Selling assets: When is the whole worth more than the sum of its parts?

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Abstract

Should a firm sell its assets separately or bundled? We study this issue in a setting where competing bidders are potentially asymmetric and their number is determined endogenously. Bidders are less likely to enter when they anticipate entry by an efficient (higher expected value) bidder. The endogenous reduction in competition lowers revenue for the seller in spite of the higher value the efficient bidder brings to the table. The reduction in revenue, however, depends on how the assets are sold: when the level of dominance of the efficient bidder is high enough the seller prefers to sell the assets bundled.

Keywords: Asset sales; Free entry; Bidder dominance

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1 Introduction

The market for asset sales is substantial, with asset being sold for a variety of reasons ranging from changes in the firm’s focus to the need to raise financing.\footnote{For instance, see Edmans and Mann (2018), who report SDC data on asset sales of 131 Billion by non-financial US firms in 2012. This is about 1.5 times the volume of Seasoned Equity Offerings in the same year. We contrast our paper with other theoretical contributions later in this section.} Entire firms are frequently sold off piecemeal, with each asset sold to the highest bidder. Evidence suggests this practice was fairly commonplace during the 1970s and 1980s, the decades of the “corporate raiders” that would buy companies for the primary purpose of breaking them up and selling the assets individually, a practice often referred to as “asset stripping.” Of course, the piecemeal asset sale strategy can also be employed by the firm’s own management. A recent prominent example is that of General Electric selling off its various assets under its new CEO Larry Culp. The logic for stripping the firm in this way is presumably that there is a high likelihood that buyers with high values can be found for individual assets, but it is less likely that any given buyer values highly all assets that comprise the firm. The most value can therefore be generated though a piecemeal sale, so that the sum of the parts winds up being worth more than the whole.

Despite this often cited pattern, many firms regularly turn down opportunities to sell their own assets individually, even when it is apparent that there are buyers who value the individual assets highly but may be less interested in the entire collection of assets that comprises a firm. A salient example is the case of Blackberry Ltd., whose board in November of 2013 rejected proposals from several technology companies for various assets, arguing that breaking up the assets was not in the best interest of the company’s stakeholders. This action was considered striking by market observers given Blackberry’s obvious need for cash and restructuring, and given the apparent interest in Blackberry’s high value patents by companies such as Microsoft Corp. and Apple Inc. Blackberry’s decision to remain whole and continue to pursue options to either recapitalize or sell the entire company as a going concern suggests a view that the greatest value may not always be obtained through a
piecemeal sale or liquidation.

In this paper, we study the optimal way of selling a firm’s assets, comparing the revenue obtained from selling a set of assets together (e.g., a firm in its entirety) with the revenue obtained from selling the assets individually. Our starting point is the observation above that the value of individual assets that can be redeployed to a better use is likely to be higher than the value of the collection of assets. In other words, the buyer who values some specific asset the most will not necessarily have the highest value for all the assets. Hence, it should be more efficient to sell the assets individually. In line with this, we allow for the possibility that potential buyers will be intrinsically asymmetric, with this asymmetry affecting their valuation for some assets but not necessarily for others. For instance, buyer asymmetry could come from different costs of putting together an offer, different values for the target, collusion among a subset of bidders, degree of overlap in specific lines of business, etc. Buyers, consequently, are not competing on a level playing field. We call buyers that are more likely ex ante to have a higher value for an asset “efficient.” All other buyers we refer to as “regular.”

Central to our analysis is that potential buyers face an important participation decision, and may choose not to invest resources – in performing due diligence, acquiring information, lining up financing, etc. – if they view the likelihood of obtaining the asset too low. Likewise, they may be less willing to participate if they anticipate having to pay too high a price to acquire the assets. To study the participation decision, we model the competitive bidding process as a second price auction subject to a costly entry decision by each potential buyer. Given the existence of efficient bidders with higher expected values, the participation of less advantaged bidders is not a foregone conclusion. Among the considerations that can affect bidders’ participation decisions are the sale mechanism being employed, the degree of asymmetry, and the costs of participation. Therefore, one important component of our analysis is that we endogenize the entry of possible buyers: each buyer must incur an upfront and fixed cost in order to learn his value for the asset for sale and submit a bid. This
framework is thus well suited for studying situations where due diligence is important and costly, such as for corporate acquisitions, or when bidding might require a filing fee or a search cost, for instance.\textsuperscript{2}

We first show that the presence of an efficient bidder has two countervailing effects. First, the efficient bidder brings more value to the table. If the other buyers were not to change their entry decisions, the seller would benefit by an increase in revenue. Second, however, other buyers will be less willing to incur the entry cost in order to submit an offer. The reason is that buyers recognize that they are less likely to successfully acquire the asset when they are competing against an efficient buyer. The presence of a efficient, higher-value buyer therefore endogenously reduces competition for the assets. We show that the second effect is always the more important one, meaning that the reduction in competition reduces the expected premium the seller receives on his asset. Moreover, we show that this result is very general and holds for just about any distribution of values for both the efficient and the regular bidders. All that matters is that the efficient bidder’s value is drawn from a distribution that is superior (in a well-defined sense: first order stochastic dominance) to that of the other bidders in order for seller revenue to be reduced, with revenue being progressively lower the more dominant is the efficient bidder.

However, we also show that, while an increase in the dominance of an efficient bidder always reduces revenue, it has a larger effect in the case of an individual asset sale than when a collection of assets is sold jointly. The reason is that when assets are sold jointly, an efficient bidder’s likely higher value for one asset has a smaller effect on other buyers’ possibilities of winning the collection of assets. Therefore, increases in efficiency have a smaller impact on competition when assets are sold jointly than when they are all sold separately. In fact, we show that when the degree of dominance by the efficient buyer(s) is sufficiently large, selling the assets jointly is optimal: it generates more revenue for the seller than selling them

\textsuperscript{2}Athey et al. (2011) find that entry costs for bidders in U.S. timber auctions are often substantial, being as high as 8\% of the eventual purchase price in an auction. Using similar data, Roberts and Sweeting (2013) find somewhat lower entry costs, although still a substantial portion of the overall value of the object for sale (e.g., timber).
individually.

Our analysis has implications for the optimal bundling of assets for sale. We abstract from other considerations that may drive acquisition premiums, such as complementarity across the assets, correlation in values, etc., in order to focus on how competition for the assets may be affected by the way in which they are packaged. The main implication is that competition may depend, endogenously, on whether the assets are sold together or separately, which, in turn, has important consequences for the premium a seller receives. One main implication is that having highly efficient buyers for parts of a firm reduces the likelihood that a firm will sell its parts separately to these bidders, instead preferring to sell itself as a package.

The literature on asset sales that focuses on joint versus separate sales is somewhat limited other than in the context of price discrimination, which we discuss below. In the context of bankruptcy auctions, Eckbo and Thorburn (2008) find empirically that “fire sale discounts” are observed when assets are liquidated in a piecemeal fashion, but not when they are sold jointly so that the firm is acquired as a going concern. For the case where the sale of assets is voluntary, Hite et al. (1987) find that partial firm sell-offs lead to small abnormal returns for sellers. By contrast, the abnormal returns are much larger and significant for proposals to liquidate the entire firm. A different focus is found in Schlingemann et al. (2002), who study corporate divestitures as a function of the liquidity of the market for assets. However, to the best of our knowledge, no work has studied how bidder heterogeneity may be a determinant of how assets are sold. From a theory perspective, the closest work to ours is Chakraborty (1999), who studies the question of when “bundling” two assets in an auction is preferable to conducting two separate auctions for the individual assets. Our work builds on his, as well as on Chakraborty (2006), to study the implications of having bidders with higher than average values participate in the auction, an aspect missing from the literature.

An important component of our analysis is that competition is endogenous. Most analyses of auctions (with some exceptions discussed below) take the number of bidders as exogenous and analyze the consequences of varying the auction format, bidders’ information sets, or the
number of bidders. Moreover, most of the literature that analyzes the effects of changes in the number of bidders assumes that all bidders are symmetric and focuses either on limiting behavior as the number of bidders grows arbitrarily large, or on changes in the costs of participation. This literature therefore ignores how the presence of more dominant bidders may deter entry and thus have important consequences for equilibrium prices, particularly in settings where due diligence and information acquisition considerations are important. Asymmetries among bidders is quite natural, and the literature on mergers and acquisitions has analyzed the consequences of asymmetries in various settings although, to the best of our knowledge, it has not focused on the entry decision.³

On the issue of entry, McAfee and McMillan (1987) consider a setting where bidders must incur a cost in order to learn their private valuations and enter the bidding.⁴ They show that in such a setting, the use of an entry fee by the seller is not optimal as it further reduces entry and leads to a lower sales price. Levin and Smith (1994) also analyze a similar setting, but study the case where there is a fixed number of potential bidders and focus on the symmetric entry equilibrium among the possible bidders, so that each bidder has a positive probability of bidding. They show that entry may sometimes be socially excessive (as in Mankiw and Whinston, 1986), and that increasing the number of potential bidders may actually reduce seller revenue. These papers do not, however, study how bidder asymmetries can have exactly the opposite effect under free entry than when the number of bidders is fixed exogenously. Marquez and Singh (2013) analyze a related issue in the context of “club bidding” by private equity firms. None of these papers, however, allow for the possibility that the seller may break up the asset into separate parts as a way of generating more revenue.

The question of whether bundled or individual sales may be preferable has been studied


⁴Harstad (1990) studies a similar setup for common value auctions. Lu and Ye (2013) consider the question of revenue maximization and auction design in the context of free entry, where bidders must pay “information acquisition” costs, much as in our setup.
theoretically in the context of price discrimination. Various authors (e.g., Adams and Yellen, 1976, McAfee and Whinston, 1989, Wilson, 1993, Armstrong, 1996, and Rochet and Choné, 1998, among others) have shown that bundling may help price discriminate among buyers, allowing firms to sell their assets for a greater total price and obtain greater revenue. Our work abstracts from issues related to price discrimination and focuses instead on how entry decisions by buyers may drive the decision on how best to sell assets.

Empirically, Klemperer (2002) provides a number of examples where the likely presence of a dominant bidder seems to have reduced the incentive of other bidders to enter, and conjectures that this reduced entry could prevent the seller from raising as much revenue as should otherwise be possible. Similarly, recent work by Athey et al. (2011) has demonstrated that auction design can affect bidder entry and, as a consequence, the competitiveness of an auction. Our work complements this view by arguing that which bidders are attracted may also have important implications for competitiveness and seller revenue.

The focus of our paper differs from those of existing papers analyzing the sale of assets in the context of information asymmetry between the seller and buyers. For instance, Eisfeldt (2004) and Kurlat (2013) examine issues of endogenous liquidity related to asset sales, Edmans and Mann (2018) and Nanda and Narayanan (1999) examine the choice between equity issuance and asset sales as a source of financing, and Bond and Leitner (2015) examine the incentives of buyers who already own similar assets to bid up prices. The distinguishing feature in this strand of literature that builds on Myers and Majluf (1984) focuses on the superior information possessed by the seller. In our model, the seller’s value is common knowledge and the tension arises from the effect of the difference in synergies amongst bidders on their entry decisions.

The paper proceeds as follows. The next section presents the basic model of sale with endogenous entry for a single asset. We analyze the consequences of having a fixed, exogenous number of bidders in Section 3, and in Section 4 we study how these results are affected once the entry decision is endogenous. Section 5 then extends the results to multiple assets and
establishes when selling them together is optimal. We study some issues related to robustness in Section 6 as well as in the Appendix. Section 7 concludes.

2 Model

We start with a setting where the seller is putting up an asset for sale to potentially many buyers. Specifically, assume that the number of potential buyers for the object for sale is very large, and that the sale for the good takes place in two stages. First, each buyer must decide whether to incur a participation cost \( c > 0 \). The participation cost is incurred ex ante, before buyers learn about their values for the object. Second, all buyers that paid the cost \( c \) and learned their values can enter an auction and bid for the good. We do not explicitly model the entry game but instead assume that the number of buyers will be determined by a standard zero profit (free entry) condition, that is, the number of participating buyers must be consistent with each buyer earning zero profits in equilibrium.\(^5\) All offers in the second stage are made simultaneously. The buyer with the highest offer then pays the next highest price that was offered. Since the sales mechanism is modeled as a second price auction, we will interchangeably refer to the buyers also as “bidders.”

The asset itself has a value \( V \) to the seller, which can be viewed as a value common to all potential bidders. All bidders are risk neutral and all except one are symmetric ex ante. A symmetric bidder’s value for the asset is \( V + x_i \), where is \( x_i \) drawn from the distribution \( F(\cdot) \), with support in \([0, 1]\). The component \( x_i \) is therefore the additional component of value for bidder \( i \) above the value to the seller and can hence be viewed as \( i \)’s synergy with the asset. One of the bidders is efficient with value \( V + y \), where \( y \) is drawn from a distribution \( G(y) \) which dominates that of the other symmetric (also referred to as regular) bidders in

\(^5\)This is a common approach in the industrial organization literature on free entry, e.g., see Tirole (1988), Mankiw and Whinston (1986), or more recently Ghosh and Morita (2007). In an auction setting, McAfee and McMillan (1987) determine the number of bidders using a similar approach. See also Athey et al. (2011) for a discussion of alternative ways of modeling the entry game. Our results continue to hold in other models of entry, such as the symmetric mixed strategy entry decision studied in Levin and Smith (1994), which has been widely used in the literature on auctions with endogenous entry. These results are available upon request.
the sense of first order stochastic dominance (FOSD): \( G(x) \leq F(x) \) for all \( x \in [0,1] \), with the inequality strict for some positive measure of \( x \). Hence, we at times refer to the efficient bidder as the dominant bidder.

For ease of exposition, we assume \( V \) is common knowledge and normalize it to zero. Given the normalization we refer to the additional private component of value to the buyer – the synergy – as simply the buyer’s value. For the remainder of the paper we use the terms buyer’s private value and the synergy in buyer’s value interchangeably. Similarly, we refer to expected premium as the expected revenue or just the revenue from the sale.

3 Analysis with a fixed number of bidders

We begin with a standard construction of the equilibrium profits for the bidders, as well as the seller’s revenue for a given number of bidders \( N \). We later endogenize \( N \).

It is a dominant strategy for bidders to bid their actual value. In other words, letting \( b_i(x_i) \) represent bidder \( i \)’s strategy as a function of his value \( x_i \), we have that \( b_i = x_i \). We focus on this equilibrium in dominant strategies. Given a specific realization of values \( (x_1, \ldots, x_N, y) \), the \( N \)th bidder’s profit is given by \( x_N - \max \{x_1, \ldots, x_{N-1}, y\} \) if \( x_N > \max \{x_1, \ldots, x_{N-1}, y\} \) and 0 otherwise. Given \( N \) regular bidders and an efficient bidder, we can take expectations to obtain a regular bidder’s expected profits to be

\[
\pi(N|G) = \int (1 - F(x)) F^{N-1}(x) G(x) \, dx. \tag{1}
\]

All integrals, unless the limits are specified, are taken over the unit interval. Similarly, we can obtain the expected profits for the efficient bidder as

\[
\Pi(N|G) = \int (1 - G(x)) F^N(x) \, dx. \tag{2}
\]

We can now calculate the seller’s expected revenue, \( R \), by recognizing that the sum of
the seller’s revenue plus the profits of all bidders - $N$ regular bidders and the efficient bidder - equals the total surplus of the auction. Since the good is always sold, the total surplus must simply be equal to the maximum value of all bidders, $\max \{x_1, \ldots, x_N, y\}$. The seller’s revenue is then the total surplus minus the profit of each of the bidders. Taking expectations, we obtain

$$R(N|G) = E[\max \{x_1, \ldots, x_N, y\}] - \Pi(N|G) - N\pi(N|G).$$  \hspace{1cm} (3)$$

How does a change in the number of regular bidders or in the distribution function of the efficient bidder affects bidder profits and seller revenue? An increase in the number of bidders, $N$, increases competition and leads to lower expected profits for all bidders. The benefit of higher competition flows to the seller since his revenue is the expectation of the second highest realization of $\{x_1, \ldots, x_N, y\}$, which is higher for a higher $N$. An important implication of FOSD is that making the efficient bidder more dominant, i.e., changing $G$ to $G'$ such that $G'(y) \gg_{FOSD} G(y)$, reduces the regular bidders’ profits. This can be seen directly from (1), where replacing $G$ with $G'$ reduces $\pi$, for all $N$. However, the expected profits of the efficient bidder increases. The net effect is higher expected revenue for the seller. We summarize this discussion in the following result.

**Lemma 1** A regular bidder’s profit is decreasing in $N$, i.e., $\pi(N'|G) < \pi(N|G)$ $\forall N' > N$. The seller’s revenue is increasing in the number of regular bidders, $N$, and in the degree of dominance of the efficient bidder. Specifically, $R(N'|G) > R(N|G)$ $\forall N' > N$ and $R(N|G') > R(N|G)$ whenever $G'(y) \gg_{FOSD} G(y)$.

## 4 Endogenous participation and the seller’s revenue

We next endogenize the number of bidders through a standard participation condition for the regular bidders, that the number of regular bidders satisfy $\pi \geq c$, or that the profits each regular bidder makes be no less than what it costs it to learn its value and participate. In what follows, we proceed by treating the number of firms as a continuous variable, so that
the zero profit condition will be satisfied with equality (this is an assumption often used in much of the literature on entry and provides significant tractability; see, e.g., Ghosh and Morita, 2007, or Erkal and Piccinin, 2010). We can therefore use the following condition to characterize the equilibrium number of regular bidders, $N^*$:

$$\pi (N^*|G) = c. \quad (4)$$

From Lemma 1 we know that, for a given number of regular bidders $N$, each bidder’s profit $\pi$ is decreasing in the dominance of the efficient bidder. Therefore, in order to satisfy (4), the equilibrium number of regular bidders, $N^*$, must decrease as the efficient bidder gets more dominant. We summarize this in the following result.

**Lemma 2** The number of regular bidders, $N^*$, in the endogenous-participation equilibrium is smaller if the efficient bidder is more dominant. Specifically, for distribution functions $G$ and $G'$, if $G'(y) \succ_{FOSD} G(y)$ so that $\pi (N|G) > \pi (N|G')$, then $N^*(G) > N^*(G').$

Empirically, Klemperer (2002) provide various examples suggesting that the presence of a dominant bidder has been seen to affect other bidders’ participation decisions. The key question is whether the impact of the endogenous reduction in competition is significant enough to overwhelm the benefit to the seller that comes from an increase in dominance, as shown in Lemma 1. For that, we have one of our main results.

**Proposition 1** With endogenous participation, the seller’s revenue goes down if the efficient bidder is more dominant. Specifically, given $c$, if $G'(y) \succ_{FOSD} G(y)$, we have $R(N^*(G')|G') < R(N^*(G)|G)$ if $N^*(G') > 1$.

The proposition establishes that when the number of bidders is endogenous, with bidders subject to a participation condition, then making the efficient bidder more dominant leads to lower revenue for the seller. It is useful to contrast this result with the change in seller revenue in the case when bidder participation is exogenously fixed. As we show in Lemma 1,
seller revenue increases as the dominance of the efficient bidder increases. With endogenous participation the seller no longer benefits from the increased value (and consequently high bid) of the efficient bidder. It bears noting as well that Proposition 1 holds for any participation cost \( c > 0 \), even if arbitrarily small so that the potential for competition is very large. As long as the participation cost is not strictly equal to zero, increases in the dominance of the efficient bidder will \textit{always} reduce the revenue to the seller.

Proposition 1 has important implications for the revenue that might be obtained from asset sales though a competitive process. In effect, the proposition says that when a potential buyer is likely to have a value higher than other possible buyers, the seller will be worse off as a result. Moreover, the higher the anticipated value of a particular seller, the lower price will the seller receive for the sale of her asset when potential buyers face real costs in deciding whether or not to submit an offer for the asset in question. Instances where such asymmetries might be expected to arise, as mentioned earlier, can be found in the sale of assets to bidder pools comprised of both strategic and financial (e.g., private equity firms) bidders, for example (see Gorbenko and Malenko, 2014 for further discussion).

The result in Proposition 1 is robust in the sense that it holds for any non-degenerate distribution function \( F \) for the regular bidders. The assumed sales mechanism – a second price auction – has two main advantages. First, it is strategically equivalent to an ascending price auction where bidders drop out as the price increases, so it represents a good framework for studying the “offer, counter-offer” or sequential bidding that often takes place for asset sales. Second, it allows us to focus on simple, detail-free mechanisms, where bidders’ optimal strategies do not depend on how many other bidders are competing or on the underlying distributions for their values. More complicated mechanisms, such as first price auctions, are known to be difficult to implement. Moreover, for the case where bidders are heterogeneous ex ante, first price auctions have no known solutions even in settings where the number of bidders is ex ante fixed.\footnote{Given that the general analysis of asymmetric first price auctions remains currently an unsolved problem, a literature studying these auctions through numerical methods has developed. In ongoing work, we have}
To gain intuition for why the competitive effect of increased dominance always leads to decreased seller revenue despite the greater value obtained from the more dominant efficient bidder, it is useful to view the problem from the perspective of the impact of the efficient bidder and what it is able to capture. To do this, we derive an equivalence result across possible distribution functions $G$ for the efficient bidder, along with the notion of competitively equivalent distributions for the efficient bidder, as follows.

**Proposition 2** For any distribution function $G(y)$ for the efficient bidder, there exists $\varphi > 0$ such that $G(y)$ is competitively equivalent to a distribution $J(y) = F^{1+\varphi}(y)$, in the following sense: $\pi(N_G|G) = \pi(N_G|J) = c$ and $R(N_G|G) = R(N_G|J)$, where $N_G$ is the equilibrium number of regular bidders, i.e., $N_G = N^*(G)$.

Proposition 2 shows that for any asymmetric second price auction with $N$ regular bidders and one that is different, there exists a related auction that is equivalent in the sense that it provides the same profit to the regular bidders as well as the same revenue to the seller. More importantly, in this equivalent auction, which we term competitively equivalent because bidders’ profits and seller revenue are the same, the distribution function for the efficient bidder’s value, $G$, can be represented as a power function of the regular bidders’ distribution, $J(y) = F^{1+\varphi}(y)$, where $\varphi = N_G - N_F$. In other words, any asymmetric auction where the regular bidders have distribution function $F$ and the other bidder has distribution $G$ can be equivalently represented by assuming that the efficient bidder draws his value from a distribution function that is a power function of $F$.

We now have a simple way of gaining intuition for the result in Proposition 1. Consider the case where going from distribution $G$ to $G'$ for the efficient bidder drops the number of regular bidders by 1, for instance. If distribution $G$ for the efficient bidder is competitively equivalent to a distribution $F^{1+\varphi}(y)$, then $G'$ will be competitively equivalent to a distribution $F^{2+\varphi}$. Now we can easily compare the following two scenarios: (i) $N_G$ regular bidders and an efficient

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extended our results on bidder entry and seller revenue to first price auctions using these methods, and have found similar results. This analysis is available upon request.
bidder with value distributed as $F^{1+\varphi}(y)$; and (ii) $N_G - 1$ regular bidders and an efficient bidder with value distributed as $F^{2+\varphi}(y)$. The second scenario is equivalently obtained by removing 1 regular bidder and transforming the value of the efficient bidder by $\max \{y, x_{N_G}\}$, i.e., it is as if the efficient bidder has “merged” with one of the regular bidders, and his value is now the greater of his own value, $y$, and that of the bidder with whom he has merged, $x_{N_G}$. The total surplus generated stays the same as before, but now the efficient bidder benefits in all states of the world in which either $y$ or $x_{N_G}$ are the highest or the second highest values. In other words, the efficient bidder has captured the contribution to total value of one of the regular bidders. It is clear that the “captured” bidder’s inability to provide competition to the efficient bidder or the other $N_G - 1$ regular bidders is what hurts the seller.\footnote{The argument here is similar to the intuitive argument provided in Cantillon (2008) to explain the result of Theorem 2 in that paper.} Thus, the decrease in competition increases the profits of the efficient bidder at the expense of the seller.

A benefit of the equivalent distributions defined in Proposition 2, which are power functions of $F(\cdot)$, is that they allow for significant tractability since a unit increase in the exponent implies an equal offsetting decrease in the number of regular bidders, i.e., $\frac{dN^*}{d\varphi} = -1$. This feature will prove useful in parts of the analysis presented below and in the appendix. An additional benefit is that the exponential form for the efficient bidder provides an intuitive parametrization of efficiency or dominance, allowing for clear comparative statics with respect to the parameter $\varphi$ and an ordering of what we mean by a “better” or “more dominant” bidder.

So far we have shown that the presence of an efficient bidder has a first-order effect on the seller’s revenue due to a decrease in competition. A natural question to ask is whether the seller can offset this by using entry subsidies to encourage participation. Another possibility for the seller could be to use reserve prices to reduce the profit of the efficient bidder and thus increase his revenue. In the appendix we show that even if the seller can provide entry subsidies or commit to non-discriminatory reserve prices, she is still worse off as a result of
the presence of the efficient bidder when bidder entry is endogenous.

4.1 A simple example

Here, we offer a simple example to illustrate the mechanism and, more importantly, to show that the fall in seller revenue can be substantial even for modest increases in dominance by the efficient bidder. Suppose that $F(x) = x$, so that a regular bidder’s value is drawn from a uniform distribution, while the efficient bidder’s value is drawn from $G_1(y) = y^2$, for $x, y \in [0, 1]$. Clearly, $G_1(y) < F(y)$ for all $y < 1$, so that the expected value of the efficient bidder’s synergy is higher than that of any regular bidder. Assume also that the entry cost is $c = \frac{1}{30}$, and conjecture that three regular bidder choose to incur the cost and participate, which is easy to verify – given our parameter choices, we can calculate the expected profit of the regular bidders to be exactly $\frac{1}{30}$. Hence, the regular bidders break even when three of them choose to participate, and have no incentive to deviate. Given three regular bidders and one efficient bidder, the revenue obtained by the seller is $\frac{13}{20}$.

We now ask what happens if the efficient bidder’s distribution of value improves, in the sense that he becomes more dominant. Specifically, suppose instead that the value for the efficient bidder is drawn from $G_2(y) = y^3$, which clearly dominates $F$, and moreover also dominates $G_1$, so that the bidder can be thought of as being even more efficient. In this case, were three regular bidders to participate, their expected profit would go down to $\frac{1}{42}$, which is less than the cost of entry, $c = \frac{1}{30}$. Hence, three bidders participating cannot be an equilibrium outcome. However, if only two regular bidders participate, their expected profit is again exactly $\frac{1}{30}$, so that the equilibrium number of regular bidders is now two.

It is not surprising that the presence of a more dominant bidder has a chilling effect on competition, since this effect stems directly from the fact that the profits of each regular bidder will be lower. The key question is what is the overall impact on seller revenue. For the case where we have two regular bidders and one efficient bidder whose value is drawn from $G_2$, we can easily compute seller revenue to be $\frac{3}{5}$, which is clearly less than the revenue
obtained with \( G(y) = y^2 \).

To get a sense of the economic magnitude of this reduction in revenue, note that the increase in dominance from \( G_1 \) to \( G_2 \) represents an increase in expected value for the efficient bidder of around 12.4\%. In other words, the efficient bidder’s value when drawn from \( G_2 \), which is \( E[y|G_2] = 0.75 \), is a little more than 12\% higher than when his value is drawn from \( G_1 \), which is \( E[y|G_1] = 0.67 \). The drop in revenue stemming from the reduction in competition, however, represents a 7.7\% drop: from revenue of \( \frac{13}{20} \) to revenue of \( \frac{3}{5} \). The effect is even larger when viewed from the perspective of the increase in total value if the number of bidders were to remain fixed at 3 (rather than endogenous), with the only change stemming from the improvement in the efficient bidder’s distribution function. For this, we can calculate the expected value of the maximum of \( y \) and \( x_1, x_2, \) and \( x_3 \), and calculate the percent change. When \( y \sim G_1, E[\max \{x_1, x_2, x_3, y\}] = \frac{5}{6} \), whereas when \( y \sim G_2, E[\max \{x_1, x_2, x_3, y\}] = \frac{6}{7} \). The percent increase is therefore approximately 2.8\%, but because of the reduction in the number of bidders and, hence, competition, leads to a 7.7\% drop in revenue for the seller.

### 4.2 Multiple efficient bidders

The results so far assume that there is a single efficient bidder. It is also possible that there may be more than one efficient bidder, competing not only with the regular bidders but also with each other. Indeed, that is precisely how we view the question of joint asset sales we study below, with (for simplicity) two efficient bidders. Can the existence of multiple efficient bidders allow the seller to benefit from the increase in value brought to the table by these bidders? Here, we show that the main result extends to the case where there are various such bidders, whose values are drawn from distribution functions that dominate those of the other, regular bidders.

To establish this, consider the case where there are \( N \) regular bidders whose values are drawn from \( F(\cdot) \). \( N \) will be obtained in equilibrium from these bidders’ zero profit condition, as before. In addition, there are \( K \) efficient bidders whose values \( y_i \) are drawn from a family
of distributions $G(z|\varphi_i)$ indexed by $\varphi_i$ and we assume that $G(z|\varphi_i) = F(z)^{1+\varphi_i}$. Without loss of generality, we order these bidders so that $\varphi_1 \geq \varphi_2 \geq \ldots \geq \varphi_K \geq 0$, implying that $G(z|\varphi_1) \leq G(z|\varphi_2) \leq \ldots \leq G(z|\varphi_K) \leq F(z)$ for all $z$.

With $K$ efficient bidders and $N$ regular bidders, expected profit for a regular bidder becomes

$$\pi = \int (1 - F(z)) F(z)^{N-1} \prod_{i=1}^{K} G(z|\varphi_i) \, dz$$

$$= \int (1 - F(z)) F(z)^{N-1+K+\sum_{i=1}^{K} \varphi_i} \, dz,$$

where in the second line we use the fact that $\prod_{i=1}^{K} G(z|\varphi_i) = F(z)^{K+\sum_{i=1}^{K} \varphi_i}$. The profit for efficient bidder $i$ is

$$\Pi_i = \int F(z)^N \prod_{j \neq i} G(z|\varphi_j) (1 - G(z|\varphi_i)) \, dz$$

$$= \int (1 - F(z)^{1+\varphi_i}) F(z)^{N+K-1+\sum_{j \neq i} \varphi_j} \, dz.$$

Finally, seller revenue is given by

$$R = 1 - \int F(z)^N \prod_{j \neq 1} G(z|\varphi_j) \, dz - \sum_{j \neq 1} \Pi_j - N\pi$$

$$= 1 - \int F(z)^{N+K-1+\sum_{j \neq i} \varphi_j} \, dz - \sum_{j \neq i} \Pi_j - N\pi \quad \forall i \in \{1, K\}.$$

We can now state the following result.

**Proposition 3** Assume that the participation cost, $c$, is low enough that $N^* > 1$ in equilibrium. Then, under endogenous participation for the regular bidders, the seller’s revenue $R\left(N^*|\{\varphi_i\}_{i=1}^{K}\right)$ is decreasing in the dominance of any efficient bidder: $\frac{dR(N^*|\{\varphi_i\}_{i=1}^{K})}{d\varphi_i} < 0 \quad \forall i$.

Proposition 3 establishes that the main result from Proposition 1, that increases in the

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The results of Proposition 2 were derived assuming a single efficient bidder. Hence, the assumption of $G$ being a power function of $F$ may be a restriction that is not without loss of generality.
dominance of an efficient bidder reduce seller revenue, holds even when there is a large number of such bidders. Moreover, Proposition 3 shows that $\frac{dR}{d\varphi_i} < 0$ for any $i$, or in other words, no matter which of the efficient bidders becomes more dominant. In effect, an increase in the dominance of any of the $K$ efficient bidders harms the seller by reducing his revenue. Furthermore, since this is true for any efficient bidder, it will also be true if we consider an increase that affects all $K$ efficient bidders simultaneously. For instance, if the $K$ efficient bidders all had a degree of dominance given by $\varphi_i = \varphi$ for all $i = 1, ..., K$, then a marginal increase in $\varphi$ would lead to a reduction in seller revenue $R$, as long as at least some symmetric bidders are not deterred from participating (i.e., as long as $N^* > 1$). Therefore, the result from Proposition 1 does not depend on the existence of a single bidder that is efficient.

5 Asset sales

The results above demonstrate that the composition of the pool of potential buyers is as important, if not more, than the total number of possible buyers in the pool. While these results apply for the sale of any asset where potential buyers face a costly entry and information acquisition decision, such as when due diligence is important, here we show that they are of particular relevance when a firm has multiple assets for sale and must decide whether to sell them jointly, as a bundle, or to sell them separately. The former can be viewed as the sale of entire divisions or even of entire firms, while the latter would be individual asset sales, separating the whole and selling the individual pieces. To study this issue, we extend the model slightly, as follows.

5.1 A model of joint asset sales

Suppose that there is a firm that owns two assets, $A$ and $B$. These assets can be interpreted as divisions of the firm, or as property, plant and equipment (PPE), or other productive assets. The question is whether to sell the company as a whole (i.e., sell both assets jointly),
or to sell the assets separately.

For each asset \( k \in \{A, B\} \), as above there is a large pool of risk neutral buyers potentially interested in buying it. Asset \( k \) has a value \( V_k \) to the seller. Buyer \( i \)'s value is \( V_k + x_k^i \), where \( x_k^i \) represents the additional private value to the buyer over what it is worth to the seller, i.e., \( x_k^i \) is buyer \( i \)'s synergy for asset \( k \). All but two buyers are “regular” in the sense defined above: \( x_k^i \) for asset \( k \) is drawn from the distribution \( F(.) \), with support in \([0,1]\). Buyer \( i \)'s value for the two assets combined is therefore \( X^i = V_A + V_B + x_A^i + x_B^i \). The other two potential buyers, which we denote by \( \alpha \) and \( \beta \), are “efficient”, also as above: the value of asset \( A \) for buyer \( \alpha \) is \( V_A + y^{\alpha} \), where \( y^{\alpha} \) is drawn from a distribution \( G(.)|\varphi \) with support also in \([0,1]\), and his value for asset \( B \) is \( V_B + x^{\alpha} \), where \( x^{\alpha} \) is drawn from \( F(.) \). Symmetrically, the private component of value for buyer \( \beta \) for asset \( B \) is denoted \( y^{\beta} \) and is drawn from distribution \( G(.)|\varphi \), while the synergy \( x^{\beta} \) for asset \( A \) is drawn from distribution \( F(.) \). \( \varphi \) is the parameter characterizing the degree of dominance of the efficient bidders, and we assume that \( G(z|0) = F(z) \), but also that \( \frac{\partial G}{\partial \varphi} < 0 \), so that \( G(z|\varphi) < F(z) \) for all \( z \in (0,1) \) when \( \varphi > 0 \). Letting \( \varphi^H \leq \infty \) be the maximal value of \( \varphi \), we also assume that \( \lim_{\varphi \to \varphi^H} G(z|\varphi) = 0 \) for all \( z < 1 \). None of the potential buyers know their true values for asset \( k \in \{A, B\} \) and must first incur a cost \( c \) to learn their values and be able to make a bid to acquire the asset. This implies that any bidder must pay a cost \( 2c \) to learn his value and bid for the entire firm.

However the firm is sold - either as a whole or each asset separately - the sales mechanism is a standard second price auction. As above, we normalize \( V_k \) to zero and refer to the additional private component of value to the buyer - the synergy - as simply the buyer’s value. Finally, we assume no complementarity between the assets, and no difference in the pools of possible buyers for the two assets, so that there is no ex ante reason why they should be sold jointly or separately. Our goal is to isolate the effect of dominance of particular buyers on the total revenue that might be generated by the sale without introducing a natural bias on how the assets should be sold.
5.2 Selling the assets together

When the assets are sold jointly, each regular bidder draws a value for each division, and these values are added together to form their value for the pooled assets. The two efficient bidders draw one value from a dominant distribution for one division, and the other from the regular distribution.

Specifically, this means that the value of regular bidder \( i \) for the pool of assets \( A \) and \( B \) is simply \( X^i = x^i_A + x^i_B \). Since both \( x^i_A \) and \( x^i_B \) are drawn from the same distribution \( F \), the distribution of their sum can be obtained by taking the convolution as

\[
q(z) = \int_0^1 f(z - y) f(y) \, dy.
\]

Integrating to get the distribution function \( Q(z) \) yields

\[
Q(z) = \begin{cases} 
\int_0^z F(z - y) f(y) \, dy & \text{for } 0 \leq z \leq 1 \\
F(z - 1) + \int_{z-1}^1 F(z - y) f(y) \, dy & \text{for } 1 < z \leq 2
\end{cases}
\]

For efficient bidder \( \alpha \), his value for the pool of assets is \( Z^\alpha = y^\alpha + x^\alpha \), where \( y^\alpha \) is drawn from \( G(\cdot | \varphi) \) and \( x^\alpha \) is drawn from \( F \). Therefore, the distribution \( H \) of the sum \( Z^\alpha \) can again be obtained by taking the convolution as

\[
h(z) = \int_0^1 g(z - y) f(y) \, dy.
\]

Again integrating, we obtained the distribution function \( H \) as

\[
H(z) = \begin{cases} 
\int_0^z G(z - y) f(y) \, dy & \text{for } 0 \leq z \leq 1 \\
F(z - 1) + \int_{z-1}^1 G(z - y) f(y) \, dy & \text{for } 1 < z \leq 2
\end{cases}
\]

An identical expression obtains for efficient bidder \( \beta \). Note that, since \( G(z) \leq F(z) \) for all \( z \), we also have that \( H(z) \leq Q(z) \). In other words, \( H(z) \succeq_{FOSD} Q(z) \), so that the efficient
bidders’ combined value for both assets dominates that of the regular bidders even though the efficient bidders are only dominant for one of the assets.

We can now use these distribution functions to find the profits for a regular bidder, which we denote as \( \pi_J \) for ”joint” since the assets are being sold jointly.

\[
\pi_J = E \left[ \max \left\{ X^1, \ldots, X^N, Z^\alpha, Z^\beta \right\} \right] - E \left[ \max \left\{ X^1, \ldots, X^{N-1}, Z^\alpha, Z^\beta \right\} \right] \\
= \int_0^2 (1 - Q(z)) H(z)^2 Q(z)^{N-1} \, dz.
\]

A similar expression obtains for the profit of an efficient bidder, \( \Pi_J \) (we drop the superscripts \( \alpha \) and \( \beta \) since both efficient bidders are symmetric):

\[
\Pi_J = E \left[ \max \left\{ X^1, \ldots, X^N, Z^\alpha, Z^\beta \right\} \right] - E \left[ \max \left\{ X^1, \ldots, X^{N-1}, Z^\alpha \right\} \right] \\
= \int_0^2 (1 - H(z)) H(z) Q(z)^N \, dz.
\]

We can also obtain the expression for the seller’s revenue as

\[
R_J = E \left[ \max \left\{ X^1, \ldots, X^N, Z^\alpha, Z^\beta \right\} \right] - 2\Pi_J - N\pi_J \\
= E \left[ \max \left\{ X^1, \ldots, X^N, Z^\alpha \right\} \right] - \Pi_J - N\pi_J \\
= 2 - \int_0^2 Q(z)^N H(z) \, dz - \Pi_J - N\pi_J.
\]

5.3 Optimal bundling of assets

In this section we study when selling assets individually is optimal (i.e., raises more revenue for the seller) versus when the seller would do better by bundling the two assets together. The free entry condition for the regular bidders is similar to that studied above, with a slight modification to account for the fact that when the assets are sold jointly, any potential buyer must incur the entry/information acquisition cost for each asset. In other words, for the case of individual sales, we have the usual condition for the equilibrium number of regular
bidders, $N^*$:

$$\pi_k (N^*|G) = c. \quad (10)$$

On the other hand, when the assets are sold jointly, the equilibrium number of regular bidders, $N_J$, will be given by

$$\pi_J (N_J|H) = 2c. \quad (11)$$

The existing literature studying when it might be optimal to bundle assets together before auctioning them, or when more revenue might be obtained by selling them separately, has, to the best of our knowledge, focused on cases where there are an equal number of bidders in each case.\(^9\) The most general result of which we are aware is obtained by Chakraborty (1999), who shows that when there is a sufficient number of bidders, selling the assets individually always dominates selling them jointly when all buyers are ex ante identical (equivalent to assuming $\varphi = 0$ in the context of our model). The intuition for the result in Chakraborty (1999) is that, when selling the assets individually, a second price auction allocates the good efficiently: the buyer with the highest value acquires the asset, paying the value of the next highest buyer. By contrast, selling the assets jointly introduces a distortion and may fail to allocate the assets to those who value them most. In particular, when sold jointly the winning party may well have a high value for one asset, but not a particularly high value for the other asset. In fact, it is certainly possible for the winner to not have the highest value for either of the two assets, but to have the highest value for the sum of the assets. When there is no efficient bidder, so that there is no reduction in competition associated with his presence, selling the assets in the most efficient way also maximizes the seller’s revenue when the auction is sufficiently competitive (i.e., when $c$ is sufficiently small).

Moreover, Chakraborty (1999) shows that the number of bidders that are needed for

\(^9\)One exception is Chakraborty (2006), who studies auctions with entry and the choice of whether to bundle or not. In Chakraborty (2006), all bidders are ex ante symmetric, with the finding that if entry costs are sufficiently large bundling does no better than selling individually, and often does worse. We discuss this further below, where we show that the result no longer holds once bidders are allowed to be ex ante asymmetric.
individual sales to be optimal may be rather small, with only 3 total bidders necessary when
the distribution function $F(.)$ is symmetric and an additional regularity condition is satisfied.
Given this finding, it is also straightforward to see that these results should extend to cases
where entry is endogenous but $N_f + 2 \leq N^* + 1$, so that there are at least as many total
number of bidders when assets are sold individually, which is $N^* + 1$, as when they are sold
jointly and there are two efficient bidders in addition to the $N_f$ regular bidders.

Given the inherent advantage of efficient allocation in the case of individual sales, the
question remains whether the presence of dominant bidders can make joint sales optimal.
We turn to this question in the next result. We show that as the dominance of the efficient
bidders increases, selling the assets jointly becomes optimal. We know, from Proposition 1,
that under free entry, the presence of the efficient bidder leads to a reduction in the expected
sales price, with the depression in the price becoming worse the more dominant is the bidder.
Moreover, the result from Proposition 1, that an increase in bidder dominance reduces seller
revenue, is true whether the assets are sold individually or jointly since it holds for any
distributions $F(.)$ and $G(.)$, where $G$ FOSD $F$. However, as we next show, the effect of the
efficient bidder on competition is different depending on how the assets are sold. For that,
we denote by $R_k$ the seller’s revenue for asset $k \in \{A, B\}$ when sold separately.

**Proposition 4** Consider $\hat{\varphi}$ and $\hat{\alpha}$ such that $2R_k \geq R_J$, that is selling assets individually
results in a higher revenue than selling the assets jointly. There exists $\overline{\varphi} > \hat{\varphi}$ such that, for
$\varphi > \overline{\varphi}$, the total revenue obtained from selling both assets individually is less than the revenue
from selling the assets jointly: $2R_k < R_J$.

The proposition establishes that when the efficient bidders are sufficiently dominant for
their respective preferred assets, selling the assets jointly rather than separately will raise the
greatest revenue. In other words, when $\varphi$ is high, the whole is worth more than the sum of
the parts. This is true because the number of bidders is endogenous and depends on bidders’
decisions whether or not to pay the entry or due diligence cost $c$ for the case of an individual
asset sale, and $2c$ in the case the firm is sold as a whole.
The intuition for the result stems from the differential effect of an increase in efficiency on competition when the assets are sold individually versus when they are sold jointly. For individual sales, an increase in dominance eventually reduces the probability of a regular bidder winning to zero. However, that is not the case for joint sales. To see this consider an extreme case. Suppose the dominant bidder $\alpha$ has a value very close to 1 for asset $A$ (the top of the support for the asset) and, similarly, dominant bidder $\beta$ has a value close to 1 for asset $B$. Even in this case, the probability that the sum of values for $A$ and $B$ is close to 2 for either of these efficient bidders is not very high. Hence, regular bidders always retain a non-trivial chance of winning, leading to greater entry than in the case where the assets are sold separately.

What is surprising about the result in Proposition 4 is that selling the assets jointly is optimal precisely when the increase in social surplus from allocating assets separately to the bidder that values it the most is potentially quite high. In other words, when $\varphi$ is large, for each asset there is a bidder who likely has a very high value compared to all other possible bidders. As argued above, when sold individually, each asset is allocated to the party that values it most so that the total surplus is maximal when both assets are sold separately. Pooling the assets to sell them, therefore, introduces a distortion in the allocation of the assets, with the distortion larger the larger is $\varphi$. The tradeoff, however, is that by doing so it reduces the impact of improved efficiency on the other bidders, and limits the extent to which competition is reduced when $\varphi$ increases. At some point, the revenue from selling the assets separately goes down too much because there is too little competition, whereas competition is not quite so dramatically reduced when the assets are sold jointly. Put differently, increases in $\varphi$ have a larger impact in reducing competition in the case of an individual asset sale than when the assets are sold jointly, where the effect is attenuated. For $\varphi$ large enough, the reduction in competition dominates so that selling the assets jointly is optimal.

One possible concern related to the result in Proposition 4 is whether the finding that bundled sales are better when $\varphi$ is sufficiently large is due to the effect of bundling on
competition by the regular bidders, or whether it arises from the fact that the two efficient
bidders are being forced to compete against each other. Indeed, there are two effects at play.
When sales are bundled and the two efficient bidders compete with each other, the seller is
more likely to receive two “high” offers and hence get a better price. But increases in the
degree of efficiency of these bidders also reduces competition by reducing \( N \), the number of
regular bidders. We show next that the main result, that as \( \varphi \) increases more revenue is
obtained by selling the assets together, is primarily driven by the reduction in competition
by regular bidders. To see that, we assume that the two efficient bidders do not compete
with each other, with one of them simply sitting out. This is equivalent to assuming that one
of the “efficient” bidders is not actually efficient, but rather draws his value for each asset
from the same distribution \( F(\cdot) \). We can now state the following result.

**Corollary 1** Suppose that there is only one efficient buyer, buyer \( \alpha \). There is a value \( \varphi \) such
that, for \( \varphi > \varphi^* \), the total revenue obtained from selling both assets individually is less than
the revenue from selling the assets jointly: \( 2R_k < R_J \).

The intuition for the corollary is similar to the intuition provided for the previous propo-
sition. The result relies on the simple observation that an almost sure dominance in one asset
does not imply an almost sure dominance for both assets combined.

### 5.4 Numerical example

In this section we present numerical examples to help understand when selling the firm’s
assets jointly is optimal, showing that this occurs when the efficient bidder’s advantage is
high. Conversely, the revenue from breaking up the firm and selling the assets individually
is higher either when all potential bidders are regular and, hence, symmetric, or when the
efficient bidder’s advantage is small.

To parametrize the model in order to obtain numerical solutions, we assume that a regular
bidder’s value for each asset is drawn from a uniform distribution in the unit interval, i.e.,
\( F(x) = x \) for \( x \in [0, 1] \). The efficient bidder is assumed to sample his value from distribution \( G(x) = x^{1+\varphi} \) for \( x \in [0, 1] \), where \( \varphi \geq 0 \).

With this, we can explicitly write the joint cdf for the combined firm value as

\[
Q(z) = \begin{cases} 
\frac{z^2}{2} & \text{for } 0 \leq z \leq 1 \\
z - \frac{1}{2} - \frac{(z-1)^2}{2} & \text{for } 1 < z \leq 2
\end{cases}
\]

for the regular bidders, and

\[
H(z) = \begin{cases} 
\frac{z^{2+\varphi}}{2+\varphi} & \text{for } 0 \leq z \leq 1 \\
z - \frac{1+\varphi}{2+\varphi} - \frac{(z-1)^{2+\varphi}}{2+\varphi} & \text{for } 1 < z \leq 2
\end{cases}
\]

for the efficient bidders.

Using the expressions derived in the previous section, we can also write the expected profit for a regular bidder when bidding for the combined assets as

\[
\pi_J(\varphi, N) = \int_0^2 (1 - Q(z)) H^2(z) Q^{N-1}(z) \, dz.
\]

The equilibrium number of bidders can be determined by using the endogenous entry condition, \( \pi_J(\varphi, N) = 2c \). For a given level of dominance \( \varphi \) the expected profit of the regular bidder is strictly decreasing in \( N \). This allows us to pin down the unique number of regular bidders in equilibrium.

The expected revenue for the case of selling the assets jointly is given by

\[
R_J = 2 - \int_0^2 Q(z)^N H(z)dz - \Pi_J - N\pi_J,
\]

which can also be calculated numerically using the given values of \( \varphi \) and \( c \), and the imputed value of \( N \).

Figure 1 graphs the seller’s revenue assuming a cost of bidding of \( c = 0.005 \) (in other words, 1% of the expected value of 0.5 for a regular bidder) for various values of the dominance
Figure 1: Expected revenue as a function of the degree of dominance by the efficient bidder(s) for $c = 0.005$.

Parameter $\varphi$. We represent the efficient bidder’s level of dominance along the x-axis by calculating his expected value of the asset for a given value of $\varphi$. Specifically, for the assumed parametrization (i.e., a uniform distribution) the expected value of the efficient bidder for his preferred asset is given by $\frac{1+\varphi}{2+\varphi}$. Thus, for $\varphi = 0$, a efficient bidder’s expected value collapses to that of a regular bidder, $\frac{1}{2}$, while for $\varphi = 3$ it corresponds to an expected value of $\frac{1+\varphi}{2+\varphi} = 0.8$ (the lowest level of dominance in the figure). The red line represents the combined revenue obtained when selling both assets individually, while the blue line represents the revenue from selling the two assets jointly. While both revenue curves decrease as dominance $\varphi$ increases because the number of regular bidders decreases, it is clear that the revenue when the assets are sold individually is much more sensitive to changes in $\varphi$ than the revenue when the assets are sold jointly. For low values of $\varphi$, the firm is clearly better off selling the assets individually. However, as $\varphi$ increases the difference between the two revenue curves decreases. At around $\frac{1+\varphi}{2+\varphi} = 0.91$ the two curves intersect, and for larger values the firm is better off selling the two assets jointly.

Figure 2 and 3 illustrate a similar pattern for higher values of $c$, the entry cost. Interestingly, as $c$ increases from 0.005, as in Figure 1, to 0.01 (Figure 2) and 0.015 (Figure 3), the
value of $\frac{1+c}{2+c}$ and, as a consequence, $\varphi$, at which selling the assets jointly becomes optimal decreases - it is approximately 0.875 when $c = 0.01$ and 0.845 for $c = 0.015$. In other words, increases in the cost of bidding, so that there are greater barriers to competition, favor selling the assets jointly since they reduce the point at which joint asset sales are optimal.

**Expected revenue as a function of the degree of dominance by the efficient bidder(s).**

![Figure 2: $c = 0.010$](image1)

![Figure 3: $c = 0.015$](image2)

This pattern can be seen very clearly from Figure 4, which plots the expected value of the efficient bidder per asset, $\frac{1+c}{2+c}$, in the horizontal axis against the entry cost $c$ on the vertical axis. The green diamonds represent configurations of $(c, \varphi)$ for which selling the assets individually raises greater revenue and is thus optimal for the seller. The blue circles represent configurations for which selling the assets jointly is optimal. Since the efficient bidder’s expected value is monotonically increasing in $\varphi$, Figure 4 clearly illustrates that as $c$ increases, the threshold value of $\varphi$ beyond which selling the assets jointly is optimal (Proposition 1) decreases.

One aspect illustrated by the numerical example in this section is the stark contrast between how sensitive seller revenue is to increases in bidder dominance $\varphi$ in the joint versus individual asset sale cases. As $\varphi$ increases, revenue when the assets are sold separately decreases quite dramatically, moving quickly toward providing a vanishingly small premium to the seller. By contrast, in the joint sale case the seller’s revenue is very nearly flat over a
Figure 4: The figure displays whether joint or individual sales of assets generate higher revenue for the seller, for various pairs of \((c, \varphi)\).

broad range of values for \(\varphi\). The reason is twofold. First, as argued above, the dominance of any given efficient bidder is more diffuse when the assets are sold as a package, leading to a smaller reduction in competition. Second, since with two assets there are actually two efficient bidders, these bidders continue to compete against each other in a more symmetric fashion even as the other regular bidders become more handicapped.

6 Additional considerations

Here, we analyze how various other factors, such as financial constraints or synergies in information acquisition, affect the optimality of joint sales relative to individual asset sales when due diligence is important and, hence, bidders face a real decision on whether to enter or not.

6.1 Due diligence costs

Up to now we have assumed that a bidder’s cost associated with the due diligence process for learning the value of the combined assets is simply double the cost of learning the value
of a single asset. While likely sensible in many circumstances, the primary reason for this assumption was to allow us to focus the analysis entirely on the role played by endogenous entry and bidder heterogeneity, excluding other factors that may favor one type of asset sale versus the other. In practice, however, it is likely that in many circumstances, and particularly for larger transaction, information acquisition costs may in fact be convex in the number of assets being acquired, thus increasing faster as more assets, and hence more complexity, is added. For instance, this would be the case if a potential bidder has limited time to learn about any possible assets for sale, and evaluating two (or more) assets rather than one comes at an increasingly higher opportunity cost of the bidder’s time.\textsuperscript{10}

Suppose that, as before, the information acquisition cost to a bidder of learning his value for an asset, either $A$ or $B$, is $c$. However, the information acquisition cost for learning the value of the combined assets is $C > 2c$. Otherwise, the model is unchanged.

Define the threshold value $\overline{C}$ as the largest value of $C$ such that an efficient bidder would always find it optimal to participate in the auction for asset $j$, when there are no regular bidders participating: $\Pi_j (\varphi_H, N = 0) = \overline{C}$. One can now see that, despite the diseconomy of scale in information acquisition, so that learning about the combined assets is proportionally more difficult and costly that learning about each individual asset, selling the assets jointly is still preferable to the seller when the efficient bidders are sufficiently dominant. The reason is similar to that in Section 5.3: as the degree of dominance of the efficient bidders increases, the number of regular bidders when assets are sold individually goes to zero, thus driving the sales price (or, rather, the premium paid for the assets) to zero as well. When assets are sold jointly, however, for any given $N$ the profit of a regular bidder is bounded above zero for any value of $\varphi$, so that entry will still occur. The higher cost of information acquisition, $C$, does reduce entry by regular bidders relative to the case we considered above, but does not eliminate competition entirely as long as the cost is not so high that not even a single

\textsuperscript{10}One may imagine that some instances may be characterized instead by economies of scale, such as if the assets are similar and learning about one then reduces the cost associated with learning about the other. In that case, the cost of acquiring information for the combined assets would likely be less than $2c$. We discuss that case briefly at the end of the section.
bidder \((N = 1)\) could stand to make a profit. Finally, the constraint on \(C\), that it not be larger than \(\overline{C}\), ensures that the diseconomy of scale is not so large as to completely prevent the participation of even the efficient bidders in the sale. For \(C\) greater than \(\overline{C}\), joint asset sales generate too little competition because the cost of evaluating the assets is too high.

We summarize this discussion in the following proposition.

**Proposition 5** For \(C < \overline{C}\), there is a value \(\varphi(C)\) such that, for \(\varphi > \varphi(C)\), the total revenue obtained from selling both assets individually is less than the revenue from selling the assets together: \(2R_k < R_J\).

### 6.2 Financial constraints

An important consideration for many instances of asset sales is the ability of potential bidders to finance the acquisition. This is particularly true in the context of divisional sales for companies, where the dollar value of the acquisition is likely high and a substantial fraction of the total value of the acquiring party. Moreover, in those instances the valuation is likely a function of projected future cash flows derived from the ownership of the assets, making the information acquisition cost, and hence the endogeneity of bidder entry, an important consideration. While a full characterization of financial constraints in bidding is beyond the scope of our analysis, in this section we study an important dimension of how the financial constraints bidders may face affect a seller’s preference between bundling assets or not.

Specifically, we consider financing costs that increase in the amount of financing that might be required. We assume that every dollar that is bid has a cost of \(\alpha > 0\) associated with raising it, so that if an offer of \(W\) is made for an asset, the bidder must incur \(\alpha W\) in fund raising costs.\(^{11}\)

We can now see how such financial constraints affect seller revenue and the optimal sales mechanism. Suppose that bidder \(i\) has value \(x^i_A\) for asset \(A\), and offers a price \(b^i_A\). If he wins

\(^{11}\)This approach to considering financial constraints has been used elsewhere to study its implications for bidding behavior. See, for instance, Gorbenko and Malenko (2018).
the auction, his value is \( x_A^i - (1 + \alpha) \max \{ b_A^j | j \neq i \} \). The bidder’s payoff will be positive if \( x_A^i - (1 + \alpha) \max \{ b_A^j | j \neq i \} > 0 \Leftrightarrow \frac{x_A^i}{1+\alpha} - \max \{ b_A^j | j \neq i \} > 0 \). Therefore, defining the bidder’s adjusted value for the asset as \( \bar{x}_A^i = \frac{x_A^i}{1+\alpha} \), it is a dominant strategy for the bidder to bid his adjusted value \( \bar{x}_A^i: \bar{b}_A^i = \bar{x}_A^i = \frac{x_A^i}{1+\alpha} \). Other than this change, however, all other aspects remain the same so that, qualitatively, Proposition 1 continues to hold for the case where bidders face financial constraints that make it costly to raise financing to pay for the assets.

In other words, while financial constraints make large (i.e., joint) acquisitions proportionately more costly, increases in dominance by the efficient bidders reduces entry to a greater extent when assets are sold individually than when they are sold jointly. For sufficiently high levels of dominance \( \varphi \), joint sales become optimal despite the higher financing costs associated with them.

### 7 Conclusion

Our analysis establishes that, when the number of bidders is endogenous and subject to a participation decision, the inclusion of a bidder whose value is drawn from a distribution that dominates that of the other bidders leads to a reduction in participation and reduces seller revenue. Interestingly, the reduction in competition always overwhelms the benefit that a seller can obtain from this efficient bidder. This is in contrast to what occurs when the number of bidders is exogenously given. In the latter case, the presence of a efficient bidder always benefits the seller through a higher price for the good being sold.

The issues highlighted above concerning endogenous entry should have important implications for settings where due diligence is important, and also for the way in which assets are sold - either bundled together or sold separately. Essentially, the effect on bidder entry is lower when assets are bundled so that, as a consequence, bundling is optimal when some bidders are very dominant since it is then that entry will be lowest when assets are sold individually.
The analysis extends to incorporate other issues that might be relevant in the selling of assets. For instance, one can well imagine that diseconomies of scale in screening, performing due diligence, and fixed costs of putting together an offer may increase the costs of entry when assets are sold jointly relative to when sold individually. We argue that while this tends to favor individual asset sales, joint sales are still optimal when bidders are very dominant because of the entry deterrence effect highlighted above. An aspect that we have not considered, but which may well be relevant in practice, is that asset values may well be correlated, either positively or negatively, introducing additional reasons why assets may be bundled to exploit such correlations. These issues are likely important but, we believe, separate from the competitive concerns identified here.
Appendix A: Proofs

Proof of Lemma 1: Regular bidder profits are \( \int (1 - F(x)) F^{N-1}(x) G(x) \, dx \) from (1). Thus, for \( N' > N \) we have

\[
\pi(N|G) - \pi(N'|G) = \int (1 - F(x)) F^{N-1}(x) \left( 1 - F^{N-N}(x) \right) G(x) \, dx > 0.
\]

We show that the seller’s revenue increases with the number of regular bidders for \( N' = N+1 \), for arbitrary \( N \):

\[
R(N+1|G) - R(N|G) = \int F^N \, dx - \int F^{N+1} \, dx + N \left( \int (1 - F) F^{N-1} G \, dx \right) - (N + 1) \left( \int (1 - F) F^N G \, dx \right)
\]

\[
= \int (1 - F) F^N \, dx + N \left( \int (1 - F) F^{N-1} G \, dx \right) - (N + 1) \left( \int (1 - F) F^N G \, dx \right)
\]

\[
> \int (1 - F) F^N G \, dx + N \left( \int (1 - F) F^{N-1} G \, dx \right) - (N + 1) \left( \int (1 - F) F^N G \, dx \right)
\]

\[
= N \left[ \int (1 - F) F^{N-1} G \, dx - \int (1 - F) F^N G \, dx \right]
\]

\[
= N \int (1 - F)^2 F^{N-1} G \, dx > 0.
\]

For the effect of the efficient bidder’s distribution function \( G \) on seller revenue, consider that

\[
R(N|G) = E \left[ \max \{ x_1, \ldots, x_N, y \} \right] - \Pi(N|G) - N \pi(N|G)
\]

\[
= 1 - \int_0^1 G(z) F(z)^N \, dz - \int (1 - G(x)) F^N(x) \, dx - N \pi
\]

\[
= 1 - \int_0^1 F(z)^N \, dz - N \pi.
\]

Therefore, from above,

\[
R(N|G') - R(N|G) = N \left[ \pi(N|G) - \pi(N|G') \right].
\]
Since under FOSD $\pi (N|G) > \pi (N|G')$, we then have that $R(N|G') - R(N|G) > 0$, which establishes the result. \hfill \square

**Proof of Proposition 1:** To establish the result, we first make use of the following fact, which we present as an auxiliary result.

**Claim:** Given $N$ regular bidders with values drawn from distribution function $F(\cdot)$ and one efficient bidder with value drawn from distribution function $G(\cdot)$, seller revenue depends on $G(\cdot)$ only through it’s impact on the number of bidders participating and their expected profit in equilibrium. Specifically, $R(N|G) = E[\max\{x_1, \ldots, x_N\}] - N\pi(N|G)$.

**Proof:** To show this, note that seller revenue, by definition, is equal to

\[
R(N|G) = E[\max\{x_1, \ldots, x_N, y\}] - \Pi(N|G) - N\pi(N|G) \\
= 1 - \int_0^1 G(z)F(z)^N \, dz - \int (1 - G(x)) F^N(x) \, dx - N\pi(N|G) \\
= 1 - \int_0^1 F(z)^N \, dz - N\pi(N|G) \\
= E[\max\{x_1, \ldots, x_N\}] - N\pi(N|G),
\]

which establishes the result. \hfill \square

We can now make use of the claim to prove the main result. Under endogenous participation, $\pi(N_G|G) = c$, with $N_G \equiv N^*(G)$ being the zero profit number of regular bidders. The seller’s revenue is then

\[
R(N_G|G) = E[\max\{x_1, \ldots, x_{N_G}\}] - N_G c.
\]

Now suppose that instead of $G$, the efficient bidder’s value is drawn from a dominant distribution function $G'$ and $N'_G \equiv N^*(G')$, where $\pi(N|G') < \pi(N|G) \ \forall N > 1$. When participation is endogenous we must have $N'_G < N_G$. 

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Consider now seller revenue under $G'$, which is

$$R(N'_G|G') = E \left[ \max \{ x_1, \ldots, x_{N'_G} \} \right] - N'_G c.$$ 

To show that $R(N'_G|G') < R(N_G|G)$, it suffices to show that $R(N|G) = E \left[ \max \{ x_1, \ldots, x_N \} \right] - N \pi (N|G) = E \left[ \max \{ x_1, \ldots, x_N \} \right] - Nc$ is increasing in $N$. For that, we have

$$\frac{d}{dN} R(N|G) = \frac{d}{dN} \left( 1 - \int_0^1 F(z)^N dz - Nc \right) = -\int_0^1 F(z)^N \ln F(z) \, dz - c.$$

Substituting for $c = \pi (N|G) = \int (1 - F(z)) F^{N-1}(z) G(z) \, dz$, we have

$$\frac{d}{dN} R(N|G) = -\int_0^1 F(z)^N \ln F(z) \, dz - \int_0^1 (1 - F(z)) F^{N-1}(z) G(z) \, dz$$

$$> -\int_0^1 F(z)^N \ln F(z) \, dz - \int_0^1 (1 - F(z)) F^N(z) \, dz$$

since $G \succ_{FOSD} F$. Thus, $\int (1 - F(z)) F^{N-1}(z) G(z) \, dz = \pi (N|G) < \pi (N|F) = \int (1 - F(z)) F^N(z) \, dz.$

Grouping terms, we have

$$\frac{d}{dN} R(N|G) > -\int F(z)^N [1 - F(z) + \ln F(z)] \, dz.$$

Examine the expression in the square bracket as a function of $F$: $1 - F + \ln F$. This expression is concave in $F$ and it attains its maximum at $F = 1$. Moreover, since $F(z)$ is monotonic in $z$, this implies that the expression is maximized at $z = 1$. At $z = 1$, the expression is zero. Hence it is negative everywhere else. Therefore,

$$-\int F(z)^N [1 - F(z) + \ln F(z)] \, dz > 0,$$

which establishes that $\frac{d}{dN} R(N|G) > 0$. With this, it follows directly that $R(N_G'|G') < R(N_G|G).$
**Proof of Proposition 2:** In the case when the efficient bidder also has the distribution $F(x)$,

$$
\pi (N_F|F) = E \left[ \max \{x_1, \ldots, x_{N_F}, x_{N_F+1}\} \right] - E \left[ \max \{x_1, \ldots, x_{N_F}\} \right] \\
= 1 - \int F^{N_F+1} (x) \, dx - \left( 1 - \int F^{N_F} (x) \, dx \right) \\
= \int F (x) (1 - F (x)) F^{N_F-1} (x) \, dx = c, \tag{12}
$$

where the last equality comes from the zero-profit condition. Similarly,

$$
\pi (N_G|G) = \int G (x) (1 - F (x)) F^{N_G-1} (x) \, dx = c,
$$

by definition of $N_G$. Now consider the regular bidders’ profits when the efficient bidder has distribution $J(x)$:

$$
\pi (N_G|J) = \int J (x) (1 - F (x)) F^{N_G-1} (x) \, dx \\
= \int F^{(N_F-N_G)} (x) F (x) (1 - F (x)) F^{N_G-1} (x) \, dx \\
= \int F (x) (1 - F (x)) F (x)^{N_F-1} \, dx \\
= c,
$$

where the last equality follows from (12). Hence, $\pi (N_G|G) = \pi (N_G|J) = c$ and $N_G$ bidders participating under distribution $J(y)$ for the efficient bidder is supported in equilibrium.

To show the equivalence in revenues we have the following from the claim presented in the proof of Proposition 1:

$$
R(N_G|G) = E \left[ \max \{x_1, \ldots, x_{N_G}|G\} \right] - N_G \pi (N_G|G).
$$

Recall that, as argued above, the revenue to the seller is affected by the efficient bidder’s
distribution only through the profits of the regular bidders and the number of regular bidders. Given that \( \pi (N_G|G) = \pi (N_G|J) = c \), in equilibrium \( N_G = N^* (J) = N_J \). Thus,

\[
R (N_G|G) = R (N_G|J) = R (N_J|J),
\]

as desired.

**Proof of Proposition 3:** In what follows, we simplify notation by eliminating the “*” for equilibrium variables when there is no risk of confusion. First, note that

\[
\frac{d\pi}{d\varphi_i} = \left( \frac{dN}{d\varphi_i} + 1 \right) \int \ln (F(z)) (1 - F(z)) F(z)^{N-1+K+\sum_{i=1}^{K} \varphi_i} \, dz.
\]

At the zero profit equilibrium, \( \frac{d\pi}{d\varphi_i} \) must be equal to zero, which implies that \( \frac{dN}{d\varphi_i} = -1 \). We use this below to calculate \( \frac{d\Pi_j}{d\varphi_i} \) as follows:

\[
\frac{d\Pi_j}{d\varphi_i} = \left( \frac{dN}{d\varphi_i} + 1 \right) \int \ln (F(z)) (1 - F(z)^{1+\varphi_j}) F(z)^{N+K-1+\sum_{i\neq j} \varphi_i} \, dz = 0.
\]

We can now calculate the effect of an increase in the dominance of efficient bidder \( i \) on seller revenue:

\[
\frac{dR}{d\varphi_i} = -\frac{dN}{d\varphi_i} \int \ln F(z) F(z)^{N+K-1+\sum_{j\neq i} \varphi_j} \, dz - \sum_{j\neq i} \frac{d}{d\varphi_i} \Pi_j - \frac{dN}{d\varphi_i} - N \frac{d\pi}{d\varphi_i}
\]

\[
= \int \ln F(z) F(z)^{N+K-1+\sum_{j\neq i} \varphi_j} \, dz + \pi
\]

\[
= \int [\ln F(z) + (1 - F(z)) F(z)^{\varphi_i}] F(z)^{N+K-1+\sum_{j\neq i} \varphi_j} \, dz.
\]

By the same argument as before, \( \ln F(z) + 1 - F(z) \) is negative for all values \( F(z) < 1 \). Given that \( \varphi_i \geq 0 \) this implies \( \ln F(z) + (1 - F(z)) F(z)^{\varphi_i} \) is also negative. Hence, \( \frac{dR}{d\varphi_i} < 0 \), as desired.

**Proof of Proposition 4:** To establish this result, note from (3) that the seller’s revenue
when the assets are sold individually is positive only if the equilibrium number of regular bidders, \( N^* \), is strictly greater than zero. However, since \( G(z|\varphi) \to 0 \) as \( \varphi \to \varphi^H \), one can see from (1) that \( \lim_{\varphi \to \varphi^H} \pi_k = 0 \), implying that for any \( c > 0 \), \( N^* \) will converge to zero as \( \varphi \to \varphi^H \). Therefore, \( R_k \) also converges to zero.

By contrast, the revenue when selling the assets jointly, \( R_J \), given in (9), may be positive even for \( N = 0 \) because of the presence of the two efficient buyers. Even ignoring that, note that for a regular bidder, keeping \( N \) constant,

\[
\lim_{\varphi \to \varphi^H} \pi_J = \lim_{\varphi \to \varphi^H} \int_0^2 (1 - Q(z)) H(z)^2 Q(z)^{N-1} \, dz \\
= \int_1^2 (1 - Q(z)) F(z - 1) Q(z)^{N-1} \, dz > 0.
\]

Therefore, \( \lim_{\varphi \to \varphi^H} \pi_J > 0 \) for any finite \( N \), implying that the equilibrium number of regular bidders will be bounded away from zero even as \( \varphi \to \varphi^H \). As a result, \( \lim_{\varphi \to \varphi^H} R_J > 0 \) as well, establishing the result.

\[\square\]

**Proof of Corollary 1:** If we have only one efficient bidder, this is equivalent to saying that one of the efficient bidders has its value drawn from \( Q \) rather than from \( H \). Regular bidder profits become

\[
\pi_J = E [\max \{X^1, \ldots, X^N, Z^\alpha, Z^\beta\}] - E [\max \{X^1, \ldots, X^{N-1}, Z^\alpha, Z^\beta\}] \\
= \int_0^2 (1 - Q(z)) H(z) Q(z)^N \, dz,
\]

since bidder \( \beta \) draws both of its values from the distribution \( F \), just like the other regular bidders. As in the proof of Proposition 4, we take the limit as \( \varphi \to \varphi^H \), keeping \( N \) constant,

\[
\lim_{\varphi \to \varphi^H} \pi_J = \lim_{\varphi \to \varphi^H} \int_0^2 (1 - Q(z)) H(z) Q(z)^N \, dz \\
= \int_1^2 (1 - Q(z)) F(z - 1) Q(z)^N \, dz > 0.
\]
Since profits for the regular bidders remain strictly positive, this demonstrates that the equilibrium number of regular bidders will be bounded away from zero even as $\varphi \to \varphi^H$ and, as a consequence, that $\lim_{\varphi \to \varphi^H} R_J > 0$ as well, thus establishing the result. $\square$

Appendix B: Robustness

Here, we briefly extend the analysis above to consider various issues of robustness discussed in the text.

Increasing competition through entry subsidies

We have shown that increased dominance of the efficient bidder leads to reduced participation and lower seller revenue. Since the main effect comes from the change in the degree of competition, a natural question is whether a seller could offset that effect by encouraging greater participation through the use of an entry subsidy. Here, we study this question.

Suppose that the seller can offer an entry subsidy $s > 0$ to all bidders that enter the auction.\(^{12}\) The zero-profit condition for symmetric bidders becomes

$$\pi(N^*(G, s)|G) + s = c,$$

where we have extended the earlier notation for the equilibrium number of bidders, $N^*(G, s)$, to include a dependence on the subsidy $s$. Clearly, $N^*$ will be increasing in $s$ because $\pi$ is decreasing in $N$.

\(^{12}\)We focus on uniform, non-discriminatory mechanisms based on the notion that in many instances it may not be easy to identify the presence or identity of a dominant bidder. Similarly, in practice implementing a discriminatory mechanism may often not be feasible, particularly when the sale is not conducted as a formal auction but rather as a more unrestricted competitive bidding process. Nevertheless, the cost imposed on the seller due to the existence of the dominant bidder does not depend on all bidders (including the dominant bidder) receiving an entry subsidy. All results go through for the case where only a subset of bidders receive subsidies as long as the impact of the subsidy is not so large that it drives the dominant bidder out. Even for that case, however, the possibility of entry by a dominant bidder can still only lead to the seller being worse off compared to the case where only symmetric bidders are able to compete.
Using the arguments above, we can formulate the seller’s problem of maximizing with respect to the size of the subsidy $s$ as:

$$
\max_s R(N^*, s|G) = 1 - \int F(z)^N dz - N(\pi + s) - s.
$$

(14)

Note that, with a slight abuse of notation, we now define the revenue $R$ as net of the cost of the subsidy. We can now state the following result.

**Proposition 6** Suppose that the seller can choose an optimal subsidy $s^* \geq 0$. Under endogenous participation, the seller’s expected revenue $R(N^*, s|G)$ is lower if the dominance of the efficient bidder increases.

**Proof of Proposition 6:** Using the result from Proposition 2, any asymmetric auction where the efficient bidder’s value is drawn from arbitrary $G$ can be equivalently represented by an auction where the efficient bidder’s value is drawn from distribution $J(y) = F^{1+\varphi}(y)$, for some $\varphi > 0$. An increase in dominance is then simply equivalent to an increase in $\varphi$. Now, the FOC for (14), assuming an interior solution, is

$$
\frac{\partial R}{\partial s} + \frac{\partial R \partial N^*}{\partial N \partial s} = 0.
$$

Denoting the solution as $s^*(\varphi)$, we can write the equilibrium seller revenue as

$$
R(N^*(\varphi, s^*(\varphi)), s^*(\varphi)|\varphi).
$$

In the above expression we are explicit about the dependence on $\varphi$. We now consider the effect of an increase in $\varphi$, when the subsidy is set optimally and equal to $s^*$:

$$
\frac{dR}{d\varphi} = \frac{\partial R}{\partial \varphi} + \frac{\partial R \partial s^*}{\partial s \partial \varphi} + \frac{\partial R \partial N^*}{\partial N \partial \varphi} + \frac{\partial R \partial N^* \partial s^*}{\partial N \partial s \partial \varphi}.
$$

$$
= \frac{\partial R}{\partial \varphi} + \frac{\partial R \partial N^*}{\partial N \partial \varphi} + \left( \frac{\partial R}{\partial s} + \frac{\partial R \partial N^*}{\partial N \partial s} \right) \frac{ds^*}{d\varphi}.
$$
The last term, $\frac{\partial R}{\partial s} + \frac{\partial R}{\partial N} \frac{\partial N^*}{\partial s}$, is equal to 0 from the Envelope Theorem, leaving us with

$$\frac{dR}{d\varphi} = \frac{\partial R}{\partial \varphi} + \frac{\partial R}{\partial N} \frac{\partial N^*}{\partial \varphi}.$$  

Keeping the size of the subsidy fixed, this derivative is negative from the same arguments used in Proposition 1. Therefore, $\frac{dR}{d\varphi} < 0$, as desired.  

The proposition shows that, even if it may be optimal to provide an entry subsidy in the sense that it provides a net positive effect on revenue, this does not change the fact that the presence of a more dominant bidder reduces the seller’s revenue. Effectively, the seller is unable to compensate for the negative effect stemming from the efficient bidder’s presence by subsidizing more competition.

**Reserve prices**

Here, we study the effect of allowing the seller to set an optimal reserve price. As McAfee and McMillan (1987) show, when all bidders are symmetric the use of reserve prices is suboptimal to the extent that they reduce entry and thus may lead to lower seller revenue.\(^{13}\) However, the use of a positive reserve price may be optimal when bidders are asymmetric, even when participation is endogenous.

Consider the following slight change to the setup: assume that the seller can, before bidders decide whether to incur the participation cost $c$, choose a reserve price $\rho$ and commit not to sell below this price. The model is otherwise unchanged. For tractability, we restrict attention to cases where the efficient bidder’s value is drawn from a distribution $G$ that is a power function of $F$: $G(y|\varphi) = F(y)^{1+\varphi}$ for all $y \leq 1$ and $\varphi \geq 0$. From Proposition 2, this is without loss of generality when the reserve price is zero.\(^{14}\)

\(^{13}\)Lu and Ye (2013) study revenue maximizing entry and auction design. They show that to induce efficiency, the auction design must have a reserve price equal to zero or, more generally, to the value of the seller. They then show that maximizing revenue requires efficiency, so that indeed the reserve price should be zero. While Lu and Ye (2013) do not consider the possibility that some bidders may be “dominant” in the sense of having values drawn from different distributions, they incorporate asymmetries through heterogeneity in bidders’ costs of entry.

\(^{14}\)Whether Proposition 2 extends to the case with a positive reserve price is unknown, as we have not yet
Given a specific realization \((x_1, \ldots, x_N, y)\) and a reserve price \(\rho\), in a second price auction the \(N^{th}\) bidder’s profit is given, much as above, by:

\[
\tilde{\pi}_N (x_1, \ldots, x_N, y) = \max \{x_1, \ldots, x_N, y, \rho\} - \max \{x_1, \ldots, x_{N-1}, y, \rho\}.
\]

Thus, the ex ante expected profit for bidder \(N\), and by symmetry all other bidders, is

\[
\pi = E [\max \{x_1, \ldots, x_N, y, \rho\}] - E [\max \{x_1, \ldots, x_{N-1}, y, \rho\}]
\]

(15)

Similarly, the dominant bidder’s expected profit is

\[
\Pi = E [\max \{x_1, \ldots, x_N, y, \rho\}] - E [\max \{x_1, \ldots, x_N, \rho\}].
\]

(16)

As done above, we can calculate the seller’s revenue, \(R\), by subtracting the profits of all bidders from the total surplus. To find the total surplus, consider the existence of a “pseudo bidder” who bids \(\rho\) and has value \(\rho\). Including this additional bidder, the total surplus would be \(\max \{x_1, \ldots, x_N, y, \rho\}\). However, when \(\max \{x_1, \ldots, x_N, y, \rho\} = \rho\) the good does not get sold, so that the surplus is in fact zero. Thus, we need to subtract the probability of this occurring, which is \(G(\rho) F^N(\rho)\), from the expectation of \(\max \{x_1, \ldots, x_N, y, \rho\}\) to obtain the total surplus as \(E [\max \{x_1, \ldots, x_N, y, \rho\}] - \rho G(\rho) F^N(\rho)\). From here, we obtain the seller’s revenue as

\[
R = E [\max \{x_1, \ldots, x_N, y, \rho\}] - \rho G(\rho) F^N(\rho) - \Pi - N\pi
\]

(17)

\[
= 1 - \int_\rho^1 F(z)^N dz - \rho G(\rho) F(\rho)^N - N \int_\rho^1 F(z)^{N-1} G(z) (1 - F(z)) dz.
\]

We can now establish the following result. We extend notation slightly and express the seller’s revenue \(R\) as a function of the parameter \(\varphi\) and the endogenous variables \(\rho\) and \(N\).
Proposition 7 Suppose that the seller can choose an optimal, revenue maximizing nondiscriminatory reserve price \( \rho^* \geq 0 \). Under endogenous participation, the seller’s revenue is lower as the dominance of the efficient bidder increases. Specifically, \( R(\rho^*, N^*|\varphi) \) is decreasing in \( \varphi \): \( \frac{dR}{d\varphi} < 0 \).

Proof of Proposition 7: To establish the result, we make use of the following auxiliary lemma which considers the case of a fixed and exogenously given reserve price \( \rho \).

Lemma 3 Consider a fixed reserve price \( \rho \in (0, 1) \). Under endogenous participation, the seller’s revenue \( R(\rho, N^*|\varphi) \) is decreasing in the dominance of the efficient bidder for any \( \rho > 0 \): \( \frac{dR}{d\varphi} < 0 \).

Proof: Suppose that \( \rho > 0 \). We can rewrite (17) slightly as

\[
R = 1 - \int_\rho^1 F(z)^{N(\varphi, \rho)} \, dz - N(\varphi, \rho) \pi(\varphi, \rho, N) - \rho G(\rho|\varphi) F(\rho)^{N(\varphi, \rho)},
\]

where we note the dependency of \( N \) on \( \varphi \) and \( \rho \). Differentiating with respect to \( \varphi \) obtains

\[
\frac{dR}{d\varphi} = -\rho \frac{\partial}{\partial \varphi} G(\rho|\varphi) F(\rho)^{N(\varphi, \rho)}
- \frac{\partial N}{\partial \varphi} \left( \int_\rho^1 F(z)^{N(\varphi, \rho)} \ln F(z) \, dz + \pi(\varphi, \rho, N) + \rho G(\rho|\varphi) F(\rho)^{N(\varphi, \rho)} \ln F(\rho) \right)
- \rho \frac{\partial}{\partial \varphi} G(\rho|\varphi) F(\rho)^{N(\varphi, \rho)} - \frac{\partial N}{\partial \varphi} \rho G(\rho|\varphi) F(\rho)^{N(\varphi, \rho)} \ln F(\rho)
- \frac{\partial N}{\partial \varphi} \left( \int_\rho^1 F(z)^{N(\varphi, \rho)} \ln F(z) \, dz + \pi(\rho, N|\varphi) \right).
\]

Note, however, that for \( G = F^{1+\varphi}, \frac{\partial}{\partial \varphi} G(\rho|\varphi) = F(\rho)^{1+\varphi} \ln F(\rho) \). Substituting, we have

\[
\frac{dR}{d\varphi} = -\left( 1 + \frac{\partial N}{\partial \varphi} \right) \rho F(\rho)^{1+\varphi} F(\rho)^{N(\varphi, \rho)} \ln F(\rho) - \frac{\partial N}{\partial \varphi} \left( \int_\rho^1 F(z)^{N(\varphi, \rho)} \ln F(z) \, dz + \pi(\rho, N|\varphi) \right).
\]

(18)

For a given \( N \), a symmetric bidder’s profit when the reserve price \( \rho > 0 \) is given by

\[
\pi(\rho, N|\varphi) = \int_\rho^1 (1 - F(y)) G(y|\varphi) F(y)^{N-1} \, dy = \int_\rho^1 (1 - F(y)) F(y)^{1+\varphi} F(y)^{N-1} \, dy.
\]
der zero profits, the value of $N$ must satisfy $\pi(\rho, N|\varphi) - c \equiv 0$ for all possible $(\rho, \varphi)$: $N^*(\rho, \varphi)$.

Differentiating $\pi$ with respect to $\varphi$, keeping $\rho$ constant, therefore yields

$$\frac{d\pi}{d\varphi} = \left(1 + \frac{\partial N}{\partial \varphi}\right) \int_\rho^1 \ln F(y) \left(1 - F(y)\right) F(y)^{1+\varphi} F(y)^{N-1} dy = 0,$$

implying that $\frac{\partial N}{\partial \varphi} = -1$ in order to satisfy the zero profit condition.

Substituting $\frac{\partial N}{\partial \varphi} = -1$ into (18) and plugging in for $\pi$ then yields

$$\frac{dR}{d\varphi} = \int_\rho^1 F(z)^{N(\varphi, \rho)} \ln F(z) dz + \int_\rho^1 (1 - F(y)) F(y)^{1+\varphi} F(y)^{N-1} dy. \quad (19)$$

A similar argument as in the proof of Proposition 1 establishes that $\frac{dR}{d\varphi} < 0$, as desired. □

Now consider what happens when the reserve price $\rho$ is chosen optimally and is allowed to be a function of the degree of dominance $\varphi$: $\rho^*(\varphi)$. The optimal reserve price $\rho^*$ solves

$$\max_{\rho} R(\rho, N^*(\varphi, \rho) |\varphi).$$

For an interior solution, $\rho^* \in (0, 1)$, the necessary first order condition is

$$\frac{\partial R}{\partial \rho} + \frac{\partial R}{\partial N} \frac{\partial N^*}{\partial \rho} = 0. \quad (20)$$

We can now write the equilibrium seller revenue as

$$R(\rho^*(\varphi), N^*(\varphi, \rho^*(\varphi)) |\varphi).$$

Now consider an increase in dominance:

$$\frac{dR}{d\varphi} = \frac{\partial R}{\partial \varphi} + \frac{\partial R}{\partial \rho} \frac{d\rho^*}{d\varphi} + \frac{\partial R}{\partial N} \frac{\partial N^*}{\partial \varphi} + \frac{\partial R}{\partial N} \frac{\partial N^*}{\partial \rho} \frac{d\rho^*}{d\varphi}$$

$$= \frac{\partial R}{\partial \varphi} + \frac{\partial R}{\partial N} \frac{\partial N^*}{\partial \varphi} + \left(\frac{\partial R}{\partial \rho} + \frac{\partial R}{\partial N} \frac{\partial N^*}{\partial \rho}\right) \frac{d\rho^*}{d\varphi}. \quad (21)$$
We can now use the Envelope Theorem: by (20), the term inside the parenthesis in (21) must be zero. We can therefore simplify (21) to

$$\frac{dR}{d\varphi} = \frac{\partial R}{\partial \varphi} + \frac{\partial R}{\partial N} \frac{\partial N^*}{\partial \varphi}.$$  \hspace{1cm} (22)

However, this is just the derivative of the seller’s revenue, $R$, taking the reserve price $\rho$ as fixed. Lemma 3 establishes that this derivative is indeed negative, or in other words that an increase in dominance $\varphi$ leads to a decrease in seller revenue even for an optimally chosen reserve price.

The final step is to consider the possible corner solutions, with $\rho^* = 0$ or 1, which may not satisfy the first order condition. If the optimal reserve price is zero, then Proposition 1 already establishes the result. Conversely, $\rho^* = 1$ is clearly not optimal for any positive entry cost $c > 0$. This establishes our result. \hfill \square

As for Proposition 1, the result that increases in bidder dominance reduces seller revenue is driven by the reduction in competition that arises endogenously. Proposition 7 shows that the negative effect arising out of the reduction in competition is greater than the increase in (expected) value stemming from the dominant bidder’s improved distribution, even when the seller can use a reserve price optimally to extract as much value as possible.
References


Myers, Stewart C and Nicholas S Majluf (1984), ‘Corporate financing and investment decisions when firms have information that investors do not have’, *Journal of financial economics* 13(2), 187–221.


