Manipulation, Panic Runs, and the Short Selling Ban*

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Abstract

This paper identifies conditions under which a short selling ban improves the ex-ante firm value. Short selling improves price discovery and enables stakeholders to make better investment decisions. However, manipulative short selling can arise as a self-fulfilling equilibrium, resulting in inefficient investment decisions. The adverse effect is amplified by the firm’s vulnerability to panic runs. Overall, short selling reduces ex-ante firm value if the firm is very vulnerable to runs and the speculator’s information quality is not too good. Our results contribute to understanding of the function of short selling in the capital markets and to the controversy around the regulation of short selling.

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1 Introduction

This paper presents a model to evaluate the efficiency of banning short sales. Ever since the first regulation against short selling was enacted by the Amsterdam stock exchange in 1610, such regulations have been controversial (e.g., Bris et al. (2007)). A salient feature of restrictions on short selling is that they are often imposed on financial stocks and during crises. Historically, short selling bans were imposed after crashes, such as the Dutch stock market crash in the seventeenth century and the bursting of the South Sea Bubble in the eighteenth century. More recently, in September 2008, the SEC banned short selling of shares of 799 companies for two weeks, and the United Kingdom and Japan declared a ban on short selling for “as long as it takes” to stabilize the markets. Similarly, in August 2011, France, Spain, Italy, and Belgium imposed temporary bans on short selling for some financial stocks during the European sovereign debt crisis.

Proponents of short selling make a straightforward argument. Like other selling and buying, short selling allows investors to express their negative opinions through trading and improves the informativeness of stock prices. The price discovery, in turn, leads to better investment decisions and more efficient allocation of capital in the economy. In contrast, opponents have argued that short sellers can manipulate the market through “bear raids.” Speculators with initial short positions may employ various tactics, including creating panics, to drive down the share prices to close their initial short positions at a lower cost.

Goldstein and Guembel (2008) (hereafter referred to as GG) present a rational model in which manipulative short selling (MSS) can arise as a self-fulfilling equilibrium when the financial market feeds back to real decisions that affect firm value. An uninformed speculator can establish a short position and push down the stock price. Since the firm cannot determine whether the short sales come from informed or uninformed speculators, the firm interprets the lower price as a bad signal and may cancel its project, resulting in lower firm value and profitable short selling for the uninformed speculator. Therefore, GG justifies the concern about MSS and potential restrictions on short selling.

The rationale for banning short selling, however, remains open to debate. While MSS
reduces the firm value, banning short selling is always detrimental to firm value in GG. The firm is a single decision-maker in GG and thus can choose to ignore the information in the stock price when making decisions. A revealed preference argument proves that firm value is higher with short selling, despite the possibility of MSS. In other words, MSS is always secondary to informed short selling. The reason that MSS arises in equilibrium is exactly because short selling by the informed speculator is sufficiently informative. Banning short selling in GG is thus equivalent to throwing out the proverbial baby with the bath water.

In search of a possible rationale for banning short sales of financial stocks during crises, we relax the assumption of a single decision-maker in GG by augmenting their model with a coordination game among the firm’s stakeholders, whose decisions collectively affect the firm value. It is well known that financial institutions are subject to panic runs caused by coordination failure. Panic runs are typically driven by bad news, which can include panic created by short sellers. We find that banning short selling improves the ex-ante efficiency if the firm’s vulnerability to runs is sufficiently high and the speculator’s information quality is at the medium level (i.e., not high). Financial firms are more vulnerable to runs, whereas crises are often associated with a high degree of uncertainty in the market. Our result therefore is consistent with the anecdotal evidence that short selling bans are usually imposed during crises and on financial firms.

To see why high vulnerability to runs is necessary, note that the GG setting is actually a benchmark of ours, where the investment decision is made by a representative investor and thus the firm is not vulnerable to runs. As discussed above, the uniform short selling ban always reduces the ex-ante efficiency, despite the equilibrium existence of MSS, as the existence of MSS requires the probability of informed speculative trading to be high. High vulnerability to runs makes the firm more likely to cancel investment, even when the probability of informed trading is not high, and thus makes short selling more likely to reduce efficiency.

The medium level of the speculator’s information quality is also necessary. To see this, consider two extremes. At one extreme, if the market is populated mainly by informed speculators, then MSS arises but very infrequently. The informational benefit from frequent
informed short selling dominates the cost of infrequent MSS. As a result, the short selling ban strictly reduces the amount of information conveyed through prices and thus efficiency. At the other extreme, if the market is mainly populated by uninformed speculators, then based on the intuition of GG, MSS does not arise in equilibrium because price is very uninformative. As a result, the short selling ban removes the informed short selling and thus reduces the efficiency only when the probability that the speculator is present is intermediate or equivalently, the speculator’s information quality is at the medium level.

We now illustrate the general intuition behind our main result. As discussed above, in GG, when there is no coordination friction, banning short selling never reduces efficiency, as the reduction in efficiency of the informed speculator’s trading dominates the increase in efficiency of uninformed speculator’s MSS. The reason is that, for MSS to be an equilibrium, the probability of informed trading has to be sufficiently high. In a coordination game, the investors’ investment decisions are driven not only by information about the fundamental, as in GG, but also by the investors’ concerns about others’ actions. Specifically, when all other investors want to run on the firm, it is optimal to run, even when the fundamental is not too low, resulting in coordination failure and thus efficiency loss. When the concerns about others’ actions are sufficiently large, investors will mainly use stock prices to infer others’ actions. It is therefore easier for the uninformed speculator to “create panic” by short selling, since investors are more likely to jointly run on the firm, even when the probability of informed trading is not too high. The reason is that the probability of informed trading only affects investors’ assessment of fundamentals. When the probability of the speculator being present is not sufficiently high (i.e., the speculator’s information is not sufficiently good in expectation), the efficiency loss from uninformed speculators’ MSS dominates, which explains why a uniform short selling ban improves efficiency.

Our result is consistent with the anecdotal evidence on the effect of short selling ban on financial firms’ stocks during the financial crisis. It is well known that the significant mismatch of assets and liabilities for financial institutions results in them being vulnerable to panic-based runs that are caused by coordination failure, which has been both theoretically micro-founded (e.g., Diamond and Dybvig (1983), Goldstein and Pauzner (2005), Morris
and empirically documented (e.g., Chen et al. (2010), Gorton and Metrick (2012)). It is also well known that uncertainty is the highest during crises and after major shocks (e.g., Bloom (2009), Bloom et al. (2018)). We show that the interaction between MSS and panic-based runs generates an adverse effect on firms’ investment decisions so severe that a ban on short selling improves efficiency, in particular when there is a lot of uncertainty and when firms are more vulnerable to panic-based runs, i.e., financial firms during crisis periods.

Our paper relates to the literature on short selling. Short selling is a basic component in modern finance theories of asset pricing and portfolio choice. Most theoretical studies thus far have viewed short sales as an institutional constraint and focused on identifying the consequences (e.g., Miller (1977), Diamond and Verrecchia (1987), Duffie et al. (2002), Abreu and Brunnermeier (2003), Scheinkman and Xiong (2003)). In most of these studies, banning short sales has an adverse effect on efficiency.

A few papers have studied the ex-post consequences of short sales in the presence of rigid frictions. The study that most closely relates to ours is by GG. We build on the MSS equilibrium in GG and extend their work to model a coordination decision-making game. This extension generates our main result that banning short selling can improve efficiency, while in GG the ban cannot improve efficiency, despite removing the MSS. Brunnermeier and Oehmke (2013) show that short selling forces firms with a market-based leverage requirement to liquidate the illiquid assets. In their model, there is no informational feedback from the stock price to real decisions. The effect of the stock price on the liquidation decision is assumed. Liu (2014) also studies a coordination game with short selling. In his model, investors in the coordination game receive private information and observe the stock price as public information. Short selling is assumed to add noise into the stock price and makes the public information noisier. In contrast, short selling in our model allows the speculator’s private information to be endogenously impounded into the stock price and makes the price more informative. Liu (2016) studies the interaction between an investors’ coordination game with the interbank market trading and focuses on the feedback loop between interbank market rate and the coordination game. The interbank market serves as a provider of liquidity and banks do not learn any information from the interbank market. In contrast, we focus on the
interaction between managers learning from prices and the coordination game.

Our study also makes a methodological contribution to the literature on coordination games with market manipulation. We use the global games methodology to obtain the unique equilibrium of the coordination game to conduct welfare analysis. However, the market manipulation component from GG employs two rounds of trading and is too complicated to be combined with the coordination game. We use a one-round trading setting to simplify the market manipulation component and integrate it with the coordination game. This formulation of market manipulation may be used in other settings.

Finally, our paper relates to the literature on the welfare effects of public information in coordination games (Angeletos and Pavan (2007)). Morris and Shin (2002) is probably the first to show that, in settings with coordination motives, more precise public information may decrease welfare when private information is sufficiently precise. In our setting, there is no private information per se, so even with coordination, more public information should increase welfare. The detrimental effect of more public information comes from the interaction of coordination with feedback effect.

The rest of the paper is organized as follows. Section 2 introduces the model setup. Section 3 derives the equilibrium strategies and pins down the unique equilibrium of the coordination game, using the global games technique. Section 4 presents the main results, specifically the result that short selling could be detrimental to firm value. Section 5 discusses implications, and Section 6 concludes.

2 Model setup

Our model adds a coordination game to the MSS setting of GG. Consider a risk-neutral economy with no discounting and four dates \((t = 0, 1, 2, 3)\), one firm with an underlying project, and a continuum of investors. At \(t = 0\), a speculator may learn about the underlying state, denoted by \(\theta\), that would (partially) determine terminal firm cash flows. At \(t = 1\), the speculator can choose to trade in a financial market. At \(t = 2\), investors play a coordination game after observing the outcome from trading. At \(t = 3\), all uncertainties are realized. We
now discuss the trading and the coordination game in more detail.

The underlying state $\theta$ is either high or low with equal probability; i.e., $\theta \in \{H, L\}$ with $\Pr(\theta = H) = \frac{1}{2}$. Without loss of generality, we normalize $H \equiv 1$. We also assume that $L < 0$; i.e., in a first-best world, the firm should invest when $\theta = H$ and not invest when $\theta = L$. The only information source for investors is the stock price endogenously determined in a Kyle setting. Specifically, there are three types of traders. The first is a speculator, who learns perfectly about $\theta$ with probability $\alpha \in (0, 1]$ and nothing with the complementary probability; that is, the speculator observes a signal $s \in \{H, L, 0\}$. We call a speculator with $s = H(L)$ a positively (negatively) informed speculator and a speculator with no information ($s = 0$) an uninformed speculator. The speculator chooses an order $d(s) \in \{-1, 0, 1\}$. The second group of traders are liquidity traders, who trade for reasons orthogonal to state $\theta$. Their aggregate order is denoted as $\bar{n}$, which is normally distributed with mean zero and variance $\sigma_n^2$. Finally, the third group of traders is the market maker, who observes the total order flow $q = d + \bar{n}$ and sets price equal to the expected firm value, taking into account that the investors may learn from the stock price:

$$P = E[v|q].$$

The stock price and the order flow have identical information content. For simplicity, we assume that investors observe the order flow (instead of the stock price).

A key feature of GG is that the firm value $v$ is endogenous to the firm’s decision at $t = 2$ after observing the stock price. We maintain this feedback effect in our model but depart from GG by replacing a single decision-maker with a coordination game. Specifically, there is now a continuum $[0, 1]$ of investors. At $t = 2$, after observing a signal to be described below, each investor makes a binary investment decision; i.e., $a_i \in \{0, 1\}$ where $a_i = 1(0)$ means that investor $i$ invests (does not invest). If investor $i$ does not invest, i.e., $a_i = 0$, she receives a payoff normalized to 0. If she invests, then her payoff $u$ is jointly determined by the state $\theta$.
and the aggregate non-investing population \( l \equiv \int_{i \in [0,1]} (1 - a_i) di \):

\[
u = \theta - \delta l.
\]

The parameter \( \delta \geq 0 \) captures the degree of strategic complementarity among investors’ investment decisions. \( \delta \) is often referred to as the project’s vulnerability to runs. The project’s aggregate value is

\[
v(\theta, \{a_i\}_{i \in [0,1]}) = (1 - l) (\theta - \delta l).
\]  

The setting in GG is a special case of our model with \( \delta = 0 \).

We make a few assumptions before proceeding. For convenience, we denote

\[
\gamma \equiv -L.
\]

As is typical in the feedback literature, \( \gamma \) is a measure of the strength of the informational feedback effect. A higher \( \gamma \) indicates that the investment decision is more sensitive to information, as the downside loss of a bad project is closer to the upside gain of a good project, making information more valuable in deciding whether to invest.

We assume that

\[ A1 : \overline{\gamma} < \gamma < 1 - \delta. \]

\( A1 \) is a straightforward adaptation from GG to accommodate the coordination component we have introduced into the model. The assumption guarantees that the feedback effect is not trivial. Specifically, \( \gamma < 1 - \delta \) guarantees that, in the absence of any information, the default choice is to invest. It enhances GG’s assumption to accommodate the coordination game represented by \( \delta \). Moreover, the feedback effect has to be sufficiently strong so that the investors’ decisions can be influenced by the stock price, even in the absence of the

\[ A1 : \overline{\gamma} < \gamma < 1 - \delta. \]

Another difference between our setting and that of GG is the assumption that noise trading is normally distributed rather than discretely distributed. As will be discussed below, this assumption allows us to solve the unique equilibrium with only one round of trading, instead of two rounds of trading as in GG, resulting in tractable analysis of welfare when we introduce the coordination friction into the feedback model.
coordination friction. This is guaranteed by the first part of A1 that $\gamma$ cannot be too small. $\gamma$ is defined by equation (19) in the appendix.

In addition, we introduce two tier-breakers. First, if the negatively (positively) informed speculator is indifferent among $d \in \{-1, 0, 1\}$, he always chooses $d = -1$ ($d = 1$). This rules out a degenerate equilibrium where no trading takes place. Second, if the uninformed speculator is indifferent among $d \in \{-1, 0, 1\}$, he chooses not to trade. This assumption biases against finding equilibria where the uninformed speculator trades.

The efficiency is defined as the expected firm value that aggregates the payoffs to all investors:

$$V \equiv E[(1 - l)(\theta - \delta l)].$$

A perfect Bayesian equilibrium (PBE) of our model consists of the speculator’s trading strategy $d(s)$, each investor’s withdrawal strategy $a_i(q)$, and beliefs about the fundamental $\theta$ such that (1) both the speculator and the investors maximize their respective objective functions, given their beliefs and the strategies of others and (2) each investor uses Bayes’ Rule, if possible, to update beliefs about $\theta$.

In sum, the timeline of the events is as follows.

At $t = 0$, the speculator’s information endowment $s$ is realized.

At $t = 1$, the trading occurs. Both the stock price and the order flow are observed.

At $t = 2$, investors observe the order flow $q$ and make decisions.

At $t = 3$, the firm’s terminal cash flows are realized.

### 3 The equilibrium

#### 3.1 The first-best benchmark

Before proceeding, we first solve for the first-best benchmark. Since there are both coordination friction and informational friction in our setting, the first-best benchmark consists of a single investor (i.e., no coordination friction) who knows $\theta$ (i.e., no informational friction) and chooses investment at date 1 to maximize $V$ as in equation (2). Equivalently, she chooses
\( l \in [0, 1] \) to maximize

\[
E[(1 - l)(\theta - \delta l)|l, \theta] = (1 - l)(\theta - \delta l),
\]

with the optimal solution in Lemma 1.

**Lemma 1**

\[
\hat{l}^{FB} = \begin{cases} 
0 & \text{if } \theta = H \\
1 & \text{if } \theta = L
\end{cases},
\]

resulting in \( V^{FB} = \frac{1}{2} \).

Lemma 1 is intuitive. Since the investor knows about the state and \( H > 0 > L \), she will continue investing when \( \theta = H \) and withdraw all of her investment when \( \theta = L \). The first-best firm value therefore is the probability of \( \theta = H \) times the firm value when \( \theta = H \); i.e., \( \frac{1}{2} H \equiv \frac{1}{2} \).

We now solve for the equilibrium in the general case using backward induction.

### 3.2 The coordination subgame

We first solve for the subgame after the investors have observed the order flow \( q \). We conjecture and later verify that \( q \) satisfies the monotone likelihood ratio property (MLRP), that is, a higher \( q \) indicates that \( \theta = H \) is more likely. Intuitively, a positively informed speculator always chooses \( d = 1 \) and a negatively informed speculator always chooses \( d = -1 \). Given the conjectures, regardless of the uninformed speculator’s choice of \( d \), a higher order flow indicates that it is more likely that the order flow comes from a positively informed speculator and therefore there is a higher probability of \( \theta = H \).

Now consider the strategy of investor \( i \) when observing the order flow \( q \). If she chooses to withdraw, then she gets 0 for sure. If she chooses to continue, then her expected payoff differential is

\[
\Delta(q) = E[\theta|q] - \delta E[l|q].
\]

As is standard in the coordination game, multiple equilibria arise if \( q \) is common knowledge, making it difficult to conduct comparative statics. We apply the global games method-
ology to obtain the unique equilibrium. Specifically, we assume that each investor receives a noisy signal $q_i$ and focus on the equilibrium when the noise converges to 0:

$$q_i = q + \epsilon_i,$$

where $\epsilon_i \sim N(0, \sigma_{\epsilon}^2)$ reflects the idiosyncratic noise in each investor’s observation of the order flow $q$. Equivalently, $\epsilon_i$ can be interpreted as the individual specific difference of investors’ interpretation of the information context of the order flow $q$. Both interpretations generate the same results. As is standard in the global games literature (e.g., Morris and Shin (2000), Goldstein and Pauzner (2005), Bouvard et al. (2015) and Gao and Jiang (2018)), we will focus on the limiting case as $\sigma_{\epsilon}$ approaches 0. This results in a unique equilibrium of the coordination game.

**Lemma 2** The investors play a common threshold strategy. The common threshold $q^*$ is determined by the following equation.

$$E[\theta|q^*] - \frac{\delta}{2} = 0. \quad (3)$$

Since $\theta$ is binary, the expectation in equation (3), $E[\theta|q^*]$, is fully characterized by the conditional probability $\beta(q^*_j) \equiv \Pr(\theta = H|q^*_j)$. A higher $\beta$ means that the marginal investor has to be more optimistic about the state to invest. Lemma 2 characterizes the equilibrium common threshold in an intuitive manner. An investor’s signal $q_i$ has two roles: it helps her to forecast the state $\theta$ and other investors’ actions that collectively determine $l$. The first role is summarized in the component $E[\theta|q^*]$, the marginal investor’s expectation of the fundamental.

Investors also use $q_i$ to forecast other investors’ signals and actions. At $q_i = q^*$, she conjectures that exactly half of the other investors will get a signal higher than $q^*$ and stay, whereas the other half will get a signal lower than $q^*$ and withdraw. Therefore she expects that half of the investors will stay: $E[l|q^*] = \frac{1}{2}$.

Collecting these two expectations and imposing the equilibrium condition that the mar-
ginal investor has to be indifferent between continuing and running at the threshold \( q^* \), i.e., \( \Delta(q^*) = 0 \), we obtain equation (3), which uniquely determines the common threshold \( q^* \).

Using Bayes’ rule to express \( E[\theta|q^*] \) in terms of \( \beta(q^*) \) in equation (3) results in

\[
\beta(q^*) = \frac{\delta + \gamma}{1 + \gamma}.
\]  

(4)

Note that \( \beta(q^*) \) is an equilibrium variable and depends on the trading strategy of the speculator, to which we turn now.

### 3.3 The trading strategy of the speculator

Anticipating the unique equilibrium for the subgame of coordination, the speculator chooses his trading strategy, which then determines the price and thus investors’ investment strategies. The trading strategy and the investors’ investment decisions are then jointly determined by solving a fixed point problem.

We start with the less complicated case in which short selling is banned. The next proposition characterizes the speculator’s trading strategy. Note that we add a subscript to \( q^* \) as \( q^* \) clearly depends on whether short selling is banned, with \( q^*_B \) denoting the regime where short sales are banned and \( q^*_A \) denoting the regime where short sales are allowed.

**Proposition 1** Suppose that short selling is banned. Then the positively informed speculator buys while other speculators do not trade; i.e., \( d^*(H) = 1 \) and \( d^*(L) = d^*(\emptyset) = 0 \). The investment threshold is \( q^*_B \), which is defined in equation (12) in the appendix. \( q^*_B < \frac{1}{2} \) and is increasing in \( \alpha \) and \( \delta \).

Proposition 1 is intuitive. The positively informed speculator finds it optimal to buy, since his information will only be partially reflected in prices, resulting in a lower expected buying price and a positive expected profit. The negatively informed speculator wants to short sell but cannot because of the ban. He clearly finds it not optimal to buy and therefore chooses not to trade. Similarly, the uninformed speculator finds it not optimal to buy from Lemma 6 and therefore chooses not to trade.
We now consider the case where short selling is allowed. We focus on the situation in which short selling is the dominant strategy for the uninformed speculator. This equilibrium selection criterion removes all MSS whenever MSS is not the unique equilibrium. Therefore our analysis presents a conservative estimation of the benefit of short selling ban.

**Proposition 2** Suppose that short selling is allowed, then

1. the informed speculator trades consistent with his information: i.e., \( d^*(H) = 1 \) and \( d^*(L) = -1 \);

2. For the uninformed speculator, \( d(\emptyset) = -1 \) (i.e., short selling) is the uniquely optimal strategy if \( \alpha > \alpha(\delta) \), where \( \alpha(\delta) \) is defined in equation (14) in the appendix;

3. the investment threshold is \( q_A^* \) when \( d(\emptyset) = -1 \), defined in equation (16) in the appendix. \( q_A^* < 0 \) and is increasing in \( \alpha \) and \( \delta \).

Note that, setting \( \delta = 0 \), Proposition 2 replicates the main result of GG that MSS arises when the probability that the speculator is informed is sufficiently high. We extend this result to our setting of a coordination game in which \( \delta > 0 \).

The first part of Proposition 2 says that the informed speculator trades in the direction of his information, which is intuitive. Specifically, as in the Kyle model, the informed speculator can make a profit because his order is partially camouflaged by noise trading.

The second part of Proposition 2 shows that the uninformed speculator will optimally choose to short in the equilibrium if the informed speculator’s information quality is sufficiently large. This result shows the existence of MSS in our model and is similar to the main message of GG. In fact, as in GG, MSS is mainly driven by the feedback effect in our model.

To see this, first consider a benchmark case in which there is no feedback effect. In this case, the market maker interprets the lower order flow as a bad signal about the state and lowers the price accordingly. Since the market maker cannot distinguish whether the short selling originates from the negatively informed speculator or the uninformed speculator, the price is set as a weighted average, conditional on the speculator being negatively informed.

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The footnotes:

1. We can prove a stronger result that \( d(\emptyset) = -1 \) cannot be a unique equilibrium if \( \alpha \leq \alpha(\delta) \), but we choose not to state it since it is not used in subsequent analysis.
or uninformed. Thus short selling is profitable for the informed speculator but not for the uninformed speculator. Specifically, we can write the expected profit of MSS in the absence of feedback as

$$\Pi(\alpha) \propto \int_0^{+\infty} (\beta(q) - \frac{1}{2})dF(q) + \int_0^{0} (\beta(q) - \frac{1}{2})dF(q),$$

where $F$ is the cumulative distribution function of $q$ and $f$ is the density function. Since there is no feedback, the investment will always be carried out as the project has an ex ante positive NPV. When $q > 0$, the order flow is so high that the market maker believes that $\beta(q) > \frac{1}{2}$ and the uninformed speculator will be making money, whereas the opposite occurs when $q < 0$. We illustrate this case in the left panel of Figure 1. The solid line depicts trading profit/losses of the uninformed speculator as a function of $q$ (i.e., $\beta(q) - \frac{1}{2}$), whereas the dashed line depicts the probability of each realization of $q$ (i.e., $f(q)$). The expected trading profit, $\Pi(\alpha)$, is therefore the integration of the product of the value of the solid line and that of the dashed line. When there is no feedback effect, short selling shifts the distribution of the order flow (i.e. $f(q)$) to the left, increasing the loss region and shrinking the profit region. Therefore, in the absence of the feedback effect, the uninformed speculator will generate a negative expected profit from short selling, and short selling is therefore not optimal for him.

However, short selling also affects investment decisions through the feedback effect. Investors use the order flow to make inferences about the state and about other investors’ decisions. A lower order flow is a bad signal about the state and thus discourages investors from investing. The reduction in investment indeed reduces the firm’s terminal cash flow and creates a self-fulfilling equilibrium. Both the informed and uninformed speculator can profit from this informational feedback effect. Specifically, in the presence of the feedback effect, the project will not be carried out when $q < q^\ast$ as $\beta(q)$ becomes so small that every investor will withdraw. This implies that the price will be zero and thus the uninformed speculator makes zero profit rather than suffering a loss when $q < q^\ast$, which is the case in the absence
of feedback; i.e.,

$$\Pi(\alpha) \propto \int_0^{+\infty} (\beta(q) - \frac{1}{2})dF(q) + \int_{q^*}^0 (\beta(q) - \frac{1}{2})dF(q).$$

We illustrate this case in the right panel of Figure 1. Again the solid line depicts the trading profit/losses of the uninformed speculator as a function of $q$ (i.e., $\beta(q) - \frac{1}{2}$ or zero), whereas the dashed line depicts the probability of each realization of $q$ (i.e., $f(q)$). The feedback effect terminates the investment when $q$ is sufficiently small, resulting in profit/loss being zero. Since sufficiently small $q$ generates a loss in the absence of the feedback effect, the feedback effect reduces the loss region and thus increases the expected payoff from short selling for speculators. Figure 2 compares the expected trading profit among different values of $\alpha$. As shown in the figure, larger $\alpha$ results in more of the loss region replaced by a zero profit region and increases the expected trading profit of the uninformed speculator.\(^3\) Intuitively, higher $\alpha$ increases the informativeness of negative order flow and reduces the loss region even more. Therefore, while the negatively informed speculator always shorts, the uninformed speculator shorts when the profit from the informational feedback effect compensates for the loss from his informational disadvantage, a condition satisfied when the fraction of informed speculator is sufficiently large.

4 The main analysis

Having characterized the equilibrium when short selling is allowed and when it is banned, we are ready to present our main result about the efficiency of banning short selling.

We have defined the efficiency as the ex-ante expected firm value $V$ in equation (2). After some algebra, for a given regime $j \in \{A, B\}$, we can write it generally as

$$V = V^{FB} - \frac{1}{2}(\varepsilon^H + \gamma\varepsilon^L).$$

\(^3\)Note that we can also see from the figure that high $\alpha$ results in a higher trading loss for moderately negative values of $q$ and higher trading profit for positive values of $q$. The overall effect is to increase expected trading profit for the uninformed speculator.
Figure 1: Profit of the uninformed speculator as a function of order flow and probability of the realization of the order flow when choosing $d = -1$. The left panel shows the case in the absence of feedback, whereas the right panel shows the case in the presence of feedback effect.

Note that $V^{FB} = \frac{1}{2}$ is the efficiency in the first-best case, as stated in Lemma 1. Relative to the first-best, the efficiency is ultimately reduced by two types of errors in the investment decisions, under-investment in the good state and over-investment in the bad state, with the respective probabilities denoted as $\varepsilon^H$ and $\varepsilon^L$. Underinvestment reduces 1 unit of efficiency by foregoing the payoff $H$ in the good state, whereas overinvestment generates a loss of $L = -\gamma$ when the state is bad.

The ban affects the efficiency through its effect on the speculator’s trading strategy, the information content of the order flow, and eventually the investors’ investment decisions. Section 4.1 analyzes how the ban affects the efficiency through each effect, and Section 4.2 presents our main results regarding when banning short selling improves efficiency. Finally, Section 4.3 illustrates our main results using a numerical example.
Figure 2: Profit of the uninformed speculator as a function of order flow with different values of $\alpha$. 
4.1 The effects of the short selling ban on trading strategies and investment decisions

First, the ban affects the speculator’s trading strategy and investors’ investment decisions. It does not affect the buy order from the positively informed speculator but replaces the sell order from both the negatively informed speculator and (possibly) the uninformed speculator with no trading. Accordingly, the ban does not affect the order flow distribution when the speculator is positively informed but shifts the distribution to the right by one in other cases. Rationally anticipating the consequences of the short selling ban on the information content of the stock price, the investors adjust their investment decisions accordingly. The order flow threshold, if increasing, will increase less than 1 as the investors are not sure whether the banned short selling comes from the informed or the uninformed speculator.

Lemma 3 also shows that, counterintuitively, banning short selling may not result in an increase in the order flow threshold; i.e., $q_B^*$ is larger if and only if $\alpha$ is sufficiently large. To understand this, note that banning short selling has two effects. First, it pushes the distribution of the order flow of the negatively (and possibly uninformed) speculator to the right (the “rightward shifting effect”), therefore increasing $q_B^*$. Second, such pushing of the distribution to the right also decreases the relative informativeness of order flow (the “informativeness decrease effect”) in inferring the speculator’s information, since the distributions of the order flow of positively and negatively informed speculators are now closer. Such a decrease of the informativeness of the order flow makes the order flow a noisier signal and decreases $q_B^*$. The reason is that the default action in the absence of any information is to stay, resulting in noisier order flow and making investors more likely to stay. When $\alpha$ is sufficiently large, the order flow is sufficiently informative that the second effect is dominated by the first effect, resulting in a larger $q_B^*$.

**Lemma 3** The short selling ban changes the investment threshold as follows: $q_B^* < q_A^* + 1$. We also have $q_B^* > q_A^*$ if and only if $\alpha > \alpha_2$, where $\alpha_2$ is defined in equation (17) in the

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4 A possible reason for $q_B^*$ to be smaller is that we take the speculator’s order flow as exogenous. In principle, the speculator would trade more aggressively when $\sigma_n^2$ is larger (Kyle (1985)), which may result in $q_B^*$ always being larger. We leave this interesting question for future research.
Table 1: Investment errors in various scenarios when $\alpha > \alpha$

<table>
<thead>
<tr>
<th>$j$</th>
<th>$A$</th>
<th>$B$</th>
<th>$B - A$</th>
<th>$B - A$</th>
<th>Prob*cost</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\varepsilon^H_{Ij}$</td>
<td>$\Phi\left(\frac{q_A^* - 1}{\sigma_n}\right)$</td>
<td>$\Phi\left(\frac{q_B^* - 1}{\sigma_n}\right)$</td>
<td>$\Phi\left(\frac{q_B^* - 1}{\sigma_n}\right) - \Phi\left(\frac{q_A^* - 1}{\sigma_n}\right)$</td>
<td>+ $\alpha$</td>
<td></td>
</tr>
<tr>
<td>$\varepsilon^L_{Ij}$</td>
<td>$1 - \Phi\left(\frac{q_A^* + 1}{\sigma_n}\right)$</td>
<td>$1 - \Phi\left(\frac{q_B^*}{\sigma_n}\right)$</td>
<td>$\Phi\left(\frac{q_A^* + 1}{\sigma_n}\right) - \Phi\left(\frac{q_B^*}{\sigma_n}\right)$</td>
<td>+ $\alpha\gamma$</td>
<td></td>
</tr>
<tr>
<td>$\varepsilon^H_{Uj}$</td>
<td>$\Phi\left(\frac{q_A^* + 1}{\sigma_n}\right)$</td>
<td>$\Phi\left(\frac{q_B^*}{\sigma_n}\right)$</td>
<td>$\Phi\left(\frac{q_A^* + 1}{\sigma_n}\right) - \Phi\left(\frac{q_B^*}{\sigma_n}\right)$</td>
<td>$1 - (1 - \alpha)$</td>
<td></td>
</tr>
<tr>
<td>$\varepsilon^L_{Uj}$</td>
<td>$1 - \Phi\left(\frac{q_A^* + 1}{\sigma_n}\right)$</td>
<td>$1 - \Phi\left(\frac{q_B^*}{\sigma_n}\right)$</td>
<td>$\Phi\left(\frac{q_A^* + 1}{\sigma_n}\right) - \Phi\left(\frac{q_B^*}{\sigma_n}\right)$</td>
<td>$1 - (1 - \alpha)\gamma$</td>
<td></td>
</tr>
</tbody>
</table>

From Proposition 2, we know that MSS is the unique equilibrium when $\alpha \geq \alpha$. From Lemma 3, we know that $q_A^* < q_B^* < q_A^* + 1$ when $\alpha > \alpha_2$. However, we are not able to sign the difference of $\alpha_2$ and $\alpha$. Since the intuition is slightly different when $\alpha_2$ is bigger than $\alpha$ compared with that when $\alpha_2$ is smaller than $\alpha$, we first present the intuition when $\alpha_2$ is smaller than $\alpha$, followed by the discussion of the case when $\alpha_2$ is larger than $\alpha$.

When $\alpha_2 < \alpha$, given the order flow distribution and the investment threshold, the investment errors in various scenarios are summarized in Table 1 when $\alpha > \alpha > \alpha_2$. In state $\theta$ for a given regime $j \in \{A, B\}$, the expected investment error $\varepsilon^\theta_j$ can be decomposed by the type of the speculator, i.e., whether the speculator is informed ($I$) or uninformed ($U$):

$$\varepsilon^\theta_j = \alpha\varepsilon^\theta_{Ij} + (1 - \alpha)\varepsilon^\theta_{Uj}. $$

Consider first the informed speculator who buys in the good state and shorts in the bad state. In the good state, the order flow has a mean of 1 and variance of $\sigma_n^2$ because the positively informed speculator buys. Thus the probability of underinvestment is $\varepsilon^H_{Ij} = \Phi\left(\frac{q_J^* - 1}{\sigma_n}\right)$ for $j \in \{A, B\}$. This explains the second row of the table. Similarly, in the bad state, the order flow has a mean of $-1$ if short selling is allowed and 0 if short selling is banned. The probability of overinvestment is thus $\varepsilon^L_{Ij} = 1 - \Phi\left(\frac{q_J^* - d^*_{Ij}(L)}{\sigma_n}\right)$, where $d^*_{Ij}(L)$ is the negatively informed speculator’s trading strategy in regime $j \in \{A, B\}$. This explains the third row of the table.

Therefore, when the speculator is informed, the effect of the short selling ban on the

appendix.
quality of the investment decision is captured by

\[ \Delta \varepsilon^\theta_I = \varepsilon^\theta_{IB} - \varepsilon^\theta_{IA}. \]

Now consider the uninformed speculator who short sells when short selling is allowed and does not trade when short selling is banned. The order flow has a mean of \( d^*_{ij}(\theta) \) in regime \( j \) and state \( \theta \), and the investment errors can be expressed as in the fourth and fifth row of the table. Therefore, when the speculator is uninformed, the effect of the short selling ban on the quality of the investment decision is captured by

\[ \Delta \varepsilon^\theta_U = \varepsilon^\theta_{UB} - \varepsilon^\theta_{UA}. \]

We can now characterize the ban’s effect on investment efficiencies as follows.

**Lemma 4** When \( \alpha > \alpha > \alpha_2 \), the short selling ban affects the accuracy of the investment decisions as follows.

1. When the speculator is informed, the ban reduces the investment accuracy, that is, \( \Delta \varepsilon^\theta_I > 0 \) for any \( \theta \).
2. When the speculator is not informed, the ban increases investment accuracy in the good state and reduces investment accuracy in the bad state, that is, \( \Delta \varepsilon^H_U < 0 \) and \( \Delta \varepsilon^L_U > 0 \).
3. \( \Delta \varepsilon^L_I = \Delta \varepsilon^L_U = -\Delta \varepsilon^H_U = \Delta \varepsilon_0 \).

Lemma 4 is intuitive. First, the ban suppresses the information from the negatively informed speculator and increases overinvestment in the bad state, despite the rational adjustment by investors. The ban removes the sell order from the negatively informed speculator and thus degrades the informational value of the order flow. Hence \( \Delta \varepsilon^L_I > 0 \). Second, the ban also increases the investment error when the speculator is positively informed; that is, \( \Delta \varepsilon^H_I > 0 \). Even though the ban does not affect the equilibrium strategy of the positively informed speculator (i.e., to buy), it changes the equilibrium strategy of other types of speculators, whose order flow cannot be distinguished from the positively informed speculator. In
particular, the suppression of the sell orders dilutes the information content of the positively informed speculator’s buy order, which adversely affects the investors’ use of information in the investment decisions. Collectively, these two channels explain Part 1 of Lemma 4 and capture the conventional wisdom that the short selling ban reduces the efficiency by degrading the information value of the stock price.

When the speculator engages in MSS, that is, shorting when uninformed, how the short selling ban affects the investment accuracy depends on the state. In the good state, the ban reduces underinvestment induced by such MSS; that is, \( \Delta \varepsilon^H U < 0 \). In the bad state, the ban results in more overinvestment, as there should be no investment in the bad state; that is, \( \Delta \varepsilon^L U > 0 \). The reason is that, in the absence of information and manipulative trading, the default action is to invest. By suppressing such information (that the speculator is uninformed), the ban leads to less investment, resulting in less errors when the state is good and more errors when it is bad. This explains Part 2 of Lemma 4.

Finally, Part 3 of Lemma 4 shows an articulate relationship among the investment errors. First, the uninformed speculator’s trading strategies affect investment errors in a symmetric manner. When the ban reduces the underinvestment in the good state, it increases the overinvestment in the bad state by the same amount; that is, \( \Delta \varepsilon^H U = -\Delta \varepsilon^L U \). Second, in the bad state, both the uninformed and informed speculators use the same trading strategy across the two regimes. Thus the ban has the same effect on the investment accuracy; that is, \( \Delta \varepsilon^L U = \Delta \varepsilon^L I \).

4.2 The main results

Proposition 3 illustrates our main result, which established conditions for when banning short selling is beneficial. We find that banning short selling improves the efficiency when the vulnerability to a run is sufficiently large and when the information quality is not very high.

Proposition 3 Consider the case in which MSS is the unique equilibrium (i.e., \( \alpha > \alpha(\delta) \)). When \( \delta \) is sufficiently large, banning short selling improves the efficiency if and only if \( \alpha \in (\alpha(\delta), \alpha^*(\delta)) \), where \( \alpha^*(\delta) \) is defined in equation (25) in the appendix.
We illustrate Proposition 3 with two special cases, one in the absence of coordination friction $\delta = 0$ (i.e., the GG benchmark) and the other with the extreme coordination friction $\delta \to 1 - \gamma$. Corollary 1 shows the GG benchmark, and the proof is omitted as it is contained in the proof of Proposition 3.

**Corollary 1 (GG benchmark)** Consider the special case in which $\delta \to 0$. The short selling ban always reduces efficiency, despite the presence of MSS.

The intuition of Corollary 1 is essentially the same as that in GG. Collecting the investment error terms and weighting them by their probabilities and associated consequences, we can write out the efficiency difference across the two regimes, resulting in

$$
\Delta V \equiv 2(V_{\text{Ban}} - V_{\text{Allowed}})
\equiv [-\alpha \Delta \varepsilon_f^H - \alpha \gamma \Delta \varepsilon_0] + (1 - \alpha)(1 - \gamma) \Delta \varepsilon_0.
\equiv -\alpha \Delta \varepsilon_f^H - [\alpha \gamma - (1 - \alpha)(1 - \gamma)] \Delta \varepsilon_0.
$$

(5)

From Lemma 4, $\Delta \varepsilon_f^H > 0$ and $\Delta \varepsilon_0 > 0$. Therefore banning informed short selling decreases firm value, and banning uninformed short selling increase firm value, as indicated in the first line of equation (5). However, when $\delta \to 0$, for MSS to be an equilibrium, price has to be sufficiently informative, and therefore $\alpha$ has to be sufficiently large; i.e., $\alpha > 1 - \gamma$.

Note that $-\alpha \gamma - (1 - \alpha)(1 - \gamma)] \Delta \varepsilon_0 < 0$ when $\alpha > 1 - \gamma$. This results in the efficiency gain from banning uninformed short selling being dominated by the efficiency loss of banning informed short selling, as indicated by the term, $-\alpha \gamma - (1 - \alpha)(1 - \gamma)] \Delta \varepsilon_0$, in the second line of equation (5).

The role of the coordination game then is to decrease the lower bound of $\alpha$ for MSS to be an equilibrium and thus change the benefit-cost tradeoff of short selling ban, as indicated in the next lemma.

**Lemma 5** Both $\alpha_1(\delta)$ and $\alpha_2(\delta)$ are decreasing in $\delta$.

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5 Technically, this is correct only when $\alpha > \alpha_2$. When $\alpha < \alpha_2$, the proof is more involved but the intuition is essentially the same; i.e., MSS only exists when $\alpha$ is sufficiently large, resulting in banning short selling decreasing efficiency.
Lemma 5 is intuitive. As the coordination problem becomes more severe, investors are more pessimistic, and they are only willing to stay if the probability of the good state is sufficiently high. Anticipating this fragility, MSS is more likely. Similarly, when investors are more pessimistic, it is more likely that the short selling ban increases the order flow threshold, as the threshold will be quite high when there is no short selling ban. The reason is that a high threshold implies that the order flows are very informative, resulting in the rightward shifting effect dominating the informativeness decrease effect. When \( \delta \) becomes sufficiently big, both \( \alpha(\delta) \) and \( \alpha_2(\delta) \) become so low that the efficiency gain from banning uninformed short selling dominates, resulting in banning short selling improving the efficiency, which we illustrate now with the other benchmark.

We now look at the other benchmark case when \( \delta \rightarrow 1 - \gamma \). As \( \delta \) approaches \( 1 - \gamma \), an uninformed investor shorts, even as the fraction of informed speculators approaches 0; i.e., \( \alpha(\delta) \rightarrow 0 \) (and \( \alpha_2(\delta) \rightarrow 0 \)), as illustrated in the following Corollary. In this case, short selling ban always improves efficiency. By continuity, a ban improves efficiency when \( \delta \) is sufficiently large and \( \alpha \) not too large.

**Corollary 2** Consider the special case in which \( \delta \rightarrow 1 - \gamma \). \( \lim_{\delta \rightarrow 1 - \gamma} \alpha(\delta) = \lim_{\delta \rightarrow 1 - \gamma} \alpha_2(\delta) = 0 \). The investment thresholds satisfy \( \lim_{\delta \rightarrow 1 - \gamma} q^*_A = 0 \), \( \lim_{\delta \rightarrow 1 - \gamma} q^*_B = \frac{1}{2} \). The errors satisfy \( \lim_{\delta \rightarrow 1 - \gamma} \Delta \varepsilon^H_I = \lim_{\delta \rightarrow 1 - \gamma} \Delta \varepsilon_0 > 0 \). Short selling ban improves efficiency if \( \alpha < \frac{1 - \gamma}{2} \).

When \( \alpha \rightarrow 0 \), clearly \( \alpha \gamma - (1 - \alpha)(1 - \gamma) < 0 \); i.e., conditional on the state being bad, the benefit from preventing MSS of the uninformed speculator is dominated by the cost of preventing informed short selling, since the likelihood of informed short selling becomes sufficiently small. The second term of \( \Delta V \), as in equation (5), therefore becomes positive. If the first term is positive, then \( \Delta V \) is unambiguously positive. Even if the first term is negative, the magnitude decreases when \( \alpha \) becomes smaller, as the benefit from informed short selling decreases, regardless of whether the state is good or bad when the probability of being informed decreases. In fact, when \( \delta \rightarrow 1 - \gamma \), we can calculate that

\[
\Delta V \rightarrow 2(\frac{1 - \gamma}{2} - \alpha)[\Phi\left(\frac{1}{\sigma_n}\right) - \Phi\left(\frac{1}{2\sigma_n}\right)].
\]
Note that \( \Phi\left(\frac{1}{\sigma_n}\right) - \Phi\left(\frac{1}{2\sigma_n}\right) > 0 \); that is, the ban reduces the sensitivity of investors’ decisions to the order flow. In addition, \( \frac{1-\gamma}{2} \) and \( \alpha \) represent the respective effects of coordination and information on investors’ investment decisions. When \( \frac{1-\gamma}{2} > \alpha \), then the coordination effect dominates the information effect. In this case, the short selling ban, by mitigating the investors’ response to the order flow, improves the efficiency. Otherwise, the ban reduces the efficiency. By continuity, we can prove the more general result that the short selling ban improves the efficiency if and only if \( \alpha \) is not sufficiently high and \( \delta \) is sufficiently high; i.e., the set is non-empty when \( \delta \) is sufficiently large.

Having established the two benchmarks, we now discuss the intuition of Proposition 3. When \( \alpha_2 < \alpha \), we know from Lemma 4 that the ban reduces efficiency when the speculator is informed, regardless of the uninformed speculator’s trading strategy. The ban increases (decreases) efficiency when the uninformed speculator engages in MSS and decreases (increases) efficiency when the uninformed speculator does not trade and the state is good (bad).

As will be shown in the proof of Proposition 3, \( \Delta V \) is decreasing in \( \alpha \) when \( \alpha > \alpha \). Intuitively, the discussion above suggests that banning informed short selling always increases decision errors but banning MSS may decrease decision errors. Thus, when the probability of informed trading is higher, the efficiency loss from banning informed short selling (i.e., \( \alpha \Delta \epsilon^H_I + \alpha \gamma \Delta \epsilon_0 \)) dominates the potential efficiency loss from banning MSS (i.e., \( (1 - \alpha)(1 - \gamma)\Delta \epsilon_0 \)). Therefore \( \Delta V \) can only be positive if \( \alpha \) is not too big. This completes the discussion for the intuition when \( \alpha_2 < \alpha \).

When \( \alpha_2 \geq \alpha \), Lemma 4 still applies, and the intuition will be the same when \( \alpha > \alpha_2 \); i.e., \( \Delta V > 0 \) when \( \alpha \in (\alpha_2, \alpha^*) \) for some \( \alpha^* \). When \( \alpha \in (\alpha, \alpha_2) \), however, since \( q^*_B \leq q^*_A \), we have \( \Delta \epsilon^H_I < 0 \); i.e., the short selling ban improves efficiency even for informed trading when \( \theta = H \). (Equivalently, the first term inside the square bracket of equation (5) is positive.) Recall from the previous discussion that \( q^*_B \leq q^*_A \) implies that the “informativeness decrease effect” dominates the “rightward shifting effect.” In other words, the order flow becomes so uninformative when banning short selling that investors are more willing to follow the default action (i.e., invest). The investment threshold becomes lower, and thus there is less run when banning short selling. Since less run is good when \( \theta = H \), \( \Delta \epsilon^H_I < 0 \). Recall that
\( \Delta z_I^H > 0 \) when \( \alpha > \alpha_2 \). Intuitively, when order flow is sufficiently informative, in the good state, the ban reduces firm value, as the investment threshold is higher and there is more underinvestment. However, when order flow becomes very noisy, in the good state, the ban increases firm value by lowering the investment threshold, resulting in less run and thus less underinvestment, even when all trading is informed. As a result, conditional on \( \theta = H \), if \( \Delta V \) is positive when banning informed short selling is unambiguously bad (i.e., when \( \Delta z_I^H > 0 \)), then \( \Delta V \) must be even more positive when banning informed short selling is good (i.e., when \( \Delta z_I^H < 0 \)). This results in \( \Delta V > 0 \) when \( \alpha < \alpha \leq \alpha_2 \) and completes our discussion.

In summary, in the absence of coordination friction (i.e., when \( \delta = 0 \)), informed short selling improves efficiency and MSS, by terminating good projects, reduces efficiency. However, MSS is secondary, as it is sustained only by informed short selling. In other words, MSS is only optimal for uninformed speculators if the price is sufficiently informative; i.e., informed short selling is sufficiently strong. This results in the benefit of informed short selling always dominating the inefficiency of MSS and thus a decrease of firm value from informed short selling, as documented in GG. We introduce another source of MSS based on panic runs from coordination failure (i.e., when \( \delta > 0 \)). The introduction of panic runs reduces investment (i.e., \( \frac{\partial q_{\Delta A}}{\partial \delta} > 0 \)) even in the absence of informed short selling, thus making MSS more prevalent (i.e., \( \frac{\partial q(\delta)}{\partial \delta} < 0 \)) and not necessarily secondary. In fact, when \( \delta \) is large and when \( \alpha \) is small, our channel of MSS sustained by panic runs is strong and the channel from GG is mild, resulting in short selling reducing efficiency overall and a short selling ban increasing efficiency.

### 4.3 A numerical example

To better understand Proposition 3, we illustrate using a numerical example. For ease of notation, we write \( \Delta V \) as \( \Delta V(\alpha, \delta) \). We assume that \( \sigma_n = \frac{1}{2} \) and \( L = -0.5 \). Note that, in this case, \( 1 - \gamma = 0.5 \), so \( \delta \in [0, 0.5] \). When \( \delta = 0 \), we can numerically calculate that \( q(0) \approx 0.58 > \alpha_2(0) = 0.51 \). In addition, \( q_{SA}(\alpha(0), 0) \approx -0.24 \in (-1/2, 0) \), and \( \Delta V(\alpha(0), 0) \approx -0.0673 < 0 \); i.e., banning short selling decreases firm value, consistent with GG. When we increase \( \delta \) to 0.2, \( \alpha(0.2) \) decreases to around 0.37 > \( \alpha_2(0.2) \approx 0.34 \), which enlarges the range for short
selling to be optimal for the uninformed speculator, as \( \alpha \) only needs to be greater than 0.37 for this to be the case. When \( \alpha \) is small, the efficiency gain from banning uninformed short selling outweighs the efficiency loss from banning informed short selling, resulting in \( \Delta V > 0 \), which is absent in GG. For example, when \( \alpha = 0.4 > \underline{\alpha}(0.2) \), \( \Delta V(0.4, 0.2) = 0.0851 > 0 \). However, when \( \alpha \) becomes large, the efficiency gain from banning uninformed short selling is outweighed by the efficiency loss from banning informed short selling, resulting in \( \Delta V < 0 \). For example, when \( \alpha = 0.6 \), \( \Delta V(0.6, 0.2) = -0.0922 < 0 \); i.e., banning short selling is bad when \( \alpha = 0.6 \). As illustrated in Figure 3, \( \Delta V \) is decreasing in \( \alpha \), and banning short selling improves efficiency if \( \underline{\alpha}(0.2) < \alpha < 0.49 \).

5 Empirical and policy implications

Our results provide several empirical and policy implications.

First, as suggested in GG, in the presence of MSS, the first-best policy would always be a discriminative ban on uninformed speculators. This policy eliminates MSS while preserving the information conveyed through trading of informed speculators. However, such a policy is impractical, due to the difficulty in judging whether the speculator is informed. In the absence of such a discriminative ban, our result provides a rationale for indiscriminately banning all short selling, as the interaction of MSS with the feedback effect and strategic
complementarities reduces firm value by forgoing productive investments in good states. Such forgone productive investments may dominate the information conveyed through trading of informed speculators.

Second, our results suggest when a short selling ban is more likely to be efficient. Specifically, it is more likely to be efficient for 1) stocks of firms that exhibit strong complementarities and thus are more vulnerable to runs (e.g., financial institutions), 2) for periods when speculators do not have very precise information (e.g., during uncertain economic periods), and 3) for situations where investors are sensitive to information (e.g., when feedback effect is strong). Our results are thus consistent with the anecdotal evidence of banning short selling for financial stocks during crisis periods. Our results also predict that such a ban is more effective when firms have more to learn from the stock market, as empirically proxied in Chen et al. (2007).

6 Conclusions

We propose a model to provide justification for banning short selling. The benefit of such a ban stems from the elimination of MSS where uninformed speculators short, which can be sustained by both panic runs by investors due to strategic complementarity and the possibility of the market learning from informed speculators (i.e., the feedback effect). We show that, when strategic complementarities are sufficiently strong and market uncertainty is sufficiently high, MSS, by inducing firms to abandon positive NPV projects, can destroy so much value that the benefit of banning MSS outweighs the cost of not being able to learn from informed speculators. To the extent that financial institutions exhibit strong degree of complementarities and crisis periods exhibit high market uncertainty, our results are consistent with anecdotal evidence that short selling bans are typically imposed on financial stocks during crisis periods.

7 Appendix: Proofs

Proof of Lemma 1:
Proof. The derivative with respect to $l$ results in

$$\frac{\partial[(1-l)(\theta-\delta l)]}{\partial l} = -\theta + \delta(2l-1).$$

Thus, when $\theta = H$, $\frac{\partial[(1-l)(\theta-\delta l)]}{\partial l} < 0$, resulting in $l^F B = 0$. When $\theta = L$, $\frac{\partial[(1-l)(\theta-\delta l)]}{\partial l}$ is either always positive or initially negative but eventually positive. In the former case, $l^F B = 1$. In the latter case, the optimal solution is corner, and one has to compare $V(l = 0)$ and $V(l = 1)$. Note that $V(l = 0) = L < V(l = 1) = 0$. Therefore $l^F B = 1$ when $\theta = L$.

Proof of Lemma 2:

Proof. The proof of the Lemma is organized in three steps. We first prove that the investors play a common threshold strategy. This is proved in two steps. In the first step, we show that all investors use the same strategy. In the second step, we show that the equilibrium strategy must be a single threshold. The proof of the second step assumes that it is optimal for the positively informed speculator to choose $d = 1$ and the negatively informed speculator to choose $d = -1$. This will be proved at the very end after step 2. In the third step we derive the common threshold strategy.

Step 1: All investors use the same strategy

Suppose that investor $i$ chooses to withdraw if and only if $q_i \in S_i$, and investor $j$ chooses to run if and only if $q_j \in S_j$, where $S_i$ and $S_j$ are subsets of the real line. Suppose that $S_i \neq S_j$. This implies that at least one of the sets of $S_i$ or $S_j$ must be non-empty. Without loss of generality, suppose that $S_i$ is not empty. This implies that there exists a $q_0$ such that $q_0 \in S_i$ but $q_0 \notin S_j$. This implies that, upon observing $q_i = q_0$, investor $i$ stays but investor $j$ withdraws, i.e., $\Delta(q_i = q_0) > 0 \geq \Delta(q_j = q_0)$, which is a contradiction, as $\Delta(q_i = q_0) = \Delta(q_j = q_0)$ as $\Delta(q)$ only depends on $q$ for any fixed speculators' strategies. Therefore all investors use the same strategy.

Step 2: The equilibrium strategy must be a threshold strategy
Upon observing $q_i$, investor $i$’s expected payoff of staying relative to withdrawing is

$$\Delta(q_i) = \Pr(\theta = H|q_i)(1 - \delta E[l|q_i]) + \Pr(\theta = L|q_i)(L - \delta E[l|q_i])$$

$$= L + \Pr(\theta = H|q_i)(1 - L) - \delta E[l|q_i]$$

$$= L + \beta(q_i)(1 - L) - \delta E[l|q_i].$$

Assume now that the positively informed speculator will choose $d = +1$ and the negatively informed speculator will choose $d = -1$. Denote the uninformed speculator’s strategy by choosing $d = 1$ with probability $\rho_1$, choosing $d = -1$ with probability $\rho_{-1}$, and choosing $d = 0$ with probability $1 - \rho_1 - \rho_{-1}$. Note that it is impossible for the uninformed speculator to mix between $d = -1$ and $d = 1$, as he needs to be indifferent between choosing $d = -1$ and $d = 1$, which is only possible if $E[P|\emptyset] = E[V|\emptyset]$. This implies that the uninformed speculator is indifferent between any trading strategies. In this case, the uninformed speculator will not trade, according to the tiebreaker. Therefore we only need to consider either $\rho_1 = 0$ or $\rho_{-1} = 0$. We prove the case when $\rho_1 = 0$ as the proof when $\rho_{-1} = 0$ is essentially the same.

Denote the density of $q$ conditional on $\theta$ as $g$. Then following Bayes’ Rule,

$$\beta(q_i) = \frac{g(q_i|\theta = H, \rho_{-1}, \alpha)}{g(q_i|\theta = H, \rho_{-1}, \alpha) + g(q_i|\theta = L, \rho_{-1}, \alpha)}$$

$$= \frac{g(q_i|\theta = H, \rho_{-1}, \alpha)}{g(q_i|\theta = H, \rho_{-1}, \alpha) + g(q_i|\theta = L, \rho_{-1}, \alpha)} + 1.$$

(6)
Note that

\[
g(q_i|\theta = H) = \frac{\alpha \phi(-\frac{q_i}{\sqrt{\sigma^2 + \sigma_i^2}}) + (1-\alpha)((1-\rho_{-1})\phi(-\frac{q_i}{\sqrt{\sigma^2 + \sigma_i^2}}) + \rho_{-1}\phi(-\frac{q_i+1}{\sqrt{\sigma^2 + \sigma_i^2}}))}{\alpha \phi(-\frac{q_i}{\sqrt{\sigma^2 + \sigma_i^2}}) + (1-\alpha)((1-\rho_{-1})\phi(-\frac{q_i}{\sqrt{\sigma^2 + \sigma_i^2}}) + \rho_{-1}\phi(-\frac{q_i+1}{\sqrt{\sigma^2 + \sigma_i^2}}))}
\]

This implies that investors in the range \(q_i < q_B\) will withdraw. Note that it is possible for \(\rho_{-1} = 0\), which then implies that \(\Lambda(q_i) > 0\) if \(q > \bar{q}\). If \(\frac{(1-\alpha)\rho_{-1}}{\alpha + 2(1-\alpha)\rho_{-1}} + \frac{\alpha(1-\alpha)\rho_{-1}}{\alpha + 2(1-\alpha)\rho_{-1}}L < 0\), then the lower dominance region exists, and by continuity there exists finite \(q\) such that \(\Lambda(q_i) < 0\) if \(q < \bar{q}\). If \(\frac{(1-\alpha)\rho_{-1}}{\alpha + 2(1-\alpha)\rho_{-1}} + \frac{\alpha(1-\alpha)\rho_{-1}}{\alpha + 2(1-\alpha)\rho_{-1}}L > 0\), then the lower dominance region does not exist.

Similarly, when \(\rho_{-1} = 0\), we can show that the lower dominance region always exists and thus \(q\) exists, but the upper dominance region may not exist.

As a summary, in equilibrium we either have 1) \(\Lambda(q_i) < 0\) if \(q < \bar{q}\) and \(\Lambda(q_i) > 0\) if \(q > \bar{q}\) or 2) \(\Lambda(q_i) < 0\) if \(q < \bar{q}\) or 3) \(\Lambda(q_i) > 0\) if \(q > \bar{q}\) for some finite \(q\) and \(\bar{q}\). We now show that, under either of the three scenarios, common threshold strategy is the equilibrium.

For case 1) and 2), denote \(q_B = \sup\{q_i : \Lambda(q_i) < 0\}\), i.e., the highest signal below which an investor prefers to withdraw. Note that it is possible for \(q_B = +\infty\), which then implies that \(\Lambda(q_B) \leq 0\). If all investors use the threshold strategy, then investor \(i\) will withdraw when \(q_i < q_B\), for any \(i\). Suppose in equilibrium they do not use threshold strategy, then for investor \(i\), there exist signals smaller than \(q_B\) such that an investor observing \(q_i\) will stay. Denote \(q_A\) to be the largest of them; i.e., \(q_A = \sup\{q_i : \Lambda(q_i) \geq 0\}\). \(\bar{q} < q_A < q_B\) and thus is finite. This implies that investors in the range \([q_A, q_B]\) and \((-\infty, \bar{q})\) will withdraw.
for sure, while investors in the range of \([q, q_A]\) may choose to stay or withdraw. Denote the strategies of the investors in the range of \([q, q_A]\) by \(n(q_i) \in [0, 1]\). Since investors are indifferent upon observing \(q_A\), we have

\[
\Delta(q_A) = 0 \geq \Delta(q_B).
\]

On the other hand, note that

\[
E[l|q_A] = \Phi\left(\frac{q - q_A}{\sqrt{2}\sigma}\right) + \int_q^{q_A} n(q_j) \frac{1}{\sqrt{2}\sigma} \phi\left(\frac{q_j - q_A}{\sqrt{2}\sigma}\right) dq_j + \Phi\left(\frac{q_B - q_A}{\sqrt{2}\sigma}\right) - \frac{1}{2},
\]

and

\[
E[l|q_B] = \Phi\left(\frac{q - q_B}{\sqrt{2}\sigma}\right) + \int_q^{q_A} n(q_j) \frac{1}{\sqrt{2}\sigma} \phi\left(\frac{q_j - q_B}{\sqrt{2}\sigma}\right) dq_j + \Phi\left(\frac{q_B - q_A}{\sqrt{2}\sigma}\right) - \frac{1}{2},
\]

where we use \(\Phi(-x) = 1 - \Phi(x)\) to arrive at the last equality.

Thus

\[
E[l|q_B] - E[l|q_A] = \Phi\left(\frac{q - q_B}{\sqrt{2}\sigma}\right) - \Phi\left(\frac{q - q_A}{\sqrt{2}\sigma}\right) + \int_q^{q_A} n(q_j) \frac{1}{\sqrt{2}\sigma} \phi\left(\frac{q_j - q_B}{\sqrt{2}\sigma}\right) dq_j - \phi\left(\frac{q_j - q_A}{\sqrt{2}\sigma}\right) dq_j.
\]

Since \(q_B > q_A\), \(\phi\left(\frac{q_j - q_B}{\sqrt{2}\sigma}\right) < 0\) for any \(q_j \in [q, q_A]\). Therefore

\[
E[l|q_B] - E[l|q_A] \leq \Phi\left(\frac{q - q_B}{\sqrt{2}\sigma}\right) - \Phi\left(\frac{q - q_A}{\sqrt{2}\sigma}\right) < 0.
\]

In addition, \(\beta(q_i)\) is increasing in \(q_i\), resulting in \(\beta(q_B) > \beta(q_A)\). Correspondingly \(\Delta(q_B) > \Delta(q_A)\) and thus the contradiction. Therefore all investors use the same threshold strategy in
equilibrium.

For cases 1) and 3), denote $q_C = \inf\{q_i : \Delta(q_i) > 0\}$, i.e., the lowest signal above which an investor will stay. Note that it is possible for $q_C = -\infty$, which then implies that $\Delta(q_C) \geq 0$. If all investors use threshold strategy, then investor $i$ will stay when $q_i > q_C$, for any $i$. Suppose in equilibrium they do not use threshold strategy, then, for investor $i$, there exist signals larger than $q_C$ such that an investor observing $q_i$ will withdraw. Denote $q_D$ to be the smallest of them; i.e., $q_D = \sup\{q_i > q_C : \Delta(q_i) \leq 0\}$. $q_C < q_D < \bar{q}$ and thus is finite. This implies that investors in the range $[q_C, q_D]$ and $(\bar{q}, +\infty)$ will stay for sure, investors in the range of $(-\infty, q_C)$ will withdraw for sure, while investors in the range of $[q_D, \bar{q})$ may choose to stay or withdraw and denote their strategies by $n(q_i) \in [0, 1]$. Since investors are indifferent upon observing $q_D$, we have

$$\Delta(q_D) = 0 \leq \Delta(q_C).$$

Using similar techniques as above we can show that this inequality cannot hold. Therefore all investors use the same threshold strategy in equilibrium.

To complete step 2, we now prove that the positively informed speculator will buy and the negatively informed speculator will sell. We will prove the optimal strategy for the positively informed speculator as the proof for the optimal strategy of the negatively informed speculators is essentially the same.

Consider the strategy of the positively informed speculator. Since the speculator knows that $\theta = H$, he knows that the equity value is $(1 - l)(1 - \delta l)$. On the other hand, conditional on total order flow $q$, stock price

$$P(q) = \Pr(\theta = H|q)[1 - l(q)][1 - \delta l(q)] + \Pr(\theta = L|q)[1 - l(q)][L - \delta l(q)]$$

$$= [1 - l(q)][\Pr(\theta = H|q)(1 - \delta l(q)) + \Pr(\theta = L|q)(L - \delta l(q))].$$
Therefore, when the speculator takes action \( d \), the speculator’s profit will be

\[
\Pi(d) = d(E[V|d] - E[P|d])
\]

\[
= d(E[(1 - l)(1 - \delta l)|d] - E[(1 - l)\{\Pr(\theta = H|q)(1 - \delta l) + \Pr(\theta = L|q)(L - \delta l)\}|d])
\]

\[
= dE[(1 - l)\Pr(\theta = L|q)(1 - L)|d].
\]

Since \( E[(1 - l)\Pr(\theta = L|q)(1 - L)|d] \geq 0 \), \( \Pi(1) \geq \Pi(0) \geq \Pi(-1) \). The tiebreaker assumption then implies that \( d = 1 \) for the positively informed speculator.

**Step 3: Derive the common threshold**

Given that each investor uses a common threshold strategy, we now solve for the common threshold, denoted as \( q^* \). The threshold strategy implies that an investor will be indifferent when observing \( q^* \), i.e., \( q^* \) satisfies the indifferent condition.

\[
\Delta(q^*) = \beta(q^*) (1 - \delta E[l|q^*]) + [1 - \beta(q^*)](L - \delta E[l|q^*])
\]

\[
= \beta(q^*) (1 - L) - \delta E[l|q^*] + L
\]

\[
= 0.
\]

When \( q = q^* \),

\[
E[l|q^*] = \Pr(q_j \leq q^*|q_i = q^*)
\]

\[
= \frac{1}{2},
\]

resulting in

\[
\beta(q^*) = \frac{\delta + \gamma}{1 + \gamma}. \tag{7}
\]

As will be shown in Lemma 8, \( \beta(q^*) \) is strictly increasing in \( q^* \). Therefore equation (7) has at most one solution. The expression of \( \beta(q^*) \), however, depends on \( q^* \) and therefore the uninformed speculators’ strategies (as the informed speculator always buy when observing
\( \theta = H \) and sell when observing \( \theta = L \).

Before we go to the proof of Proposition 1, we prove three lemmas that will be useful throughout the proofs. The next lemma echoes the result from GG that the manipulation is only one-sided. It is never optimal for the speculator to buy when he is uninformed. The reason is due to the feedback effect. Since the project is positive NPV ex-ante, short selling may cause the firm to (incorrectly) infer that the state is bad and choose not to invest, resulting in a reduction of firm value. The uninformed speculator thus would profit from short selling. On the other hand, buying will cause the firm to (incorrectly) infer that the state is good and invest, resulting in a higher price and thus a loss for the uninformed speculator.

**Lemma 6** \( d = 1 \) is always a dominated strategy for the uninformed speculator. In other words, the uninformed speculator will never buy.

**Proof of Lemma 6:**

**Proof.** We prove that \( d(\emptyset) = 1 \) is dominated when short selling is allowed, as the proof for when short selling is banned is essentially the same.

As shown in the proof of Lemma 2, the uninformed speculator will not mix between \( d = 1 \) and \( d = -1 \). Therefore the market maker will conjecture that either the uninformed speculator mixes between \( d = -1 \) and \( d = 0 \) or \( d = 1 \) and \( d = 0 \). We prove the case for the conjecture being that the uninformed speculator mixes between \( d = -1 \) and \( d = 0 \) as the proof for the other case is essentially the same. Suppose the market maker conjectures that the uninformed speculator chooses \( d = -1 \) with probability \( \hat{\rho}_{-1} \) and chooses \( d = 0 \) with
probability $1 - \hat{p}_{-1}$, we can calculate from equation (6) that

$$
\beta(q, \hat{p}_{-1}, \alpha)
= \frac{g(q|\theta = H, \hat{p}_{-1}, \alpha)}{g(q|\theta = H, \hat{p}_{-1}, \alpha) + g(q|\theta = L, \hat{p}_{-1}, \alpha)}
= \frac{g(q|\theta = H, \hat{p}_{-1}, \alpha)}{g(q|\theta = H, \hat{p}_{-1}, \alpha) + \frac{\alpha\phi\left(\frac{q-1}{\sqrt{\sigma_z^2 + a^2}}\right) + (1-\alpha)\phi\left(\frac{q}{\sqrt{\sigma_z^2 + a^2}}\right) + \hat{p}_{-1}\phi\left(\frac{q+1}{\sqrt{\sigma_z^2 + a^2}}\right)}{\alpha\phi\left(\frac{q-1}{\sqrt{\sigma_z^2 + a^2}}\right) + (1-\alpha)\phi\left(\frac{q}{\sqrt{\sigma_z^2 + a^2}}\right) + \hat{p}_{-1}\phi\left(\frac{q+1}{\sqrt{\sigma_z^2 + a^2}}\right)}} + 1.
$$

(8)

To prove that $d(\emptyset) = 1$ is dominated for any conjecture $\hat{p}_{-1}$, we show that the trading profit from choosing $d(\emptyset) = 1$ given any conjecture $\hat{p}_{-1}$ is negative. More generally, for any pure strategy $d(\emptyset)$, the trading profit is

$$
\Pi(d(\emptyset), \hat{p}_{-1}, \alpha)
\equiv d(\emptyset) \{E[V|d(\emptyset)] - E[P(d(\emptyset), \hat{p}_{-1}, \alpha)|d(\emptyset)]\}.
$$

Note that, when $\sigma_z \to 0$,

$$
l(q)
= \text{Pr}(\Phi\left(\frac{q^*(\hat{p}_{-1}, \alpha) - q}{\sigma_z}\right))
\to \begin{cases} 
1 & \text{when } q < q^*(\hat{p}_{-1}, \alpha) \\
0 & \text{when } q > q^*(\hat{p}_{-1}, \alpha).
\end{cases}
$$
Therefore

\[
E[V|d(\theta)] = \frac{1}{2}[1 - \Phi(q^*(\hat{\rho}_{-1}, \alpha) - d(\theta))] + \frac{1}{2}L[1 - \Phi(q^*(\hat{\rho}_{-1}, \alpha) - d(\theta))]
\]

\[
= \frac{1}{2}(1 + L)[1 - \Phi(q^*(\hat{\rho}_{-1}, \alpha) - d(\theta))].
\]

Similarly, when \(\sigma_{\varepsilon} \to 0\), the market maker sets

\[
P(q, \hat{\rho}_{-1}) = \begin{cases} 
\beta(q, \hat{\rho}_{-1}, \alpha) + [1 - \beta(q, \hat{\rho}_{-1}, \alpha)]L & \text{if } q > q^*(\hat{\rho}_{-1}, \alpha) \\
0 & \text{if } q \leq q^*(\hat{\rho}_{-1}, \alpha) 
\end{cases}
\]

Therefore

\[
\Pi(d(\theta), \hat{\rho}_{-1}, \alpha) = d(\theta)E[V|d(\theta)] - E[P(d(\theta), \hat{\rho}_{-1}, \alpha)|d(\theta)]
\]

\[
= d(\theta) \int_{q^*(\hat{\rho}_{-1}, \alpha)}^{+\infty} \left[\frac{1}{2} - \beta(q, \hat{\rho}_{-1}, \alpha)\right](1 + L)\frac{1}{\sigma_n} \phi(q - d(\theta)) dq. \tag{9}
\]

Insert \(d(\theta) = 1\) and equation (8) into equation (9) and let \(\sigma_{\varepsilon} \to 0\) results in

\[
\Pi(1, \hat{\rho}_{-1}, \alpha) = \frac{1 - L}{2} \int_{q^*(\hat{\rho}_{-1}, \alpha)}^{+\infty} \frac{\phi(\frac{q-1}{\sigma_n})\alpha - \frac{1}{\sigma_n} [\phi(\frac{q+1}{\sigma_n}) - \phi(\frac{q-1}{\sigma_n})]}{\alpha \phi(\frac{q-1}{\sigma_n}) + \alpha \phi(\frac{q+1}{\sigma_n}) + 2(1 - \alpha)[(1 - \hat{\rho}_{-1})\phi(\frac{q}{\sigma_n}) + \hat{\rho}_{-1}\phi(\frac{q+1}{\sigma_n})]} dq
\]

\[
\equiv \frac{1 - L}{2} \int_{q^*(\hat{\rho}_{-1}, \alpha)}^{+\infty} h(\alpha, q, \hat{\rho}_{-1})[\phi(\frac{q+1}{\sigma_n}) - \phi(\frac{q-1}{\sigma_n})] dq,
\]

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where

\[
\begin{align*}
  h(\alpha, q, \rho_{-1}) &\equiv \frac{\alpha \phi(q-1)}{\phi(q+1) + \alpha \phi(q+1) + 2(1-\alpha)[(1-\rho_{-1}) \phi(q) + \rho_{-1} \phi(q+1)]} \\
  &= \frac{\alpha \phi(q-1)}{\phi(q+1) + \alpha + 2(1-\alpha)[(1-\rho_{-1}) \phi(q) + \rho_{-1}]} \\
  &= \frac{2e^{2q} \alpha}{\alpha e^{2q} + \alpha + 2(1-\alpha)[(1-\rho_{-1}) e^{2q} + \rho_{-1}]}.
\end{align*}
\]

Note that the sign of \( \Pi(1, \rho_{-1}, \alpha) \) is the same as the sign of \( \int_{q^*(\rho_{-1}, \alpha)}^{+\infty} h(\alpha, q, \rho_{-1}) \frac{1}{\sigma_n}[\phi(q+1) - \phi(q-1)]dq \).

Using integration by parts,

\[
\begin{align*}
  \int_{q^*(\rho_{-1}, \alpha)}^{+\infty} h(\alpha, q, \rho_{-1}) \frac{1}{\sigma_n}[\phi(q+1) - \phi(q-1)]dq &\equiv \int_{q^*(\rho_{-1}, \alpha)}^{+\infty} \frac{\partial h(\alpha, q, \rho_{-1})}{\partial q}[\phi(q+1) - \phi(q-1)]dq \\
  &= \int_{q^*(\rho_{-1}, \alpha)}^{+\infty} \frac{\partial h(\alpha, q, \rho_{-1})}{\partial q}[\phi(q+1) - \phi(q-1)]dq \\
  &= -h(\alpha, q^*(\rho_{-1}, \alpha), \rho_{-1})[\phi(q^*(\rho_{-1}, \alpha) + 1) - \phi(q^*(\rho_{-1}, \alpha) - 1)] \\
  &\quad - \int_{q^*(\rho_{-1}, \alpha)}^{+\infty} \frac{\partial h(\alpha, q, \rho_{-1})}{\partial q}[\phi(q+1) - \phi(q-1)]dq.
\end{align*}
\]

The first term is negative as \( \Phi(q^*(\rho_{-1}, \alpha) + 1) - \Phi(q^*(\rho_{-1}, \alpha) - 1) > 0 \) and \( h(\alpha, q^*(\rho_{-1}, \alpha), \rho_{-1}) > 0 \).

The second term is also negative as \( \Phi(q+1) - \Phi(q-1) > 0 \) and

\[
\begin{align*}
  \frac{\partial h(\alpha, q, \rho_{-1})}{\partial q} &\equiv \frac{2e^{2q} \alpha}{\alpha e^{2q} + \alpha + 2(1-\alpha)[(1-\rho_{-1}) e^{2q} + \rho_{-1}]} \\
  &= 2e^{2q} \frac{\alpha e^{2q} + \alpha + 2(1-\alpha)[(1-\rho_{-1}) e^{2q} + \rho_{-1}]}{\alpha e^{2q} + \alpha + 2(1-\alpha)[(1-\rho_{-1}) e^{2q} + \rho_{-1}]} \\
  &= 2e^{2q} \left[ 1 - \frac{1}{\alpha e^{2q} + \alpha + 2(1-\alpha)[(1-\rho_{-1}) e^{2q} + \rho_{-1}]} \right] \\
  &> 0.
\end{align*}
\]
Therefore $\Pi(1, \hat{\rho}_{-1}, \alpha) < 0$ for any $\hat{\rho}_{-1}$, making $d(\emptyset) = 1$ a dominated strategy. ■

**Lemma 7** There exists a unique $\alpha_1(\hat{\rho}_{-1}) \in [0, 1)$ such that $\int_{q^*(\hat{\rho}_{-1}, \alpha_1(\hat{\rho}_{-1}))}^{+\infty} f(\alpha_1(\hat{\rho}_{-1}), q, \hat{\rho}_{-1}, 1) dq = 0$.

**Proof of Lemma 7:**

**Proof.** We first prove the case when $\hat{\rho}_{-1} > 0$. When $\delta < 1 - \gamma$ and $\alpha$ is sufficiently small, $q^*(\hat{\rho}_{-1}, \alpha) \to -\infty$ for any $\hat{\rho}_{-1} > 0$. To see this, note that

$$\beta(q^*) = \frac{\delta + \gamma}{1 + \gamma}.$$

In addition, from Lemma 8, $\beta(q, \hat{\rho}_{-1}, \alpha)$ is increasing in $q$, resulting in

$$\beta(q, \hat{\rho}_{-1}, \alpha) > \lim_{q \to -\infty} \beta(q, \hat{\rho}_{-1}, \alpha) = \frac{(1-\alpha)\hat{\rho}_{-1}}{\alpha + (1-\alpha)\hat{\rho}_{-1}}$$

$$= \frac{(1-\alpha)\hat{\rho}_{-1}}{\alpha + (1-\alpha)\hat{\rho}_{-1}} + 1$$

$$\to \frac{1}{2} > \frac{\delta + \gamma}{1 + \gamma} \text{ when } \alpha \to 0.$$

Therefore, by continuity, $q^*(\hat{\rho}_{-1}, \alpha) \to -\infty$ when $\alpha$ is sufficiently small.

Note that we can write $\int_{-\infty}^{+\infty} f(\alpha, q, \hat{\rho}_{-1}, -1) dq$ as

$$\int_{-\infty}^{+\infty} f(\alpha, q, \hat{\rho}_{-1}, -1) dq$$

$$= \int_{-\infty}^{0} f(\alpha, q, \hat{\rho}_{-1}, -1) dq + \int_{0}^{+\infty} f(\alpha, q, \hat{\rho}_{-1}, -1) dq$$

$$= \int_{0}^{+\infty} f(\alpha, q, \hat{\rho}_{-1}, -1) dq + \int_{0}^{+\infty} f(\alpha, -q, \hat{\rho}_{-1}, -1) dq$$

$$= \int_{0}^{+\infty} [f(\alpha, q, \hat{\rho}_{-1}, -1) + f(\alpha, -q, \hat{\rho}_{-1}, -1)] dq,$$

where we used change of variables to arrive at the second inequality. Note that $f(\alpha, q, \hat{\rho}_{-1}, -1) >$
0 > f(\alpha, -q, \tilde{\rho}_{-1}, -1) \text{ if and only if } q > 0. \text{ In addition, when } q > 0,

\[ \frac{f(\alpha, q, \tilde{\rho}_{-1}, -1)}{-f(\alpha, -q, \tilde{\rho}_{-1}, -1)} = \frac{\alpha + 2\tilde{\rho}_{-1}(1 - \alpha) + \alpha e^{\frac{2q}{\sigma^2}} + 2(1 - \alpha)(1 - \tilde{\rho}_{-1})e^{\frac{1}{2\sigma^2}}}{\alpha + 2\tilde{\rho}_{-1}(1 - \alpha) + \alpha e^{\frac{2q}{\sigma^2}} + 2(1 - \alpha)(1 - \tilde{\rho}_{-1})e^{\frac{2q+1}{2\sigma^2}}} < 1, \]

as

\[ e^{\frac{2q}{\sigma^2}} < e^{\frac{2q+1}{2\sigma^2}}, \]

and

\[ e^{\frac{1}{2\sigma^2}} < e^{\frac{2q+1}{2\sigma^2}}. \]

Therefore \( f(\alpha, q, \tilde{\rho}_{-1}, -1) + f(\alpha, -q, \tilde{\rho}_{-1}, -1) < 0 \), and we have \( \int_{-\infty}^{+\infty} f(\alpha, q, \tilde{\rho}_{-1}, -1) dq < 0 \).

When \( \alpha \) is not sufficiently small, \( -\infty < q^*(1, \alpha) < 0 \).

Through changing variables, for any \( N > 0 \), we have

\[
\int_{q^*(\tilde{\rho}_{-1}, \alpha)}^{0} f(\alpha, q, \tilde{\rho}_{-1}, -1) dq = \frac{1}{N} \int_{Nq^*(\tilde{\rho}_{-1}, \alpha)}^{0} f(\alpha, \frac{q}{N}, \tilde{\rho}_{-1}, -1) dq = \frac{1}{N} \int_{Nq^*(\tilde{\rho}_{-1}, \alpha)}^{-Nq^*(\tilde{\rho}_{-1}, \alpha)} f(\alpha, -\frac{q}{N}, \tilde{\rho}_{-1}, -1) dq.
\]

Therefore

\[
\int_{q^*(\tilde{\rho}_{-1}, \alpha)}^{+\infty} f(\alpha, q, \tilde{\rho}_{-1}, -1) dq = \frac{1}{N} \int_{0}^{-Nq^*(\tilde{\rho}_{-1}, \alpha)} f(\alpha, -\frac{q}{N}, \tilde{\rho}_{-1}, -1) dq + \int_{0}^{+\infty} f(\alpha, q, \tilde{\rho}_{-1}, -1) dq
\]

\[
= \int_{0}^{-Nq^*(\tilde{\rho}_{-1}, \alpha)} \left[ \frac{1}{N} f(\alpha, -\frac{q}{N}, \tilde{\rho}_{-1}, -1) + f(\alpha, q, \tilde{\rho}_{-1}, -1) \right] dq + \int_{-Nq^*(\tilde{\rho}_{-1}, \alpha)}^{+\infty} f(\alpha, q, \tilde{\rho}_{-1}, -1) dq.
\]
In addition, $q^*(\hat{p}_{-1}, \alpha)$ is increasing in $\alpha$ as by envelope theorem,

$$\frac{\partial q^*(\hat{p}_{-1}, \alpha)}{\partial \alpha} = -\frac{\partial \beta}{\partial \alpha}|_{q=q^*(\hat{p}_{-1}, \alpha)} > 0,$$

as

$$\frac{\partial \beta}{\partial q}|_{q=q^*(\hat{p}_{-1}, \alpha)} > 0$$

from Lemma 8 and

$$\frac{\partial \beta}{\partial \alpha}|_{q=q^*(\hat{p}_{-1}, \alpha)} = \frac{(e^{2\hat{q}^*(\hat{p}_{-1}, \alpha)} - 1)[e^{2\hat{q}^*(\hat{p}_{-1}, \alpha)\alpha} (1 - \rho_{-1}) + \rho_{-1}]}{[\alpha + (1 - \alpha)(1 - \rho_{-1})e^{2\hat{q}^*(\hat{p}_{-1}, \alpha)\alpha} + (1 - \alpha)\rho_{-1}]^2} < 0,$$

where the last inequality is because $q^*(\hat{p}_{-1}, \alpha) < 0$. Therefore the derivative of $\int_{q^*(\hat{p}_{-1}, \alpha)}^{+\infty} f(\alpha, q, \hat{p}_{-1}, -1) dq$ with respect to $\alpha$ is

$$\frac{\partial}{\partial \alpha} \int_{q^*(\hat{p}_{-1}, \alpha)}^{+\infty} f(\alpha, q, \hat{p}_{-1}, -1) dq = \left[ \frac{1}{N} f(\alpha, q^*(\hat{p}_{-1}, \alpha), \hat{p}_{-1}, -1) + f(\alpha, -Nq^*(\hat{p}_{-1}, \alpha), \hat{p}_{-1}, -1) \right] \frac{\partial (q^*(\hat{p}_{-1}, \alpha))}{\partial \alpha}$$

$$+ \int_{0}^{-Nq^*(\hat{p}_{-1}, \alpha)} \frac{\partial}{\partial \alpha} \left[ \frac{1}{N} f(\alpha, -\frac{q}{N}, \hat{p}_{-1}, -1) + f(\alpha, q, \hat{p}_{-1}, -1) \right] dq$$

$$- f(\alpha, -Nq^*(\hat{p}_{-1}, \alpha), \hat{p}_{-1}, -1) N \frac{\partial (q^*(\hat{p}_{-1}, \alpha))}{\partial \alpha} + \int_{-Nq^*(\hat{p}_{-1}, \alpha)}^{+\infty} \frac{\partial}{\partial \alpha} f(\alpha, q, \hat{p}_{-1}, -1) dq$$

$$= f(\alpha, q^*(\hat{p}_{-1}, \alpha), \sigma_n) \frac{\partial (q^*(\hat{p}_{-1}, \alpha))}{\partial \alpha}$$

$$+ \int_{0}^{-Nq^*(\hat{p}_{-1}, \alpha)} \frac{\partial}{\partial \alpha} \left[ \frac{1}{N} f(\alpha, -\frac{q}{N}, \hat{p}_{-1}, -1) + f(\alpha, q, \hat{p}_{-1}, -1) \right] dq$$

$$+ \int_{-Nq^*(\hat{p}_{-1}, \alpha)}^{+\infty} \frac{\partial}{\partial \alpha} f(\alpha, q, \hat{p}_{-1}, -1) dq.$$

Now take the limit of $\frac{\partial}{\partial \alpha} \int_{q^*(\hat{p}_{-1}, \alpha)}^{+\infty} f(\alpha, q, \hat{p}_{-1}, -1) dq$ when $N \to +\infty$. The first term is positive as $f(\alpha, q^*(\hat{p}_{-1}, \alpha), \sigma_n) < 0$ and $\frac{\partial (q^*(\hat{p}_{-1}, \alpha))}{\partial \alpha} = -\frac{\partial (q^*(\hat{p}_{-1}, \alpha))}{\partial \alpha} < 0$. The second term is
positive as

$$
\lim_{N \to +\infty} \frac{\partial}{\partial \alpha} \left[ \frac{1}{N} f(\alpha, -\frac{q}{N}, \hat{\rho}_{-1}, -1) + f(\alpha, q, \hat{\rho}_{-1}, -1) \right]
= \frac{\partial}{\partial \alpha} f(\alpha, q, \hat{\rho}_{-1}, -1)
\propto \frac{2^{q+1}}{\left[ e^{\frac{2q}{\beta}} (1 - \rho_{-1}) + \rho_{-1} \right] e^{-\frac{(q+1)^2}{2\sigma^2}} \left( e^{\frac{2q}{\beta}} - 1 \right)}
\left[ \alpha(1 + e^{\frac{2q}{\beta}}) + 2(1 - \alpha)(1 - \rho_{-1}) e^{\frac{2q}{\beta}} + 2(1 - \alpha)\rho_{-1} \right]^2
> 0 \text{ when } q > 0.
$$

The third term converges to zero as $-N q^*(\hat{\rho}_{-1}, \alpha) \to +\infty$. Therefore $\frac{\partial}{\partial \alpha} \int_{q^*(\hat{\rho}_{-1}, \alpha)}^{+\infty} f(\alpha, q, \hat{\rho}_{-1}, -1) dq > 0$ when $N \to +\infty$, implying that $\frac{\partial}{\partial \alpha} \int_{q^*(\hat{\rho}_{-1}, \alpha)}^{+\infty} f(\alpha, q, \hat{\rho}_{-1}, -1) dq > 0$ as we can choose a number $N_0$ sufficiently large so that we can change variables and express

$$
\frac{\partial}{\partial \alpha} \int_{q^*(\hat{\rho}_{-1}, \alpha)}^{+\infty} f(\alpha, q, \hat{\rho}_{-1}, -1) dq
= \int_{0}^{-N_0 q^*(\hat{\rho}_{-1}, \alpha)} \left[ \frac{1}{N_0} f(\alpha, -\frac{q}{N_0}, \hat{\rho}_{-1}, -1) + f(\alpha, q, \hat{\rho}_{-1}, -1) \right] dq + \int_{-N_0 q^*(\hat{\rho}_{-1}, \alpha)}^{+\infty} f(\alpha, q, \hat{\rho}_{-1}, -1) dq.
$$

Thus $\int_{q^*(\hat{\rho}_{-1}, \alpha)}^{+\infty} f(\alpha, q, \hat{\rho}_{-1}, -1) dq$ is increasing in $\alpha$. When $\alpha$ is sufficiently small, $q^*(\hat{\rho}_{-1}, \alpha) \to -\infty$ when $\hat{\rho}_{-1} > 0$ and $\int_{q^*(\hat{\rho}_{-1}, \alpha)}^{+\infty} f(\alpha, q, \hat{\rho}_{-1}, -1) dq < 0$, as we already show. When $\alpha \to 1$, $q^*(\hat{\rho}_{-1}, \alpha) \to \frac{\sigma_0^2}{\beta} \ln \frac{\delta + \gamma}{1 - \frac{\beta}{\beta}}$. Note that, when $\delta < 1 - \gamma$, $\frac{\delta + \gamma}{1 - \frac{\beta}{\beta}} < 1$. In addition, $\frac{\delta + \gamma}{1 - \frac{\beta}{\beta}}$ increases in $\delta$, resulting in $q^*(\hat{\rho}_{-1}, \alpha)$ increasing in $\delta$. Since when fixing $\alpha$, $\int_{q^*(\hat{\rho}_{-1}, \alpha)}^{+\infty} f(\alpha, q, \hat{\rho}_{-1}, -1) dq$ increases in $q^*(\hat{\rho}_{-1}, \alpha)$ when $q^*(\hat{\rho}_{-1}, \alpha) < 0$, $\int_{q^*(\hat{\rho}_{-1}, \alpha)}^{+\infty} f(\alpha, q, \hat{\rho}_{-1}, -1) dq$ increases in $\delta$. When $\delta \to 1 - \gamma$, $\frac{\delta + \gamma}{1 - \frac{\beta}{\beta}} \to 1$ and $q^*(\hat{\rho}_{-1}, \alpha) \to 0$ and $\int_{q^*(\hat{\rho}_{-1}, \alpha)}^{+\infty} f(\alpha, q, \hat{\rho}_{-1}, -1) dq > 0$. When $\delta \to 0$, $\frac{\delta + \gamma}{1 - \frac{\beta}{\beta}} \to \gamma$, implying that $q^*(\hat{\rho}_{-1}, \alpha)$ is increasing in $\gamma$. When $\gamma \to 0$, then $q^*(\hat{\rho}_{-1}, \alpha) \to -\infty$ and $\int_{q^*(\hat{\rho}_{-1}, \alpha)}^{+\infty} f(\alpha, q, \hat{\rho}_{-1}, -1) dq < 0$. When $\gamma \to 1$ then $q^*(\hat{\rho}_{-1}, \alpha) \to 0$ and $\int_{q^*(\hat{\rho}_{-1}, \alpha)}^{+\infty} f(\alpha, q, \hat{\rho}_{-1}, -1) dq > 0$. Therefore there exists a unique $\alpha_1(\hat{\rho}_{-1}) \in (0, 1)$, which is also a function of $\gamma$ and $\delta$ such that short selling is optimal form the uninformed speculator if $\alpha > \alpha_1$ and $\gamma > \gamma_1$, where $\gamma_1$ is the unique solution to

$$
\int_{\frac{\sigma_0^2}{\beta} \ln \gamma_1}^{+\infty} f(1, q, \hat{\rho}_{-1}, -1) dq = 0,
$$

(10)
and \( \alpha_1(\hat{\rho}_{-1}) \in (0,1) \) is the unique solution to

\[
\int_{q^*(\hat{\rho}_{-1}, \alpha_1(\hat{\rho}_{-1}))}^{+\infty} f(\alpha_1(\hat{\rho}_{-1}), q, \hat{\rho}_{-1}, 1) dq = 0.
\]

Note that, since \( f(1, q, \hat{\rho}_{-1}, -1) \) is independent of \( \hat{\rho}_{-1} \), \( \gamma_1 \) and \( \alpha_1 \) are independent of \( \hat{\rho}_{-1} \).

When \( \hat{\rho}_{-1} = 0 \), we have \( \lim_{q \to -\infty} \beta(q, \hat{\rho}_{-1}, \alpha) = 0 \). Therefore \( q^*(0, \alpha) > -\infty \). However, the above proof still shows that \( \int_{q^*(0, \alpha)}^{+\infty} f(\alpha, q, 0, -1) dq = 0 \). In addition,

\[
\int_{q^*(0,0)}^{+\infty} f(0, q, 0, -1) dq = 0
\]
as \( f(0, q, 0, -1) = 0 \) \( \forall q \). Therefore \( \alpha_1(\hat{\rho}_{-1}) = 0 \) when \( \hat{\rho}_{-1} = 0 \). □

**Lemma 8** \( \beta(q) \) increases in \( q \). \( q^* \) increases in \( \delta \).

**Proof Lemma 8:**

**Proof.** From equation (8) we have

\[
\beta(q, \rho_{-1}, \alpha) = \frac{\alpha \phi\left(\frac{q - 1}{\sigma_0^2 + \sigma_2^2} + \rho_{-1}\phi\left(\frac{q + 1}{\sigma_0^2 + \sigma_2^2}\right)\right) + (1 - \alpha)\left(1 - \rho_{-1}\phi\left(\frac{q - 1}{\sigma_0^2 + \sigma_2^2}\right) + \rho_{-1}\phi\left(\frac{q + 1}{\sigma_0^2 + \sigma_2^2}\right)\right)}{\alpha \phi\left(\frac{q - 1}{\sigma_0^2 + \sigma_2^2} + \rho_{-1}\phi\left(\frac{q + 1}{\sigma_0^2 + \sigma_2^2}\right) + \rho_{-1}\phi\left(\frac{q + 1}{\sigma_0^2 + \sigma_2^2}\right)\right) + 1}.
\]

It is straightforward to verify that \( \frac{\partial \beta(q^*)}{\partial q} > 0 \). Intuitively, a higher \( q \) increases investors' posterior belief about the high state.

Note that \( q^* \) solves \( \beta(q^*) = \frac{\delta + \gamma}{1 + \gamma} \). Taking the derivative of this equation with respect to \( \delta \), we have \( \frac{\partial \beta(q^*)}{\partial q^*} \frac{\partial q^*}{\partial \delta} = \frac{\delta + \gamma}{\delta + 1 + \gamma} > 0 \). Therefore \( \frac{\partial q^*}{\partial \delta} > 0 \). □

**Proof of Proposition 1:**

**Proof.** The optimal trading strategies of the speculators follow from Lemma 2 and Lemma 6. We now derive the investment threshold, denoted as \( q^*_{B} \).
Insert $\rho_{-1} = 0$ into equation (8) and rearranging terms results in

$$
\beta(q, 0, \alpha) = \frac{\phi\left(\frac{q-1}{\sqrt{\sigma_n^2 + \sigma^2}\rho}\right)}{\alpha(1 - \phi\left(\frac{q-1}{\sqrt{\sigma_n^2 + \sigma^2}\rho}\right)) + (2 - \alpha)\phi\left(\frac{q-1}{\sqrt{\sigma_n^2 + \sigma^2}\rho}\right)}.
$$

Insert the expression of $\beta(q, 0, \alpha)$ into equation (7) and rearranging terms results in

$$
\frac{\phi\left(\frac{q_B^* - 1}{\sigma_n^2 + \sigma^2}\right)}{\alpha \phi\left(\frac{q_B^*}{\sigma_n^2 + \sigma^2}\right)} + 1 - \alpha = \frac{\delta + \gamma}{2}.
$$

Note that the left hand side of equation (11) is increasing in $q_B^*$. When $q_B^* \to +\infty$, the left hand side becomes $+\infty$, which is clearly larger than the right hand side. When $q_B^* \to -\infty$, the left hand side becomes $1 - \alpha$. Thus equation (11) has a solution if and only if $1 - \alpha < \frac{\delta + \gamma}{2}$, or $\alpha > 1 - \frac{\delta + \gamma}{2}$. Solving equation (11) results in

$$
q_B^* = \left(\sigma_n^2 + \sigma^2\right) \ln \frac{\alpha - \left(1 - \frac{\delta + \gamma}{1 - \frac{\delta + \gamma}{2}}\right)}{\alpha + 1}.
$$

Since $\ln \frac{\alpha - \left(1 - \frac{\delta + \gamma}{1 - \frac{\delta + \gamma}{2}}\right)}{\alpha + 1} < 0$, we have $q_B^* < \frac{1}{2}$. In addition, when $\alpha \leq 1 - \frac{\delta + \gamma}{2}$, we have $q_B^* = -\infty < \frac{1}{2}$.

Finally, $q_B^*$ is strictly increasing in $\alpha$ and $\delta$ as

$$
\frac{\partial q_B^*}{\partial \alpha} = \left(\sigma_n^2 + \sigma^2\right) \frac{1 - \left(1 - \frac{\delta + \gamma}{1 - \frac{\delta + \gamma}{2}}\right)}{\alpha\left(1 - \left(1 - \frac{\delta + \gamma}{1 - \frac{\delta + \gamma}{2}}\right)\right)} > 0,
$$

and

$$
\frac{\partial q_B^*}{\partial \delta} = \left(\sigma_n^2 + \sigma^2\right) \frac{\alpha}{\alpha - \left(1 - \frac{\delta + \gamma}{1 - \frac{\delta + \gamma}{2}}\right)} \frac{2(1 - L)}{(1 - \frac{\delta + \gamma}{1 - \frac{\delta + \gamma}{2}})^2} > 0.
$$

Proof of Proposition 2:

Proof. The proposition is proved in four steps. From Lemma 6, we know that $d(\emptyset) = 1$ is
never optimal, which leaves the possibility of either \( d(\emptyset) = -1 \) being optimal, \( d(\emptyset) = 0 \) being optimal, or a mixed strategy between \( d(\emptyset) = -1 \) and \( d(\emptyset) = 0 \) being optimal. We divide the proof into two steps. In step 1, we derive the conditions under which \( d(\emptyset) = -1 \) is the dominant strategy. In step 2, we derive the investment threshold when \( d(\emptyset) = -1 \).

**Step 1:** Recall that \( d(\emptyset) = -1 \) is dominant if and only if 

\[
\int_{q^*(\bar{\rho}_{-1}, \alpha)}^{+\infty} \left[ \beta(q, \bar{\rho}_{-1}, \alpha) - \frac{1}{2} (1 - L) \frac{1}{\sigma_n} \phi \left( \frac{q + 1}{\sigma_n} \right) \right] dq > 0,
\]

or, equivalently,

\[
\int_{q^*(\bar{\rho}_{-1}, \alpha)}^{+\infty} \left[ \beta(q, \bar{\rho}_{-1}, \alpha) - \frac{1}{2} \frac{1}{\sigma_n} \phi \left( \frac{q + 1}{\sigma_n} \right) \right] dq
\]

\[
\equiv \int_{q^*(\bar{\rho}_{-1}, \alpha)}^{+\infty} f(\alpha, q, \bar{\rho}_{-1}, -1) dq > 0,
\]

where

\[
f(\alpha, q, \bar{\rho}_{-1}, d) \equiv \left[ \beta(q, \bar{\rho}_{-1}, \alpha) - \frac{1}{2} \frac{1}{\sigma_n} \phi \left( \frac{q - d}{\sigma_n} \right) \right].
\]

From Lemma 7, we know \( \int_{q^*(\bar{\rho}_{-1}, \alpha)}^{+\infty} f(\alpha, q, \bar{\rho}_{-1}, -1) dq > 0 \) if and only if \( \alpha > \alpha_1(\bar{\rho}_{-1}) \), where \( \alpha_1(\bar{\rho}_{-1}) \) is a continuous function and uniquely satisfies

\[
\int_{q^*(\bar{\rho}_{-1}, \alpha_1(\bar{\rho}_{-1}))}^{+\infty} f(\alpha_1(\bar{\rho}_{-1}), q, \bar{\rho}_{-1}, -1) dq = 0. \tag{13}
\]

By the Extreme Value Theorem, we know that \( \alpha_1(\bar{\rho}_{-1}) \) attains its maximum for some \( \bar{\rho}_{-1} \in [0, 1] \). Let this maximum value be \( \bar{\alpha} \); i.e.,

\[
\bar{\alpha} = \max_{\bar{\rho}_{-1} \in [0, 1]} \alpha_1(\bar{\rho}_{-1}). \tag{14}
\]

Therefore we have that \( d(\emptyset) = -1 \) is the unique strategy for the uninformed speculator if and only if \( \alpha > \bar{\alpha} \).

**Step 2:** Derive the investment thresholds when \( \rho_{-1} = 1 \).
From equation (8) we have

\[
\beta(q, \rho_{-1}, \alpha) = \alpha \phi\left(\frac{q - 1}{\sqrt{\sigma_n^2 + \sigma_z^2}}\right) + (1 - \alpha)\phi\left(\frac{q + 1}{\sqrt{\sigma_n^2 + \sigma_z^2}}\right) - \alpha \phi\left(\frac{q - 1}{\sqrt{\sigma_n^2 + \sigma_z^2}}\right) + (1 - \alpha)\phi\left(\frac{q + 1}{\sqrt{\sigma_n^2 + \sigma_z^2}}\right) - \alpha \phi\left(\frac{q - 1}{\sqrt{\sigma_n^2 + \sigma_z^2}}\right) + (1 - \alpha)\phi\left(\frac{q + 1}{\sqrt{\sigma_n^2 + \sigma_z^2}}\right) + 1.
\]

Note that the investment threshold \(q^*\) is determined by equation (7), or equivalently, \(\beta(q^*, \rho_{-1}, \alpha) = \frac{\delta + \gamma}{1 - \frac{\delta}{2}}\).

When \(\rho_{-1} = 1\), denote the investment threshold as \(q_A^*\). Insert the expression of \(\beta(q_A^*, \rho_{-1}, \alpha)\) into equation (7) and rearranging terms result in

\[
\frac{\phi\left(\frac{q_A^* - 1}{\sqrt{\sigma_n^2 + \sigma_z^2}}\right)}{\phi\left(\frac{q_A^* + 1}{\sqrt{\sigma_n^2 + \sigma_z^2}}\right)} + 1 - \alpha = \frac{\delta}{2} + \gamma, \quad (15)
\]

First note that equation (15) has a solution \(\alpha \in (0, 1)\) only if \(1 - \alpha < \frac{\delta + \gamma}{1 - \frac{\delta}{2}}\), or equivalently, \(\alpha > 1 - \frac{\delta + \gamma}{1 - \frac{\delta}{2}}\). When \(\alpha \leq 1 - \frac{\delta + \gamma}{1 - \frac{\delta}{2}}\), then

\[
\frac{\phi\left(\frac{q_A^* - 1}{\sqrt{\sigma_n^2 + \sigma_z^2}}\right)}{\phi\left(\frac{q_A^* + 1}{\sqrt{\sigma_n^2 + \sigma_z^2}}\right)} + 1 - \alpha \geq \frac{\delta}{2} + \gamma, \quad (15)
\]

implying that it is always optimal for the investors to stay, or equivalently, \(q_A^* = -\infty\).

When \(\alpha > 1 - \frac{\delta + \gamma}{1 - \frac{\delta}{2}}\), one can also solve for a closed-form solution of \(q_A^*\) from equation (15) to be

\[
q_A^* = -\frac{1}{2}\left[\ln(\alpha - (1 - \frac{\delta}{2} + \gamma)) - \ln \alpha\right] + \frac{\sigma_n^2 + \sigma_z^2}{2}\left[\ln(\alpha - (1 - \frac{\delta}{2} + \gamma)) - \ln \alpha\right]
\]

\[
= q_B^* - \frac{1}{2}, \quad (16)
\]

where the last equality comes from comparing \(q_A^*\) and \(q_B^*\) from equation (12). Again since \(\ln(\alpha - (1 - \frac{\delta}{2} + \gamma)) - \ln \alpha < 0\), \(q_A^* < 0\).
Since \( q_A^* = \frac{q_B^* - \frac{1}{2}}{2} \),
\[
\frac{\partial q_A^*}{\partial \alpha} = \frac{1}{2} \frac{\partial q_B^*}{\partial \alpha} > 0,
\]
and
\[
\frac{\partial q_A^*}{\partial \delta} = \frac{1}{2} \frac{\partial q_B^*}{\partial \delta} > 0,
\]
where the comparative statics for \( q_B^* \) come from Proposition 1. ■

**Proof of Lemma 3:**

**Proof.** Recall from equation (16) that

\[
q_B^* = 2q_A^* + \frac{1}{2}.
\]

Therefore \( q_A^* < q_B^* \) if and only if

\[
q_A^* > -\frac{1}{2}.
\]

Recall that when \( \sigma \to 0 \),
\[
q_A^* = \frac{\sigma_n^2}{2} \ln[1 - \frac{1 - \gamma - \delta}{\alpha}]\]

is increasing in \( \alpha \). Thus \(-\frac{1}{2} < q_A^* < 0\) if and only if

\[
\alpha > \alpha_2 \equiv \frac{1 - \gamma - \delta}{(1 - \frac{\delta}{2})(1 - e^{-\frac{1}{\sigma_n^2}})} > 1 - \frac{\delta}{2} + \frac{\gamma}{1 - \frac{\delta}{2}}.
\]

(17)

Note that equation (17) will only be satisfied if

\[
1 - \gamma - \delta < (1 - \frac{\delta}{2})(1 - e^{-\frac{1}{\sigma_n^2}}) < 1.
\]

(18)

Note that \( \frac{1 - \gamma - \delta}{(1 - \frac{\delta}{2})(1 - e^{-\frac{1}{\sigma_n^2}})} \) is decreasing in \( \delta \) so equation (18) will be satisfied if

\[
1 - \gamma < 1 - e^{-\frac{1}{\sigma_n^2}},
\]

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or equivalently,
\[ \gamma > e^{-\frac{1}{\sigma_n^2}}. \]

Now define
\[ \overline{\gamma} = \max(\gamma_1, e^{-\frac{1}{\sigma_n^2}}). \]  \hfill (19)

Thus, when \( \gamma > \overline{\gamma} \) and \( \alpha > \alpha_2 \), \( q_A^* > -\frac{1}{2} \).

In addition, since \( q_A^* < 0 \), we also have \( q_B^* < q_A^* + 1 \), resulting in \( q_A^* < q_B^* < q_A^* + 1 \) when \( \gamma > \overline{\gamma} \).

We then have that when \( \gamma > \overline{\gamma} \) and \( \alpha > \alpha_2 \), \( q_A^* < q_B^* < q_A^* + 1 \) and Proposition 2 holds.

Finally, note that \( q_B^* = 2q_A^* + \frac{1}{2} \) and \( -\frac{1}{2} < q_A^* < 0 \) implies that \( -\frac{1}{2} < q_B^* < \frac{1}{2} \). This also implies that
\[ \phi(\frac{q_B^*}{\sigma_n}) > \phi(\frac{q_A^* + 1}{\sigma_n}), \]

as
\[ |q_B^*| < \frac{1}{2} < q_A^* + 1, \]

which is used in the proof of Proposition 3. \( \blacksquare \)

**Proof of Lemma 4:**

**Proof.** Note that when \( \alpha_1 > \alpha > \alpha_2 \),

\[ \Delta \varepsilon_i^H \]
\[ = \varepsilon_{IB}^H - \varepsilon_{IA}^H \]
\[ = \Phi(\frac{q_B^* - 1}{\sigma_n}) - \Phi(\frac{q_A^* - 1}{\sigma_n}) > 0, \]

and

\[ \Delta \varepsilon_i^L \]
\[ = \varepsilon_{IB}^L - \varepsilon_{IA}^L \]
\[ = \Phi(\frac{q_A^* + 1}{\sigma_n}) - \Phi(\frac{q_B^*}{\sigma_n}) > 0, \]

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as \( q_A^* < q_B^* < q_A^* + 1 \).

Part 1 is thus proved.

In addition,

\[
\Delta \varepsilon_U^H = \varepsilon_{UB}^H - \varepsilon_{UA}^H = \Phi(q_B^*/\sigma_n) - \Phi(q_A^*/\sigma_n) < 0,
\]

as \( q_B^* < q_A^* + 1 \).

Finally,

\[
\Delta \varepsilon_L^L = \varepsilon_{UB}^L - \varepsilon_{UA}^L = -\Phi(q_B^*/\sigma_n) + \Phi(q_A^*/\sigma_n) = \Delta \varepsilon_U^H > 0.
\]

Thus, parts 2 and 3 are proved. ■

**Proof of Lemma 5:**

**Proof.** Firstly, since \( \alpha_2(\delta) = \frac{1-\gamma-\delta}{(1-\frac{1}{2})(1-\frac{1}{\sigma_n^2})} \), it is straightforward to show that \( \alpha_2(\delta) \) is decreasing in \( \delta \) as the derivative \( \frac{\partial \alpha_2(\delta)}{\partial \delta} = -\frac{2}{(2-\delta)^2(1-\frac{1}{\sigma_n^2})} < 0 \).

We now prove that \( \alpha(\delta) \) is decreasing in \( \delta \). We divide the proof into two steps. In the first step, we show that \( \alpha_1(\tilde{\rho}_{-1}, \delta) \) is a decreasing function of \( \delta \) for any \( \tilde{\rho}_{-1} \). In the second step, we show that \( \alpha(\delta) = \max_{\tilde{\rho}_{-1}} \alpha_1(\tilde{\rho}_{-1}, \delta) \) is a decreasing function of \( \delta \).

**Step 1**

Remember that \( \alpha_1(\tilde{\rho}_{-1}, \delta) \) uniquely satisfies equation (13), or equivalently

\[
\int_{q^*(\tilde{\rho}_{-1}, \alpha_1, \delta)}^{+\infty} \left( \beta(q, \tilde{\rho}_{-1}, \alpha_1) - \frac{1}{2} \frac{1}{\sigma_n} \phi(q/\sigma_n) \right) dq = 0.
\]

Taking derivative with respect to \( \delta \) results in

\[
\frac{\partial}{\partial \alpha_1} \int_{q^*(\tilde{\rho}_{-1}, \alpha_1, \delta)}^{+\infty} (\beta(q, \tilde{\rho}_{-1}, \alpha_1) - \frac{1}{2} \frac{1}{\sigma_n} \phi(q/\sigma_n)) dq \frac{\partial \alpha_1}{\partial \delta} - \frac{1}{2} \frac{1}{\sigma_n} \phi(q^*/\sigma_n) \frac{\partial q^*}{\partial \delta} = 0.
\]
Rearranging terms results in

\[ \frac{\partial \alpha_1(\delta)}{\partial \delta} = \frac{\partial}{\partial \alpha_1} \int_{\hat{q}^-}^{+\infty} \beta(q^*, \hat{\rho}_{-1}, \alpha_1) - \frac{1}{2} \frac{1}{\sigma_n} \phi\left(\frac{q^*+1}{\sigma_n}\right) dq. \tag{20} \]

Note that, based on equation (7), \( \beta(q^*, \hat{\rho}_{-1}, \alpha_1) = \frac{\delta + \gamma - 1}{(1 + \gamma)} \). Therefore \( \beta(q^*, \hat{\rho}_{-1}, \alpha_1) - \frac{1}{2} = \frac{\delta + \gamma - 1}{2(1 + \gamma)} < 0 \), where the inequality comes from Assumption A2.

The numerator of equation (20) is thus negative because \( \frac{\partial q^*}{\partial \delta} > 0 \), according to Lemma 8. The denominator of equation is (20) is positive based on Lemma 7. Therefore \( \frac{\partial \alpha_1(\delta)}{\partial \delta} < 0 \).

This completes the proof of step 1.

**Step 2**

To show that \( \underline{\alpha}(\delta) \) decreases in \( \delta \), it is equivalent to show that, for any \( \delta_1 < \delta_2 \), \( \underline{\alpha}(\delta_1) > \underline{\alpha}(\delta_2) \).

By definition, there exists \( \rho_1 \in [0, 1] \) and \( \rho_2 \in [0, 1] \) such that \( \alpha_1(\rho_1, \delta_1) = \overline{\alpha}(\delta_1) \), and \( \alpha_1(\rho_2, \delta_2) = \underline{\alpha}(\delta_2) \). We further have \( \overline{\alpha}(\delta_1) = \alpha_1(\rho_1, \delta_1) \geq \alpha_1(\rho_2, \delta_1) > \alpha_1(\rho_2, \delta_2) = \underline{\alpha}(\delta_2) \).

The first inequality is due to the definition of \( \rho_1 \), and the second is due to the monotonicity of \( \alpha_1(\hat{\rho}_{-1}, \delta) \) with respect to \( \delta \). Therefore \( \underline{\alpha}(\delta) \) decreases in \( \delta \). \( \blacksquare \)

**Proof of Proposition 3:**

**Proof.** We first prove the proposition when \( \alpha_2 \leq \overline{\alpha} \), resulting in \( q^*_A > -\frac{1}{2} \) when \( \alpha > \overline{\alpha} \). It is proved in four steps. In the final step, we prove the proposition when \( \alpha_2 > \alpha \).

**Step 1:** Prove that short selling is uniquely optimal for the uninformed speculator if \( \alpha > \overline{\alpha}(\delta) \).

This follows from Proposition 2.

**Step 2:** Prove that \( \frac{\partial \Delta V}{\partial \beta} < 0 \) and \( \frac{\partial \Delta V}{\partial \alpha} < 0 \) when \( \alpha > \overline{\alpha} \).

Recall from equation (5) that

\[ \Delta V = -\{\alpha \Delta \varepsilon^H_I + [\alpha(1-\gamma)] \Delta \varepsilon_0 \} = -\{\alpha \Delta \varepsilon^H_I + [\alpha(1-\gamma)] \Delta \varepsilon_0 \}. \]
Therefore

\[ \frac{\partial \Delta V}{\partial \delta} \approx -\alpha \frac{\partial \Delta}{} - [\alpha - (1 - \gamma)] \frac{\partial \Delta}{\partial \delta} \]

\[ = - \frac{1}{\sigma_n} \frac{\partial q^*_A}{\partial \delta} \{ \alpha[2\phi(\frac{q^*_B - 1}{\sigma_n}) - \phi(\frac{q^*_A - 1}{\sigma_n})] + [\alpha - (1 - \gamma)](\phi(\frac{q^*_A + 1}{\sigma_n}) - 2\phi(\frac{q^*_B}{\sigma_n})) \}. \]

Since \( \frac{\partial q^*_A}{\partial \delta} > 0 \) from Proposition 2,

\[ sgn(\frac{\partial \Delta V}{\partial \delta}) = sgn\{-\alpha[2\phi(\frac{q^*_B - 1}{\sigma_n}) - \phi(\frac{q^*_A - 1}{\sigma_n})] - [\alpha - (1 - \gamma)](\phi(\frac{q^*_A + 1}{\sigma_n}) - 2\phi(\frac{q^*_B}{\sigma_n})) \}. \]

Note that

\[ -\alpha[2\phi(\frac{q^*_B - 1}{\sigma_n}) - \phi(\frac{q^*_A - 1}{\sigma_n})] - [\alpha - (1 - \gamma)](\phi(\frac{q^*_A + 1}{\sigma_n}) - 2\phi(\frac{q^*_B}{\sigma_n})) \]

\[ = -2\alpha(\frac{d}{\sigma_n} + 2)[\alpha - (1 - \gamma)]\phi(\frac{q^*_B}{\sigma_n}) + \alpha\phi(\frac{q^*_A - 1}{\sigma_n}) - [\alpha - (1 - \gamma)]\phi(\frac{q^*_A + 1}{\sigma_n}) \].

Rearranging terms in equations (11) and (15) results in

\[ \frac{\phi(q^*_A - 1)}{\phi(q^*_A + 1)} = \frac{1}{\alpha \left( 1 - \frac{\delta^2}{2} \right)} - (1 - \alpha) \].

(22)

and

\[ \frac{\phi(q^*_B - 1)}{\phi(q^*_B)} = \frac{1}{\alpha \left( 1 - \frac{\delta^2}{2} \right)} - (1 - \alpha) \].

(23)
Inserting equations (22) and (23) into equation (21) and rearranging terms result in

\[-2\alpha \phi\left(\frac{q_B^* - 1}{\sigma_n}\right) + 2[\alpha - (1 - \gamma)]\phi\left(\frac{q_B^*}{\sigma_n}\right) + \alpha \phi\left(\frac{q_A^* - 1}{\sigma_n}\right) - [\alpha - (1 - \gamma)]\phi\left(\frac{q_A^* + 1}{\sigma_n}\right)\]

\[= 2\phi\left(\frac{q_B^*}{\sigma_n}\right)\{-\alpha \frac{1}{\alpha} \frac{\delta + \gamma}{2} - (1 - \alpha)\} + \alpha - (1 - \gamma)\} + \phi\left(\frac{q_A^* + 1}{\sigma_n}\right)\{\alpha \frac{1}{\alpha} \frac{\delta + \gamma}{2} - (1 - \alpha)\} - [\alpha - (1 - \gamma)]\}

\[= 2\phi\left(\frac{q_B^*}{\sigma_n}\right)(\gamma - \frac{\delta + \gamma}{1 - \frac{\delta}{2}}) - \phi\left(\frac{q_A^* + 1}{\sigma_n}\right)(\gamma - \frac{\delta + \gamma}{1 - \frac{\delta}{2}})\]

\[= [2\phi\left(\frac{q_B^*}{\sigma_n}\right) - \phi\left(\frac{q_A^* + 1}{\sigma_n}\right)](\gamma - \frac{\delta + \gamma}{1 - \frac{\delta}{2}}) < 0,\]

as \(2\phi\left(\frac{q_B^*}{\sigma_n}\right) - \phi\left(\frac{q_A^* + 1}{\sigma_n}\right) > 0\) from Lemma 3 and \(\gamma - \frac{\delta + \gamma}{1 - \frac{\delta}{2}} < 0\). Therefore

\[\frac{\partial \Delta V}{\partial \alpha} < 0.\]

Taking the derivative of \(\Delta V\) with respect to \(\alpha\) results in

\[\frac{\partial \Delta V}{\partial \alpha} \propto -\alpha \frac{\partial \Delta \hat{\gamma}_H}{\partial \alpha} + [\alpha - (1 - \gamma)]\frac{\partial \Delta \varepsilon_0}{\partial \alpha} - \Delta \varepsilon_H - \Delta \varepsilon_0\]

\[= -\alpha \frac{\partial \Delta \hat{\gamma}_H}{\partial \alpha} + [\alpha - (1 - \gamma)]\frac{\partial \Delta \varepsilon_0}{\partial \alpha} - \Delta \varepsilon_H - \Delta \varepsilon_0\]

\[= -\frac{1}{\sigma_n} \frac{\partial q_A^*}{\partial \alpha}\{\alpha[2\phi\left(\frac{q_B^* - 1}{\sigma_n}\right) - \phi\left(\frac{q_A^* - 1}{\sigma_n}\right)] + [\alpha - (1 - \gamma)]\phi\left(\frac{q_A^* + 1}{\sigma_n}\right) - 2\phi\left(\frac{q_B^*}{\sigma_n}\right)\} - \Delta \varepsilon_H - \Delta \varepsilon_0.\]

Again, since \(\frac{\partial q_A^*}{\partial \alpha} > 0\) from Proposition 2,

\[\text{sgn}\{-\frac{1}{\sigma_n} \frac{\partial q_A^*}{\partial \alpha}\{\alpha[2\phi\left(\frac{q_B^* - 1}{\sigma_n}\right) - \phi\left(\frac{q_A^* - 1}{\sigma_n}\right)] + [\alpha - (1 - \gamma)]\phi\left(\frac{q_A^* + 1}{\sigma_n}\right) - 2\phi\left(\frac{q_B^*}{\sigma_n}\right)\}\}

\[= \text{sgn}\{-\alpha[2\phi\left(\frac{q_B^* - 1}{\sigma_n}\right) - \phi\left(\frac{q_A^* - 1}{\sigma_n}\right)] + [\alpha - (1 - \gamma)]\phi\left(\frac{q_A^* + 1}{\sigma_n}\right) - 2\phi\left(\frac{q_B^*}{\sigma_n}\right)\}. \tag{24}\]
Inserting equations (22) and (23) into equation (24) and rearranging terms result in

\[-\alpha [2\phi(q_B^* - 1) - \phi(q_A^* - 1)] - [\alpha - (1 - \gamma)][\phi(q_A^* + 1) - 2\phi(q_B^*)]\]

\[= -2\alpha\phi(q_B^* - 1) + 2[\alpha - (1 - \gamma)]\phi(q_B^*) + \alpha\phi(q_A^* - 1) - [\alpha - (1 - \gamma)]\phi(q_A^* + 1)\]

\[= 2\phi(q_B^*)(\gamma - \frac{\delta + \gamma}{1 - \frac{\delta}{2}} - (1 - \alpha) + \alpha - (1 - \gamma)) + \phi(q_A^*)\alpha[\frac{\delta + \gamma}{1 - \frac{\delta}{2}} - (1 - \alpha)] - [\alpha - (1 - \gamma)]\]

\[= 2\phi(q_B^*)(\gamma - \frac{\delta + \gamma}{1 - \frac{\delta}{2}}) - \phi(q_A^*)(\gamma - \frac{\delta + \gamma}{1 - \frac{\delta}{2}})\]

\[< 0,
\]

where we used equation (21) to arrive at the first equality.

Therefore \(\frac{\partial V}{\partial \alpha} < 0\) as the rest two terms of \(\frac{\partial V}{\partial \alpha}\) are clearly negative.

**Step 3: Prove that** \(\Delta V(\delta = 0, \alpha = \alpha(0) + \varepsilon) < 0, \Delta V(\delta = 1 - \gamma, \alpha = \alpha(1 - \gamma) + \varepsilon) > 0\) **and** \(\Delta V(\delta = 1 - \gamma, \alpha = 1) < 0\) **for** \(\varepsilon\) **sufficiently small.**

When \(\delta = 0\), there is no coordination problem among investors. We prove the first inequality and therefore Corollary 1 for the case in which MSS is the unique equilibrium.

The proof for the case in which the uninformed speculator plays a mixed strategy is very similar and is therefore omitted here. We first write down the firm value when short selling is allowed:

\[V_{Allowed} = (\alpha(1 - \Phi(q_A^* - 1)) + (1 - \alpha)(1 - \Phi(q_A^* + 1))) + L(1 - \Phi(q_A^* + 1))\]

\[\geq (\alpha(1 - \Phi(q_B^* - 1)) + (1 - \alpha)(1 - \Phi(q_B^* + 1))) + L(1 - \Phi(q_B^* + 1)).\]

The inequality is the first key of the proof. It says that, by changing the investment threshold from \(q_A^*\) to \(q_B^*\), firm value should decrease. This is because, when \(\delta = 0\), each investor is behaving as if he or she is choosing a threshold \(q^*\) to maximize firm value. Since the optimal threshold is \(q_A^*\), firm value should be weakly lower if the threshold is \(q_B^*\).

The second key of the proof is the fact that \(1 - \alpha - \gamma < 0\) when MSS is the unique
This follows from the proof of Lemma 7. In fact, if \(1 - \alpha - \gamma > 0\), then MSS does not exist, as the existence of MSS requires that \(\alpha > 1 - \frac{\delta + \gamma}{1 - \frac{\delta}{2}}\), which reduces to \(\alpha > 1 - \gamma\) when \(\delta = 0\). Hence we have

\[
1 - \alpha - \gamma < 0
\]

\[
\Leftrightarrow 1 - \alpha + L < 0
\]

\[
\Leftrightarrow (1 - \alpha) + L < 0
\]

\[
\Leftrightarrow (1 - \alpha)[\Phi(q^*_{B} + 1) - \Phi(q^*_{B})] + L[\Phi(q^*_{B} + 1) - \Phi(q^*_{B})] < 0
\]

\[
\Leftrightarrow (1 - \alpha)[1 - \Phi(q^*_{B} + 1)] + L[1 - \Phi(q^*_{B} + 1)] > (1 - \alpha)[1 - \Phi(q^*_{B})] + L[1 - \Phi(q^*_{B})].
\]

As a result,

\[
V_{Allowed} \geq \{\alpha[1 - \Phi(\frac{q^*_{B} - 1}{\sigma_n})] + (1 - \alpha)[1 - \Phi(\frac{q^*_{B} + 1}{\sigma_n})]\} + L[1 - \Phi(\frac{q^*_{B} + 1}{\sigma_n})]
\]

\[
> \{\alpha[1 - \Phi(\frac{q^*_{B} - 1}{\sigma_n})] + (1 - \alpha)[1 - \Phi(\frac{q^*_{B}}{\sigma_n})]\} + L[1 - \Phi(\frac{q^*_{B}}{\sigma_n})]
\]

\[
= V_{Ban}.
\]

This suggests that, in the absence of the coordination game, banning short selling always reduces firm value. Furthermore, by continuity, \(\Delta V(\delta = 0, \alpha = \alpha(0) + \varepsilon) < 0\) for sufficiently small \(\varepsilon\).

When \(\delta \to 1 - \gamma\), \(q^*_A \to 0\), \(\alpha \to 0\) and \(q^*_B \to \frac{1}{2}\). In addition,

\[
\Delta \varepsilon^H \to \Phi(-\frac{1}{2\sigma_n}) - \Phi(-\frac{1}{\sigma_n}),
\]

and

\[
\Delta \varepsilon_0 \to \Phi(\frac{1}{\sigma_n}) - \Phi(\frac{1}{2\sigma_n}).
\]
Note that
\[
\Phi\left(-\frac{1}{2} \sigma_n\right) - \Phi\left(-\frac{1}{\sigma_n}\right) \\
= \left[1 - \Phi\left(\frac{1}{2} \sigma_n\right)\right] - \left[1 - \Phi\left(\frac{1}{\sigma_n}\right)\right] \\
= \Phi\left(\frac{1}{\sigma_n}\right) - \Phi\left(\frac{1}{2} \sigma_n\right),
\]
where we use \(\Phi(-x) = 1 - \Phi(x)\) to arrive at the second equality. Thus
\[
\Delta V \rightarrow \alpha [\Phi(-\frac{1}{2} \sigma_n) - \Phi(-\frac{1}{\sigma_n})] - [\alpha - (1 - \gamma)] [\Phi\left(\frac{1}{\sigma_n}\right) - \Phi\left(\frac{1}{2 \sigma_n}\right)] \\
= (1 - 2\alpha - \gamma) [\Phi\left(\frac{1}{\sigma_n}\right) - \Phi\left(\frac{1}{2 \sigma_n}\right)],
\]
which is positive if and only if \(\alpha < \frac{1}{2}(1 - \gamma)\). Therefore \(\Delta V(\delta = 1 - \gamma, \alpha = \alpha(1 - \gamma) + \varepsilon) > 0\) when \(\varepsilon < \frac{1}{2}(1 - \gamma)\).

Finally,
\[
\Delta V(\delta = 1 - \gamma, \alpha = 1) = [-1 - \gamma] [\Phi\left(\frac{1}{\sigma_n}\right) - \Phi\left(\frac{1}{2 \sigma_n}\right)] < 0.
\]

Thus, step 3 is proved.

**Step 4: Prove the Proposition for the case where \(\alpha(\delta) > \alpha_2(\delta)\).**

First, based on step 2 and step 3, when \(\delta = 1 - \gamma\), because of the Intermediate Value Theorem, there exists \(\alpha^*(\delta)\), which is larger than \(\alpha(\delta)\), such that \(\Delta V > 0\) if and only if \(\alpha < \alpha^*(\delta)\).

Second, since \(\Delta V\) is a continuous function of \(\delta\) and \(\alpha\), we know that \(\alpha^*(\delta)\) should also exist when \(\delta\) is sufficiently close to \(1 - \gamma\). \(\alpha^*(\delta)\) is uniquely determined by the following equation:
\[
\Delta V(\delta, \alpha^*(\delta)) = 0. \tag{25}
\]

Third, when \(\delta\) is sufficiently small and close to 0, we know from step 3 that \(\Delta V < 0\) for any \(\alpha\). Hence banning short selling always reduces efficiency in this case.

Step 4 is thus proved.

**Step 5: Prove the Proposition for the case where \(\alpha(\delta) < \alpha_2(\delta)\).**
Note that $q_A^* > -\frac{1}{2}$ when $\alpha > \alpha_2(\delta)$. Thus, following steps 1 to 4, we can establish that $\Delta V > 0$ if $\alpha \in (\alpha_2(\delta), \alpha^*(\delta))$, and $\Delta V < 0$ if $\alpha > \alpha^*(\delta)$, where $\alpha^*(\delta)$ is the unique solution of equation (25). Note that, when $\delta$ is sufficiently close to $1 - \gamma$, $\alpha_2(\delta) \to 0$. Thus, by continuity, $(\alpha_2(\delta), \alpha^*(\delta))$ is not empty when $\delta$ is sufficiently large.

We still need to show that, when $\delta$ is sufficiently large, $\Delta V > 0$ if $\alpha \in (\bar{\alpha}(\delta), \alpha_2(\delta))$. Recall that

$$
\Delta V
= \{-\alpha \Delta \xi^H - \alpha \gamma - (1 - \alpha)(1 - \gamma)\} \Delta \xi_0
= \{-\alpha \Delta \xi^H - \alpha (1 - \gamma)\} \Delta \xi_0.
$$

When $\delta$ is sufficiently large such that $\alpha_2(\delta) < 1 - \gamma$, we should have $\alpha < 1 - \gamma$ when $\alpha \in (\bar{\alpha}(\delta), \alpha_2(\delta))$. Furthermore, since $\Delta \xi_0 > 0$ and $\Delta \xi^H \leq 0$ when $\alpha \in (\bar{\alpha}(\delta), \alpha_2(\delta))$, we have $\Delta V > 0$ when $\alpha \in (\alpha(\delta), \alpha_2(\delta)]$. 

Proof of Corollary 2:

Proof. When $\delta \to 1 - \gamma$, we know from the proof of Lemma 7 that $q_A^* \to 0$, and that

$$
\int_{q_A^*}^{+\infty} \beta(q, \tilde{\rho} - 1, \alpha_1) - \frac{1}{2} \tilde{\sigma} \phi \left( \frac{q + 1}{\sigma_n} \right) dq = \int_{\alpha(\delta)}^{\alpha^*(\delta)} f(\alpha_1, q, \tilde{\rho} - 1, -1) dq > 0
$$

always holds. Hence

$$
\alpha_1(\tilde{\rho} - 1, \delta) \to 0
$$

when $\delta \to 1 - \gamma$, which implies $\bar{\alpha}(\delta) = \max_{\tilde{\rho} - 1} \alpha_1(\tilde{\rho} - 1, \delta) \to 0$.

Based on equation (12) and equation (17), we can also easily derive that when $\delta \to 1 - \gamma$, $q_B^* \to \frac{1}{2}$ and $\alpha_2(\delta) \to 0$. The proof of the final part of the Corollary, i.e., short selling ban improves efficiency if $\alpha < \frac{1 - \gamma}{2}$, is contained in the main text and thus omitted here.

References


