Online Appendix to “Optimal Regulation of Financial Intermediaries”

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September 2016

Abstract

In Section 1 I develop the contract environment in detail, and provide a verification theorem for the HJB equation. In Section 2 I show how the competitive equilibrium and the planner’s problem can be characterized with a system of second order PDEs that can be easily solved numerically. In Section 3 I illustrate the numerical methods and the paper’s results with a concrete numerical application where the economy is hit by uncertainty shocks that drive up idiosyncratic risk.

1 Optimal Contracts

This section develops the contractual environment in detail. It is useful for the applications and numerical solutions to include an exogenous retirement time $\tau_i$ which arrives with independent Poisson intensity $\theta$ (this will yield a stationary distribution). After retirement the agent cannot manage capital any longer, and the contract just delivers a terminal utility by giving the agent consumption (he becomes a household). The results in the main paper can be obtained by letting $\theta \to 0$.

1.1 Setting

Let $(\Omega, P, \mathcal{F})$ be a complete probability space. Throughout this appendix, all stochastic processes are adapted to the filtration $\mathcal{F}_t$ generated by the aggregate brownian motion $Z$, the idiosyncratic brownian motion $W_i$, and an idiosyncratic Poisson process $N_i$ with arrival rate $\theta$ associated with retirement of intermediary $i$. I will use a weak formulation of the
problem, which is equivalent to the strong one in the paper. Prices and coefficients depend only on the history of aggregate shocks, e.g. \( r, \pi, \sigma, \nu, \phi, q, g, \iota(g), \tau^k, \zeta \). In what follows I will drop the reference \( i \) to the intermediary in order to simplify notation.

There is a complete financial market, with risk free interest rate \( r \) and price of aggregate risk \( \pi \) (and associated equivalent martingale measure \( Q \)). The agent can continuously trade capital \( k \) at price \( q \), and obtains a return per dollar invested in capital

\[
dR_t = \left( \frac{a-t_t(g_t)}{q_t} + g_t + \mu_{q,t} + \sigma_t\sigma'_{q,t} - \tau^k_t \right) dt + (\sigma_t + \sigma_{q,t})dZ_t + \nu_t dW^s_t
\]

The agent starts with net worth \( n_0 \) and signs a contract \( C = (c, \bar{U}, k) \) with full commitment. The contract specifies consumption \( c = \{c_t \geq 0; t \leq \tau \} \), a terminal utility \( \bar{U} = \{\bar{U}_t; t \leq \tau \} \), and capital under management \( k = \{k_t \geq 0; t \leq \tau \} \). After retirement the agent cannot manage capital any longer, and the principal delivers utility \( \bar{U}_\tau \).

After signing the contract the intermediary can choose a hidden action \( s = \{s_t; t \leq \tau \} \) which we interpret as a stealing process. Stealing changes the distribution of observed outcomes from \( P \) to \( P^s \) so that the return can be written

\[
dR_t = \left( \frac{a-t_t(g_t)}{q_t} + g_t + \mu_{q,t} + \sigma_t\sigma'_{q,t} - \tau^k_t + s_t \right) dt + (\sigma_t + \sigma_{q,t})dZ_t + \nu_t dW^s_t
\]

where \( W^s_t = W_t + \int_0^t\frac{s_u}{\nu_u}du \) is a brownian motion under \( P^s \). For each dollar stolen, the intermediary keeps a fraction \( \phi_t \in (0,1) \) which he adds to his consumption: \( \tilde{c} = c + \phi q k s \) (the intermediary doesn’t have access to hidden savings). As a result he gets utility \( U^s(\tilde{c}) = U^s_0 \), where the utility process \( U^s = \{U^s_t; t \leq \tau \} \) is given by

\[
U^s_t = \mathbb{E}_t^s \left[ \int_t^\tau f(\tilde{c}_s, U^s_t)ds + \bar{U}_\tau \right]
\]

In this environment it is always optimal to implement no stealing \( s = 0 \), for the same reasons as in DeMarzo and Sannikov (2006) for example.\(^1\) The principals’ objective is to minimize the cost of delivering utility \( u_0 \) to the intermediary \( F(C) = F_0 \) where the continuation cost of the contract \( F = \{F_t; t \leq \tau \} \) is

\[
F_t = \mathbb{E}_t^Q \left[ \int_t^\tau e^{-\int_t^u r_md\mu} \left( c_u - q_u k_u \alpha_u \right) du + e^{-\int_t^\tau r_md\mu} \bar{F}_\tau(\bar{U}_\tau) \right]
\]

\(^1\)If a contract implements stealing \( s \) in equilibrium, then the principal can do better by giving \( c' = c + \phi q k s \) to the agent as legitimate consumption, and implementing no stealing \( s' = 0 \).
and \( \alpha_t \equiv \frac{a - u(w)}{q_t} + g_t + \mu_{q,t} + \sigma_t \sigma_{q,t}' - r_t - \tau_t - (\sigma_t + \sigma_{q,t}) \pi_t \). Here \( \bar{F}_t(\bar{U}) \) is the cost of delivering utility \( \bar{U} \) to the intermediary who has retired and cannot manage capital any longer. The intermediary has in fact become a household, so the cost of delivering utility takes the form

\[
\bar{F}_t(\bar{U}) = \zeta_t^{-1} \left( (1 - \gamma) \bar{U} \right) ^{1/\gamma}
\]

and we assume \( \zeta_t^{-1} > 0 \) is bounded.

We say a contract \( C = (c, \bar{U}, k) \) is admissible if 1) there is a solution \( U^0 \) to (1), with

\[
E \left[ \left( \int_0^t f(c_u, U^0_u) du \right)^2 + (U^0_t)^2 \right] < \infty
\]

for all \( t \), and 2)

\[
E_Q \left[ \int_0^\tau e^{-\int_0^\tau r_m dm} \left| c_t - q_t k_t \alpha_t \right| dt + \sup_{u \leq \tau} e^{-\int_0^\tau r_m dm} \bar{F}_u(U_u) \right] < \infty
\]

Given a contract \( C \), we say that a stealing process \( s \) is valid if 1) there is a \( U^s \) solution to (1), and 2) \( \frac{s}{p} \geq 0 \) is bounded and there is a constant \( T \in \mathbb{R}_+ \) such that \( s_t = 0 \ \forall \ t \geq T \). Let \( \mathcal{S}(C) \) be the set of valid stealing plans given contract \( C \). We say an admissible contract \( C \) is incentive compatible if

\[
0 \in \arg\max_{s \in \mathcal{S}(C)} U^s(c + \phi q k s)
\]

Let \( \mathcal{I}C \) be the set of incentive compatible contracts. For an initial utility \( u_0 \) for the agent, an incentive compatible contract is optimal if it minimizes the cost of delivering initial utility \( u_0 \) to the agent, that is

\[
J(u_0) = \min_{C \in \mathcal{I}C} F(C)
\]

\[
\text{st : } U^0(C) \geq u_0
\]

By changing \( u_0 \) we can trace the Pareto frontier for this problem. In particular, at time \( t = 0 \) the intermediary has net worth \( n_0 \) which he gives to the principal in exchange for the contract. We set \( u_0 \) so that the principal breaks even

\[
n_0 + J(u_0) = 0
\]

\[\text{Note the intermediary can choose } T \text{ as large as desired, as well as the bound on } \frac{\mu_{q,t}}{q_t}. \text{ These regularity conditions can be relaxed with some work, but are economically innocuous.}\]

\[\text{Notice } A(C) \neq \emptyset \text{ because } 0 \in \mathcal{S}(C) \text{ for an admissible contract.}\]
Let $J_t = F(C^*)$ be the continuation cost of an optimal contract $C^*$. This is how much it would cost the agent to “buy into” the contract at that time, and therefore this is the net worth of the intermediary at time $t$.

1.2 Recursive formulation

We can use the continuation utility of the intermediary as a state variable in order to provide incentives for not stealing. First we obtain a law of motion for continuation utility.

**Lemma 1.** For any admissible contract $C = (c, \bar{U}, k)$, the intermediary’s continuation utility $U^0$ satisfies

$$dU^0_t = -f(c_t, U^0_t)dt + \sigma_{U,t}dZ_t + \tilde{\sigma}_{U,t}dW_t - \lambda_t (dN_t - \theta dt)$$

for some $\sigma_U$ and $\tilde{\sigma}_U$ in $L^2$, and $\lambda_t = U^0_{t^-} - \bar{U}_t$.

**Proof.** Consider the process

$$Y_t = \mathbb{E}_t \left[ \int_0^\tau f(c_u, U^0_u)du + \bar{U}_\tau \right] = \int_0^t f(c_u, U^0_u)du + U^0_t$$

on $\{t \leq \tau\}$. Since $Y$ is an $\mathbb{F}$-adapted $P$-martingale, and $\mathbb{F}$ is generated by $Z, W$ and $N$, we can apply a martingale representation theorem to obtain

$$dY_t = f(c_t, U^0_t)dt + dU^0_t = \sigma_{U,t}dZ_t + \tilde{\sigma}_{U,t}dW_t - \lambda_t (dN_t - \theta dt)$$

Rearranging we get (4). Since $U_\tau = \tilde{U}_\tau$ it must be that $\lambda_t = U^0_{t^-} - \bar{U}_t$, and from admissibility of $C$ we get $\mathbb{E} [Y^2_t] < \infty$ for all $t$, so therefore $\sigma_U$ and $\tilde{\sigma}_U$ are in $L^2$.

Notice that because retirement is contractible, the agent’s utility can in principle “jump” when the agent retires. However, if $U^0_t$ jumps down when the agent retires, for example, then while he doesn’t retire it must drift up to compensate the agent. To obtain equation (6) in the main text, we just drop the jump term.

Faced with contract $C$, the intermediary can consider a valid stealing process $s \in S(C)$, getting consumption $\tilde{c} = c + \phi q k s$ under probability $P^s$. The following lemma gives neces-

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4In this context, $L^2$ is the set of $\mathbb{F}$-adapted processes $x$ such that $\mathbb{E} \left[ \int_0^t x_u^2 du \right] < \infty$ for any $t$.

5Notice that for $t \geq \tau$, $Y_t$ is constant, so $\sigma_{U,t}$, $\tilde{\sigma}_{U,t}$, and $\lambda_t$ are zero, but this is not relevant for our purposes.
sary and sufficient conditions for an admissible contract to be incentive compatible, for the parameter configuration that is of interest to us.

**Lemma 2.** If \( EIS \psi > 1 \) and the risk aversion \( \gamma > 1 \), an admissible contract \( C = (c, \bar{U}, k) \) is incentive compatible if and only if

\[
0 \in \arg \max_{s \geq 0} f(c_t + \phi_t q_t k_t s, U_t^0) - \tilde{\sigma}_{U,t} \frac{s_t}{\nu_t} - f(c_t, U_t^0) \quad (5)
\]

**Remark.** The result of this lemma can be extended to other combinations of \( \psi \) and \( \gamma \).

**Proof.** Suppose the agent picks a valid stealing plan \( s \). His utility is \( U^s \), defined by

\[
U^s_t = \mathbb{E}_t^s \left[ \int_t^T f(c_u + \phi_t q_t k_u s_u, U_u^s) du + \bar{U}_T \right]
\]

We would like to compare this with the utility from good behavior \( U^0 \). To do this, it’s useful to first express \( U^0 \) as an expectation under \( P^s \). Using (4) and \( dW_t = dW^s_t - \frac{s_t}{\nu_t} dt \), we obtain

\[
dU^0_t = \left( -f(c_t, U^0_t) - \frac{s_t}{\nu_t} \right) dt + \sigma_{U,t} dZ_t + \tilde{\sigma}_{U,t} dW^s_t - \lambda_t (dN_t - \theta dt)
\]

Now we can integrate, bearing in mind that since \( \frac{s}{\nu} \) is bounded \( \sigma_U \) and \( \tilde{\sigma}_U \) are both in \( L^2(P^s) \) as well. We get:

\[
U^0_t = \mathbb{E}_t^s \left[ \int_t^{T \wedge \tau} \left( f(c_u + \phi_t q_t k_u s_u, U_u^0) + \frac{s_u}{\nu_u} \tilde{\sigma}_{U,u} \right) du + U^0_{T \wedge \tau} \right]
\]

where \( s_t = 0 \) for all \( t \geq T \). Notice \( U^s_{T \wedge \tau} = U^0_{T \wedge \tau} \). Now we can obtain

\[
U^s_t - U^0_t = \mathbb{E}_t^s \left[ \int_t^{T \wedge \tau} \left( f(c_u + \phi_t q_t k_u s_u, U_u^s) - f(c_u, U_u^0) - \frac{s_u}{\nu_u} \tilde{\sigma}_{U,u} \right) du \right] \quad (6)
\]

on \( \{ t \leq T \wedge \tau \} \).

To prove necessity, suppose (5) fails. Pick a bounded stealing strategy such that

\[
f(c_t + \phi_t q_t k_t s, U_t^0) - \tilde{\sigma}_{U,t} \frac{s_t}{\nu_t} - f(c_t, U_t^0) > 0
\]

on a set of positive measure \( A \), and zero outside (we can pick \( T \) as large as desired). Look
at the integrand in \(^{\text{6}}\), and write

\[
\begin{align*}
    f(c_u + \phi_t q_u k_u s_u, U^s_u) - f(c_u, U^0_u) - \bar{\sigma}_{U,u} \frac{s_u}{\nu_u} &= f(c_u + \phi_t q_u k_u s_u, U^s_u) - f(c_u + \phi_t q_u k_u s_u, U^0_u) \\
    &\quad + f(c_u, U^0_u) - f(c_u, U^0_u) - \bar{\sigma}_{U,u} \frac{s_u}{\nu_u} \\
    &\geq f(c_u + \phi_t q_u k_u s_u, U^s_u) - f(c_u + \phi_t q_u k_u s_u, U^0_u)
\end{align*}
\]

with strict inequality on \(A\). Now we use an interesting fact about the EZ aggregator \(f(c, U)\):

if \(\gamma > 1\) and \(\psi > 1\), then there is a constant \(\kappa > 0\) such that \(f(c, y) - f(c, x) \leq \kappa(y - x)\) for \(y \geq x\), and any \(c \in [0, \infty)\). We can then write

\[
\begin{align*}
    f(c_u + \phi_t q_u k_u s_u, U^s_u) - f(c_u, U^0_u) - \bar{\sigma}_{U,u} \frac{s_u}{\nu_u} &\geq \kappa(U^s_u - U^0_u) \quad \text{when } U^s_u - U^0_u \leq 0
\end{align*}
\]

and the inequality is strict on \(A\). Now define the process \(M_t = U^s_t - U^0_t\) and rewrite the previous condition as

\[
M_t = U^s_t - U^0_t = \mathbb{E}^s_t \left[ \int_{0}^{T \land \tau} H_u \, du \right] \quad \text{with } H_t \geq \kappa M_t \text{ whenever } M_t \leq 0
\]

We can now use a generalized version of Skiadas’ Lemma\(^{\text{7}}\) to obtain that \(M_t = U^s_t - U^0_t \geq 0\) as follows. Let \(\tau_0 = \inf\{t \geq 0 : U^0_t \leq U^s_t\}\) and write

\[
\begin{align*}
    M_t \mathbf{1}_{\{\tau_0 > t\}} &\geq \mathbb{E}^t_s \left[ \int_{t}^{\tau_0 \land T \land \tau} \kappa M_u \mathbf{1}_{\{\tau_0 > t\}} \, du + M_{\tau_0 \land T \land \tau} \mathbf{1}_{\{\tau_0 > t\}} \right] \\
    M_t \mathbf{1}_{\{\tau_0 > t\}} &\geq \mathbb{E}^t_s \left[ \int_{t}^{T \land \tau} \kappa M_u \mathbf{1}_{\{\tau_0 > u\}} \, du + M_{\tau_0 \land T \land \tau} \mathbf{1}_{\{\tau_0 > t\}} \right] = \mathbb{E}^t_s \left[ \int_{t}^{T \land \tau} \kappa M_u \mathbf{1}_{\{\tau_0 > u\}} \, du \right]
\end{align*}
\]

Applying the stochastic Gronwall-Bellman inequality\(^{\text{8}}\) we get that \(M_t \mathbf{1}_{\{\tau_0 > t\}} \geq 0\) for \(0 \leq t \leq T \land \tau\). Since \(M_0 \mathbf{1}_{\{\tau_0 = 0\}} \geq 0\), we conclude \(M_0 \geq 0\). We can apply a similar argument for any \(u\) (redefining the stopping time \(\tau_u = \inf\{t \geq u : U^0_t \leq U^s_t\}\)) and get \(M_u \geq 0\) for all \(0 < u < T \land \tau\). For \(u \geq T \land \tau\) we already know that \(M_{u \land \tau} = 0\).

Now to make the inequality strict, if \(M_t = 0\) a.e. on \([0, \tau] \times \Omega\), then \(H_t \geq \kappa M_t = 0\),

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\(^{\text{6}}\)see Proposition 3.2 in Kraft et al. (2011)\(^{\text{5}}\).

\(^{\text{7}}\)The strategy is similar to Theorem A.2 in Kraft et al. (2011).

\(^{\text{8}}\)See Duffie and Epstein (1992).
and the inequality is strict on positive measure subset $A$, and therefore $M_0 > 0$. If $M_t > 0$ for at least some $(\omega, t)$ with $t < \tau$, with positive probability, we do the following. For some small $\epsilon > 0$, let $\tau^\epsilon = \inf \{ t : M_t \geq \epsilon \}$. If we take $\epsilon$ small enough, the probability that we get to such a point before $\tau$ is positive: $P^s(\{ \tau^\epsilon \land \tau < \tau \}) > 0$ (for any $P^s$ since they are equivalent). It must be that there is some stealing going on after this, since otherwise $M_{\tau^\epsilon}$ would be zero. Now consider the alternative stealing plan $s'$ that steals only until $\tau^\epsilon$ and then stops, that is $s'_t = s_t$ for $t < \tau^\epsilon$ and $s' = 0$ after this. By a similar argument as before,

$$U_{00} \leq U_{s'} \tau^\epsilon$$

if $\tau^\epsilon \land \tau < \tau$, and equal otherwise. Now if we compare $s$ and $s'$, both plans induce the same probability measure until $\tau^\epsilon \land \tau$, and the same consumption stream, but the payoff at $\tau^\epsilon \land T$ is larger for $s$ (strictly so with positive probability):

$$U_{s'} = E_t [\int_{\tau^\epsilon}^{\tau \land T} f(c_t + \phi_t q_t k_t s_t U_{00} t) \, du] \geq E_t [\int_{\tau^\epsilon}^{\tau \land T} f(c_t + \phi_t q_t k_t s'_{t \land \tau} U_{s'} \tau^\epsilon) \, du]$$

By strict monotonicity of EZ preferences with respect to terminal value, we get $U_{00} > U_{s'} \tau^\epsilon = U_{00}$. This proves stealing is attractive if (5) fails.

For sufficiency, suppose (5) holds, so that for any valid stealing plan we have

$$f(c_t + \phi_t q_t k_t s_t U_{00} t) - \tilde{\sigma}_{U_t} s_t - f(c_t, U_{00} t) \leq 0$$

Using the same properties of the EZ aggregator as before but with the opposite inequalities, we get

$$M_t = U_{t} - U_{0} = E_t [\int_{t}^{T \land \tau} H_u du] \quad \text{with } H_t \leq \kappa M_t \quad \text{whenever } M_t \geq 0$$

The same reasoning as before now yields $M_0 = U_{0} - U_{0} \leq 0$, so the contract is indeed incentive compatible.

The FOC for (5) are

$$\partial_t f(c_t, U_{0} t) \phi_t q_t k_t = \tilde{\sigma}_{U_t} \frac{1}{\nu_t}$$
which yields the “skin in the game” condition (7) in the main text

\[ \tilde{\sigma}_{U,t} = \partial_c f(c_t, U_t) \phi_t q_t k_t \nu_t \]  

(7)

1.3 HJB equation

Because of homothetic preferences, the value function for the principal’s cost minimization problem takes the form \( J_t = \xi_t x_t \). Here \( x_t = ((1 - \gamma)U_t)^{1 - \gamma} > 0 \) is a monotone (and concave) transformation of the intermediary’s continuation utility. As a result, we can also interpret it as the intermediary’s continuation utility. The stochastic process \( \xi = \{ \xi_t; t \leq \tau \} \) captures the investment opportunity set, and has a law of motion

\[ \frac{d\xi_t}{\xi_t} = \mu_{\xi,t} dt + \sigma_{\xi,t} dZ_t + ((\xi_t \zeta_t)^{-1} - 1) dN_t \]  

(8)

It depends only on the aggregate shocks \( Z \) that affect market prices, and on whether the intermediary has retired. The last term ensures that \( \xi_\tau = \zeta_\tau^{-1} \).

Use the following normalization: \( k_t = \hat{k}_t x_t \), \( c_t = \hat{c}_t x_t \), \( \sigma_{U,t} = \sigma_{x,t}(1 - \gamma)U^0_t \), \( \tilde{\sigma}_{U,t} = \hat{c}^{\frac{1}{\gamma}} \phi_t \hat{k}_t \nu_t (1 - \gamma)U^0_t \), and \( \lambda_t = \hat{\lambda}_t (1 - \gamma)U^0_t \). Then we can write

\[ \frac{dx_t}{x_t} = \left( \frac{1}{1 - \hat{c}^\frac{1}{\gamma}} (\rho - \hat{c}^\frac{1}{\gamma}) + \frac{1}{2} \gamma \sigma_{x,t}^2 + \frac{1}{2} \gamma (\hat{c}^{\frac{1}{\gamma}} \phi_t \hat{k}_t \nu_t)^2 + \theta \lambda_t \right) dt \\
+ \sigma_{x,t} dZ_t + (\hat{c}^{\frac{1}{\gamma}} \phi_t q_t \hat{k}_t \nu_t) dW_t + \left( (1 - \hat{\lambda}(1 - \gamma))^{1 - \gamma} - 1 \right) dN_t \]  

(9)

The principal’s cost minimization problem is now a standard optimal control problem. The associated HJB BSDE is

\[ r_t J_t dt = \min_{c_t, k_t, \sigma_t, \lambda_t} (c_t - q_t k_t \alpha_t) dt + \mathbb{E}_t^Q [dJ_t] \]  

(10)

subject to (8), as well as \( c \geq 0 \) and \( k \geq 0 \). Using the normalization of the controls, the form of the value function, and the fact that \( Z_t = Z_t^Q - \int_0^t \pi_u du \), where \( Z^Q \) is a brownian motion under \( Q \), we can rewrite the HJB equation

\[ r_t \xi_t = \min_{c_t, k_t, \sigma_t, \lambda_t} \hat{c}_t - q_t \hat{k}_t \alpha_t + \xi_t \left\{ \frac{1}{1 - \gamma} (\rho - \hat{c}_t^{\frac{1}{\gamma}}) - \sigma_{x,t} \pi_t + \mu_{\xi,t} - \sigma_{\xi,t} \pi_t \right\} \]  

(11)
This should be interpreted together with (8). The expression on the right hand side is convex if $\psi > 2$, and the FOC are sufficient. If $\psi \leq 2$ then the optimal contract does not exist, unless capital pays zero excess return, as explained in the paper. Focusing on the $\psi > 2$ case, we get the following FOC

$$\xi_t \hat{c}_t^{\frac{1}{\psi}} + \xi_t \gamma \left( \phi_t q_t \hat{k}_t \nu_t \right)^2 \hat{c}_t^{-\frac{2}{\psi} - 1} = 1$$

(12)

$$\frac{a - \ell_t(g_t)}{q_t} + g_t + \mu_{q,t} + \sigma_t \sigma_{q,t} - (r_t + \tau^k_t) = \underbrace{(\sigma_t + \sigma_{q,t}) \pi_t}_{\text{agg. risk premium}} + \gamma \xi_t \left( \hat{c}_t^{\frac{1}{\psi}} \phi_t \nu_t \right)^2 q_t \hat{k}_t$$

(13)

$$\sigma_{x,t} = \frac{\pi_t}{\gamma} - \frac{\sigma_{\xi,t}}{\gamma}$$

(14)

$$\left(1 - \hat{\lambda}(1 - \gamma)\right)^{\frac{1}{\gamma - 1}} \frac{1}{\xi_t \hat{\xi}_t} = 1$$

(15)

The FOC for $\hat{c}$ has the cost of delivering consumption to the intermediary on the right hand side, and the benefit of a lower promised utility on the left hand side. This is the standard tradeoff we would expect. In addition, however, there is another benefit to giving consumption to the agent: it relaxes the “skin in the game constraint”. By front loading consumption, the principal can reduce the marginal private benefit of stealing and consuming, and therefore improve idiosyncratic risk sharing. As a result, there is a tradeoff between distortions in intertemporal consumption and idiosyncratic risk sharing.

The FOC for $\hat{k}$ gives us a pricing equation for capital. As usual, capital pays an excess return because it is exposed to aggregate risk with a market price of $\pi_t$. But in addition, capital must also pay an excess return for its exposure to idiosyncratic risk, even though this risk is not priced by the financial market. The reason for this is that the principal knows that if he gives more capital to the intermediary, he will have to expose him to risk for incentive reasons, and this is costly because the intermediary is risk averse.

The FOC for $\sigma_x$ has the following interpretation. Delivering utility to the intermediary is costly for the principal. He would therefore prefer to promise him more utility in states
of the world where it is relatively cheaper. This can happen because the value of a unit of consumption in that state is lower (captured by the $\pi_t$ term) or because the cost of delivering utility in that state is lower (captured by the $\sigma_{\xi,t}$ term). A similar logic explains the FOC for $\hat{\lambda}$. After retirement the intermediary cannot manage capital any longer, so delivering utility to him is more costly. This is captured by $\zeta_t^{-1} \geq \xi_t$. As a result, the optimal contract has $\hat{\lambda} \geq 0$. The principal prefers to promise the intermediary less utility after he retires (when it is more costly to deliver utility to him) even if it means promising him more utility while he doesn’t retire.

We can plug in the FOC into the HJB equation (11). If we find a solution $\xi$ to the HJB equation, we can use it to build the optimal contract using the policy functions $\hat{c}$, $\hat{k}$, $\sigma_x$, and $\hat{\lambda}$ (so the HJB holds with equality) and the law of motion of $x$, (9) with initial condition $x_0 = ((1 - \gamma)u_0)^{\frac{1}{1 - \gamma}}$. Let $C^* = (c^*, \bar{U}^*, k^*)$ be the candidate optimal contract thus constructed, with associated state $x^*$. We have the following verification theorem.

**Theorem 1.** Let $\xi$ be a strictly positive solution to the HJB equation (11) bounded above by $\zeta^{-1}$. Then,

1) For any incentive compatible contract $C$ that delivers at least utility $u_0$ to the agent, we have $\xi_0 (1 - \gamma)u_0)^{\frac{1}{1 - \gamma}} \leq F_0(C)$.

2) Let $C^*$ be a candidate contract constructed as described above. If $C^*$ is admissible and delivers utility $u_0$ to the agent, then it is optimal, with cost $F_0(C^*) = J_0(u_0) = \xi_0 (1 - \gamma)u_0)^{\frac{1}{1 - \gamma}}$.

**Proof.** For the first part, consider an incentive compatible contract $C = (c, \bar{U}, k)$ that delivers utility $u_0$ to the agent, and has an associated state variable $x$. Use the HJB equation to obtain

$$e^{-\int_0^{\tau \land \tau_r} ru du} \xi^{\tau \land \tau_r} x^{\tau \land \tau_r} \geq \xi_0 x_0 - \int_0^{\tau \land \tau_r} e^{-\int_0^{\tau_r} ru du} (c_t - q_t k_t \alpha_t) dt$$

$$+ \int_0^{\tau \land \tau_r} e^{-\int_0^{\tau_r} ru du} \xi_t x_t (\sigma_{\xi,t} + \sigma_{x,t}) dZ_t^Q + \int_0^{\tau \land \tau_r} e^{-\int_0^{\tau_r} ru du} \xi_t x_t \left( \hat{c}_t^{\frac{1}{\nu}} \phi q_t \hat{k}_t \nu_t \right) dW_t$$

$$+ \int_0^{\tau \land \tau_r} e^{-\int_0^{\tau_r} ru du} \xi_t x_t \theta \left( (1 - \hat{\lambda}(1 - \gamma))^{\frac{1}{1 - \gamma}} \frac{1}{\xi_t \hat{\xi}_t} - 1 \right) (dN_t - \theta dt)$$
Here we are using the localizing sequence \( \{\tau^n\}_{n \in \mathbb{N}} \):

\[
\tau^n = \inf \left\{ T \geq 0 : \int_0^T \left| e^{-\int_0^t r_u du} \xi_t x_t \left( \sigma_{x,t} + \sigma_{t} \right) \right|^2 dt + \int_0^T \left| e^{-\int_0^t r_u du} \xi_t x_t \left( \hat{c}_t - \hat{\phi}_t \kappa_t \nu_t \right) \right|^2 dt \geq n \right\}
\]

The stochastic integrals are therefore martingales, so we can take expectations under \( Q \) to obtain

\[
\mathbb{E}_0^Q \left[ e^{-\int_0^{\tau^n \wedge \tau} r_u du} \xi_{\tau^n \wedge \tau} x_{\tau^n \wedge \tau} \right] \geq \mathbb{E}_0^Q \left[ \int_0^{\tau^n \wedge \tau} e^{-\int_0^t r_u du} (c_t - q_t k_t \alpha_t) dt \right]
\]

Now we will use the dominated convergence theorem to take the limit \( n \to \infty \). First,

\[
\left| \int_0^{\tau^n \wedge \tau} e^{-\int_0^t r_u du} (c_t - q_t k_t \alpha_t) dt \right| \leq \int_0^{\tau^n \wedge \tau} e^{-\int_0^t r_u du} |c_t - q_t k_t \alpha_t| dt
\]

which is integrable because \( \mathcal{C} \) is admissible. Second,

\[
e^{-\int_0^{\tau^n \wedge \tau} r_u du} \xi_{\tau^n \wedge \tau} x_{\tau^n \wedge \tau} \leq \sup_{t \leq \tau} e^{-\int_0^t r_u du} \xi_t x_t \leq \sup_{t \leq \tau} e^{-\int_0^t r_u du} \zeta_t^{-1} x_t
\]

which is also integrable because \( \mathcal{C} \) is admissible. So letting \( n \to \infty \), we get \( \tau^n \wedge \tau \to \tau \) a.s. and therefore using \( \xi_{\tau} = \zeta_{\tau}^{-1} \):

\[
\mathbb{E}_0^Q \left[ e^{-\int_0^{\tau \wedge \tau} r_u du} \xi_{\tau} x_{\tau} \right] \geq \mathbb{E}_0^Q \left[ \int_0^{\tau} e^{-\int_0^t r_u du} (c_t - q_t k_t \alpha_t) dt \right]
\]

Rearranging we get the desired result.

For the second part, let \( \mathcal{C}^* \) be the candidate optimal contract with associated utility process \( x^* \). By construction, the HJB holds with equality, so the same argument shows that \( \mathcal{C}^* \) in fact has a cost \( F_0(\mathcal{C}^*) = \xi_0 \left( (1 - \gamma) u_0 \right)^{\frac{1}{1-\gamma}} \). We know \( \mathcal{C}^* \) delivers utility \( u_0 \) by assumption, and Lemma 2 ensures it is incentive compatible.

Finally, from (10) we can easily see that the net worth of the agent \( n_t = J_t = \xi_t x_t \) follows the law of motion

\[
dn_t = (r_t n_t + q_t k_t \alpha_t - c_t + \sigma_{n,t} n_t \pi_t) dt + \kappa_t n_t \nu_t d\Phi_t + \hat{\phi}_t q_t k_t \nu_t dW_t + \lambda_{n,t} n_t (dN_t - \theta dt)
\]
where \( \bar{\phi}_t = \xi_t \hat{c}^{-1/\psi} \phi \), and \( \lambda_{n,t} = \hat{\lambda}_t (\gamma - 1) \). With \( \theta = 0 \) we obtain equation (5) in the main body.

## 2 Numerical solution

In this section I show how the competitive equilibrium and the planner’s allocation can be solved as a system of PDEs. In Section 3 I illustrate the method with a concrete example where the economy is hit by uncertainty shocks. It is useful to introduce retirement among intermediaries as explained in Section 1 in order to obtain a non-degenerate stationary distribution for the economy. Retirement arrives with Poison arrival rate \( \theta \), at which point the intermediary becomes a household. If we set \( \theta \to 0 \) we obtain the setting in the paper.

### 2.1 Competitive equilibrium

The HJB equation for intermediaries with retirement is given by (11), with FOCs (12), (13), (14), and (15). These are the same as in the paper, except for the FOC for \( \hat{\lambda}_t \).

The representative household’s HJB and FOC are unchanged, as are the market clearing conditions. The only equilibrium condition that needs to be tweaked is the drift of the endogenous state variable

\[
X_t = \int_{x_t} (x_{t} (x_{t})^{2}) d\mu - \int_{y_t} (y_{t})^2 d\sigma
\]

\[
\mu_X = \frac{\rho}{1 - 1/\psi} - \hat{c}^{-1/\psi} \phi_{\theta_1} (g)_{\nu} X + \frac{\gamma}{2} \sigma_{XX} \sigma_{\theta_{1}} (g)_{\nu} X - g - \sigma_{X} + \sigma^2 + \theta \left( \hat{\lambda} - 1 \right)
\]

(16)

Notice how with \( \theta = 0 \) we obtain equation (18) in the paper. With \( \theta > 0 \), because each intermediary’s utility will jump down on retirement, it must drift up while they don’t retire to compensate them. As a result, \( X \) gains a positive drift \( \theta \hat{\lambda} \). On the other hand, when intermediaries retire their continuation utility is not counted in \( X \) any longer, since they are now households, so we get the term \(-\theta\).

The strategy to solve the competitive equilibrium is to use Ito’s lemma to transform the problem into a system of PDEs for \( q, \xi \), and \( \zeta \). Suppose we are given these functions. We can build \( S = \zeta(q - \xi X) \) and \( \Lambda = \xi \zeta \), and use Ito’s lemma to compute the drift and volatility of all these objects, in terms of \( \mu_X \) and \( \sigma_X \), which we still don’t know, and \( \mu_Y \) and \( \sigma_Y \), which are exogenously given. From the equilibrium condition for the allocation of aggregate risk, (??), we obtain

\[
\sigma_X - \sigma_S = -\frac{1}{\gamma} \sigma_{\Lambda}
\]
From Ito’s lemma we get
\[
\sigma_{\Lambda} = \frac{\Lambda}{\Lambda} \sigma_{X} \sigma_{X} + \frac{\Lambda_{Y}}{\Lambda} \sigma_{Y} \quad \sigma_{S} = \frac{S_{X}}{S} \sigma_{X} \sigma_{X} + \frac{S_{Y}}{S} \sigma_{Y}
\]

There is a two-way feedback loop: $\sigma_{X}$ depends on how the MRT $\Lambda$, responds to aggregate shocks, $\sigma_{\Lambda}$; but $\sigma_{\Lambda}$ is an endogenous object that depends, among other things, on how $X$ responds to aggregate shocks, $\sigma_{X}$. We can solve for a fixed point for $\sigma_{X}$ to obtain
\[
\sigma_{X} = \frac{\frac{S_{Y}}{S} - \frac{1}{\gamma} \frac{\Lambda_{Y}}{\Lambda}}{1 - \left(\frac{S_{X}}{S} - \frac{1}{\gamma} \frac{\Lambda_{X}}{X}\right)} \sigma_{Y}
\]

At this point we have $\sigma_{X}$ and therefore the volatility of $q$, $\xi$, $\zeta$ (and therefore $S$ and $\Lambda$) in terms of their first and second derivatives. Now we can use the definition of $\sigma_{X}$ to write $\sigma_{x} = \sigma_{X} + \sigma$, and use the FOC for $\sigma_{x}$ to write $\pi = \gamma \sigma_{x} + \sigma_{\xi}$. Then using the FOC for $\sigma_{w}$, in households’ HJB, we get $\sigma_{w} = \frac{x}{\gamma} + \frac{1}{\gamma} \sigma_{\zeta}$. Now we need to compute the drifts. First, use the FOCs to obtain $\hat{c}$ and $\hat{c}_{h}$, and we can use the definition of $\mu_{X}$ to compute it. With this we have the drift of $q$, $\xi$, $\zeta$ (and therefore $S$ and $\Lambda$).

Finally, use households’ HJB to compute $r$. We end up with intermediaries’ HJB (with $\hat{k} = X^{-1}$ from market clearing for capital), the FOC for capital, and the market clearing condition for consumption goods. This is a system of two second order PDEs and an algebraic constraint (the market clearing for consumption goods) for $q$, $\xi$, and $\zeta$.

**Boundary conditions.** We don’t need to impose conditions at the boundary of the domain. Rather, we have global conditions. We are looking for a solution with $p$, $\xi$, and $\zeta$ strictly positive, as well as $q - \xi X$ (households’ wealth). $\xi$ and $\zeta$ should be bounded away from zero (so $\xi^{1-\gamma}$ and $\zeta^{1-\gamma}$ are bounded above), and we want the resulting process for $X$ and $S$ to remain positive, and intermediaries’ and households’ plans to be admissible and deliver utility $\frac{(Sk)^{1-\gamma}}{1-\gamma}$ and $\frac{(Xk)^{1-\gamma}}{1-\gamma}$. If we find a solution with these properties, then we have a competitive equilibrium. We know the HJB, their FOCs, and the market clearing conditions hold by construction. We only need to make sure these plans are truly optimal: with $\xi^{1-\gamma}$ and $\zeta^{1-\gamma}$ bounded above this is guaranteed (Theorem 1 for intermediaries and standard arguments for households).
**Numerical algorithm.** The system of equations can be solved by adding a fictitious finite time horizon \( T \), with some terminal values for these functions. A time derivative must be added to the computation of all drifts, and we can then solve backwards in time. In this respect we have a system of first order ODEs with respect to time, which can be solved with a standard integrator, such as Runge-Kutta 4 for example. If the time derivatives vanish as we solve backwards, we have a solution to the system of PDEs we were interested in (infinite horizon). Terminal conditions are not important as long as the time derivatives vanish in the limit. Since the market clearing condition for consumption is an algebraic constraint, it is easier to differentiate it with respect to time to obtain a differential equation. We just need to make sure that terminal conditions are consistent with market clearing for consumption goods, and the algorithm will preserve this as we solve backwards. We can also verify ex-post that this condition is satisfied by the solution.

There are two complications. The first is that the FOC for \( \hat{c} \) cannot be solved analytically, and solving it numerically at each step would make the algorithm much slower. What we can do is add \( \hat{c} \) as a function to be solved for, and differentiate the FOC for \( \hat{c} \) with respect to time, like we did for market clearing for consumption. We get an extra unknown but also an extra differential equation, and terminal conditions must be chosen so that the FOC for \( \hat{c} \) is satisfied. This can also be verified ex-post (the benefit is we only solve the FOC for \( \hat{c} \) once at the beginning).

The second complication is that the domain of the system \((X, Y) \in D \subset \mathbb{R}^2_+\) is unknown. Basically, for a given \( Y \) we know that \( X \in (0, \bar{X}(Y)) \), but we don’t know what is the maximum utility that can be delivered to intermediaries for each exogenous state \( Y \). To deal with this we can do a change of variables, such as \( \tilde{X} = \frac{X}{X + \xi(q - \xi X)} \in (0, 1) \), and solve the resulting system.

### 2.2 Planner’s problem

Retirement requires modifying the planner’s HJB equation and the law of motion of \( X \). First, \( X \) now captures the continuation utility of currently remaining intermediaries. Likewise, \( S \) is the continuation utility of current households, including previously retired intermediaries. The HJB becomes

\[
\frac{\rho}{1 - \frac{1}{\psi}} = \max_{g, \hat{c}, \sigma, \lambda} \left( a - \psi(g) - \hat{c}X \right)^{1 - \frac{1}{\psi}} \left( S^{\frac{1}{\psi}} - 1 \right) + \mu S + g - \frac{\gamma \sigma^2}{2} S - \frac{\gamma \sigma^2}{2} + (1 - \gamma) \sigma S \sigma \tag{18}
\]
The $\theta$ term captures the fact that current households only get a part of future continuation utility of households in the future, because future households include current intermediaries that will retire in the meantime. Because retirement for each intermediary is observable and contractible, their continuation utility can jump down on retirement. The $\hat{\lambda}$ term captures this. Likewise, $\mu_X$ is the same as in the competitive equilibrium, (16). We use Ito’s lemma to obtain $\mu_S$ and $\sigma_S$, and plug it into the HJB equation:

$$
- \theta X S \left( 1 - \hat{\lambda}(1 - \gamma) \right)^{\frac{1}{1-\gamma}} 
$$

(19)

$$
\mu_S = \frac{S_Y}{S} \mu_Y + \frac{S_X}{S} \mu_X X + \frac{1}{2} \frac{S_{YY}}{S} \sigma_Y^2 + \frac{1}{2} \frac{S_{XX}}{S} (\sigma_X X)^2 + \frac{S_{XY}}{S} \sigma_X X \sigma_Y 
$$

(20)

$$
\sigma_S = \frac{S_X}{S} \sigma_X X + \frac{S_Y}{S} \sigma_Y 
$$

(21)

Now we have an extra FOC for $\hat{\lambda}$:

$$
\frac{S'_X}{S} X + \frac{X}{S} \left( 1 - \hat{\lambda}(1 - \gamma) \right)^{\frac{1}{1-\gamma}-1} = 0 
$$

$$
\Rightarrow \hat{\lambda} = \frac{1 - \Lambda^{\frac{1-\gamma}{1-\gamma}}}{1-\gamma} 
$$

All the other FOC are unchanged. Notice that the planner’s FOC for $\hat{\lambda}$ coincides with the private FOC [15], using $\Lambda = \xi \zeta$. As a result, it is still the case that the only inefficiency is in the FOC for $g$, because of the hidden trade. Retirement does not introduce any source of inefficiency.

**Numerical solution.** The planner’s HJB is a PDE for $S(X,Y)$. As in the competitive equilibrium, we can solve it by adding a fictitious finite horizon $T$. This requires us to add a time derivative when computing $\mu_S$. We can then solve backward for arbitrary terminal conditions. If the time derivative vanishes as we solve back, we found the original PDE.

Just like in the competitive equilibrium case, we need to deal with two complications. The first is that now the FOC for both $\hat{c}$ and $g$ are difficult to solve analytically, so we add both as functions of $(X,Y)$ and differentiate the FOCs with respect to time to obtain two more PDEs. We just need to ensure that terminal conditions satisfy the FOCs (the benefit, as before, is that we only solve them numerically once). We can check at the end that the FOC are satisfied. The second problem is that as before we don’t know the
domain, so we need to do a change of variables as in the competitive equilibrium, such as
\[ \tilde{X} = \frac{X}{X+\tilde{s}} \in (0, 1), \] and solve the resulting system.

### 3 Application: Uncertainty Shocks

We can illustrate the general results with a concrete numerical application. Consider an
economy hit by TFP shocks to the effective level of capital, \( \sigma > 0 \), and uncertainty shocks
to idiosyncratic risk as in Di Tella (2013). The idiosyncratic risk of capital, \( \nu_t = Y_t \), now
follows an autoregressive stochastic process

\[
d\nu_t = \beta(\bar{\nu} - \nu_t) dt + \sqrt{\nu_t \sigma_\nu} d\tilde{Z}_t
\]

By convention, we will take \( \sigma_\nu < 0 \), so that we may think of \( Z \) as a “good” shock that drives
idiosyncratic risk \( \nu_t \) down. The aggregate risk of capital is still fixed \( \sigma \). For simplicity we
consider only one aggregate shock (\( Z \) is unidimensional). If \( \sigma > 0 \) the good shock \( Z \) also
improves TFP, but there is no loss in intuition from setting \( \sigma = 0 \).

**Parameter values:** For the numerical exercise I use the following parameter values.
They do not constitute a careful calibration, but rather a numerical example. *Preferences:*
\( \gamma = 5, \psi = 3, \rho = 0.1, \theta = 0.1; *technology: *a = 1, \sigma = 0.0125, \nu(g) = 200g^2; *moral hazard: *
\( \phi = 0.2; *uncertainty shock: *\beta = 0.69, \bar{\nu} = 0.25, \sigma_\nu = -0.17. \)

### 3.1 Competitive Equilibrium vs. social Planner.

Figure[1] shows that the social planner can deliver more utility to households \( S \) for any level
of utility for intermediaries \( X \). He achieves this by reducing asset prices \( q_t \). This improves
idiosyncratic risk sharing, which allows him to deliver utility to intermediaries at a lower
cost, but comes at the cost of lower investment and growth. In both the unregulated competi-
tive equilibrium and the planner’s allocation, asset prices \( q \) (and therefore investment)
are lower when experts’ continuation utility \( X \) is low, and when idiosyncratic risk \( \nu \) is high.
It is costly to provide incentives to intermediaries when capital is very risky relative to their
continuation utility.

---

This is a CIR process. If \( 2\beta\bar{\nu} \geq \sigma_\nu^2 \), then \( \nu_t > 0 \) always, and it has a long-run distribution with mean \( \bar{\nu} \).
We can see in Figure 2 that in the competitive equilibrium $\sigma_X > 0$ throughout, which means that after an uncertainty shock $X$ goes down. So the price of capital falls both because $\nu$ rises, and because $X$ falls. We can understand this in terms of a financial amplification channel. Financial losses are concentrated on the balance sheets of intermediaries; $\sigma_n - \sigma_w > 0$ throughout. Although agents are free to share aggregate risk, the cost of intermediaries’ utility $\Lambda$ is low after an uncertainty shock, so optimal contracts concentrate financial losses on intermediaries. Utility losses, however, are concentrated on households; $\sigma_x - \sigma_h$ is negative throughout. Although there is no inefficiency in aggregate risk sharing, the allocation of aggregate risk in the competitive equilibrium is different than the planner’s solution. In the planner’s solution, financial losses are less concentrated on intermediaries, and they might even be concentrated on households. In this sense, we could say the competitive equilibrium features an excessive concentration of risk on intermediaries, and therefore an inefficient financial amplification channel. Utility losses, on the other hand, are excessively concentrated on households. However, there is no need to regulate aggregate risk sharing.

Figure 3 shows the MRT $\Lambda$ for the competitive equilibrium and the planner’s allocation. Since both allocation are different, the MRT $\Lambda$ is different and reacts differently to aggregate shocks – this lies behind the differences in the allocation of aggregate risk. The reason for this is that the externality $\eta$ also depends on both $\nu$ and $X$ and is therefore affected by the
Figure 2: Exposure of endogenous state variable $X$ to uncertainty shocks $\sigma_X$, concentration of financial risk $\sigma_n - \sigma_w$, and concentration of utility risk $\sigma_x - \sigma_h$, as functions of $X$ for a fixed $\nu = 0.25$ (above), and as functions of $\nu$ for a fixed $X = 4.86$ (below). Solid line is the CE, dashed line is the SP.

Figure 3: MRT between intermediaries’ and households utility $\Lambda$, externality $\eta$, and volatility of total wealth $\text{vol}(qk + Tk)$, as functions of $X$ for a fixed $\nu = 0.25$ (above), and as functions of $\nu$ for a fixed $X = 4.86$ (below). Solid line is the CE, dashed line is the SP.
uncertainty shock. The externality $\eta$ can be computed both in the competitive equilibrium and the planner’s allocation; in the planner’s allocation it is internalized by a tax on assets (or reserve requirements) with present value $T/q = \eta$. The externality $\eta$ is higher during periods of high idiosyncratic risk $\nu$ and low continuation utility for intermediaries $X$; i.e. after an uncertainty shock. This means that in the optimal allocation, $T/q$ goes up during downturns. While this may seem counterintuitive, notice that total wealth is $(q+T)k$, and it is less volatile in the optimal allocation.

References


