ e^{-r\tau_n} (J^*_n - J_{\tau_n}) \right]
\]

Because the original contract was admissible, \( \lim_{n \to \infty} E^Q \left[ e^{-r\tau_n} J^*_n \right] = 0 \). Since in addition the agent’s response was feasible, the new contract is also admissible, and we get \( \lim_{n \to \infty} E^Q \left[ e^{-r\tau_n} J^*_n \right] = 0 \) as well. This shows the new contract is admissible, and the cost for the principal is not greater than under the old contract. This completes the proof.

We can then simplify the contract to \( C = (c, k) \), and say an admissible contract is incentive compatible if the agent’s optimal strategy is \((c, 0, 0, 0)\), or \((c, 0)\) for short.

**Incentive compatibility**

Since the contract can depend on the history of aggregate shocks \( \tilde{Z} \), so can his continuation utility \( U^{c,0} \) and his consumption \( c \). However, because the agent is not responsible for aggregate shocks, incentive compatibility does not place any constraints on his exposure to aggregate risk. On the other hand, since the agent can invest his hidden savings, his Euler equation needs to be modified appropriately. The discounted marginal utility of a hidden dollar must be a supermartingale under any feasible hidden investment strategy, since otherwise the agent could save a dollar instead of consuming it, invest it in aggregate risk and his private technology, and consume it later when the marginal utility is expected to be higher.
Lemma 20. Take the agent’s hidden investment possibility set $H$ as given. If $C = (c,k)$ is an incentive compatible contract, the agent’s continuation utility $U_{c,0}$ and consumption $c$ satisfy the laws of motion

$$dU_{t}^{c,0} = \left( \rho U_{t}^{c,0} - \frac{c_{t}^{1-\gamma}}{1-\gamma} \right) dt + \Delta_{t} \sigma dZ_{t} + \hat{\sigma}_{t}^{u} d\hat{Z}_{t}$$ (55)

$$\frac{dc_{t}}{c_{t}} = \left( \frac{r - \rho}{\gamma} + \frac{1+\gamma}{2} (\sigma_{t}^{c})^{2} + \frac{1+\gamma}{2} (\hat{\sigma}_{t}^{c})^{2} \right) dt + \sigma_{t}^c dZ_{t} + \hat{\sigma}_{t}^c d\hat{Z}_{t} + dL_{t}$$ (56)

for some $\Delta, \hat{\sigma}^u, \sigma^c, \hat{\sigma}^c$, and a weakly increasing processes $L$, such that

$$\Delta_{t} \geq c_{t}^{-\gamma} \phi k_{t}$$ (57)

$$z(\alpha - \sigma_{t}^{c} \sigma \gamma) \leq 0 \quad \forall z \in H$$ (58)

$$\hat{\sigma}^c = \frac{\pi}{\gamma}$$ (59)

Proof. The proof of (55) and (57) are similar to Lemma 4 and 2, where the $d\hat{Z}$ term appears because the contract can depend on the history of aggregate shocks. For (56), the proof is analogous to Lemma 2, but now we need the discounted marginal utility

$$Y_{t} = e^{\int_{0}^{t} (-\rho + z_{s} (\alpha + \pi \hat{\sigma}) + \sigma_{s} z_{s} + \frac{1}{2} (z_{s} \sigma)^{2} - \frac{1}{2} (z_{s} \hat{\sigma} + \hat{\pi} z_{s})^{2} + \int_{0}^{s} (z_{s} \sigma) dZ_{s} + \int_{0}^{s} (z_{s} \hat{\sigma} + \hat{\pi} z_{s}) d\hat{Z}_{s}) ds} c_{t}^{-\gamma}$$ (60)

to be a supermartingale for any investment strategy $\tilde{z}_{t} \in \mathbb{R}$ and $z_{t} \in H$. Using the Doob-Meyer decomposition, the Martingale Representation theorem, and Ito’s lemma, we can write

$$\frac{dc_{t}}{c_{t}} = \mu_{t}^{c} dt + \sigma_{t}^{c} dZ_{t} + \hat{\sigma}_{t}^c d\hat{Z}_{t} + dL_{t}$$

Since the drift of expression (60) must be weakly negative, we get

$$\left( r - \rho - \gamma \mu_{t}^{c} + \frac{1}{2} ((1+\gamma) \sigma_{t}^{c})^{2} + \frac{1}{2} ((1+\gamma) \hat{\sigma}_{t}^{c})^{2} \right) dt + (z(\alpha + \pi \hat{\sigma}) + \pi \tilde{z} - \gamma \sigma_{t}^{c} z \sigma - \gamma \hat{\sigma}_{t}^{c} (\tilde{z} \hat{\sigma} + \hat{\pi} \tilde{z})) dt - \gamma dL_{t} \leq 0$$ (61)

Taking $z = \tilde{z} = 0$, which are always allowed, we obtain the expression for $\mu_{t}^{c}$ in (56), and $L$ weakly increasing. Once we plug this into (61), and using that $\tilde{z}_{t}$ can be both positive or negative, we get (59). Condition (58) is therefore necessary to ensure (61) holds.

The IC constraint (58) depends on whether the agent is allowed to have a hidden investment in his own private technology. If hidden investment in the agent’s private
technology is not allowed, \( H = \{0\} \) so condition (58) drops out. If instead hidden investment in the agent’s private technology is allowed, \( H = \mathbb{R}_+ \), so condition (58) reduces to \( \sigma^r \geq \frac{\alpha}{\sigma_\gamma} \).

**State Space**

We can still use the the state variables \( x \) and \( \hat{c} \). Their laws of motion are

\[
\frac{dx_t}{x_t} = \left( \frac{\rho - \hat{c}_t^{1-\gamma}}{1-\gamma} + \frac{\gamma r}{2} (\sigma_x^r)^2 + \frac{\gamma}{2} (\hat{\sigma}_t^x)^2 \right) dt + \sigma_x^r dZ_t + \tilde{\sigma}_t^x d\tilde{Z}_t \tag{62}
\]

\[
\frac{d\hat{c}_t}{\hat{c}_t} = \left( \frac{r - \rho - \hat{c}_t^{1-\gamma}}{\gamma} + \frac{(\sigma_x^r)^2}{2} + \gamma \sigma_x^r \sigma_t^\hat{c} + \frac{1 + \gamma \sigma_t^\hat{c}^2}{2} \sigma_t^\hat{c} \right) dt + \sigma_t^\hat{c} dZ_t + \tilde{\sigma}_t^\hat{c} d\tilde{Z}_t + dL_t \tag{63}
\]

and the incentive compatibility constraints can be written

\[
\sigma_t^x = \hat{c}_t^{-\gamma} \phi \hat{k}_t \sigma
\]

\[
z(\alpha - (\sigma_t^\hat{c} + \sigma_x^r)\sigma_t^\sigma) \leq 0 \quad \forall z \in H \tag{64}
\]

\[
\hat{\sigma}_t^\hat{c} + \hat{\sigma}_t^x = \frac{\pi}{\gamma} \tag{65}
\]

As before, \( \hat{c} \) has an upper bound \( \hat{c}_h \), which must be modified to take into account that it is not incentive compatible to give the agent a perfectly safe consumption stream.

**Lemma 21.** Take the agent’s hidden investment possibility set \( H \) as given. For any incentive compatible contract \( C \), \( \hat{c}_t \in (0, \hat{c}_h] \) at all times, where \( \hat{c}_h \) is given by

\[
\hat{c}_h \equiv \max_{\sigma^r \geq 0} \left( \frac{\rho - r (1-\gamma)}{\gamma} - \frac{1 - \gamma (\sigma^x)^2}{2} - \frac{1 - \gamma \left( \frac{\pi}{\gamma} \right)^2}{2} \right)^{\frac{1}{1-\gamma}} \tag{67}
\]

\[
st : \quad z(\alpha - \sigma_t^\sigma \sigma_t^\gamma) \leq 0 \quad \forall z \in H
\]

If ever \( \hat{c}_t = \hat{c}_h \), then the continuation contract satisfies \( \hat{c}_{t+s} = \hat{c}_h \) and \( \hat{k}_t = \frac{\sigma^r \hat{c}_h}{\sigma_t^\sigma} \) for all future times \( t + s \), and \( x_t \) follows the law of motion (62), where \( \sigma^r \) is the optimizing choice in (67) and \( \hat{\sigma}^x = \frac{\pi}{\gamma} \). The cost of this continuation contract is \( \hat{v}_h \).
Proof. We know from Lemma 3 that \( \hat{c} \) has an upper bound for any incentive compatible contract. This requires \( \hat{\sigma} = \hat{\sigma} \leq 0, \mu \leq 0 \), and \( dL_t = 0 \) and that point. Using the law of motion of \( \hat{c} \) we get

\[
\frac{r - \rho}{\gamma} - \frac{1 - \gamma}{2} \sigma_x^2 + \frac{1 + \gamma}{2} (\hat{\sigma}^x)^2 - \pi \hat{\sigma}^x \leq 0
\]

where \( \hat{\sigma}^x = \frac{\pi}{\gamma} \). If we choose \( \sigma^x \) to minimize the rhs of this expression, subject to the IC constraints imposed by hidden investment, we can then solve for the largest \( \hat{c} \) that can be attained by an incentive compatible contract. Since these choices are independent of \( \hat{c} \), this is equivalent to the maximization problem in expression (67). It follows that \( \hat{c}_h \) is an absorbing boundary. Using the IC constraint (64) and the law of motion of \( x \) we obtain the desired result. This completes the proof.

\[ \square \]

The upper bound \( \hat{c}_h \) restricts the principal’s ability to promise safety in the future. Even if the agent cannot invest his hidden savings in his private technology, \( H = \{0\} \), he can still invest in aggregate risk. In this case the maximizing choice is \( \sigma^x = 0 \) and we get \( \hat{c}_h = \left( \frac{r - \rho(1 - \gamma)}{\gamma} - \frac{1 - \gamma}{2} \left( \frac{\pi}{\gamma} \right)^2 \right)^{\frac{1}{1 - \gamma}} \). Notice that if \( \pi = 0 \) this boils down to expression (8) in the baseline setting without aggregate risk or hidden investment. If the agent can also invest his hidden savings in his own private technology, \( H = \mathbb{R}_+ \), then the maximizing choice is \( \sigma^x = \frac{\alpha}{\sigma^y} \), and \( \hat{c}_h = \left( \frac{r - \rho(1 - \gamma)}{\gamma} - \frac{1 - \gamma}{2} \left( \frac{\alpha}{\sigma^y} \right)^2 - \frac{1 - \gamma}{2} \left( \frac{\pi}{\gamma} \right)^2 \right)^{\frac{1}{1 - \gamma}} \) is lower.

The HJB equation

The optimal contract can be characterized with the same HJB equation as in the case without hidden investment, appropriately extended to incorporate aggregate risk and the new incentive compatibility constraints.

\[
0 = \min_{\sigma^x, \hat{\sigma}^x} \left( \hat{c} - r\hat{v} - \sigma^x \hat{\sigma}^x \frac{\alpha}{\phi^y} + \hat{v} \left( \frac{r - \rho(1 - \gamma)}{\gamma} + \frac{1 + \gamma}{2} (\sigma^x)^2 + \frac{1 + \gamma}{2} \left( \hat{\sigma}^x \right)^2 - \pi \hat{\sigma}^x \right) + \hat{v}' \hat{c} \right)
\]

subject to \( \sigma^x \geq 0 \) and (65) and (66).
Using (66) to eliminate \( \tilde{\sigma}^\hat{c} \), and taking FOC for \( \tilde{\sigma}^x \), we obtain

\[
\tilde{\sigma}^x = \frac{\pi}{\gamma} \quad \tilde{\sigma} = 0
\]

This is the first best exposure to aggregate risk. The principal and the agent don’t have any conflict about aggregate risk, and the principal cannot use it to relax the moral hazard problem, so they implement the first best aggregate risk sharing.\(^{20}\)

The FOC for \( \sigma^x \) and \( \sigma^\hat{c} \) depend on whether the agent can invest his hidden savings in his private technology. Without hidden investment, the FOCs are the same as in the baseline, (16) and (17). With hidden investment, the IC constraint (65) could be binding in some region of the state space. The shape of the contract, however, is the same as in the baseline without hidden investment.

**Theorem 6.** Take the agent’s hidden investment possibility set \( H \) as given. The principal’s cost function \( \hat{v}(\hat{c}) \) has a flat portion on \([0, \hat{c}_l] \) and a strictly increasing portion on \([\hat{c}_l, \hat{c}_h] \), for some \( \hat{c}_l \in (0, \hat{c}_h) \). The HJB equation (68) holds with equality for \( \hat{c} \geq \hat{c}_l \). For \( \hat{c} < \hat{c}_l \), we have \( \hat{v}(\hat{c}) = \hat{v}(\hat{c}_l) \equiv \hat{v}_l \) and the HJB holds as an inequality

\[
A(\hat{c}, \hat{v}_l) \equiv \min_{\sigma^x} \hat{c} - \sigma^x \hat{c}^\gamma \frac{\alpha}{\phi \sigma} - r \hat{v}_l + \hat{v}_l \left( \frac{\rho - \hat{c}^{1-\gamma}}{1 - \gamma} + \frac{\gamma}{2} (\sigma^x)^2 - \gamma \left( \frac{\pi}{\gamma} \right)^2 \right) \quad \forall \hat{c} < \hat{c}_l. \quad (69)
\]

At \( \hat{c}_l \) the cost function \( \hat{v}(\hat{c}) \) satisfies \( \hat{v}'(\hat{c}_l) = 0 \), \( \hat{v}''(\hat{c}_l) > 0 \), and \( A(\hat{c}_l, \hat{v}_l) = 0 \).

The optimal contract has \( \hat{\sigma}^x_l = \frac{\pi}{\gamma} \) and \( \hat{\sigma}^c_l = 0 \) always. It starts at \( \hat{c}_0 = \hat{c}_l \), where \( \sigma^x_0 \) is chosen without taking into account its effect on the agent’s precautionary saving motive, to maximize

\[
\sigma^x \hat{c}_l^\gamma \frac{\alpha}{\phi \sigma} - \hat{v}(\hat{c}_l) \frac{\gamma}{2} (\sigma^x)^2 \quad (70)
\]

At \( \hat{c}_l \) we have \( \mu^\hat{c}(\hat{c}_l) > 0 \) and \( \sigma^\hat{c}(\hat{c}_l) = 0 \). For all \( t > 0 \), we have \( \hat{c}_t \geq \hat{c}_l \), \( \sigma^c \leq 0 \) and \( \sigma^x \geq 0 \).

**Proof.** The proof is similar to Theorem 1, except we use the more general definition of \( A(\hat{c}, \hat{v}) \)

\[
A(\hat{c}, \hat{v}) = \hat{c} - r \hat{v} - \frac{1}{2} \hat{v} \left( \frac{\phi \sigma \alpha}{\phi \sigma} \right) + \hat{v} \left( \frac{\rho - \hat{c}^{1-\gamma}}{1 - \gamma} - \frac{1}{2} \left( \frac{\pi}{\gamma} \right)^2 \right)
\]

where \( A(\hat{c}, \hat{v}) = 0 \) is the HJB equation if \( \hat{v}' = \hat{v}'' = 0 \) and the IC constraint (65) is not binding. Notice we already know from the FOCs that \( \hat{\sigma}^x = \pi/\gamma \) and \( \hat{\sigma}^c = 0 \).

\(^{20}\)If the agent didn’t have access to hidden investment in aggregate risk, and the agent’s private technology is exposed to aggregate risk \( \hat{\sigma} \neq 0 \), then the principal could potentially use the agent’s exposure to aggregate risk to relax the moral hazard problem.
Part 1 goes through without any change. In part 2, to show that the region where the HJB holds cannot have flat parts, we notice that with $\hat{v}' = \hat{v}'' = 0$, $\sigma^\epsilon$ drops out of the HJB, and can be used to ensure that the IC constraint (65) holds for any choice of $\sigma^\epsilon$. Therefore at a flat part $A(\hat{c}, \hat{v}) = 0$ should hold, which we know cannot be the case.

In part 3, the proof that $\hat{v}'(\hat{c}_1) = 0$ goes through with small modifications, as well as $\hat{v}''(\hat{c}_1 + \epsilon) \geq 0$. Indeed, if at the upper end of a flat region $\hat{c}_2$ we have a kink, with the right derivative $\hat{v}'(\hat{c}_2) > 0$, we must have $\sigma^\epsilon(\hat{c}_2 + \epsilon) \to 0$ as $\epsilon \to 0$, since otherwise we would cross into the flat region where the HJB doesn’t hold. If the drift is strictly positive there, we can start at $\hat{c}_2 - \delta$, with the same $\sigma^\epsilon$ that we were using at $\hat{c}_2$. The IC constraint (65) does not depend on $\hat{c}$ so it is still satisfied. Since this extends the solution and $\hat{v}'(\hat{c}_2) > 0$, we would get a lower cost, which contradicts the flat region immediately below $\hat{c}_2$. The proof for the case with zero drift is unchanged.

With $\hat{v}'(\hat{c}_2) = 0$, we must have $\hat{v}''(\hat{c}_2 + \epsilon) \geq 0$, or else $\hat{v}'(\hat{c} + \epsilon) < 0$ for small $\epsilon$. At $\hat{c}_2$ we must have $A(\hat{c}_2 - \epsilon, \hat{v}(\hat{c}_2)) \geq 0$ because otherwise we could do better by lingering below $\hat{c}_2$ before jumping up to $\hat{c}_2$. We must also have $A(\hat{c}_2, \hat{v}(\hat{c}_2)) \leq 0$, because at $\hat{c}_2$ we have $\hat{v}' = 0$, and $\hat{v}'' \geq 0$, the RHS of the HJB will be at least as large as $A(\hat{c}_2, \hat{v}(\hat{c}_2))$ (it could be strictly larger if the IC constraint (65) is binding). Since $A$ is continuous in $\hat{c}$, we must have $A(\hat{c}_2, \hat{v}(\hat{c}_2)) = 0$. Since this is true in particular with $\hat{c}_1 = 0$ and $\hat{c}_2 = \hat{c}_1$, we have proven $\hat{v}'(\hat{c}_1) = 0$ and $A(\hat{c}_1, \hat{v}_1) = 0$. The rest of part 3 goes through without changes.

Part 4 goes through with natural modifications. In the case $A_2(\hat{c}_1, \hat{v}_1) < 0$, we consider setting $\sigma^\epsilon = 0$, and $\sigma^\alpha = \frac{\alpha}{\sigma^\gamma \hat{c}_1}$. This is consistent with IC constraint (65) because $\hat{c}_1^\alpha \geq \hat{v}_1$ from Lemma $\beta$. We then obtain the first order ODE

$$A(\hat{c}, \hat{v}_{fo}) + \hat{v}_{fo} \hat{c} \left( \frac{r - \rho - \rho - \hat{c}^{1-\gamma}}{\hat{c} - \hat{c}^{1-\gamma}} + \frac{(\alpha \hat{\gamma})^2}{\hat{\gamma}^2} + \frac{1}{2} \frac{\pi^2}{\gamma^2} \right) = 0$$

and the rest of the proof goes through, with the only exception that we use the more general Lemma 30 to establish $\mu^\epsilon > 0$. So we get $\hat{v}''(\hat{c}_1) > 0$.

Part 5 goes through and ignoring the IC constraint (65) we obtain $\sigma^\epsilon(\hat{c}_1) = 0$ and $\sigma^\epsilon(\hat{c}_1) = \frac{\alpha}{\sigma^\gamma \hat{c}_1}$, which maximize (70). These satisfy the IC constraint (65) because $\hat{v}_1 \leq \hat{c}_1^\alpha$. Lemma 30 implies $\mu^\epsilon(\hat{c}_1) > 0$. It only remains to show that $\sigma^\epsilon(\hat{c}_1) \leq 0$ always. We already know this is the case when the IC constraint (65) is not binding. If it is binding, then $\sigma^\epsilon = \frac{\alpha}{\sigma^\gamma} - \sigma^\alpha$, and the FOC for $\sigma^\alpha$ yields:

$$\sigma^\alpha = \frac{\hat{c}^{1-\gamma} \sigma + \hat{v}'' \sigma^2 \sigma}{\hat{v}'(\gamma - \hat{v}'' \hat{c}) + \hat{v}'' \hat{c}^2} \geq \frac{\alpha}{\sigma^\gamma}$$

which implies $\sigma^\epsilon \leq 0$. To see this inequality, multiply both sides by the denominator, which must be positive (second order condition) to get

$$\hat{c}^{1-\gamma} \frac{\alpha \sigma}{\sigma^{\gamma}} + \hat{v}'' \sigma^2 \frac{\sigma}{\sigma^{\gamma}} \geq \frac{\alpha}{\sigma^\gamma} (\gamma - \hat{v}'' \hat{c}) + \hat{v}'' \hat{c}^2$$

$$\hat{c}^{1-\gamma} \frac{\alpha \sigma}{\sigma^{\gamma}} \geq \frac{\alpha}{\sigma} (\gamma - \hat{v}'' \hat{c})$$

$$\hat{c}^{1-\gamma} \frac{\alpha \sigma}{\sigma^{\gamma}} \geq \frac{\alpha}{\sigma} (\gamma - \hat{v}'' \hat{c})$$

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where the next to last inequality uses $\hat{v}' \geq 0$, and the last inequality uses $\hat{v} \leq \hat{v}'$. We conclude that $\sigma_t \leq 0$ always. The rest of the proof is unchanged.

We also have a verification theorem for the HJB equation.

**Theorem 7.** Take the agent’s hidden investment possibility set $H$ as given. Let $\hat{v}(\hat{c}) : [\hat{c}_l, \hat{c}_h] \to [\hat{v}_l, \hat{v}_h]$ be a strictly increasing $C^2$ solution to the HJB equation (68) for some $\hat{c}_l \in (0, \hat{c}_h)$, such that $\hat{v}_l \equiv \hat{v}(\hat{c}_l) \in (0, \hat{v}_h]$, $\hat{v}'(\hat{c}_l) = 0$, $\hat{v}''(\hat{c}_l) > 0$ and $\hat{v}(\hat{c}_h) = \hat{v}_h$. Assume also that $\hat{v}(\hat{c}) \leq \hat{c}^\gamma$ for $\hat{c} \in [\hat{c}_l, \hat{c}_h]$, and, if $\gamma < \frac{1}{2}$ that

$$1 - \hat{v}_l \left( \hat{c}_l^{-\gamma} + \hat{c}_l^{2\gamma-1} \alpha^2 (\rho \sigma)^{-2} \hat{v}_l^{-2} \right) \leq 0$$

Then,

1) For any incentive compatible contract $C = (c, k)$ that delivers at least utility $u_0$ to the agent, we have $\hat{v}(\hat{c}_l)((1 - \gamma)u_0)^{\frac{1}{1-\gamma}} \leq J_0(C)$.

2) Let $C^*$ be a contract generated by the policy functions of the HJB. Specifically, the state variables $x^*$ and $\check{c}^*$ are solutions to (62) and (63) (with potential absorption at $\hat{c}_h$), with initial values $x_0^* = ((1 - \gamma)u_0)^{\frac{1}{1-\gamma}}$ and $\check{c}_0^* = \check{c}_l$. If $C^*$ is admissible and $\sigma_{\check{c}^*}$ bounded, then $C^*$ is an optimal contract, with cost $J_0(C^*) = \hat{v}(\hat{c}_l)((1 - \gamma)u_0)^{\frac{1}{1-\gamma}}$.

**Proof.** The proof is very similar to Theorem 2, but we use the definition

$$A(\hat{c}, \hat{v}) = \check{c} - r\hat{v} - \frac{1}{2} \frac{\hat{v}_{c}\sigma}{\hat{v}_c} + \hat{v} \left( \rho - \hat{c}^{1-\gamma} - \frac{1}{1-\gamma} - \frac{\pi^2}{2\gamma} \right)$$

and use it to show that the HJB holds as an inequality below $\hat{c}_l$. Notice that the optimal policies in the HJB imply $\hat{\sigma} = \pi/\gamma$ and $\hat{\check{c}} = 0$ throughout. With $\hat{v}'(\hat{c}_l) = 0$ and $\hat{v}''(\hat{c}_l) > 0$, the IC constraint (65) is not binding at $\hat{c}_l$, because $\hat{v}(\hat{c}_l) \leq \check{c}$, so we get $\sigma_x(\hat{c}_l) = \frac{\pi^2}{2\gamma \sigma \hat{c}_l}$ and $\sigma_{\check{c}}(\hat{c}_l) = 0$ and therefore $A(\hat{c}_l, \hat{v}_l) = 0$. Lemma 30 then shows that $\mu'(\hat{c}_l) > 0$. We also want to show that $\sigma_{\check{c}} \geq 0$ and $\sigma_x \leq 0$ for all $\check{c} \in [\hat{c}_l, \hat{c}_h]$. When know this is true when the IC constraint (65) is not binding. Using $\hat{v}(\hat{c}) \leq \check{c}$ we can show this is also the case if the constraint is binding, as in Part 5 of Theorem 6. We can then use Theorem 4 to establish the global incentive compatibility of the candidate optimal contract. The rest of the proof is unchanged.

The following result is useful to verify admissibility.

**Lemma 22.** If the candidate contract $C^*$ constructed in Theorem 7 has $\mu_{x^*} < r + \pi^2/\gamma$, then $C^*$ is admissible and delivers utility $u_0$ to the agent.
Proof. We know that $\hat{c}_t^* \in [\hat{c}_l, \hat{c}_h]$ and recall that $\hat{c}_h > 0$. Then an upper bounded $\mu^x < r + \pi^2 / \gamma$ implies a bounded $0 \leq \sigma^x \leq \bar{\sigma}_X$, and we also know that $\bar{\sigma} = \pi / \gamma$. Then

$$\mathbb{E}_Q \left[ \int_0^\infty e^{-rt} (|c^*_t| + |k^*_t|) dt \right] \leq 2 \max \left\{ \hat{c}_h, \frac{\bar{\sigma}_X \hat{c}_1^*}{\phi \sigma} \right\} \mathbb{E}_Q \left[ \int_0^\infty e^{-rt} x^*_t dt \right] < \infty$$

where the last inequality follows from $\mu^x < r + \pi^2 / \gamma$ (notice the expectations is taken under $Q$). Let $U^* = \left( x^* \right)^{1-\gamma}$, so using the law of motion of $x^*$, (9), we get

$$U^*_0 = \mathbb{E} \left[ \int_0^{\tau_n} e^{-rt} \frac{c^*_t^{1-\gamma}}{1-\gamma} dt + e^{-\rho \tau_n} U^*_{\tau_n} \right]$$

with $\tau_n \to \infty$ a.s. Use the monotone convergence theorem and notice that

$$\lim_{n \to \infty} \mathbb{E} \left[ e^{-\rho \tau_n} U^*_{\tau_n} \right] = 0$$

because $\rho - (1 - \gamma)(\mu^x - \frac{2}{\gamma}(\sigma^x)^2) - \frac{2}{\gamma}(\frac{\pi}{\gamma})^2 = c^{1-\gamma} \geq \min\{c_l^{1-\gamma}, c_h^{1-\gamma}\} > 0$. We then get that $U^*_0 = U^*_0 = u_0$. We conclude that the contract is indeed admissible.

\[ \square \]

Verifying global incentive compatibility

We can extend Theorem 3 to verify global incentive compatibility.

**Theorem 4.** Let $C = (c, k)$ be an admissible contract with associated processes $x$ and $\hat{c}$ satisfying (38) and (39), and (40), (41), and in the case of hidden investment (42), with bounded $\mu^x, \mu^\hat{c}, \sigma^\hat{c},$ and $\bar{\sigma}^\hat{c}$, and with $\hat{c}$ uniformly bounded away for zero and bounded above by $\hat{c}_h$. Suppose that the contract satisfies the following property

$$\sigma^\hat{c}_t \leq 0$$

Then for any feasible strategy $(\hat{c}, a, z, \hat{z})$, with associated hidden savings $h$, we have the following upper bound on the agent’s utility, after any history

$$U_t^\hat{c}, a, z, \hat{z} \leq \left( 1 + \frac{h_t}{x_t} \hat{c}_t^{1-\gamma} \right)^{1-\gamma} U_t^c, 0$$

In particular, since $h_0 = 0$, for any feasible strategy $U_t^\hat{c}, a, z, \hat{z} \leq U_t^c, 0$, and the contract $C$ is therefore incentive compatible.

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Proof. As in the proof of Lemma 3, we get

$$e^{-pt} \left( U^{c,a,z,\hat{z}} - F(\hat{h}_t, \hat{c}_t)U^{c,0}_t \right) = \mathbb{E}^p_t \left[ \int_t^{\tau_n} e^{-r_s}(1-\gamma)U^{c,0}_uY_u \, du + e^{-r_{\tau_n}} \left( U^{c,a,z,\hat{z}} - F_{c,a}U^{c,0}_\tau \right) \right]$$

where $Y_t = A_t + B_t + \hat{B}_t + C_t$ must be modified to 1) incorporate hidden investment in term $B_t$, and 2) incorporate aggregate risk in term $\hat{B}_t$. Terms $A_t$ and $C_t$ are not changed, and we already know they are non-positive from the proof of Theorem 3. We only need to show that $B_t + \hat{B}_t \leq 0$ as well.

Start with $B_t$. Because the agent can now invest his hidden savings, we have the following expression:

$$B_t = \frac{F_{c,t}}{\sigma_t} \hat{c}_t \left( \frac{1}{2} \gamma (\sigma_t^2 + \sigma_t^\alpha)^2 - \gamma \frac{1}{2} (\sigma_t^2)^2 - \sigma_t^\alpha \right) + \frac{F_{c,t}}{\sigma_t} \frac{1}{(1-\gamma)} \gamma (\sigma_t^2)^2 - \sigma_t^\alpha \frac{1}{\gamma} \hat{c}_t$$

$$+ \frac{F_{c,t}}{\sigma_t} \frac{1}{(1-\gamma)} \hat{h}_t \left( z_t \gamma (\sigma_t^2 - \sigma_t^\alpha \gamma) + \frac{F_{c,t}}{\sigma_t} \frac{1}{(1-\gamma)} \gamma (\sigma_t^2 - \sigma_t^\alpha \gamma) \hat{h}_t \right)$$

Notice that when the agent invests in his private technology, we get an expected return $z_t(\alpha + \pi \sigma)$, but we have only included the term $z_t \gamma$ in the expression for $B_t$. We will include the premium for aggregate risk $\pi \sigma$ in the term $B_t$, which takes an analogous form:

$$\hat{B}_t = \frac{F_{c,t}}{\sigma_t} \hat{c}_t \left( \frac{1}{2} \gamma (\sigma_t^2 + \sigma_t^\alpha)^2 - \gamma \frac{1}{2} (\sigma_t^2)^2 - \sigma_t^\alpha \right) + \frac{F_{c,t}}{\sigma_t} \frac{1}{(1-\gamma)} \gamma (\sigma_t^2)^2 - \sigma_t^\alpha \frac{1}{\gamma} \hat{c}_t$$

$$+ \frac{F_{c,t}}{\sigma_t} \frac{1}{(1-\gamma)} \hat{h}_t \left( z_t \gamma (\sigma_t^2 - \sigma_t^\alpha \gamma) + \frac{F_{c,t}}{\sigma_t} \frac{1}{(1-\gamma)} \gamma (\sigma_t^2 - \sigma_t^\alpha \gamma) \hat{h}_t \right)$$

The two terms $B_t$ and $\hat{B}_t$ are very similar. The only difference is that instead of $z_t \sigma$ - the exposure to idiosyncratic risk $Z$ induced by hidden investment in the private technology - we have $z_t \gamma + \hat{z}_t$ - the exposure to aggregate risk from investment in both private technology and aggregate risk; and instead of $z_t \sigma^\alpha$ we have $(z_t \gamma + \hat{z}_t) \gamma$. In addition, all the volatilities are with respect to aggregate risk $\hat{Z}$ instead of $Z$.

Now let’s re-write these terms in a more convenient form. Take $\hat{B}_t$ and simplify to obtain:

$$B_t = \frac{F_{c,t}}{\sigma_t} \hat{c}_t \left( \frac{1}{2} \gamma (\sigma_t^2 + \sigma_t^\alpha)^2 - \gamma \frac{1}{2} (\sigma_t^2)^2 - \sigma_t^\alpha \gamma \right) + \frac{F_{c,t}}{\sigma_t} \frac{1}{(1-\gamma)} \hat{h}_t \left( z_t \gamma (\sigma_t^2 - \sigma_t^\alpha \gamma) \right)$$

$$+ \frac{F_{c,t}}{\sigma_t} \frac{1}{(1-\gamma)} \hat{h}_t \left( z_t \gamma (\sigma_t^2 - \sigma_t^\alpha \gamma) \right)$$

We know from the proof of Theorem 3 that with $z_t = 0$ this will be

$$B_{0,t} = -\frac{(\sigma_t^2 + \sigma_t^\alpha)^2}{2} \gamma \hat{h}_t^2 \hat{c}_t^{1-\gamma} \left( 1 + \hat{h}_t \hat{c}_t^{-\gamma} \right)^{-\gamma-1} \hat{c}_t^{-\gamma-1} \leq 0$$

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So let’s gather all the terms with $z_t$ into $B_{1,t}$ so that $B_t = B_{0,t} + B_{1,t}$

$$B_{1,t} = \frac{F_{h,t}^2}{(1-\gamma)} \dot{h}_t (z_t \alpha - \gamma \sigma_t^x z_t \sigma_t + \frac{2\sigma_t^x z_t \sigma_t \dot{h}_t F_{\dot{h},t} + ((z_t \sigma_t)^2 - 2z_t \sigma_t \gamma) \dot{h}_t^2 F_{\dot{h},t}}{2(1-\gamma)}$$

Now plug in the formula for $F_{\dot{h},t}$, $F_{\dot{h},t}$, and $F_{\dot{h},t}$ to obtain:

$$B_{1,t} = \dot{c}_t^{-\gamma} \left( 1 + \dot{h}_t \dot{c}_t^{-\gamma} \right)^{-\gamma} \dot{h}_t z_t (\alpha - \gamma \sigma_t \dot{c}_t)$$

$$+ \frac{1}{2} \left( 2\sigma_t^x z_t \sigma_t \dot{c}_t \dot{h}_t \left( -\gamma \dot{c}_t^{-\gamma-1} \left( 1 + \dot{h}_t \dot{c}_t^{-\gamma} \right)^{-\gamma} + \gamma^2 \dot{h}_t \dot{c}_t^{-2\gamma-1} \left( 1 + \dot{h}_t \dot{c}_t^{-\gamma} \right)^{-\gamma-1} \right) \right)$$

$$- \frac{1}{2} \left( (z_t \sigma_t)^2 - 2z_t \sigma_t \gamma \dot{h}_t^2 \gamma \dot{c}_t^{-2\gamma} \left( 1 + \dot{h}_t \dot{c}_t^{-\gamma} \right)^{-\gamma-1} \right)$$

Reorganize to get:

$$B_{1,t} = \dot{c}_t^{-\gamma} \dot{h}_t z_t \left( \alpha - \gamma \sigma_t (\sigma_t^x + \sigma_t^x) \right) \left( 1 + \dot{h}_t \dot{c}_t^{-\gamma} \right)^{-\gamma}$$

$$+ \frac{1}{2} \left( 2\sigma_t^x z_t \sigma_t \dot{c}_t \dot{h}_t \gamma \dot{h}_t \dot{c}_t^{-2\gamma} - ((z_t \sigma_t)^2 - 2z_t \sigma_t \gamma) \dot{h}_t^2 \gamma \dot{c}_t^{-2\gamma} \left( 1 + \dot{h}_t \dot{c}_t^{-\gamma} \right)^{-\gamma-1} \right)$$

$$= \dot{c}_t^{-\gamma} \dot{h}_t z_t \left( \frac{\alpha}{\sigma} - \gamma (\sigma_t^x + \sigma_t^x) \right) \left( 1 + \dot{h}_t \dot{c}_t^{-\gamma} \right)^{-\gamma-1} \frac{1}{2} \left( \sigma_t^x + \sigma_t^x \gamma - z_t \sigma \right)^2$$

The second term is always positive, so we only need to concern ourselves with the first one. Performing the same algebraic steps on $B_t$, and taking advantage of the structural similarities, we obtain

$$\dot{B}_t = \dot{c}_t^{-\gamma} \dot{h}_t (z_t \dot{\sigma} + \dot{z}_t) \left( \pi - \gamma (\dot{\sigma}_t^x + \dot{\sigma}_t^x) \right) \left( 1 + \dot{h}_t \dot{c}_t^{-\gamma} \right)^{-\gamma}$$

$$- \dot{h}_t^2 \gamma \dot{c}_t^{-2\gamma} \left( 1 + \dot{h}_t \dot{c}_t^{-\gamma} \right)^{-\gamma-1} \frac{1}{2} \left( \dot{\sigma}_t^x + \dot{\sigma}_t^x \gamma - (z_t \dot{\sigma} + \dot{z}_t) \right)^2$$

where the second term is also positive. Finally, we can add the two first terms

$$\dot{c}_t^{-\gamma} \dot{h}_t \left( 1 + \dot{h}_t \dot{c}_t^{-\gamma} \right)^{-\gamma} \left[ z_t \left( \alpha - \gamma \sigma_t (\sigma_t^x + \sigma_t^x) \right) + (z_t \dot{\sigma} + \dot{z}_t) \left( \pi - \gamma (\dot{\sigma}_t^x + \dot{\sigma}_t^x) \right) \right]$$

$$= \dot{c}_t^{-\gamma} \dot{h}_t \left( 1 + \dot{h}_t \dot{c}_t^{-\gamma} \right)^{-\gamma} \left[ z_t (\alpha + \pi \dot{\sigma}) + \pi \dot{z}_t - \gamma (\dot{\sigma}_t^x + \dot{\sigma}_t^x) z_t \sigma - \gamma (\dot{\sigma}_t^x + \dot{\sigma}_t^x) (z_t \dot{\sigma} + \dot{z}_t) \right] \leq 0$$

where the last inequality follows from the incentive compatibility conditions (65) and (66). Notice this argument works for any contract where the discounted marginal utility of consumption is a supermartingale under any valid trading strategy. The rest of the proof dealing with the terminal term follows the same
Steps as in Theorem 3.

**Implementation**

The implementation as a portfolio problem is still valid, but now the agent also had exposure \( \tilde{\sigma}^n \) to aggregate risk \( \tilde{Z} \)

\[
\frac{dn_t}{n_t} = \left( r + \lambda_t \alpha / \tilde{\phi}_t + \tilde{\sigma}_t^n \pi - \theta_t \right) dt + \lambda_t \sigma_t dZ_t + \tilde{\sigma}_t^n d\tilde{Z}.
\]

(71)

The capital structure \( S = (\lambda, \tilde{\phi}, \theta, \tilde{\sigma}^n) \) with associated process for \( n \) that solves (71) implements the contract \( C = (c, k) \) if \( c_t = \theta_t \times n_t \) and \( k_t = \lambda_t (n_t + e_t) = \lambda_t / \tilde{\phi}_t \times n_t \). Expressions (23), (24), and (25) are still valid, and we have in addition \( \tilde{\sigma}_n = \pi / \gamma \).

**Lemma 23.** The optimal contract has a unique implementation in the family of standard capital structures, with inside equity share \( \tilde{\phi}_t = \frac{n_t}{n_t + e_t} \) given by (23), leverage \( \lambda_t = \frac{k_t}{n_t + e_t} \) given by (24), a payout policy \( \theta_t = \frac{\alpha}{\tilde{\phi}_t} \) given by (25), exposure to aggregate risk \( \tilde{\sigma}^n = \pi / \gamma \) and assets given by (21). The marginal value of inside equity is larger than the marginal value of hidden savings, \( \hat{v}(\hat{c})^{-1} > \hat{c}^{-\gamma} \), and the inside equity share \( \tilde{\phi}(\hat{c}) < \phi \) for all \( \hat{c} \in [\hat{c}_l, \hat{c}_h] \).

**Proof.** The proof is the same as in Lemma 5 and 6, but using the HJB equation (68), and with the addition that \( \tilde{\sigma}^n = \hat{\sigma} = \pi / \gamma \).

Optimal portfolio with \( \tilde{\phi} = \phi \). The optimal portfolio with \( \tilde{\phi} = \phi \) can be adapted to the presence of aggregate risk.

\[
\lambda_p = \frac{\alpha}{\gamma \phi \sigma^2}, \quad \theta_p = \frac{\rho - r(1 - \gamma)}{\gamma} - \frac{1 - \gamma}{2} \left( \frac{\alpha}{\phi \sigma \gamma} \right)^2 - \frac{1 - \gamma}{2} \left( \frac{\pi}{\gamma} \right) \quad \tilde{\sigma}^n = \pi / \gamma \quad \text{and} \quad \hat{v}_p = \theta_p \hat{\gamma}^\gamma.
\]

The resulting contract \( C_p = (c_p, k_p) \) corresponds to the black dot in Figures 5, 6. It has a constant \( \hat{c}_p = \theta_p \hat{\gamma}^\gamma \) and \( \hat{v}_p = \theta_p \hat{\gamma}^\gamma \), and is therefore incentive compatible by Theorem 4.

**Lemma 24.** The contract implemented with a constant equity constraint \( C_p \) is an incentive compatible contract.
Proof. The proof follows the same lines as that of Lemma 7, but uses Lemma 25 to establish the incentive compatibility of the resulting contract.

Stationary contracts  Set \( \mu^\hat{c} = \sigma^\hat{c} = \tilde{\sigma}^\hat{c} = 0 \). To do this we need to set

\[
\sigma^x = \sigma^x_r(\hat{c}) \equiv \sqrt{2} \sqrt{\frac{\rho - \hat{c}^{-1} - \gamma}{1-\gamma} + \frac{\rho - r}{\gamma} - \frac{1}{2} \left( \frac{\pi}{\gamma} \right)^2},
\]  

(72)

and \( \tilde{\sigma}^x = \pi/\gamma \) which implies \( \tilde{\sigma}^\hat{c} = 0 \). We can find a stationary contract for any \( \hat{c} \in (\hat{c}_s, \hat{c}_h] \), where

\[
\hat{c}_s = \left( \frac{2\gamma}{1 + \gamma} \right)^{\frac{1}{1-\gamma}} \left( \frac{\rho - r (1 - \gamma)}{\gamma} - \frac{1 - \gamma}{2} \left( \frac{\pi}{\gamma} \right)^2 \right)^{\frac{1}{1-\gamma}} \in (0, \hat{c}_h)
\]

Lemma 25. For any \( \hat{c} \in (\hat{c}_s, \hat{c}_h] \), the corresponding stationary contract with \( \sigma^\hat{c} = 0 \) and \( \sigma^x = \sigma^x_r(\hat{c}) \) given by (72) is globally incentive compatible and has cost \( \hat{v}_r(\hat{c}) x_0 \), where

\[
\hat{v}_r(\hat{c}) = \frac{\hat{c} - \frac{\alpha}{\delta \sigma} \hat{c}^\gamma \sigma^x_r(\hat{c})}{2r - \rho - (1 + \gamma) \frac{\rho - \hat{c}^{-1} - \gamma}{1-\gamma} + \gamma (\pi/\gamma)^2}. 
\]  

(73)

Since stationary contracts are incentive compatible, we have \( \hat{v}(\hat{c}) \leq \hat{v}_r(\hat{c}) \).

Proof. The proof is analogous to that of Lemma 8, using the HJB (68). First, using \( \alpha < \bar{\alpha} \), we can verify that \( 0 \leq \hat{c}^* \leq \hat{c}_h \), regardless of whether the agent can invest in his hidden savings. Second, \( \hat{v}_r(\hat{c}) > 0 \) for all \( \hat{c} \in (\hat{c}_s, \hat{c}_h) \) from Lemma 29. The same argument as in Theorem 2 shows that \( \hat{v}_r(\hat{c}) \) from (73) is the cost corresponding to the stationary contract with \( \hat{c} \) and \( \sigma^x \) given by (72), as long as the contract is indeed admissible and delivers utility \( u_0 \) to the agent. We can check that \( \mu^x < r + \frac{\pi^2}{\gamma} \) for the stationary contract if and only if \( \hat{c} > \hat{c}_* \). In this case, since \( \mu^x < r + \frac{\pi^2}{\gamma} \) arguing as in the proof of Lemma 22 we can show that the stationary contract is admissible and delivers utility \( u_0 \) to the agent if and only if \( \hat{c} > \hat{c}_* \). Since the contract satisfies (62), (63), and (64), and (66) by construction, we only need to check that (65) holds too. It’s easy to see this is the case because \( \hat{c} \leq \hat{c}_h \). Lemma 4 then ensures that it is incentive compatible. This completes the proof.

Lemma 26. Any stationary contract can be implemented with a constant capital structure with

\[
\tilde{\phi}_r(\hat{c}) = \frac{\hat{v}_r(\hat{c})}{\hat{c}^\gamma}, \quad \lambda_r(\hat{c}) = \frac{\sigma^x_r(\hat{c})}{\sigma} \quad \theta_r(\hat{c}) = \frac{\hat{c}}{\hat{v}_r(\hat{c})} \quad \text{and} \quad \tilde{\sigma}^n(\hat{c}) = \pi/\gamma
\]  

(74)

81
The contract implemented with a constant equity constraint \( C_p \) is the stationary contract corresponding to \( \hat{c}_p \), but is not optimal. The best stationary contract \( C_{p}^{min} \) is less risky for the agent, i.e. we have \( \hat{c}_p < c_p < C_{p}^{min} \). For all \( \hat{c} \in (\hat{c}_p, \hat{c}_h) \) the marginal value of equity is larger than the marginal value of hidden savings, \( \hat{v}_r^{-1}(\hat{c}) > \hat{c}^{-\gamma} \), and the inside equity share \( \tilde{\phi}_r(\hat{c}) < \phi \).

Proof. The capital structure in (74) is a special case of the implementation in (23), (24), (25), and \( \tilde{\sigma}^n = \pi/\gamma \). The same argument as in Lemma 23 works here. We already know that the contract implemented with a constant equity stake \( \phi, C_p \) is stationary. Lemma 17 ensures that \( \hat{c}_p \in [\hat{c}_*, \hat{c}_h] \) and Lemma 25 that it is incentive compatible.

The best stationary contract \( C_{p}^{min} \) has \( \hat{c}_p^{min} > \hat{c}_p > \hat{c}_* \) from part 1) of Lemma 17. So \( C_p \) is not the best stationary contract. Part 2) of Lemma 17 shows that \( \hat{v}_r^{-1}(\hat{c}) > \hat{c}^{-\gamma} \) for all \( \hat{c} \in (\hat{c}_p, \hat{c}_h) \), with equality at \( \hat{c}_p \) and \( \left( \frac{\pi - r(1 - \gamma)}{\gamma} - \frac{1 - \gamma}{\gamma} \left( \frac{\pi}{\pi} \right)^2 \right)^{-\frac{1}{\gamma}} > \hat{c}_h \). The expression for \( \tilde{\phi}_r(\hat{c}) \) in (74) then shows that \( \tilde{\phi}_r(\hat{c}) < \phi \) in this region.

\[ \tilde{\phi}_r(\hat{c}) = \phi \left( \frac{\hat{v}(\hat{c})}{\hat{c}} + \frac{\hat{v}'(\hat{c})}{\hat{c}} \times \frac{\hat{c}}{\hat{c}} \right) \leq \tilde{\phi}_r(\hat{c}). \] (75)

For any given \( \hat{c} \in [\hat{c}_*, \hat{c}_h] \), we have \( \hat{v}(\hat{c}) \leq \hat{v}_r(\hat{c}) \), because stationary contracts are incentive compatible and always available as continuation contracts. Furthermore, \( \hat{v}'(\hat{c}) \geq 0 \) and \( \hat{v}_r(\hat{c}) \leq 0 \). Using the definition of \( \tilde{\phi}_r(\hat{c}) \) in (23) and \( \tilde{\phi}_r(\hat{c}) \) in (74), we obtain \( \tilde{\phi}_r(\hat{c}) \leq \phi_r(\hat{c}) \).

Lemma 28. If the agent has access to hidden investment, \( H = \mathbb{R}_+ \) and \( \phi = 1 \), the optimal contract is the optimal portfolio with an equity constraint \( \phi = \phi, C_p \).

Proof. \( C_p \) is both admissible and incentive compatible by lemma 24. Since \( \hat{c}_h = \hat{c}_p = \left( \frac{\pi - r(1 - \gamma)}{\gamma} - \frac{1 - \gamma}{\gamma} \left( \frac{\pi}{\pi} \right)^2 \right), \) we can use the same verification argument as in Theorem 7, using the flat value function \( \hat{v}(\hat{c}) = \hat{v}_p \) for all \( \hat{c} \in (0, \hat{c}_h) \). For the argument to go through, it must be the case that the HJB holds as an inequality for all \( \hat{c} \leq \hat{c}_p \):

\[ A(\hat{c}, \hat{c}_p) = \hat{c} - r\hat{v}_p - \frac{1}{2} \left( \frac{\hat{v}'(\hat{c})}{\hat{v}_p} \right) + \hat{v}_p \left( \frac{\hat{c} - \hat{c}_1^{-\gamma}}{1 - \gamma} - 1 \right)^2 > 0 \]

This is true because \( \hat{v}_p = \hat{c}_p^2 \), and from lemma 17 we know that \( \partial_r A(\hat{c}_p, \hat{c}_p) < 0 \). From Lemma 13 we know that \( A(\hat{c}, \hat{v}) \) is positive near 0 and either has one root in \( \hat{c} \) if \( \gamma \geq 1/2 \), or is convex with at most two roots if \( \hat{c} \leq 1/2 \). This means that \( A(\hat{c}, \hat{c}_p^2) > 0 \) for all \( \hat{c} \in (0, \hat{c}_p) \).

\[ \square \]
Intermediate results

**Lemma 29.** The cost function of stationary contracts \( \hat{v}_r(\hat{c}) \) defined by (73) is strictly positive for all \( \hat{c} \in (\hat{c}_s, \hat{c}_h) \) if and only if \( \alpha < \bar{\alpha} \).

**Proof.** We need to check the numerator in (73), since the denominator is positive for all \( \hat{c} \geq \hat{c}_s \):

\[
\hat{c} \left( 1 - \frac{\alpha}{\phi \sigma} \sqrt{2} \sqrt{1 + \gamma \frac{1}{2\gamma} \left( \frac{\rho - r(1 - \gamma)}{\gamma} - \frac{1 - \gamma}{2} \left( \frac{\pi}{\gamma} \right)^2 \right)} \right)^{-1}
\]

The rest of the proof consists of evaluating this expression at \( \hat{c} = \hat{c}_s \) and showing it is non-positive if the bound is violated, since the expression is increasing in \( \hat{c} \). We get \( \hat{c} \times \left( 1 - \frac{\alpha}{\phi \sigma} \sqrt{2} \right)^{-1} \)

So if \( \alpha \geq \bar{\alpha} \) the numerator is non-positive, and if \( \alpha < \bar{\alpha} \) then it’s strictly positive. This completes the proof.

\[\square\]

**Lemma 30.** Let \( \hat{c}_l \in (0, \hat{c}_h) \) and \( \hat{v}_l \leq \hat{v}_p \). If \( \sigma^\hat{c} = \tilde{\sigma}^\hat{c} = 0 \), \( \sigma^x = \frac{\alpha \hat{c}^\gamma}{\hat{c} \hat{v}_p \phi} \), and \( \tilde{\sigma}^x = \pi/\gamma \), and \( A(\hat{c}_l, \hat{v}_l) = 0 \), where

\[
A(\hat{c}, \hat{v}) \equiv \hat{c} - r\hat{v} - \frac{1}{2} \left( \frac{\hat{c}^\gamma \alpha}{\hat{v} \phi \sigma} \right)^2 + \hat{v} \left( \frac{\rho - \hat{c}^{1-\gamma}}{1 - \gamma} - \frac{1}{2} \frac{\pi^2}{\gamma} \right)
\]

then

\[
\mu^\hat{c} = \frac{r - \rho}{\gamma} - \frac{\rho - \hat{c}^{1-\gamma}}{1 - \gamma} + \frac{1}{2} (\sigma^x)^2 + \frac{1}{2} \left( \frac{\pi}{\gamma} \right)^2 > 0
\]

**Proof.** Looking at (63), with \( \sigma^\hat{c} = \tilde{\sigma}^\hat{c} = 0 \) we get for the drift

\[
\mu^\hat{c} = \frac{r - \rho}{\gamma} + \frac{1}{2} (\sigma^x)^2 + \frac{1}{2} (\tilde{\sigma}^x)^2 - \frac{\rho - \hat{c}^{1-\gamma}}{1 - \gamma}
\]

So \( \mu^\hat{c} > 0 \) implies

\[
\frac{1}{2} (\sigma^x)^2 + \frac{1}{2} (\tilde{\sigma}^x)^2 > \frac{r - \rho}{\gamma} + \frac{\rho - \hat{c}^{1-\gamma}}{1 - \gamma}
\]

Since we also want \( A(\hat{c}_l, \hat{v}_l) = 0 \), we get

\[
0 = \hat{c} - r\hat{v} + \hat{v} \left( \frac{\rho - \hat{c}^{1-\gamma}}{1 - \gamma} - \frac{\gamma}{2} (\sigma^x)^2 - \frac{\gamma}{2} \left( \frac{\pi}{\gamma} \right)^2 \right)
\]

\[
< \hat{c} - \hat{v} \hat{c}^{1-\gamma} \equiv M
\]
Notice that if \( \hat{v} = \hat{c}^\gamma \) we have \( M = 0 \). If \( \hat{v} > \hat{c}^\gamma \) we have \( M < 0 \) and if \( \hat{v} < \hat{c}^\gamma \) we have \( M > 0 \). So for \( A(\hat{c}; \hat{v}) = 0 \) and \( \mu^c > 0 \) we need \( \hat{v} < \hat{c}^\gamma \). In fact, if \( \hat{v} = \hat{c}^\gamma \) and in addition

\[
\frac{1}{2} \left( \frac{\alpha}{\phi \sigma} \right)^2 + \frac{1}{2} \left( \frac{\pi}{\gamma} \right)^2 = \frac{\rho - \hat{c}_1^{1-\gamma}}{1 - \gamma} + \frac{\rho - r}{\gamma}
\]

(76)

then we have \( A = 0 \) and \( \mu^c = 0 \). In this case, because we have \( \mu^c = 0 \) we therefore have the value of a stationary contract, i.e. \( \hat{v} = \hat{v}_v(\hat{c}) \) given by (73). This point corresponds to the optimal portfolio with \( \hat{c} = \hat{c}_v, (\hat{c}_v, \hat{v}_v) \). We know from Lemma 17 that \( \hat{c}_p \in [\hat{c}_v, \hat{c}_b] \). By assumption, \( \hat{v}_1 \leq \hat{v}_p \).

First we will show that \( \mu^c \geq 0 \), and then make the inequality strict. Towards contradiction, suppose \( \mu^c < 0 \) at \( \hat{c}_1 \). Then it must be the case that \( \hat{v}_1 > \hat{c}^\gamma_1 \) because we have \( A(\hat{c}_1, \hat{v}_1) = 0 \). We will show that \( A(\hat{c}_1, \hat{v}_1) > 0 \) and get a contradiction. First take the derivative of \( A \):

\[
A'_c(\hat{c}_1, \hat{v}_1) = 1 - \hat{v}_1 \left( \hat{c}_1^{1-\gamma} + \hat{c}_1^{2\gamma-1} \left( \frac{\alpha}{\phi \sigma} \right)^2 \frac{1}{\hat{c}_1^2} \right) < 0
\]

where the inequality holds for all \( \hat{c} < \hat{c}_1^{1/2} \). So \( A(\hat{c}_1, \hat{v}_1) > A(\hat{c}_1^{1/2}, \hat{v}_1) \). Letting \( \hat{c}_m = \hat{c}_1^{1/2} \) we get

\[
A(\hat{c}_1, \hat{v}_1) \geq \hat{c}_m - \hat{v}_1 \left( \hat{c}_1^{1-\gamma} + \hat{c}_1^{2\gamma-1} \left( \frac{\alpha}{\phi \sigma} \right)^2 \frac{1}{\hat{c}_1^2} \right) = \hat{c}_m - \hat{v}_1 \left( \hat{c}_1^{1-\gamma} + \hat{c}_1^{2\gamma-1} \left( \frac{\alpha}{\phi \sigma} \right)^2 \frac{1}{\hat{c}_1^2} \right)
\]

\[
= \hat{c}_m - \hat{v}_1 \left( \hat{c}_1^{1-\gamma} + \hat{c}_1^{2\gamma-1} \left( \frac{\alpha}{\phi \sigma} \right)^2 \frac{1}{\hat{c}_1^2} \right)
\]

(77)

where the last equality uses \( \hat{v}_1 = \hat{c}_m \), and the last inequality uses \( \hat{c}_m = \hat{v}_1^{1/2} \leq \hat{c}_p^{1/2} = \hat{c}_p \). This is a contradiction, and therefore it must be the case that \( \mu^c \geq 0 \) at \( \hat{c}_1 \).

It’s clear from the previous argument that \( \mu^c(\hat{c}_1) = 0 \) only if \( (\hat{c}_1, \hat{v}_1) = (\hat{c}_p, \hat{v}_p) \). We will show this cannot be the case because \( \alpha > 0 \). First, note that \( (\hat{c}_p, \hat{v}_p) \) is a tangency point where \( \hat{v}_v(\hat{c}) \) touches the locus \( \hat{v}_b(\hat{c}) \) defined by \( A(\hat{c}; \hat{v}_b(\hat{c})) = 0 \). If \( (\hat{c}_1, \hat{v}_1) = (\hat{c}_p, \hat{v}_p) \) then this must be the minimum point for \( \hat{v}_v(\hat{c}) \), so the derivative of both \( \hat{v}_v(\hat{c}) \) and \( \hat{v}_b(\hat{c}) \) must be zero. This means that \( A'_c(\hat{c}_1, \hat{v}_1) = 0 \). However,

\[
1 - \hat{v}_1 \left( \hat{c}_1^{1-\gamma} + \hat{c}_1^{2\gamma-1} \left( \frac{\alpha}{\phi \sigma} \right)^2 \frac{1}{\hat{c}_1^2} \right) < 0
\]

where the inequality follows from \( \hat{v}_1 = \hat{v}_p = \hat{c}_p^{1/2} \) (note that \( \hat{c}_1 > 0 \) because as Lemma 13 shows \( A(\hat{c}, \hat{v}) \) is strictly positive for \( \hat{c} \) near 0). This can’t be a minimum of \( \hat{v}_v(\hat{c}) \). Therefore \( (\hat{c}_1, \hat{v}_1) \neq (\hat{c}_p, \hat{v}_p) \) and \( \mu^c(\hat{c}_1) > 0 \). This completes the proof.

\[\square\]