Excess Volatility: Beyond Discount Rates*

Stefano Giglio      Bryan Kelly
University of Chicago and NBER

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Abstract

We document a form of excess volatility that is irreconcilable with standard models of prices, even after accounting for variation in discount rates. We compare prices of claims on the same cash flow stream but with different maturities. Standard models impose precise internal consistency conditions on the joint behavior of long and short maturity claims and these are strongly rejected in the data. In particular, long-maturity prices are significantly more variable than justified by the behavior at short maturities. Our findings are pervasive. We reject internal consistency conditions in all term structures that we study, including equity options, currency options, credit default swaps, volatility swaps, and inflation swaps.

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1 Introduction

Term structure analysis is a powerful setting for evaluating a model’s ability to describe asset price data for two reasons. First, any model that satisfies a minimal requirement—that it rules out arbitrage opportunities—imposes strict testable restrictions on the joint behavior of prices along the term structure. Specifically, no-arbitrage prices must obey the law of iterated expectations, thus the prices of long maturity claims must reflect investors’ expectations about the future value of short maturity claims.\(^1\) This places tight consistency conditions on the extent of covariation between prices at different maturities that is admissible within a given model. Too much (or too little) covariation between long and short maturity prices, relative to the covariation allowable within a model, can rule out a model as a viable descriptor of the economy. Second, term structure data are unique in economics in how accurately they are described with parsimonious models. For example, a three-factor model explains the panel of Treasury interest rates for maturities of one up to thirty years with an \(R^2\) in excess of 99%. Data with such high signal to noise ratios have great power for discriminating between alternative models.

In this paper, we document a form of excess volatility in term structure prices that is irreconcilable with “standard” asset pricing models. Our central finding is that price fluctuations at different points in the term structure are internally inconsistent with each other—prices on the long end of the term structure are far more variable than justified by the behavior of short end prices—given usual modeling assumptions. The consistency violations are highly significant both statistically and economically. Perhaps most interestingly, excess volatility of long maturity prices is evident in a large number of asset classes, including claims to equity and currency volatility, sovereign and corporate credit default risk, and inflation. Only for the term structure of treasuries do we find that violation of the no arbitrage restrictions are economically small, consistent with the findings of a large literature on term structure models.\(^2\)

We define as “standard” any model in which cash flows are driven by a vector autoregression under the risk-neutral pricing measure, a class of models that we refer to as “affine-\(Q\).”\(^3\) This class encompasses many leading asset pricing paradigms, from structural equilibrium models with long run risks (Bansal and Yaron, 2004) or variable rare disasters (Gabaix,

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\(^1\)For seminal work on the role of cross-equation restrictions and the law of iterated values in rational models, see Hansen and Sargent (1980); Hansen and Richard (1987); Anderson, Hansen and Sargent (2003); Hansen and Scheinkman (2009); Hansen (2012).

\(^2\)See for example Joslin, Singleton and Zhu (2011).

\(^3\)Common notation refers to the real-world statistical measure as “\(\mathbb{P}\)” and the risk-adjusted pricing measure as “\(\mathbb{Q}\).” Section 2 discusses the origins and interpretation of the \(\mathbb{Q}\) measure in detail.
to the reduced-form affine no-arbitrage models that are ubiquitous in fixed income and derivatives pricing (Duffie, Pan and Singleton, 2000). The affine-\(Q\) class has become pervasive precisely due to its convenience in delivering tractable closed-form solutions in diverse valuation settings.\(^4\)

We focus on risk-neutral, or \(\text{“}Q\text{”}\), models based on a feature that is crucial for thinking about excess price volatility. By its definition, the \(Q\) measure incorporates all potential variation in discount rates.\(^5\) Therefore, any inference regarding price volatility based on estimates of the \(Q\) measure explicitly accounts for discount rate behavior. This is in contrast to the notion of excess volatility famously documented by Shiller (1979, 1981) and others, in which price fluctuations are deemed excessive relative to predictions from a specific model—one with constant discount rates. A potential resolution of the Shiller puzzle is to recognize that discount rates are variable, an insight that lies at the foundation of leading frameworks of modern finance.\(^6\) By using the \(Q\) representation of models in our analysis, any excessive volatility that we document must be coming from sources other than the types of discount rate variation that can be represented within an affine-\(Q\) model. In short, we choose the affine-\(Q\) specification as the null model for our analysis on the basis of its great flexibility for nesting many leading economic frameworks and for helping us to rule out what has become the de facto explanation for excess volatility, time-varying discount rates.

1.1 A One-factor Example

Our main empirical finding is that in all asset classes we examine long maturity prices overreact to short maturity price fluctuations, relative to the predictions of an affine-\(Q\) model. A simple example illustrates the nature of this overreaction.

Consider a term structure of claims to the one-factor cash flow process \(x_t\). Under the pricing measure \(Q\), cash flows evolve according to

\[
x_t = \rho x_{t-1} + \Gamma \epsilon_t
\]

(we abstract from constants and risk-free rate adjustments in this example in the interest of simplicity). The price of a \(j\)-maturity forward claim on these cash flows is \(f_{t,n} = E^Q_t[x_{t+n}]\).

\(^4\)Furthermore, the affine-\(Q\) class nests a wide range of dynamics by allowing the data to choose the appropriate number of driving factors. This flexibility often allows the affine-\(Q\) class to accurately approximate nonlinear models as well, a property that we illustrate empirically in Section 4.

\(^5\)More specifically, the \(Q\) measure incorporates variation in risk premia, which is the primary driver of total discount rate variation. Throughout we refer to discount rates and risk premia interchangeably.

The term structure of forward prices at maturities 1, ..., N is therefore given by

\[ f_{t,1} = \rho x_t, \quad f_{t,2} = \rho^2 x_t, \quad \ldots, \quad f_{t,n} = \rho^N x_t. \]  

(1)

The key cross-equation restrictions in this model require that the term structure of prices obeys a strict one-factor structure, and that the only admissible shape for the price curve is one in which the factor loadings follow a geometric progression in \( \rho \), which governs cash flow dynamics under \( Q \). This restriction is equivalently represented with prices of cumulative claims (defined by \( p_{t,n} = E_t^Q [x_{t+1} + \ldots + x_{t+n}] \)), in which case the term structure takes the form:

\[ p_{t,n} = (\rho + \rho^2 + \ldots + \rho^N)x_t. \]

Tests of this model’s restrictions hinge on an estimate of \( \rho \). Fortunately, \( \rho \) is easily estimated from regressions of prices onto prices. For example, let the first maturity claim price, \( f_{t,1} \), stand in for the latent factor \( x_t \). Let \( b_2 \) denote the (population) slope coefficient in a regression of the price at maturity two, \( f_{t,2} \), onto \( f_{t,1} \). According to Equation (1), \( b_2 \) exactly recovers \( \rho \). This regression is intuitive. Investors’ relative valuation of the first two claims perfectly reveals the cash flow persistence that they perceive. If investors price assets as though \( x_t \) is very persistent, a rise in the short price \( f_{t,1} \) will coincide with a rise in \( f_{t,2} \) of nearly the same magnitude, which indicates that \( \rho \) is near one under the investors’ subjective pricing measure.

If we project prices for remaining maturities 3, ..., \( N \) onto the short price, we recover a sequence of “unrestricted” regression coefficients denoted \( b_3, \ldots, b_N \). Note that these unrestricted regressions do not impose the slope coefficient restrictions in (1). These regressions can however be recast in their “restricted” form. The model restriction relating, for example, \( b_N \) to \( b_2 \) is

\[ b_N = (b_2)^{N-1}. \]  

(2)

We convert this restriction into a test of excess volatility by constructing a variance ratio statistic for maturity \( N \):

\[ VR_N = \frac{Var(b_N f_{t,1})}{Var((b_2)^{N-1} f_{t,1})}. \]

The numerator, \( Var(b_N f_{t,1}) \), is the explained variance in the unrestricted regression of long-end prices \( (f_{t,N}) \) onto the short end \( (f_{t,1}) \). The denominator, \( Var((b_2)^{N-1} f_{t,1}) \), is the explained variance of the same regression under restriction (2). Under the null model, the restricted and unrestricted variances are the same and \( VR_N = 1 \). If the ratio statistic significantly exceeds one, the price at maturity \( N \) varies more than is justified by the behavior of
Figure 1: Variance Swap Tests

Factors=2, $R^2=99.6\%$

Note. The figure plots the standard deviation of prices under the unrestricted factor model (solid line) and under the restricted model (dashed line). The circles in the unrestricted line represent the maturities we observe in the data. The numbers next to each circle are the Variance Ratios at each maturity. The shaded area encloses the 97.5\textsuperscript{th} and 2.5\textsuperscript{th} percentiles of the model-implied variance in bootstrap simulations. The left axis reports the volatility of prices.

the short end of the term structure. Note that the same variance ratio test can be applied to cumulative claims as well.

This one-factor example is intentionally simplified to illustrate our approach for testing excess volatility along the term structure. In Section 2, we develop an estimation and inference approach for $VR_{N}$ in general $K$-factor affine specifications.

1.2 A Representative Term Structure

Figure 1 illustrates the behavior of variance ratios in one of our datasets—the term structure of variance swaps—which are claims to the cumulative variance of the S&P 500 index over the life of the contract.\textsuperscript{7} An unrestricted two-factor model provides an excellent description of the term structure, delivering an $R^2$ of 99.6\% for the panel of prices. The solid black line plots the explained swap price volatility from an unrestricted regression of each long maturity claim on the first two short maturity claims. The dashed line plots the explained variation

\textsuperscript{7}These data are described in detail in Section 3.
from the regression that imposes the model restrictions. The variance ratio statistic for each maturity is printed above the unconstrained volatility estimates and the blue shaded region represents the point-wise 95% bootstrap confidence interval of the test.

At 24 months, the variance ratio statistic reaches 2.15, meaning that the variability in long maturity prices is more than twice as large as that allowed by the affine model restriction, and is highly statistically significant. The high variance ratio can be thought of in the following way. The concave shape of price volatility at the short end of the curve suggests that cash flows mean revert fairly quickly under $Q$. But this is appears inconsistent with indications of much higher persistence implied from the long end. As a result, unrestricted price volatility increases with maturity at a much faster rate than the price volatility predicted by the model. Note that both curves represent “explained” price volatilities from regressions of long prices onto short prices. That is, both the unrestricted and restricted model describe the portion of long-maturity price fluctuations that is captured by behavior at the short end. The high variance ratio therefore indicates that the prices at the long end of the curve react to the short end much more strongly in the data than affine model dynamics allow.

The excess volatility of long-maturity claims cannot be explained by movements in discount rates alone. Any discount rate variation that is describable within the affine class is subsumed by our model. Our price-on-price regressions estimate the dynamics of the latent factors under the pricing measure, and we allow the data to determine the appropriate number of factors driving the term structure. This lends our approach the flexibility to consistently estimate effectively any specification in the affine-$Q$ class, regardless of whether the factors are driven by discount rate variation or physical cash flows. Nor can high variance ratios be explained by a poor fit from the factor model. The $R^2$ from the factor specification is nearly 100%, meaning that the unconstrained linear factor model does an excellent job describing the data. Instead, the high variance ratio is a violation of the cross-equation restrictions of the affine model. That is, the data are exceedingly well described by a linear factor model, but with factor loadings that differ from the loadings implied by model restrictions. Behavior of the variance swap term structure is representative of our broader empirical findings. In all asset classes that we study we document excess volatility of long-maturity prices similar to that in Figure 1.

1.3 Potential Explanations

In Section 4, we examine four potential explanations for the excess volatility of long maturity claims: omitted factors, non-linear dynamics, long memory dynamics, and violations of no-arbitrage.
First, if the true data generating process is a $K$-factor affine model but we use fewer than $K$ factors in our analysis, the variance ratio statistic is likely to diverge significantly from one. However, omitted factors are unlikely to explain our findings because an unrestricted factor model explains more than 99% of the variation in each term structure we study. We show that it is quantitatively infeasible for a model with too few factors to generate a variance ratio far above one while at the same time producing an unrestricted $R^2$ over 99%. Additionally, we conduct robustness checks that gradually increase the number of factors used in our tests. This pushes the factor model $R^2$ even closer to 100% yet still produces variance ratios significantly in excess of one.

Second, we explore a large class of non-linear dynamic specifications known as smooth-transitioning autoregressive (STAR) models. In most parameterizations, STAR models are very closely approximated by a low-dimension affine model and therefore do not produce variance ratios above one. For the most extreme non-linear specifications it is possible to generate variance ratios that statistically reject the affine restrictions, but even in these cases the variance ratios are substantially smaller than those found in the data.

Third, we explore a wide range of long memory models that fall into the stationary ARFIMA family. These models can exhibit persistence that decays much more slowly than the autoregressive structure assumed in affine-$Q$ specifications. The vast majority of ARFIMA specifications appear well approximated by simple affine models and do not lead to high variance ratios. However, as the long memory parameter reaches the boundary of the non-stationary range, we show that it is possible to generate variance ratios as high as three at the 24 month maturity while at the same time producing unrestricted factor model $R^2$ values above 99%.

Fourth, we show that investor overreaction leads to no-arbitrage violations that produce high variance ratios at long maturities similar to those in the data. We discuss how, in some ways, long memory behavior and arbitrage violations induced by overreaction are observationally equivalent. To disentangle this ambiguity, we construct a trading strategy that helps distinguish between alternative explanations for excessive volatility—a misspecified econometric model on the one hand versus genuine mispricing on the other. The strategy takes the view that the estimated affine model reflects the true value of claims, so that any excessive fluctuations of long-maturity prices are profitable arbitrage opportunities. The trade is implemented by buying (selling) long maturity claims when they are undervalued (overvalued) relative to the affine model, and hedges this position by selling (buying) short maturity claims in the exact proportion dictated by the estimated model. If the true data generating process does not admit arbitrage, then the trading strategy will perform poorly,
and we show that this is the case in simulations. In the variance swap market, however, we find that the strategy is highly profitable, suggesting that at least part of the affine-$Q$ violation we document is due to arbitrage opportunities arising from investor overreaction at the long end of the term structure. We then show that the trading strategy returns are unexplained by standard risk factors or by exposure to factor risk in the variance swap term structure, which further supports the overreaction interpretation of our results.

Lastly, while excess volatility is statistically significant in the market for Treasuries, it is the only market in which the effect appears economically small. This fact is potentially compatible with the mispricing interpretation. The sheer size of the Treasury market relative to the other markets we study make it a likely habitat for arbitrageurs that rapidly correct overreaction errors. Our results therefore support the validity and empirical success of affine-$Q$ models in modeling the term structure of interest rates, while at the same time highlighting the presence of significant violations of no-arbitrage and the law of iterated values in all other term structures we study.

1.4 Literature Review

A close predecessor of our paper—especially given its focus on the term structure of volatility—is Stein (1989), who compares the volatility of short and long maturity one-year S&P 100 options. He finds excess volatility of one-year option prices and interprets it as evidence of investor overreaction. Our paper builds on Stein’s original insight with a few key differences. First, he analyzes comovement of long and short maturity prices relative to cash flow persistence estimated from the $P$ measure. In other words, the reference model of Stein (1989) does not account for discount rate variation, nor do the interest rate volatility tests of Shiller (1979) or the equity volatility tests of Shiller (1981). Our excess volatility test explicitly accounts for discount rate variation by estimating cash flow dynamics under the $Q$ measure. In addition, Stein (1989) uses a one-factor model for volatility, while our approach allows for an arbitrary number of factors. Lastly, our tests are extended to a wide range of asset classes.

Our findings are also related to Gurkaynak, Sack and Swanson (2005), who show that the responsiveness of long run Treasury bond yields to macroeconomic announcements is excessive relative to established “new-Keynesian” DSGE models. As in Shiller (1979), Shiller (1981) and Stein (1989), this reference model does not account for rational discount rate variation. More recently, Hanson and Stein (2015) study overreaction at the long end of the Treasury yield curve focusing on FOMC announcement days. An interesting distinction from our work is that long maturity Treasury rates exhibit by far the least excess volatility
among the asset classes we study.

The Treasury yield curve has been subject of a large literature literature. Early contributions by Shiller (1979) and Singleton (1980) showed excess volatility of long-term bonds relative to the expectations hypothesis model, while later literature has worked extensively with affine-$Q$ specifications that explicitly account for time variation in discount rates. For a review and recent contributions, see for example Ang and Piazzesi (2003); Dai and Singleton (2002); Duffee (2002); Le, Singleton and Dai (2010); Piazzesi (2010); Joslin, Singleton and Zhu (2011). That literature has typically found that no-arbitrage restrictions hold quite well in the interest rate market. We confirm this fact by showing that model violations in the treasury market are economically small compared to violations in derivatives markets.

While observing prices and their comovement allows us to detect overreaction at the long end of the curve, it does not allow us to determine the underlying mechanism driving this overreaction. Our evidence lends support to recent efforts to understand the key role of expectations formation in financial markets (for example, Barberis et al. (2015b,a); Bordalo, Gennaioli and Shleifer (2015); Greenwood and Shleifer (2014); Gennaioli, Shleifer and Ma (2015)), and our trading strategy analysis in Section 4.5 highlights the potential costliness of overreaction by investors. Our findings point to the importance of future research into how agents might form internally inconsistent expectations over multiple horizons.

2 Asset Term Structures in Linear Models

In this section we develop our general approach to testing the internal consistency of asset term structures in the affine-$Q$ setting.

2.1 Claims by Maturity

Our focus is on the joint price behavior of claims to the same underlying cash flow process but with different maturities. Let $x_t$ denote a scalar cash flow. For most of our analysis, we focus on linear claims to the $x_t$ process. We study the extension to exponential-linear claims Section 2.5.1.

At time $t$, a linear $n$-maturity forward claim promises a one-time stochastic cash flow of $x_{t+n}$ to be paid in period $t+n$. Under the weak assumption that a model admits no arbitrage opportunities, there exists a pricing measure $Q$ under which prices of such claims are expectations of future cash flows discounted at the risk-free interest rate. We assume
that no-arbitrage is satisfied, thus the \( n \)-maturity forward price is representable as

\[
f_{t,n} = E^Q_t \left[ x_{t+n} \frac{S_t}{S_{t+n}} \right]
\]  

(3)

where \( S_t \) is the value of a risk-free account that pays the instantaneous short-term rate.

In our empirical analysis, risk free rate variation is negligible compared to risky asset price variation in almost all asset classes.\(^8\) So, to reduce notation in the remainder of this section, we assume that \( S_t \) is constant and equal to one. We return to a detailed analysis of risk-free rates and associated robustness checks in Appendix C.

The pricing of forward claims is easily recast in terms of linear cumulative claims that promise a sequence of cash flows through maturity. The time \( t \) price of an \( n \)-maturity cumulative claim is a sum of forward prices,

\[
p_{t,n} = E^Q_t [x_{t+1} + \ldots + x_{t+n}] = f_{t,1} + \ldots + f_{t,n}.
\]

Under no-arbitrage, the pricing measure possesses a martingale property that binds prices together across time and maturity,

\[
f_{t,n} = E^Q_t [f_{t+1,n-1}] \quad \text{and} \quad p_{t,n} = E^Q_t [p_{t+1,n-1}] + f_{t,1},
\]

which follows from the law of iterated expectations,

\[
f_{t,n} = E^Q_t [x_{t+n}] = E^Q_t [E^Q_{t+1} [x_{t+n}]] = E^Q_t [f_{t+1,n-1}].
\]

### 2.2 Cash Flow Dynamics Under \( \mathbb{Q} \)

Affine models assume that the cash flow process obeys a linear factor structure, and that these factors evolve as a vector autoregression (VAR) under \( \mathbb{Q} \). In particular, let \( H_t \) be a vector of \( K \) factors with \( \mathbb{Q}\)-dynamics given by

\[
H_t = \rho H_{t-1} + \Gamma \epsilon_t.
\]

(4)

Under \( \mathbb{Q} \), the \( K \times K \) parameter matrices \( \rho \) and \( \Gamma \) govern transition probabilities and \( \epsilon_t \) is mean zero and orthogonal to \( H_{t-1} \). Cash flows are determined by the factors according to

\[
x_t = \delta_0 + \delta'_t H_t
\]

\(^8\)The obvious exception is the Treasury bond market, in which case we account for risk-free rate variation in the standard way.
where $\delta_0$ is a scalar and $\delta_1$ is a $K \times 1$ vector.

Since the factors, $H_t$, are latent in our setting, model identification requires a normalization of model parameters. In particular, we impose the normalization that the matrix $\rho$ is diagonal (so that its diagonal elements, which are also its eigenvectors, easily reveal the decay rates of the factors under $Q$) and that $\delta_1$ is a vector of ones. These identification assumptions impose no economic restrictions, but ensure that the model we bring to the data has exactly as many parameters as there are observables. For a detailed discussion of our normalization choices, see Joslin, Singleton and Zhu (2011) and Hamilton and Wu (2012).

Finally, for notational ease in this section, we set $\delta_0 = 0$. This is not without a loss of generality, as $\delta_0$ determines the overall price level of claims in the term structure. But, for our purpose of understanding the volatility of prices, $\delta_0$ is constant and eventually drops from our analysis.

We can now rewrite the cash flow $x_t$ as

$$x_t = 1' H_t.$$  \hfill (5)

We refer to the class of models satisfying Equations (4) and (5) as “affine-$Q$.” Models with this structure are ubiquitous in the asset pricing literature due to their convenience for describing prices of linear (and exponentially-linear) cash flow claims.

### 2.3 Term Structure of Prices

Given (4) and (5), the price of a linear forward claim with maturity $n$ is

$$f_{t,n} = 1' \rho^n H_t.$$  \hfill (6)

Equation (6) contains a set of cross-equation restrictions implied by the affine-$Q$ model. Prices at all maturities must obey a strict factor structure so that any and all comovement among prices must be due to $H_t$. Furthermore, the loadings at each maturity must abide by a specific structure—they must follow a geometric progression in $\rho$.

It is also easy to see that this specification satisfies the property of law of iterated expectation reported above:

$$E_t^Q[f_{t+1,n-1}] = E_t^Q[1' \rho^{n-1} H_{t+1}] = 1' \rho^{n-1} E_t^Q[H_{t+1}]$$

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9In the exponential-affine setting, we require the additional assumption that $\epsilon_t$ is Gaussian, as in Hamilton and Wu (2012). In this setting, the covariance matrix of the errors, $\Gamma$, is the same under $P$ and $Q$.\]
\[ 1' \rho^{n-1} \rho H_t = 1' \rho^n H_t = f_{t,n} \]

Our empirical test investigates the extent to which observed term structures adhere to the model restrictions.

### 2.4 A Convenient Recursive Representation

Equation (6) also implies that the latent factors \( H_t \) are exactly recoverable from any set of \( K \) prices (either forwards or cumulative claims) at different maturities. In turn, this also implies that the price at any maturity \( j \) can be represented as an exact linear function of a set of \( K \) different prices at any maturity other than \( j \).

In particular, denote the \( K \times 1 \) vector of time \( t \) prices for forwards with maturities 1 to \( K \) as \( F_{t,1:K} = (f_{t,K}, \ldots, f_{t,1})' \), and likewise for \( F_{t,2:K+1} \), \( F_{t,3:K+2} \), and so forth. Define \( b = (b_1, \ldots, b_K)' \) to be the coefficient in a projection of \( f_{t,K+1} \) onto \( F_{t,1:K} \). In this model, the projection is exact so there is no residual,

\[ f_{t,K+1} = b' F_{t,1:K}. \tag{7} \]

This equation simply states that in a linear model with \( K \) factors, the \((K + 1)\) forward can be expressed as an exact linear combination of the maturities 1 to \( K \). Because the vector \( F_{t,1:K} \) plays a special role the rest of the paper, we refer to it simply as the “short end” of the term structure: the set of short-term maturities that exactly span the full term structure.

This equation only links maturities 1 through \( K + 1 \). We can derive a recursive relation that links the entire price curve to the short end in a convenient way. In particular, any two blocks of \( K \) consecutive forward prices with maturity shifted by one period (for example, \( F_{t,1:K} \) and \( F_{t,2:K+1} \)) are linked by the equation:

\[ F_{t,j+1:K+j} = BF_{t,j:K+j-1}, \quad B = \begin{bmatrix} b_K & b_{K-1} & \ldots & b_2 & b_1 \\ 1 & 0 & \ldots & 0 & 0 \\ 0 & 1 & \ldots & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \ldots & 1 & 0 \end{bmatrix}. \tag{8} \]

By the definition of \( b \) in (7), the relationship in (8) holds for \( j = 1 \). It follows from the law of iterated expectations that (8) holds for \( j = 2 \) because

\[ E_t^Q[F_{t+1,2:K+1}] = BE_t^Q[F_{t+1,1:K}] \iff F_{t,3:K+2} = BF_{t,2:K+1}. \]

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A recursive argument therefore establishes (8). It pins down the price of any forward on the term structure with the prices at the \( K \) immediate neighboring maturities via the matrix \( B \). Iteratively substituting (8) into itself implies

\[
F_{t,j+1:K+j} = BF_{t,j+1:K} = B^2F_{t,j+1:K-1} = \ldots = B^jF_{t,1:K}.
\] (9)

The geometric recursion in (9) further shows that prices at any maturity are pinned down by any \( K \) prices, even those at distant maturities. In particular, the equation links any price to the “short-end” vector \( F_{t,1:K} \), where the coefficients are entirely determined by the powers of \( B \).

Equation (9) is merely a restatement of the cross-equation restrictions summarized by Equation (6). However, the restrictions in (6) face the practical difficulty that they relate the restrictions to unobserved factors. What makes (9) powerful is that the restrictions are restated only in terms of observable prices. Specifically, the affine model structure requires not only that prices are perfectly correlated with the rest of the maturity curve, but limits the admissible shapes of the curve to those with geometrically decaying loadings \((B^j)\) in regressions involving prices at different maturities.

While forwards are more convenient in mathematical derivations, it is more convenient to work with cumulative prices in empirical analyses (we discuss this further below). The representation of restriction (9) in terms of prices of cumulative claims is

\[
P_{t,j+1:K+j} = (I + B + \ldots + B^j)R^{-1}P_{t,1:K}.
\] (10)

where \( P_{t,j:m} = (p_{t,m},p_{t,m-1},\ldots,p_{t,j})' \). The \( B \) matrix is from Equation (8) and \( R \) is the \( K \times K \) upper-triangular matrix of ones, which facilitates the algebraic adjustment from forwards to sums of forwards.\(^{10}\) Representations (9) and (10) are exactly equivalent and we work interchangeably with the two.

### 2.5 Testing for Excess Comovement

We present now our general test for overreaction. For our analysis, we use prices for the first \( K \) maturities on the short end of the term structure to represent the \( K \) latent factors. In the preceding discussion we considered population projections, but to formalize the test we

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\(^{10}\)Because \( p_{t,n} = p_{t,n-1} + f_{t,n} \) for all \( n \), we can write \( P_{t,2:K+1} = P_{t,1:K} + F_{t,2:K+1} \), \( P_{t,1:K} = RF_{t,1:K} \) because cumulative prices are sums of forwards, and \( F_{t,2:K+1} = BF_{t,1:K} \) by iterated expectations. Substituting, we reach \( P_{t,2:K+1} = RF_{t,1:K} + BF_{t,1:K} = (R + B)F_{t,1:K} \), and one more substitution arrives at \( P_{t,2:K+1} = (R + B)R^{-1}P_{t,1:K} = (I + B)R^{-1}P_{t,1:K} \).
work with sample regressions. First, for each maturity \( j = K + 1, \ldots, N \), we regress \( f_{t,j} \) on to \( F_{t,1:K} \),

\[
f_{t,j} = \hat{a}_j + \hat{b}'_j F_{t,1:K} + u_{t,j}.
\]

(11)

We allow for a measurement error term \( u_{t,j} \) to avoid stochastic singularity following the term structure literature.

We construct a test of overreaction in the form of a variance ratio statistic at each maturity \( j \). The coefficient \( \hat{b}_j \) in Equation (11) is the unconstrained OLS regression estimate. The numerator of the variance ratio statistic for any maturity \( j > K + 1 \) is the explained variance in the unconstrained regression and equals \( \hat{b}'_j \hat{\Sigma}_{1:K} \hat{b}_j \), where \( \hat{\Sigma}_{1:K} \) is the sample covariance matrix of \( F_{t,1:K} \).

The denominator of the variance ratio is the explained variance in the constrained version of (11), where the constraint is the cross-equation restriction in Equation (9). The estimate \( \hat{B} \) for the recursions in Equation (8) and (9) is obtained by using the estimated \( \hat{b}_{K+1} \) as the first row of \( B \) and leaving all other rows unchanged. The constrained loading of \( f_{t,j} \) on \( F_{t,1:K} \), denoted \( \tilde{b}_j \), is the first row of the matrix \( \hat{B} \) raised to the power \( j - K \):

\[
\tilde{b}_j = e_1(\hat{B}^{j-K}), \quad e_1 = (1, 0, \ldots, 0).
\]

(12)

The explained variance in the constrained regression is therefore \( \tilde{b}'_j \tilde{\Sigma}_{1:K} \tilde{b}_j \), and the test statistic is

\[
VR_j = \frac{\hat{b}'_j \hat{\Sigma}_{1:K} \hat{b}_j}{\tilde{b}'_j \tilde{\Sigma}_{1:K} \tilde{b}_j}.
\]

(13)

As we consider the time series variation of some long maturity price \( f_{t,j} \), we wonder the extent to which this variation is consistent with variation at other maturities, from the point of view of an affine \( K \)-factor model. The \( VR_j \) statistic calculates the unconditional covariation of the long and short end prices and reports the fraction of this variation consistent with the model’s cross-equation restrictions. Under the null of an affine \( K \)-factor model, \( VR_j = 1 \). Any deviation from unity (above and beyond that due to sampling variation) indicates a violation of the model’s restrictions. Variance ratios that are significantly greater than unity indicate that long maturity prices overreact to movements at the short end, relative to model predictions.

The variance ratio in (13) is based on forward prices, but the test is equivalently formulated from prices of cumulative claims. The test structure is identical, only the unrestricted and restricted regression coefficients (\( \hat{b} \) and \( \tilde{b} \)) need modification. In analogy with (11), let \( \hat{d}_j \) be the OLS slope estimate from an unconstrained regression of \( p_{t,j} \) on \( P_{t,1:K} \). The constrained
regression coefficient, denoted \( \tilde{d}_j \), comes from the cross-equation restriction in Equation (10):

\[
\tilde{d}_j = e_1(I + B + \ldots + B^{j-K})R^{-1}
\]  

and the variance ratio test statistic is

\[
VR_j = \frac{\hat{d}_j^\prime \hat{\Sigma}_{1:K} \hat{d}_j}{\hat{d}_j^\prime \hat{\Sigma}_{1:K} \hat{d}_j}. 
\]  

Our empirical work uses the cumulative form in (15) for the following reason. If the eigenvalues of the risk-neutral cash flow persistence matrix \( \rho \) are bounded below one in absolute value, then the system is stationary. In this case, cumulative prices at long maturities converge to a constant (and the denominator of \( VR_j \) converges to 0) because cash flows mean revert under the pricing measure. So, in theory, infinite maturity assets have undefined variance ratios under the null. While this is not a pressing practical concern (most claims have maturities up to a few years), it is avoided by the test for cumulative claims in (15).

There are many potential ways to formulate tests of the affine model’s restrictions, and many of these are asymptotically equivalent. Our specific test construction has the attractive interpretation as a measure of excess volatility relative to a benchmark model. Our test choice is inspired by, and designed to remain comparable with, the rich history of excess volatility tests studied by Shiller (1979), Shiller (1981), Stein (1989), Campbell and Shiller (1988a), Campbell (1991), Cochrane (1992), and many others.

Under the null of an affine no-arbitrage model, the restricted and unrestricted loading vectors \( \hat{b}_j \) should be equal \( \tilde{b}_j \) element-by-element. When there is more than one factor in the model, it raises the question of how to best evaluate the joint restrictions that apply to multiple loadings. An attractive feature of the variance ratio test is that it offers a sensible aggregation of all of the loading comparisons. The total explained variance in the restricted and unrestricted models are

\[
\sum_{k=1}^{K} \sum_{l=1}^{K} \hat{b}_{j,k} \tilde{b}_{j,l} \hat{\sigma}_{k,l} \quad \text{and} \quad \sum_{k=1}^{K} \sum_{l=1}^{K} \hat{b}_{j,k} \hat{b}_{j,l} \hat{\sigma}_{k,l}.
\]

Rather than comparing loadings element-wise, the variance ratio sums loadings into a scalar in order to compare alternative models. The weights assigned to elements in the sum are based on the (co)variances of the short maturity prices. Those prices that (co)vary most strongly are most informative about the dynamics of the model, and loadings on these factors receive the largest weights in the test.
We derive the asymptotic distribution of $VR_j$ to conduct statistical inference. The details are in Appendix B.1, including feasible inference calculations that are robust to heteroskedasticity and serial correlation in the residuals of regression (11). These calculations answer the question, “How likely are we to observe a given variance ratio given the sampling error of model parameter estimates?” The appendix also describes a bootstrap procedure for conducting inference in small samples that we use to construct confidence intervals in our main analysis.

2.5.1 Exponential-affine Models

The linear claim structure of Equation 3 is particularly useful for modeling variance claims (Egloff, Leippold and Wu, 2010; Ait-Sahalia, Karaman and Mancini, 2014; Dew-Becker et al., 2015), which include several of the term structures we study.

Claims in other asset classes, such as interest rates or credit default swaps (CDS), are more naturally modeled as exponential-affine claims; in this case, it is the log of $x_t$ that is linear in factors $H_t$.

The model restrictions and testing procedures we derived above also apply in the exponential-affine setting under two additional assumptions regarding the distribution of factor innovations, $\Gamma \epsilon_t$, in Equation (4). First, $\epsilon_t$ follows a Gaussian distribution under $Q$. Second, $\Gamma \epsilon_t$ is homoskedastic or, alternatively, it is heteroskedastic but conditional volatility is uncorrelated with the factors (as in unspanned volatility models).

In exponential-affine models, the price of a cumulative claim is:

$$p_{t,n} = \mathbb{E}_t^Q [\exp (x_{t+1} + \ldots + x_{t+n})].$$

The leading examples in this class are interest rate claims, where $x_t = -r_t$ is the instantaneous interest rate. In this case prices are related to factors according to

$$\log p_{t,n} = 1' [\rho + \rho^2 + \ldots + \rho^n] H_t + \text{constant}. \quad (16)$$

(for a discussion of the unspanned volatility case, see for example, Collin-Dufresne and Goldstein (2002), Dai and Singleton (2003), Joslin (2006), Bikbov and Chernov (2009), Creal and Wu (2015)).

For some claims it is preferable to model individual forwards with an affine-exponential

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11A minor adaptation for the case of bonds is that powers of \( \rho \) range from 0 to \( n - 1 \) rather than from 1 to \( n \), though this is inconsequential for our variance ratio test.
form:
\[ \log f_{t,n} = \log E_t^{Q} [\exp (x_{t+n})] = 1'\rho H_t + \text{constant}. \]  

(17)

The pricing formulas of Equations (16) and (17) differ from the simple affine form in (6) only by a constant due to assumptions on the distribution of factor innovations. Thus, (16) and (17) recover all the necessary structure to perform estimation and testing as described above, subject to the modification that we analyze log prices rather than price levels.

In the remainder of the paper, we focus on the homoskedastic case for three reasons. First, the majority of the asset classes we analyze (in particular, all of the variance claims) are typically modeled as claims to the level of variance, in which case heteroskedasticity does not affect pricing (conditional variance enters only through the Jensen inequality term in exponential models).

Second, conditional heteroskedasticity affects the loadings on the factors in exponential-affine models only to the extent that the factors themselves span the volatility of the errors. The term structure literature finds evidence of a large unspanned volatility component in interest rates (see for example Collin-Dufresne and Goldstein (2002)). So-called unspanned volatility models fix the loadings of bond prices on volatility factors to be zero. In this case, the factor loadings for log prices follow the same recursion as in standard homoskedastic models (Creal and Wu (2015)).

In the remaining case where factor shock volatility is in fact spanned by prices, the magnitude of the effect on factor loadings is shown in the bond market to be small relative to the part of the loading coming from the claim’s direct exposure to factors. Nonetheless, spanned volatility models can potentially affect our variance ratio test and Appendix C.2 performs robustness tests that directly account for heteroskedasticity. Our main conclusion from this check is that heteroskedasticity of factor innovations is not a central driver of our results.

3 Empirical Findings

This section presents our main empirical findings. We study many term structures of variance swaps, equity options, currency options, credit default swaps, inflation swaps, and Treasury bonds (see also Backus, Boyarchenko and Chernov (2015) for a study of risk premia across a similar group of term structures).\(^{12}\)

\[^{12}\text{We focus our empirical analysis on term structures with available data, and for which we observe maturities that extend from the short end to the long end; we leave other term structures, such as that of dividend strips (van Binsbergen, Brandt and Koijen, 2012; van Binsbergen et al., 2013; van Binsbergen and Koijen, 2015) or housing (Giglio, Maggiori and Stroebel, 2015a,b), for future research.}\]
3.1 Implementation

For each asset class, we map the pricing specific to that term structure to either the linear or the exponentially-affine specification (as discussed below case by case). To minimize the potential confounding effects of illiquidity in these term structures, for each asset class we focus on the most liquid contracts available.

A key input to our tests is an estimate for the number of factors, \( K \). For each term structure, we choose \( K \) based on the number of principal components necessary to explain at least 99% of the variation in the panel of prices at all available maturities.\(^\text{13}\) We make an exception for Treasury yields by directly assuming three factors, even though two principal components explain 99.9% of the variation in the panel, based on standard practice in the interest rate literature. In later sections, we explore the sensitivity of our results to different choices for \( K \).

From here, we use the first \( K \) short maturity prices to represent the factor space. Then, we regress the \( K + 1 \) maturity price on the first \( K \) maturities to estimate the baseline loadings (vector \( b \) in equation (7)). These serve as the basis for restricted regression coefficients for maturities \( K + 2 \) through \( N \).\(^\text{14}\)

3.2 Term Structure Tests by Asset Class

For each asset class, we describe the data and discuss any contract-specific or institutional features that need to be considered in the empirical analysis. Appendix D provides further in-depth descriptions of our data.

3.2.1 S&P 500 Variance Swaps

The first market we study is that for variance swaps on the S&P 500 index. The variance swap market has the fascinating feature that it allows investors to trade direct claims on the riskiness of equities. A long variance swap position receives cash flows at maturity proportional to the sample variance of the S&P 500 over the life of the contract. Let \( RV_t \) denote the sum of squared daily log index returns during calendar month \( t \). The payoff of an

\(^{13}\)We conduct this principal components step using variance-standardized prices, so that all points in the term structure are on equal footing in determining the number of factors.

\(^{14}\)In Section 5, we discuss why treating the first \( K \) prices as an exact representation of the latent factor space is a powerful approach for detecting violations of the model’s internal consistency conditions. It differs from a common practice in the term structure literature of estimating factors as principal components using price data at all available maturities, which is motivated by arguments of efficiency and overcoming potential measurement error in prices. Section 5.1 explains why model comparisons that rely on data from the full term structure can have limited power to detect the patterns of no-arbitrage violations we uncover in this paper. Section C.3 explains why our results are inconsistent with effects of measurement error.
$n$–maturity variance swap is $\sum_{j=1}^{n} RV_{t+j}$. Ignoring risk-free rate variation (as is typical in this literature), the price of a variance swap corresponds to the $Q$-expectation of the payoff:

$$p_{t,n} = E_t^Q \left[ \sum_{j=1}^{n} RV_{t+j} \right]$$

This structure maps directly into the simple affine framework of Section 2 with $x_t = RV_t$.

Variance swaps are traded in a liquid over-the-counter market with a total outstanding notional of around $4$ billion in “vega” at the end of 2013, meaning that a movement of one point in volatility would result in $4$ billion changing hands (see Dew-Becker et al. (2015)). As discussed in Dew-Becker et al. (2015), bid-ask spreads for maturities up to 24 months are relatively low, at around 1-2%. In addition, the liquidity of the swap market is also supported by option market liquidity. Variance swaps are anchored to the prices of S&P 500 index options by a no-arbitrage relationship because options can be used to synthetically replicate the swap.\footnote{Dew-Becker et al. (2015) show that the term structure of variance swap prices indeed closely matches the term structure of options-based synthetic swaps (more commonly known as the VIX).}

We use daily price data for cumulative claims at all available maturities (1, 2, 3, 6, 12, and 24 months) during the period 1995 to 2013. Our baseline test uses $K = 2$, as two components explain 99.6% of the variance in the panel, a choice supported by existing literature (e.g. Egloff, Leippold and Wu (2010), Ait-Sahalia, Karaman and Mancini (2014), Dew-Becker et al. (2015)).

Our main findings for variance swaps are reported in Figure 1 (in the Introduction). The horizontal axis shows maturity of claims in months and the vertical axis shows the time series volatility (standard deviation) of daily swap prices. The solid black line plots the explained swap price volatility from an unrestricted regression of each long maturity claim on the first $K$ short maturity claims—this is the square root of the numerator in the variance ratio test. Points corresponding to observed maturities are marked with a circle. The dashed line plots the explained variation from the restricted regression that imposes the affine model’s consistency conditions based on coefficient estimates in a regression of price $K + 1$ on prices for the first $K$ maturities—this the square root of the test’s denominator. The variance ratio statistic for each maturity is printed above the unrestricted volatility estimates. The test statistic is only available for maturity $K + 2$ and higher because the first $K + 1$ maturities are used to estimate model parameters. Finally, the blue shaded region represents the point-wise 95% bootstrap confidence interval of the test. If the unrestricted estimate lies beyond this region, the cross-equation restriction of the affine model is rejected at the 5% significance.
level or better.

Plotting price variability in terms of standard deviation is convenient for visualizing the degree of cash flow persistence under the pricing measure. For a cumulative claim, the coefficient in a regression of long prices onto short prices is a geometric series in the persistence parameter, $\rho$. For example, in a one-factor model, the model-based standard deviation of an $n$-maturity claim is $\left(\sum_{j=1}^{n} \rho^j\right) \sqrt{\text{Var}(p_{t+1})}$. If cash flows are integrated ($\rho = 1$) under the pricing measure, then the standard deviation is a linear function of maturity. On the other hand, if the $Q$-persistence of cash flows is in $(0,1)$, then the standard deviation of price is a concave function of maturity.

For variance swaps (indeed for all other term structures we study), the unconstrained estimate of price volatility is concave in maturity, indicating stationarity of cash flows under the pricing measure. This is a first suggestion that variability on the long end is inconsistent with integrated or explosive model dynamics under $Q$.

As described in the Introduction, the unrestricted price variance at 24 months more than doubles the variance allowed under the affine pricing model’s restriction. Evidently, comovement among prices at the short end of the curve suggests that cash flows mean revert relatively quickly under $Q$. But this is not borne out on the long end—model-restricted volatilities increase with maturity at a much slower rate than the unrestricted volatility. Recall that these are “explained” price volatilities from regressing onto short-end prices. The high variance ratio therefore indicates that the prices at the long end of the curve react to the short end much more strongly than the affine model dynamics allow—these facts are about overreaction of the long end relative to the short end, and relative to the estimated affine model.

Figure 2 offers another visualization of how overreaction drives observed price volatility. It plots estimated loadings of prices at each maturity on the model’s two factors, for both the restricted and the unrestricted model. The figure shows that long maturity prices overreact because they load too heavily onto both factors, relative to the loadings predicted by the null model.

Two points warrant special emphasis regarding these results. First, the excess volatility of long-maturity claims cannot be explained by movements in discount rates alone, as any discount rate variation that is describable within the affine class is subsumed by the $Q$ model. Second, the data are exceedingly well described by a linear factor model (evident from an unrestricted $R^2$ near 100%), but with factor loadings that sharply differ from those implied by model restrictions.

Our data vendor provides a second variance swap data set with extended maturities up
to 120 months but only covering the 2008 to 2013 period. Figure 3 reports results for this data, which again show significant excess volatility at all maturities above 18 months. The variance ratio at five years reaches 6.13, implying that almost 90% of the total variance at this maturity and higher is due to overreaction relative to the model.

### 3.2.2 Equity Implied Variance

A well known result in option pricing establishes that variance swaps can be synthesized from a portfolio of put and call options with different strike prices.\textsuperscript{16} Synthetic variances swaps are frequently encountered in practice. A prominent example is the VIX index maintained by the Chicago Board Options Exchange, whose squared value approximates the price of a variance swap on the S&P 500 index. Indeed, when we repeat the analysis of Figure 1 using the VIX term structure, we find results that are essentially identical to the results using

\textsuperscript{16}See Britten-Jones and Neuberger (2000) and Jiang and Tian (2005).
Figure 3: Variance Swap Tests: 10 Years

Factors=3, $R^2=99.9\%$

Note. See Figure 1.

actual swaps.

For many option underlyings, however, a reliable VIX construction is unavailable due to the lack of deep out-of-the-money options. As an alternative, we study term structures of at-the-money (ATM) option implied volatilities. Partly justified by Carr and Lee (2009) who show that ATM implied volatilities approximate prices of claims to realized volatility ($\sqrt{RV}$), we treat implied variances as proxies for the price of a claim to realized variance. This is the same approach taken in Stein (1989)'s seminal work on excess volatility in the options market. Figures 4 and 5 report results of variance ratio tests for the term structures of equity options. We report results for three individual stocks (Apple, Citigroup, and IBM), two domestic stock indices (S&P 500 and NASDAQ), and three international stock indices (STOXX 50, FTSE 100, and DAX).

The results corroborate those observed for variance swaps. Variance ratios at the longest maturities (from 18 to 30 months) range between 1.5 and 3.5. The only exception is the NASDAQ, for which the variance ratio is 1.1 at the long end.
3.2.3 Currency Implied Variances

We next study the term structure of currency options. As in the case of equity options, we treat implied variances at different maturities as proxies for variance swaps, and apply the same variance ratio test for linear claims that we used to study variance swaps. Our data are for three of the highest volume currency pairs (GBP-USD, GBP-JPY, and USD-CHF) from JP Morgan, covering the period 1998-2014, with maturities up to 24 months. Variance ratio tests based on currency options are plotted in Figure 6, and share the same patterns found in term structures of other volatility claims.

3.2.4 Interest Rates

US government bond prices are among the most well studied data in all of economics. Our US bond data comes from Gurkaynak, Sack and Wright (2006). The data consist of zero-coupon
nominal rates with maturities of 1 to 15 years for the period 1971 to 2014, and is available at the daily frequency (we do not use higher maturities since they became available only later). The term structure is bootstrapped from coupon bonds and uses only interpolation so that a maturity will only be present if enough coupon bonds are available for interpolation at that maturity.

The pricing model we use for interest rates is the standard homoskedastic exponential-affine model. We discuss this specification in detail in Appendix C and show empirically that heteroskedasticity plays a minor role in the variance ratios we estimate. We use three factors as a baseline. It is well known from the term structure literature that three factors produce an extremely good fit of the model. Our estimates confirm that three factors produce more than 99.9% of the common variations among log yields.

Our variance ratios tests also show that yields deviate only slightly from the affine model restrictions. The maximum variance ratio is 1.2 at 15 years. While this variance ratio is sta-
tistically greater than unity and thus corroborates the excess volatility results of Gurkaynak, Sack and Swanson (2005), it is economically the smallest excess volatility effect that we find among all of the term structures that we study. It is interesting that affine term structure models were originally formulated to describe the Treasury market, and this is the market in which affine models indeed appear to perform best.

### 3.2.5 Credit default swaps

Credit default swaps (CDS) are the primary security used to trade and hedge default risk of corporations and sovereigns. As of December 2014, the outstanding notional value of single-name CDS was $10.8 trillion. Our CDS data is from MarkIt. The CDS data is daily and includes maturities of 1, 3, 5, 7, 10, 15, 20, and 30 years, though not all maturities are necessarily traded with liquidity. We focus on the most liquid single-names and sovereigns.

Among the different CDS contracts written on the same reference entity, we choose those with highest liquidity. In particular, we choose CDS written on senior bonds, with modified-restructuring (MR) clause, and denominated in US dollars. Since there was little CDS activity before the financial crisis and most of these contracts had low liquidity, we focus on the period from January 2007 onwards. We choose the three most traded sovereigns (Italy, Brazil, Russia) and the three most traded corporates (JP Morgan, Morgan Stanley, Bank of America) according to 2008 volume. In the plots below, we focus on maturities up to 15 years for individual names, and on 30 years for sovereign CDS. Confidential DTCC data indicate that following 2008 there is positive volume at 15 years for our corporate names and 30 years for our sovereign names.

In Appendix D we describe how CDS prices can be approximated in the framework of

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17 For sovereigns, we use contracts with the CR clause, as more data is available than for the MR contracts.
18 See Fitch (2009).
Section 2. The link to the affine setup is constructed by exploiting the exponential-affine specification for defaultable bonds obtained by Duffie and Singleton (1999), and noting that the CDS spread can be expressed as an approximate linear function of the yield of a defaultable bond.

Figure 8 reports variance ratio tests for CDS markets. For individual names, variance ratios are as high as 2.7 at 15 years (with the exception of JP Morgan, whose long maturity variance ratio is insignificantly different from one). For sovereign CDS, data show variance ratios in excess of six at long maturities. Overall, CDS results indicate a qualitatively and quantitatively significant overreaction similar to other asset classes considered in this paper.

3.2.6 Inflation swaps

We obtain inflation swaps data from Bloomberg. We observe the full term structure between one and 30 years at the daily frequency over the period 2004 to 2014. As reported in Fleming and Sporn (2013), “Despite a low level of activity and its over-the-counter nature, the U.S. inflation swap market is reasonably liquid and transparent. That is, transaction prices for this market are quite close to widely available end-of-day quoted prices, and realized bid-ask spreads are modest.” This data is used to study deflation risk by Fleckenstein, Longstaff...
Figure 8: Credit Default Swaps

Note. See Figure 1.

and Lustig (2013).

The term structure model for inflation swaps fall neatly within the exponential-affine specification of Section 2 (with additional model details in Appendix D). Empirically, the variance ratio pattern for inflation swaps differs from other asset classes in that price volatility is at first strongly concave in maturity, but then volatility rises rapidly between maturities of 15 and 30 years. It is no surprise, then, that variance ratios at the long end exceed 6.0 and are inconsistent with affine model restrictions.

3.3 Nature of Violation

This section reports pervasive evidence that long-term prices co-move with short-end prices in a manner inconsistent with linear pricing models. Our tests embed a simple decomposition of price variance at each maturity into three orthogonal components. The first component is the fraction of the variance explained by the model (the denominator of the variance ratio test). The second is the fraction of the price that is explained by the comovement with the short-end prices, but is in excess of the volatility predicted by the model (the difference between the numerator and the denominator of the variance ratio test). The last component is the fraction of the price variation at each maturity that is orthogonal to the short-end
Table 1: Variance decomposition for Variance Swaps

<table>
<thead>
<tr>
<th>Maturity n</th>
<th>$VR_n$</th>
<th>% explained by factors, consistent with model</th>
<th>% explained by factors, but in excess of model</th>
<th>% unexplained by factors</th>
</tr>
</thead>
<tbody>
<tr>
<td>24</td>
<td>2.15</td>
<td>38%</td>
<td>44%</td>
<td>18%</td>
</tr>
<tr>
<td>12</td>
<td>1.22</td>
<td>75%</td>
<td>16%</td>
<td>9%</td>
</tr>
<tr>
<td>6</td>
<td>1.0017</td>
<td>97%</td>
<td>0%</td>
<td>3%</td>
</tr>
</tbody>
</table>
4 Potential Sources of Violation

In this section we explore potential explanations behind the pervasive evidence of excess price volatility relative to the affine-$Q$ model. We classify possibilities into four forms of model misspecification, i) omitted factors, ii) long memory $Q$-dynamics of cash flows, iii) non-linear $Q$-dynamics, and iv) arbitrage opportunities along the term structure. We argue that missing factors and non-linear dynamics are unlikely to generate patterns that we see in the data. We show that long-range dependence in cash flows can generate high variance ratios along with high $R^2$, but allowing for additional factors generally pushes ratios back to one. Violations of no-arbitrage due to overreaction emerge as the most likely driver of high variance ratios at long maturities. While long memory and overreaction share some similar attributes, they in principle can be distinguished by the fact that one admits arbitrage while the other does not. We document a highly profitable trading strategy that exploits excessive fluctuations of long-maturity prices, which suggests that at least part of the affine-$Q$ violation is due to arbitrage opportunities arising from investor overreaction at the long end of the term structure.

4.1 When the Affine Model is Misspecified

We start with a general characterization of our tests under model misspecification. Our estimator assumes a $K$-factor affine-$Q$ model of prices along the term structure. If this is not the true data generating process, then the population projection in Equation (7) becomes

$$f_{t,K+1} = b'F_{t,1:K} + u_t \quad (18)$$

or, in analogy to the matrix recursion in (8),

$$F_{t,2:K+1} = BF_{t,1:K} + U_t, \quad (19)$$

with $B$ taking the same structure as earlier and $U_t = (u_t, 0, ..., 0)'$. Equation (18) now contains a residual that is solely due to specification error.

Under misspecification, the coefficient $B$ in (19) is no longer fixed and instead becomes specific to the maturities used in the projection. For other maturities, the projection coefficient generally takes a different value. This reflects the fact that cross-equation restrictions of the affine model in (9) are only satisfied when the model is correctly specified.

A key question is whether the violations of the cross-equation restrictions observed in the data can tell us anything about the nature of the model misspecification. We arrived at
the no-arbitrage restrictions in (9) by iterating expectations in the price-on-price projection equation. Repeating this using the representation of Equation (18) and imposing the no-
arbitrage condition that $E_t^Q[f_{t+1,j}] = f_{t,j+1}$, we find for all $j > 1$ that

$$F_{t,j+1,K+j} = B^j F_{t,1,K} + \sum_{l=0}^{j} B^l E_t^Q[U_{t+l}].$$

Equation (20) is an exact representation of prices at all maturities regardless of misspecification (assuming there is no arbitrage). The first term on the right-hand side captures the variation in $F_{t,j+1,K+j}$ that is consistent with the affine model restrictions given projection (18). The second term captures the deviation from the model. We can decompose the behavior of this deviation by projecting it onto $F_{t,1,K}$. All elements of the vector $U_{t+1}$ other than the first are zero, so we write this projection as

$$e_1 \sum_{l=0}^{j} B^l E_t^Q[u_{t+l}] = \gamma_{K+j} F_{t,1,K} + \zeta_{t,K+j},$$

where $\gamma_{K+j}$ is a K-vector and $\zeta_{t,K+j}$ is scalar. This decomposition allows us to write (20) as

$$F_{t,j+1,K+j} = (B^j + \gamma_{K+j}) F_{t,1,K} + \zeta_{t,K+j}$$

where the projection residual $\zeta_{t,K+j}$ is orthogonal to the first $K$ prices, $F_{t,1,K}$. When testing model restrictions, we estimate the unrestricted linear projection of $F_{t,j+1,K+j}$ on to $F_{t,1,K}$ in (21) and compare the estimated projection coefficient, $(B^j + \gamma_{K+j})$, to the affine-model-restricted coefficient, $B^j$.

The behavior of the unrestricted projection is informative about the nature of the misspecification. Two stark empirical facts emerge uniformly from data in all asset classes. First, the unrestricted linear factor model (21) provides an excellent fit of the data, with $R^2$ approaching 100%. Second, variance ratios are significantly greater than one. Together, these facts reveal interesting behavior in the specification error term, $\sum_{l=0}^{j} B^l E_t^Q[U_{t+l}]$.

High variance ratios means that the total variation of the specification error, $Var(\sum_{l=0}^{j} B^l E_t^Q[U_{t+l}])$, must be large. At the same time, an unrestricted $R^2$ approaching 100% means that the portion of the specification error that is uncorrelated with the short maturity prices, $Var(\zeta_{t,K+j})$, must be tiny. In other words, the specification error must be nearly perfectly correlated with the factors from the short end. This is evidently the case, as high variance ratios are equivalent to the unrestricted projection coefficients being significantly larger in magnitude than the model restriction allows—the $\gamma_{K+j}$ coefficients
Note. The figure reports the variance ratios and \( R^2 \) obtained when the true model is a two-factor model but only one factor is used in estimating the \( Q \) dynamics. Results are for various combinations of the variance of the second factor relative to the first (\( \sigma_2^2/\sigma_1^2 \)) and persistence of the second factor (\( \rho_2 \)).

are far from zero (as found in Figure 2). As we discuss below, this appears consistent with investor overreaction.

4.2 Missing Factors

Even if the true model were an affine factor model, prices might appear excessively volatile if the estimated model has too few factors relative to the truth. Two pieces of evidence indicate that omitted factors are unlikely to explain our findings.

Any omitted factor that is consistent with our estimation results must have a particular set of traits described in Section 4.1. It must be volatile and persistent enough to generate high variance ratio at long maturities. Yet it must also be highly correlated with the other factors, as too much unique variation in the factor will pull the \( R^2 \) below the values found in the data.

A calibration shows that such a factor is essentially infeasible from a quantitative standpoint. Consider a term structure whose data-generating process is a two-factor model of cash flows. Factor \( i \) has variance \( \sigma_i^2 \) and persistence of \( \rho_i \), \( i = 1, 2 \), and the factors have a correlation of \( \phi \). What happens when we estimate an affine model with \( K = 1 \), thereby misspecifying the model to have too few factors? Figure 10 shows the possible scenarios for
Figure 11: Variance swaps: varying the number of factors

Note. See Figure 1.

the (population) $R^2$ and variance ratio statistic at a maturity of 24 periods. The calculations are based on a range of values for the persistence of the second factor ($\rho_2$) and how correlated the factors are ($\phi$). We fix the monthly persistence of the first factor to $\rho_1 = 0.5$, and fix the variability of the second factor is relative to the first at $\sigma_2^2/\sigma_1^2 = 0.10$ in the left panel and 0.25 in the right panel.

We find no combination of parameters that can simultaneously generate an $R^2$ over 99% and a variance ratio that is meaningfully greater than one. The best chance comes when the second factor is extremely persistent ($\rho_2 \to 1$) and highly correlated with the first factor ($\phi \to 1$). This rather strange second factor influences long-end variances due to its strong serial correlation, yet it is masked by the first factor due to their high correlation, which allows the model to achieve a very high $R^2$ with a single factor. However, even in this “best” case, the variance ratios from the misspecified model rise only a few percentage points above one so long as the $R^2$ is near 99%.\footnote{The quantitative results in this example are quite general. Considering more factors, allowing for a higher ratio of $\sigma_2^2/\sigma_1^2$, or allowing for greater persistence in the first factor typically make it even less likely that a missing factor can explain our findings.}

Second, if a missing factor were driving our results, we can account for it in our empirical analysis with a simple robustness check that allows for additional factors in the model. Figure 11 shows a sequence of variance swap test plots with the number of factors increasing from one to three.

With one factor, the model $R^2$ is 94.1%, and the variance ratio at 24 months is 5.61. The two-factor case is the main result reported in Figure 1, which has an $R^2$ of 99.6% and a long-end variance ratio of 2.15 at 24 months. Finally, with three factors, the $R^2$ exceeds 99.9%, and continues to produce large economic and statistical rejections of the affine model ($VR_{24} = 2.16$). We see this type of behavior throughout the asset classes we study. Table
A6 in Appendix E reports variance ratios for all other asset classes as we expand the number of factors beyond those that explain at least 99% of the total variation (the benchmark case in Section 3). The table documents similarly high and significant variance ratios as we gradually expand the number of factors.\footnote{There is of course always a factor model that delivers variance ratios equal to one—it is the model with a number of factors equal to the number of maturities observed in the term structure. While this model is obviously uninteresting, it reminds us that the modeler’s objective is to maximize the variety of phenomena explained by a model while minimizing the number of inputs and parameters necessary to do so. Adding factors eats up valuable cross-equation restrictions that give the model its economic and statistical content. Besides the evident inability of additional factors to reconcile the data with affine models, resorting to richer parameterizations when a great majority of data variation is already explained is scientifically unsatisfying. 

Duffee (2010) raises an additional concern about using too many factors. He shows that overfitting of the interest rate term structure with more than 3 factors leads to implausibly high Sharpe ratios for some fixed-income portfolios.}

4.3 Long Memory

Excessive volatility of long-lived claims intuitively raises the possibility that our findings are due to long memory cash flow dynamics that are poorly captured by the more rapid, geometric mean reversion inherent in affine models.

Our data suggest that cash flows are stationary under $\mathbb{Q}$ in all asset classes we study. This is evident from the concave shape of plots of price volatility versus maturity. If cash flows were integrated of order one, we would expect price volatility to rise linearly with maturity. Furthermore, if the model is integrated, the transition matrix $\rho$ in Equation (4) will have an eigenvalue of one. In Appendix A, we show how to invert the regression loadings for the restricted model described in Section 2 to obtain the eigenvalues of $\rho$. In all term structures, we estimate eigenvalues of $\rho$ that are below one in absolute value.

It is possible that cash flows are stationary under $\mathbb{Q}$ yet they mean revert more slowly than an autoregression would suggest. Granger and Joyeux (1980) propose the broad class of fractionally integrated, or ARFIMA, models to capture precisely this type of long memory behavior. An ARFIMA process is indexed by a parameter $d$ that determines its degree of long-range dependence. When $d$ is in the interval $(0,0.5)$, it is positively fractionally integrated yet stationary (the special case of $d = 0$ corresponds to a standard ARMA process).

We investigate the effect of estimating an affine (short memory) model when the data is in fact fractionally integrated. No-arbitrage term structure prices become intractable to derive analytically in the ARFIMA setting, but are easily evaluated via simulation. We simulate term structure prices assuming an ARFIMA$(1,d,0)$ model using a grid of values for $d \in (0,0.5)$ and values of the AR coefficient of 0.25, 0.50, or 0.75. Figure 12 demonstrates the range of long-memory behavior that is embedded in our simulated term structure.
Figure 12: **Long Memory Mean Reversion**

![Figure 12: Long Memory Mean Reversion](image)

**Note.** ARFIMA(1,d,0) reversion from a one standard deviation shock to the process’s mean value of zero over 25 periods, assuming an AR(1) coefficient of 0.75 and d values of 0, 0.10, 0.30, and 0.49.

extremely slow decay for the case \(d = 0.49\) illustrates how an ARFIMA process is difficult to distinguish from an integrated process as \(d\) approaches the upper limit of the stationary range.

We calculate prices at maturities up to 24 periods and use a time series sample size of 1,000 periods. Then we estimate and construct variance ratio tests using the misspecified, short memory affine model with either one, two, or three factors. Results reported in Table 2 show that it is uncommon to find a model that produces an \(R^2\) greater than 99% along with a variance ratio above two. When this does occur, it is because the long memory behavior is close to non-stationary. However, in these cases, inclusion of additional factors brings variance ratios close to one. Evidently, despite its incorrect specification, the affine model with two or three factors is an accurate enough approximation of the ARFIMA process that the misspecification can go undetected. The ability of additional seemingly unnecessary factors to drive variance ratios toward one in simulations is an important difference versus the data. In the missing factor robustness checks of Table A6, we find variance ratios in the data that remain well above one despite inclusion of additional unnecessary factors.

Overall, these simulations suggest that long-memory dynamics may have some role in explaining our findings. Below, we further consider the plausibility of long memory dynamics in the context of a trading strategy, and our the results cast doubt on the long memory
Table 2: Effects of Long Memory

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Note. Variance ratios and $R^2$ computed in simulations of an ARFIMA(1,d,0) model. $d$ corresponds to the order of integration; K is the number of factors used in the variance ratio test. $VR_{12}$ is the variance ratio at 12 months maturity, and $VR_{24}$ is the test at 24 months. AR(1) is the autoregressive coefficient in the ARFIMA model.

4.4 Non-linearities

A third potential explanation of our findings is that cash flows evolve non-linearly. In this section, we explore the effects of estimating and testing restrictions of a misspecified affine model when the true cash flow process has non-linear dynamics.

We study a class of processes known as smooth transition autoregressive (STAR) models.\(^{21}\) As emphasized by Granger and Terasvirta (1993), STAR models encompass a broad variety of non-linear dynamics that have proven successful in modeling economic time series.

Term structure prices are intractable to derive analytically for STAR models, but are easy to calculate via simulation. We assume that cash flows evolve according the one-factor non-linear process

$$x_t = \rho x_{t-1} \left(1 - \left(1 + e^{-\gamma(x_{t-1} - c)}\right)^{-1}\right) + \left(1 - \rho\right)x_{t-1} \left(1 + e^{-\gamma(x_{t-1} - c)}\right)^{-1} + \epsilon_t.$$ \(^{(22)}\)

\(^{21}\)Terasvirta (1994) provides an excellent econometric treatment of STAR models.
Equation (22) is the most commonly used variant in the STAR class and is known as the logistic STAR model. It nests the standard linear autoregression, but allows for the process to transition between high and low serial correlation depending on the state of the process. The degree of non-linearity is governed both by $\rho$ and $\gamma$.

Figure 13 illustrates the forms of non-linearity captured by the STAR model at various parameter values by plotting the relationship between $x_t$ and $E_{t}^Q[x_{t+1}]$. When $\rho$ is close to either 0 or 1, the model exhibits extreme state-dependence in cash flows, transitioning between dynamics that are very persistent in some periods and nearly i.i.d. in others. For a given value of $\rho$, higher $\gamma$ produces higher curvature and can even mimic a kink when $\gamma$ is very large.

We calculate no-arbitrage prices in the STAR model at maturities up to 24 periods and use a time series sample size of 1,000 periods. Then we estimate and construct variance ratio tests using the misspecified affine model with up to three factors. The results are reported in Table 3. In this large family of non-linear models (including rather extreme non-linearities under certain parameterizations), the variance ratio does not rise far above one in any specification. In other words, the affine specification is a very good approximation to the true non-linear $Q$-dynamics and the variance ratio does not detect significant violations of cross-equation restrictions. This suggests that non-linearities are an unlikely driver of the variance ratios we observe in the data.
Table 3: Effects of Non-linearity

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Note. Variance ratios and \( R^2 \) computed in simulations of a logistic STAR model with parameters \( \gamma \) and \( \rho \). \( K \) is the number of factors used in the variance ratio test. \( VR_{12} \) is the variance ratio at 12 months maturity, and \( VR_{24} \) is the test at 24 months.

4.5 Overreaction and Other Expectation Errors

A fourth possibility for explaining variance ratios greater than one is that the affine model is indeed an accurate description of the true value of claims, but that some of these claims are subject to temporary mispricing. This amounts to profitable violations of no-arbitrage along the term structure.

We can characterize what any mispricing must look like if prices along the term structure follow an (unrestricted) linear factor model. For illustration, suppose that a term structure of forward prices is exactly described by a one-factor model, so that (suppressing constants)

\[
f_{t,1} = b_1 x_t, \quad f_{t,2} = b_2 x_t, \quad \ldots, \quad f_{t,n} = b_n x_t. \tag{23}\]

In order for this term structure to satisfy no-arbitrage, it must be the case that the loadings follow a geometric progression in some constant \( \rho \):

\[
b_1 = \rho, \quad b_2 = \rho^2, \quad \ldots, \quad b_n = \rho^n. \tag{24}\]

If the factor loadings behave in any other way, they must violate the law of iterated expectations. Forward prices have to satisfy two properties. First, they must be linear in the factor by assumption, meaning that \( f_{t,j} = E_t^Q[x_{t+j}] = b_j x_t \). Second, by the law of iterated
expectations, any forward price today also represents an expectation of tomorrow’s price of a forward with shorter-maturity.

\[ f_{t,j} = E^Q_t[x_{t+j}] = E^Q_t[E^Q_{t+1}[x_{t+j}]] = E^Q_t[f_{t+1,j-1}] \]

If we fix the initial coefficient to \( b_1 = \rho \), these two properties together imply that \( b_2 = b_1^2 = \rho^2 \), which in turn implies \( b_3 = \rho^3 \), and so forth. This argument proves the following proposition in the one-factor case, and easily generalizes to the case of multiple factors.

**Proposition.** If a term structure of prices obeys an exact affine factor model, then arbitrage exists along the term structure if and only if factor loadings have non-geometric decay.

This simple result is powerful for understanding the nature of affine model violations documented above. The unrestricted model in our variance ratio tests takes a linear factor form much like (23), and we find that this model provides an excellent description of the data with \( R^2 \) values near 100%. At the same time, the high variance ratios reveal that the \( b_j \) coefficients decay at a less than geometric rate. This violates the structure in (24), suggesting that the law of iterated expectations is violated thus admitting arbitrage opportunities. The empirical fact that the loadings decay more slowly than the affine model allows tells us that the nature of the model violation is one of overreaction at the long end of the term structure.

An important caveat is that the term structure \( R^2 \) for the unrestricted linear model is not identically 100%, which means that the conditions of the proposition are not exactly satisfied, and thus the slow decay in coefficients detected by high variance ratios is potentially due to the econometricians using a misspecified model. That is, we face a form of the joint hypothesis problem that arises in any asset pricing model test: Is a rejection indicating that the null model is incorrect or that asset prices sometimes deviate from “true” value? This issue makes it difficult to discern whether the affine model is violated due to misspecification as in Sections 4.2 to 4.4, or due to mispricings arising from other investor behaviors (such as a tendency to commit errors when iterating expectations).

Three questions arise as we consider the possibility that prices occasionally reflect expectation errors. First, can we find evidence that favors this view over the alternative of an incorrect econometric model with no mispricing? Second, what type of investor behavior might lead to arbitrage opportunities? Third, is there evidence outside of asset markets that is consistent with our term structure pricing results? We address each of these questions in turn.
4.5.1 Extrapolation and Arbitrage

A foundational assumption in behavioral economics is that investors over-extrapolate when forming expectations. Barberis (2013) explains,

This assumption is usually motivated by Kahneman and Tversky (1974)’s representativeness heuristic. According to this heuristic, people expect even small samples of data to reflect the properties of the parent population. As a result, they draw overly strong inferences from these small samples, and this can lead to over-extrapolation.

A number recent models explore the usefulness of extrapolative expectations in matching a variety of asset pricing phenomena, including excess price volatility in equity and credit markets. These models do not examine how expectation formation can vary with the horizon of the expectation, and in particular have not explored the implications that extrapolation may have for price volatility along a term structure. Yet given that the affine model inconsistency we document stems from long maturity factor loadings appearing too high—which makes the long end of the price curve overreact—extrapolation is a natural candidate for a behavioral bias that might produce systematic mispricing along the term structure.

We now present a stylized example of a model in which investor behavior implies an exact affine factor model for term structure prices, but with factor loadings that decay non-geometrically. In this model the cash flow process that establishes “correct” prices is a first order autoregression:

\[ x_{t+1} = (1 - \rho)\mu + \rho x_t + \epsilon_{t+1}. \]

We assume that investors, however, form biased expectations due to extrapolation. Their extrapolative expectations are summarized by replacing the long run mean of cash flows, \( \mu \), with a distorted mean,

\[ \mu^\theta_t = \mu + \theta(x_t - \mu). \]

The distortion represents the investor’s tendency to over-emphasize recent data when contemplating the cash flow distribution. If recent cash flows exceed the long run mean, investors believe that this mean is higher than in fact it is, and they vice versa when \( x_t \) is below \( \mu \). This leads to a term structure of forward prices that violates the law of iterated expectations:

\[ f_{t,n} = E^\theta_t [x_{t+n}] = E_t[\mu^\theta_t + \rho(x_{t+n-1} - \mu^\theta_t)] = \mu(1 - \rho)(1 - \theta) + x_t(\rho^n[1 - \theta] + \theta). \]

\footnote{For example, Barberis and Shleifer (2003); Greenwood and Shleifer (2014); Barberis et al. (2015b,a); Bordalo, Gennaioli and Shleifer (2015); Gennaioli, Shleifer and Ma (2015).}

\footnote{Furthermore, assets markets that have typically been modeled using extrapolation, such as stocks, mortgages, and corporate bonds, are long-duration assets. Thus, excess volatility in these markets is likely to be a similar phenomenon to the long maturity excess volatility that we document in many other markets.}
This form of expectation error produces a term structure of prices that is exactly described by an affine one-factor model, but with factor loadings that decay slower than geometrically with maturity. So, by the proposition above, this term structure admits arbitrage.\textsuperscript{24}

4.5.2 A Trading Strategy Test

An approach that begins to address the joint hypothesis problem is to understand whether model deviations appear excessively profitable. If there exists a strategy that exploits deviations from the null model to earn large trading profits while taking on little risk, it is evidence in favor of the mispricing interpretation of the data.

Under the null hypothesis of a $K$-factor affine model, we can determine at any point in time whether a long maturity claim is overpriced or underpriced by comparing it to the price predicted by the estimated model (and determined as a function of the first $K$ short maturity prices as in Equation (10)). Our evidence of overreaction at the long end suggests that a large increase in short maturity prices is likely to temporarily drive long maturity prices above their correct value. Similarly, a drop in the short end may push long-end prices below their correct value, producing temporary undervaluation of those claims.

The logic of the strategy begins with the presumption that the estimated affine model is correct on average, so that observed price deviations from the model are temporary and expected to correct. Thus, consider taking a position at time $t$ in a claim with maturity $N_1 > K$ and holding this position for $n$ periods. At $t + n$, the maturity of the position has shortened to $N_2 = N_1 - n$, and is expected to have a correct price (based on the model) of

$$p_{t+n,N_2} = a_{N_2} + (b_{N_2})'P_{t+n,1:K} \quad (25)$$

where $a_{N_2}$ and $b_{N_2}$ are model-implied coefficients as in Equation (10). And, over the $n$-period investment period, the claim has paid out cash flows of $x_{t+1}, \ldots, x_{t+n}$.

Construction of the strategy works backward from $t + n$, when the trade is unwound, to initiation of the trade at time $t$. In particular, we seek a trade that is expected to have zero liquidation value at $t + n$, but that generates a positive cash flow at initiation. Equation (25) suggests comparing the prices of two portfolios at time $t$. Portfolio $A$ simply buys the $N_1$-maturity claim at a price of $p_{t,N_1}$. After holding $A$ for $n$-periods, it has yielded cash flows $\text{24}$

Finally, we note that a series of papers have explored the relationship between learning and volatility, showing that rational models of learning can generate observed excess price volatility and seemingly extrapolative behavior (Timmermann, 1993; Barsky and De Long, 1993; Veronesi, 1999; Pástor and Veronesi, 2003, 2009b,a). The effect of learning is often summarized by an additional linear factor affecting prices, thus the models still fall within the affine class. And, being fully rational, they do not generate violations of the law of iterated values like those we observe in the data.
of \( x_{t+1}, \ldots, x_{t+n} \) and has ongoing value of \( p_{t+n, N_2} \).

Portfolio \( B \) is designed to replicate the right-hand-side of Equation (25). First, it invests the present value of \( a_{N_2} \) in the \( n \)-maturity risk-free bond (for simplicity let us assume that the risk-free rate is zero). Next, it buys all claims with maturities of \( n+1, \ldots, n+K \), corresponding to the price vector \( P_{t,n+1:n+K} \). The exact number of shares purchased in each claim is given by the vector \( b_{N_2} \). Third, it buys \((1 - (b_{N_2})'1)\) shares of an \( n \) maturity claim with price \( p_{t,n} \).

After \( n \) periods, the risk-free bond has matured with a value of \( a_{N_2} \) and the position \((b_{N_2})'P_{t,n+1:n+K}\) has ongoing value of \((b_{N_2})'P_{t+n,1:K}\). The \( n \)-maturity claim has expired with no remaining value, but has ensured that the intermediate cash flows generated over the life of the trade are exactly \( x_{t+1}, \ldots, x_{t+n} \). In short, portfolio \( B \) exactly replicates the expected future value of portfolio \( A \) and exactly matches all intermediate cash flows generated by \( A \).

Because portfolio \( B \) is an exact hedge to portfolio \( A \) according to the model, any difference in the prices of the time \( t \) initiation prices of \( A \) and \( B \) represents an arbitrage. If the price of \( B \) exceeds that of \( A \), the strategy establishes a long position in \( A \) and a short position in \( B \), and vice versa. This strategy generates a strictly positive cash flow at time \( t \), exactly offsets all intermediate cash flows, and has zero liquidation value in expectation.\(^{25}\)

We compute the return to this strategy taking into account realistic constraints on capital and margining of short positions. In particular, we assume each trade is allocated \( C \) dollars of capital to invest. The capital must cover the long position plus a fraction \( M \) of the short position in the form of margin. We denote \( q \) as the number of units we trade, which we solve for given initial capital and margin requirements. \( Z_S \) the per-unit revenue from the short position, and \( Z_L \) the per-unit cost of the long position. We write \( Z_L = Z_S - \Pi \), where \( \Pi > 0 \) is the immediate per-unit profit realized from the trade (no-arbitrage is equivalent to \( \Pi = 0 \)).

Capital plus initial revenue, \( qZ_S \), must cover the cost of the long position, \( qZ_L \), plus the margin on the short position. Therefore, the number of units traded, \( q \), must satisfy

\[
C + qZ_S \geq qZ_L + MqZ_S
\]

or therefore

\[
q \leq \frac{C}{MZ_S - \Pi}.
\]

This caps the number of units that can be traded depending on capital and margin. Larger

\(^{25}\)In practice, the liquidation equation (25) does not hold exactly, so it is a risky arbitrage. To minimize the liquidation risk, \( a_{K_2} \) and \( b_{K_2} \) are based on unrestricted regressions of \( N_2 \)-maturity prices on prices for maturities 1 through \( K \). This minimizes the squared liquidation error.
positions can be taken when more capital is available and when haircuts are smaller. These constraints also have the attractive feature that the size of the trade is increasing in the size of the initial profit, \( \Pi \).

We implement the trading strategy in the variance swap market. The strategy is implemented purely out-of-sample. That is, when deciding on a trade at time \( t \), estimated model parameters (particularly those of \( a_{N_t} \) and \( b_{N_t} \)) and position choices only use data that an investor would have access to in real time, in particular the history of term structure prices through date \( t \). We re-estimate the model each day using the most recent 250 trading days. The haircut is assumed to be \( M = 1/2 \). We only trade on days when the initiation profit \( \Pi \) is sufficiently large, which avoids trading on small mispricings that are indistinguishable from estimation noise. We examine thresholds based on the historical distribution of \( \Pi \) during the rolling estimation window. Therefore, at each date \( t \), the initial profit is being compared only with backward looking information and the trading choice genuinely preserves the out-of-sample character of the trade.

The variance swaps column in Table 4 reports the annualized Sharpe ratios of a trading strategy with one month holding period and with various choices for the maturity of the long-end claim being traded (\( N_1 \) of 4, 6, 12, 18, or 24 months) and the trading threshold for initiation profits (equal to the 50\(^{th} \), 75\(^{th} \), or 90\(^{th} \) historical percentile for \( \Pi \)).

We obtain consistently high Sharpe ratios in all cases, often above 1, and we find high Sharpe ratio especially in cases where \( \Pi \) is particularly large (cases in which the model identifies a large mispricing).

As highlighted in the previous section, variance ratios above one might indicate expectation errors (mispricing along the term structure), but could also be due to misspecification of the true model. In the latter case, trading against a correct model based on a misspecification should not lead to trading profits. To confirm this intuition, we also report results of our trading strategy applied in simulated models. We compare against three models in which long-maturity variance ratios are greater than one because the estimated affine model is misspecified, but in which the simulated prices in fact satisfy no-arbitrage. These include

1. the two factor affine model with \( \rho_1 = 0.9 \) and \( \rho_2 = 0.5 \), but estimated assuming a single factor structure
2. the long memory ARFIMA model with \( d = 0.49 \) and AR(1) coefficient 0.75
3. the non-linear logistic STAR model with parameters \( \rho = 0.01 \) and \( \gamma = 0.5 \).  

\(^{26}\)The threshold maps approximately into the fraction of days traded, with the 50\(^{th} \) percentile trade triggered about half of the time and 90\(^{th} \) percentile trade is initiated roughly one day in ten.
Table 4: Trading Strategy Sharpe Ratios

<table>
<thead>
<tr>
<th>Mispricing Threshold</th>
<th>Longest Maturity Traded</th>
<th>Variance Swaps</th>
<th>Missing Factor</th>
<th>Long Memory</th>
<th>Non-linear</th>
<th>Arbitrage</th>
</tr>
</thead>
<tbody>
<tr>
<td>50</td>
<td>4</td>
<td>0.79</td>
<td>0.07</td>
<td>-0.34</td>
<td>-0.01</td>
<td>0.17</td>
</tr>
<tr>
<td>50</td>
<td>6</td>
<td>1.63</td>
<td>0.08</td>
<td>-0.24</td>
<td>0.06</td>
<td>0.08</td>
</tr>
<tr>
<td>50</td>
<td>12</td>
<td>0.78</td>
<td>0.09</td>
<td>0.30</td>
<td>0.04</td>
<td>4.87</td>
</tr>
<tr>
<td>50</td>
<td>18</td>
<td>1.18</td>
<td>0.10</td>
<td>0.40</td>
<td>0.04</td>
<td>5.45</td>
</tr>
<tr>
<td>50</td>
<td>24</td>
<td>0.66</td>
<td>0.10</td>
<td>0.37</td>
<td>0.06</td>
<td>5.98</td>
</tr>
<tr>
<td>75</td>
<td>4</td>
<td>1.06</td>
<td>-0.13</td>
<td>-0.23</td>
<td>-0.16</td>
<td>0.21</td>
</tr>
<tr>
<td>75</td>
<td>6</td>
<td>1.99</td>
<td>-0.12</td>
<td>-0.40</td>
<td>-0.14</td>
<td>0.12</td>
</tr>
<tr>
<td>75</td>
<td>12</td>
<td>0.81</td>
<td>-0.11</td>
<td>0.26</td>
<td>-0.06</td>
<td>5.50</td>
</tr>
<tr>
<td>75</td>
<td>18</td>
<td>1.27</td>
<td>-0.11</td>
<td>0.37</td>
<td>-0.01</td>
<td>6.23</td>
</tr>
<tr>
<td>75</td>
<td>24</td>
<td>0.83</td>
<td>-0.11</td>
<td>0.41</td>
<td>0.02</td>
<td>6.93</td>
</tr>
<tr>
<td>90</td>
<td>4</td>
<td>1.52</td>
<td>0.15</td>
<td>-0.26</td>
<td>-0.06</td>
<td>0.34</td>
</tr>
<tr>
<td>90</td>
<td>6</td>
<td>1.45</td>
<td>0.15</td>
<td>-0.36</td>
<td>-0.10</td>
<td>0.39</td>
</tr>
<tr>
<td>90</td>
<td>12</td>
<td>1.09</td>
<td>0.15</td>
<td>0.18</td>
<td>0.02</td>
<td>5.90</td>
</tr>
<tr>
<td>90</td>
<td>18</td>
<td>2.03</td>
<td>0.15</td>
<td>0.38</td>
<td>0.04</td>
<td>6.69</td>
</tr>
<tr>
<td>90</td>
<td>24</td>
<td>0.38</td>
<td>0.14</td>
<td>0.42</td>
<td>0.09</td>
<td>7.49</td>
</tr>
<tr>
<td>Average</td>
<td></td>
<td>1.17</td>
<td>0.04</td>
<td>0.08</td>
<td>-0.01</td>
<td>3.76</td>
</tr>
</tbody>
</table>

**Note.** The table reports annualized Sharpe ratios for trading strategies that exploit the mispricing relative to the affine-$Q$ model. All strategies are implemented using information available to the investor at the time of the trade, and involve 1 month holding period horizons for each trade. The first column reports at what level of mispricing (relative to the historical distribution) a trade is executed. The second column reports which maturity $N_1$ the trading occurs on. The third column reports the trading strategy applied on actual variance swap data, while the remaining columns implement the trading strategy on different simulated dataset. Columns 4-6 feature models that are not exactly represented as an affine-$Q$ model and therefore are misspecified by the investor operating the trading strategy. The last column is a simulated dataset with an arbitrage opportunity similar to those described in this paper.

In each of these cases, we simulate a single sample of 1,000 daily term structure observations, and run the same out-of-sample trading strategy used for the variance swap data. Sharpe ratios in these cases are uniformly close to zero.

Lastly, we simulate a term structure that exactly obeys an unrestricted linear factor structure but that admits arbitrage. In particular, prices on the short end of the curve have factor loadings of $(\sum_{i=1}^{n} \rho^S_i)$ for $n = 1, \ldots, 11$ while loadings on the long end have loadings $(\sum_{i=1}^{n} \rho^L_i)$ for $n = 12, \ldots, 24$, with $\rho^S = 0.9$ and $\rho^L = 0.925$. In this case, trading strategies that focus on long maturities identify a bona fide arbitrage and deliver out-of-sample Sharpe ratios as high as 7.5.\textsuperscript{27} On the short end of the curve, prices are internally consistent, so

\textsuperscript{27}Sharpe ratios are not infinite because the size of the profit each period is random and the recursive nature of the trading strategy realistically incorporates estimation noise in model parameters.
Sharpe ratios are close to zero when trading at maturities less than 12 months.

The large trading strategy profits in the data, combined with counterfactual evidence from simulations, suggest that the overreaction in long maturity swaps is a violation of no-arbitrage. It is possible, however, that a trading strategy based on a misspecified model would yield high risk-adjusted returns by inadvertently loading heavily on risk factors that are not well captured by the affine model. In this case, the trading strategy might have high Sharpe ratios merely due to high correlation with priced risk factors.

To test whether this is the case, we compute the alpha of the trading strategy relative to various asset pricing factors. We scale the trading strategy to have a yearly standard deviation of 20%, comparable with the market, and focus on the 18-month case with a mispricing threshold of 50%. The average annual return of this strategy is 24% with an annual Sharpe ratio of 1.18. The alpha relative to the Fama and French (1993) three-factor model is 23% per annum, meaning that essentially none of the strategy’s performance is captured by exposure to the Fama-French factors. We obtain nearly identical results from adding factors extracted directly from the term structure of variance swaps (capturing shocks to the level and slope of the variance swap curve, see (Dew-Becker et al., 2015) for details). The Sharpe ratios associated with this trading strategy thus do not seem explained by exposure to risk factors, consistent with the interpretation that variance ratio tests are identifying mispricings at the long end of the curve.

4.5.3 Overreaction in Survey Expectations

The Survey of Professional Forecasters publishes quarterly forecasts of inflation at different horizons, for the upcoming 4 quarters (starting in 1981). Starting in 1991, the SPF also publishes 10-year forecasts. Using the panel of cumulative inflation forecasts for 1, 2, 3, 4 and 40 quarters, we estimate an exponential-affine model for inflation using the first 4 quarters (i.e., estimating 3 latent factors), and test the model using maturity 40.

Figure 14 reports the results. By construction, the model perfectly fits volatility of the first three forecasts. However, consistent with our results for prices, the forecast of 10-year inflation comoves too strongly with the short-term forecasts, yielding a variance ratio above 2, aligning with our results for term structures of prices. Overreaction in the term structure of inflation forecasts suggests that errors in expectation formation not only arise during

---

28 We describe the data in detail in Appendix D.5. In 2005, the SPF started reporting also 5-year forecasts; we do not use this additional data here because the sample, at the quarterly frequency, becomes extremely short. See also Chernov and Mueller (2012) for a recent study that uses this data in the context of affine- models. Other recent work on the term structure of inflation forecasts includes Capistrán and Timmermann (2009) and Patton and Timmermann (2012).
The figure reproduces the analysis of figure 1 using the term structure of inflation forecasts from the Survey of Professional Forecasters.

valuation but are potentially a more widespread phenomenon.

5 Robustness

In this section we examine alternative formulations for the test of cross-equation restrictions, we show that our results are insensitive to measurement error in prices, and we provide additional evidence of excess volatility in subsample analysis.

5.1 Why Test Long Maturities?

Are long maturity claims excessively volatile relative to the affine model, or are short maturity claims not volatile enough? In this subsection we discuss our choice to focus our tests on long maturity price volatility.

5.1.1 Comparing Prices to Physical Cash Flows

The first reason for our emphasis on long maturity excess volatility is that prices on the short end of the term structure do not appear to fluctuate excessively when compared to
underlying physical cash flows, while long maturity prices do.

Variance swaps provide a valuable case study because the underlying cash flow process is observable. Payoffs to these securities are determined by the variance of S&P 500 index returns that is realized over the life of the contract. That is, realized variance ($RV_t$) corresponds to the cash flow variable $x_t$ in our model. Because realized variance is public information, it serves as a natural anchor for understanding potential over or underreaction of swap prices.

Suppose, for illustration, that the $Q$-dynamics of realized variance are described by a one-factor model

$$RV_{t+1} = c + \rho RV_t + \epsilon_{t+1},$$

(an unrestricted one-factor model explains 97% of the variation in the variance swap term structure). A regression of the two month swap on the one month swap implies a persistence estimate of $\hat{\rho} = 0.83$. To understand the sensitivity of prices to fluctuations in realized variance, we can scale the price of the $j$-maturity claim by the model-predicted loading, $p_{t,j}/\sum_{i=1}^{j} \hat{\rho}^i$, and regress this on $RV_t$. If the model is correctly specified, the scaled price equals $RV_t$ plus a constant and the regression coefficient will equal one (up to sampling error).

---

$^{29}$Contrast this with, for example, CDS term structures for which the underlying $x_t$ corresponds to an unobservable default intensity.
The left panel of Figure 15 plots the results of these sensitivity regressions. At the short end of the curve, the estimated sensitivity coefficient is 0.95, and the 95% confidence interval includes 1.0, indicating that the one month swap price reacts to realized variance in a manner entirely consistent with the one-factor model. At longer maturities, sensitivities rise sharply above one, suggesting that long-maturity prices overreact to fluctuations in realized S&P 500 return variance given a one-factor model.

Another asset class with an observable underlying cash flow process is the inflation swap term structure. These claims pay off realized CPI inflation over the life of the contract. In a one-factor affine model for inflation swaps, the estimated \( \hat{\rho} \)-persistence parameter for annual inflation is \( \hat{\rho} = 0.46 \). Regressing the scaled inflation swap price, \( p_{t,j}/\sum_{i=1}^{j} \hat{\rho}_i \), on realized inflation delivers a sensitivity coefficient of 1.08 for the one year contract and a 95% confidence interval that includes 1.0. The sensitivity estimates increase with maturity and the confidence intervals beyond four years no longer include one. In summary, the evidence in Figure 15 is more supportive of overreaction in prices of long maturity claims than of underreaction at the short-end.

5.1.2 Alternative Formulations of the Test

While Figure 15 motivates our test’s emphasis on long maturity excess volatility, there are a multitude of ways to formulate tests of cross-equation restrictions. One natural alternative to our approach is to estimate model parameters from the long end of the term structure, and perform variance ratio tests on the short end. This alternative is statistically equivalent to the test that we propose, but can conceal important model violations.

For illustration, consider a setting where prices in fact obey a strict one-factor structure, but where the no-arbitrage cross-equation restrictions are violated. In particular, suppose that prices on the short end of the term structure \((j=1,2)\) behave according to \( p_{t,j} = (\rho_S + \ldots + \rho_S^j)x_t \) while prices on the long end \((j=N-1,N)\) are \( p_{t,j} = (\rho_L + \ldots + \rho_L^j)x_t \), with \( \rho_L > \rho_S > 0 \).

In the population version of our baseline test, we estimate the model parameter from a regression of \( p_{t,2} \) on \( p_{t,1} \) and therefore recover \( \rho_S \), which we use to impose model restrictions. Next we estimate an unrestricted regression of long maturity price \( p_{t,N} \) on \( p_{t,1} \), which has a coefficient of \( \text{Cov}(p_{t,N}, p_{t,1})/\text{Var}(p_{t,1}) = (\rho_L + \ldots + \rho_L^N)/\rho_S \). We compare this to the restricted regression of \( p_{t,N} \) on \( p_{t,1} \) imposing \( \rho_L = \rho_S \), which implies a coefficient of \( (\rho_S + \ldots + \rho_S^N)/\rho_S \). The variance ratio for the long maturity test is therefore

\[
VR_N = \left( \frac{\rho_L + \ldots + \rho_L^N}{\rho_S + \ldots + \rho_S^N} \right)^2.
\]
In the alternative approach of estimating from the long end and testing on the short end, the model parameter is derived from regressing $p_{t,N}$ on $p_{t,N-1}$, yielding an estimate equal to $\rho_L$. The unrestricted regression coefficient of the short maturity price $p_{t,1}$ on $p_{t,N}$ is $\rho_S/(\rho_L + \ldots + \rho^N_L)$ and the restricted coefficient is $\rho_L/(\rho_L + \ldots + \rho^N_L)$. The variance ratio for the short maturity test is therefore

$$VR_1 = \left(\frac{\rho_S}{\rho_L}\right)^2.$$

Clearly, tests based on $VR_1$ and $VR_N$ are equivalent as deviations from unity occur in both cases if and only if $\rho_L \neq \rho_S$. An important difference between the two tests is how they aggregate specification errors along the term structure. A value of $VR_1$ near to but just below one may indicate an important model violation. For example, if $\rho_S = 0.97$ and $\rho_L = 0.99$ and we are considering maturities up to 24 periods, then $VR_1 = 0.92$ and $VR_{24} = 2.49$. In this example, the model violation is one of high duration. It’s impact on the behavior of short maturity claims is limited, as indicated by the small deviation of $VR_1$ from one, while it is a crucial violation for understanding the pricing of long maturity claims.

This example is representative of price behavior in all asset classes we study. Term structure data very broadly imply high cash flow persistence, so the most useful securities for identifying model violations are those with long maturities. Prices of these claims aggregate parameter discrepancies over long horizons, making it particularly easy to visualize the internal inconsistency of prices for a given model, as in Figure 1.

Another testing approach is to use a likelihood ratio or other distance metric to compare pricing errors between two models—one model that imposes pricing restrictions versus a more general model with weaker restrictions—using prices from the entire term structure for estimation (as in Bekaert and Hodrick, 2001). In our setting, a natural implementation of this approach would estimate latent factors by extracting principal components from the panel of all maturities. Then, parameters of the null model (e.g., a one-factor affine no-arbitrage model) are estimated by minimizing pricing errors, and the overall fit is compared to that of a specific alternative (e.g., a two-factor model). This test has the benefit of using information throughout the term structure and generally has excellent power for distinguishing between alternative models.

We find, however, that this approach often lacks power to reject the null model in the presence of long-maturity overreaction like that documented Section 3. A simulation is helpful for understanding how standard model comparison tests can fail to detect overreaction. We generate data from a one-factor model with maturities of up to 24 periods. Simulated
Table 5: Model Comparison Using Full Term Structure

<table>
<thead>
<tr>
<th>Misspecification</th>
<th>MSE1/MSE2</th>
<th>BIC1</th>
<th>BIC2</th>
<th>VR24</th>
</tr>
</thead>
<tbody>
<tr>
<td>ρ_S 0.75</td>
<td>1.020</td>
<td>-866</td>
<td>-909</td>
<td>1.76</td>
</tr>
<tr>
<td>ρ_L 0.75</td>
<td>1.057</td>
<td>-674</td>
<td>-688</td>
<td>3.43</td>
</tr>
<tr>
<td>ρ_L 0.75</td>
<td>1.078</td>
<td>-580</td>
<td>-558</td>
<td>6.43</td>
</tr>
<tr>
<td>ρ_S 0.85</td>
<td>1.011</td>
<td>-795</td>
<td>-768</td>
<td>2.23</td>
</tr>
<tr>
<td>ρ_L 0.85</td>
<td>1.024</td>
<td>-634</td>
<td>-551</td>
<td>5.87</td>
</tr>
<tr>
<td>ρ_L 0.85</td>
<td>1.023</td>
<td>-575</td>
<td>-453</td>
<td>14.61</td>
</tr>
<tr>
<td>ρ_S 0.90</td>
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<td>-796</td>
<td>-707</td>
<td>2.64</td>
</tr>
<tr>
<td>ρ_L 0.90</td>
<td>1.010</td>
<td>-667</td>
<td>-544</td>
<td>6.56</td>
</tr>
<tr>
<td>ρ_L 0.90</td>
<td>1.002</td>
<td>-847</td>
<td>-724</td>
<td>2.49</td>
</tr>
</tbody>
</table>

Note. The table reports statistical tests of an affine model with violation of arbitrage. The model is a one factor model, but specified such that for maturities up to 12, Q-persistence is ρ_S, while for maturities above 12 it is ρ_L. The table reports the ratio of mean squared pricing errors for a 1 and 2 factor model as well as the BIC criterion for one and two factors. The last column reports the variance ratio at 24 months using the number of factors selected by the BIC criterion.

Prices will behave very similarly to an affine model except that long-maturity prices overreact and therefore violate the no-arbitrage internal consistency conditions. In particular, for short maturities (j=1,...,12), factor loadings are given by ∑_{i=1}^{j} ρ_i S, while for long maturities (j=13,...,24) the loadings are ∑_{i=1}^{j} ρ_i L, where ρ_L > ρ_S.

In each simulation, we generate 10 years of monthly data. From simulated prices, we estimate a one-factor model that extracts a single component from the full panel of prices, then estimate the model’s single model parameter ρ by minimizing the sum of squared pricing errors at all maturities. We compare this fit to an otherwise identical model that allows for two principal component factors, again estimating this model’s two persistence parameters by minimizing pricing errors. We then compare the models in two ways. First, we calculate the ratio of mean squared pricing errors for the one-factor and two factor model (MSE1/MSE2). Adding factors can only improve the model’s fit. High values indicate that moving to two factors produces a large improvement in fit. We also report the Bayesian Information Criterion (BIC) for each model assuming that errors are normally distributed. The BIC trades off model fit versus parameterization, with lower values of BIC indicating a superior model. Because the BIC is based on log likelihoods of the estimated models, BIC comparison is conceptually similar to conducting a likelihood ratio test.

Table 5 shows simulation results. We consider various degrees of model misspecification described by a given combination of ρ_S and ρ_L, and report the average of each model statistic across 1,000 simulations. Overall, simulations show that it is difficult to reject the one-factor
model based by comparing it with an encompassing two-factor model. The improvements in mean squared error are small, usually less than a few percent. And, in most of the cases we consider, the BIC prefers the one-factor model (superior BIC values are shown in bold).

For comparison, we also report our one-factor variance ratio test, which estimates the model’s single parameter from the first two maturities and tests cross-equation restrictions for the longest maturity (24 months). In contrast to the full term structure model comparison approach, our variance ratio test easily detects the internal consistency violation with variance ratios above two in all cases but one. The reason for the discrepancy between the two approaches is that our test is explicitly designed to detect the type of overreaction found in the data and built into the data generating process (DGP) for these simulations. The information criterion, on the other hand, relies on assessing the one-factor model solely based on its performance relative to the two-factor model. But, in this example, both models are misspecified, so the likelihood of rejecting the null is small. Of course, if the alternative specification matched the DGP, the BIC would always select the alternative over the null. In reality, the exact nature of the misspecification is unknown. The variance ratio test appears well suited to detect overreaction without the need to specify a particular alternative.

5.2 Linear Versus Exponential Representations

In modeling the market for volatility claims we have followed the literature in writing the payoff as a linear function of underlying factors. We now explore the robustness of our results to a common non-linear functional form. We study an alternative model for volatility claims
in which realized variance is modeled is a exponentially linear in the factors:

\[ RV_t = \exp(x_t), \quad x_t = \delta_0 + \delta_1 H_t \]

with \( H_t \) conditionally normally distributed and homoskedastic (we treat the heteroskedastic case below). In this case, the log price of a forward claim to one period of variance at time \( t+n \) is

\[ \ln p_{n,t} = E^Q_t[\exp(x_{t+n})] = 1 \rho^n H_t + \text{constant}. \]

We construct psuedo-cumulative claims whose prices are the sum of the log prices of the individual cash flows,

\[ \tilde{p}_{t,n} = \sum_{j=1}^{n} \ln p_{t,n} \]

These do not correspond to log prices of tradable cumulative contracts, but instead are a way to aggregate the log forward prices into a form for which our variance ratio tests are applicable.

Figure 16 reports variance ratios for cumulative variance swap prices when realized variance is assumed to be affine in levels as in Figure 1 (left panel) or affine in logs (right panel).\(^{30}\) The figure shows that there is little difference between variance ratio tests in the two contexts. In both cases, the null model is significantly rejected with variance ratios above 2.0 at long maturities.

### 5.3 Stability in Subsamples

One advantage of our test is that it only uses the comovement of prices to estimate and test the model. These covariances are rather precisely estimated even when a short time series is available, therefore our test can be conducted within rolling subsample windows. In Figure 17 we report variance ratio estimates in the variance swap market obtained for a four-year rolling window. Variance ratios at 24 months are far above one for the majority of the sample, and reach peaks of nearly 8 in some subperiods. This demonstrates first that our findings are robust to alternative data samples. Perhaps more importantly, it illustrates that our main results are unlikely to be driven by instability of the affine model. For rolling estimation windows as short as six months we find results quantitatively similar to our full sample estimates.

\(^{30}\)To construct the log prices of variance forwards, we interpolate the variance swap curve using a cubic spline. Our original test on variance swaps does not require any interpolation, though working with forwards does. The results of Figure 16 are robust to different methods of interpolation.
Note. The figure plots the VR at maturities 6 months, 12 months, and 24 months obtained using a rolling 4-year window for estimation and testing.

6 Conclusion

We document excess volatility for long maturity claims in a large cross-section of asset classes. Our tests of excess volatility exploit the strict overidentification restrictions from term structure asset pricing.

We use the short end of the term structure to learn the implied cash flow dynamics perceived by investors under the pricing measure $Q$. This delivers a model of expectations at all maturities that are linked by the law of iterated expectations and the implied dynamics of the factors driving the cash flows.

We find that prices of long-maturity claims are dramatically more variable than justified by standard models. This excess volatility cannot be explained by time variation in discount rates, which that is already accounted for by the risk-neutral expectations that we extract from the short end of the term structure. Our results therefore show that the excess volatility puzzle, first highlighted by Shiller (1981), cannot be fully resolved by rational variation in discount rates.
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A Model Identification and Estimation

In this appendix we show how to estimate the matrix $B$ of size $K \times K$ in a setting in which: 1) any $G \geq K$ maturities at the short end of the curve are observed and are used to construct the $K$ factors, and 2) the $G$ maturities observed are not necessarily consecutive (for example, one wants to extract $K = 2$ factor using maturities 1, 2, 4 or 1, 2, 4, 6). If $G > K$, the first $K$ principal components of the $G$ observed maturities are used as factors $H_t$ (which will still be a $K$-element vector). We proceed with the derivation assuming cumulative claims are used, but an equivalent derivation holds when using forwards.

We refer to the $G$ maturities observed at the short end of the curve as $n_1, \ldots, n_G$, and to the vector of those prices as $P_{t,G}$. We assume that each individual observed price in the term structure has potential measurement error:

$$p_{t,n} = 1' \left[ \rho + \rho^2 + \ldots + \rho^n \right] H_t + u_{t,n}$$

while the first $K$ principal components of $P_{t,G}$ are observed without error:

$$\bar{P}_t = f \cdot P_{t,G}$$

where $f$ is an $K \times G$ matrix selecting the first $K$ principal components of $P_{t,G}$ (referred to as $\bar{P}_t$). Naturally, this nests the case (studied in the paper) in which exactly the first $K$ maturities are observed without error and used as factors ($K = G$): in that case, $f$ is simply the identity matrix.

We can prove the following Proposition, that shows how to recover the matrix $\rho$ in this setting (and therefore in turn recover all loadings of long-term prices onto the short-end factors $\bar{P}_t$ under the model).

**Proposition 1.** Consider the regression of a price $p_{t,n_{G+1}}$ onto the factors $\bar{P}_t$:

$$p_{t,n_{G+1}} = d + c' \bar{P}_t + u_{t,n_{G+1}}$$

All eigenvalues $\rho_i$ of $\rho$ are among the roots of the polynomial equation

$$[1 + \rho_i + \ldots + (\rho_i)^{n_{G+1}-1}] = \tilde{c}_1 [1 + \rho_i + \ldots + (\rho_i)^{n_1-1}] + \ldots + \tilde{c}_G [1 + \rho_i + \ldots + (\rho_i)^{n_G-1}]$$

where all of the coefficients $\tilde{c}$ depend exclusively on the factor loadings $f$ and on the regression coefficients $c$.

**Proof.** Start by defining

$$S^n \equiv 1' (\rho + \rho^2 + \ldots + \rho^n)$$

$S^n$ is a $1 \times K$ vector that depends only on the diagonal matrix $\rho$. We can therefore write for each price:

$$p_{t,n} = S^n H_t$$
Note that calling $\rho_i$ the $i$th element of the diagonal of $\rho$, we can rewrite $S^n$ as:

$$S^n = \begin{bmatrix} \rho_1 + \ldots + \rho_1^n \\
\rho_2 + \ldots + \rho_2^n \\
\vdots \\
\rho_K + \ldots + \rho_K^n \end{bmatrix}$$

The assumption that the principal components $P_t$ are observed without error yields:

$$P_t = f \cdot \begin{bmatrix} S_{n1} \\
S_{n2} \\
\vdots \\
S_{nG} \end{bmatrix} H_t$$

Consider now the regression:

$$p_{t,nG+1} = d + c'P_t + u_{t,nG+1}$$

and project each side of the equation on $H_t$ (noting that $u_{t,nG+1}$ is orthogonal to $H_t$). The loadings on $H_t$ on the two sides of the equation must match. Therefore:

$$S^{nG+1} = c' \begin{bmatrix} S_{n1} \\
S_{n2} \\
\vdots \\
S_{nG} \end{bmatrix} = \tilde{c} \begin{bmatrix} S_{n1} \\
S_{n2} \\
\vdots \\
S_{nG} \end{bmatrix}$$

where the last equality is obtained by defining $\tilde{c} = c'f$, a $1 \times H$ vector that depends only the factor loadings $f$ and the regression coefficients $c$. We can then write:

$$S^{nG+1} = \tilde{c}_1 S_{n1} + \tilde{c}_2 S_{n2} + \ldots + \tilde{c}_G S_{nG}$$

Now, given that as shown above each element $i$ of $S^n$ depends only on element $i$ of the diagonal of $\rho$, this is a system of $K$ independent equations, each of the form:

$$[\rho_i + \rho_i + \ldots + \rho_i^{nG+1-1}] = \tilde{c}_1 [\rho_i + \rho_i + \ldots + \rho_i^{n-1}] + \ldots + \tilde{c}_G [\rho_i + \rho_i + \ldots + \rho_i^{n-1}]$$

Finally, we can divide by $\rho_i$ throughout (assuming $\rho_i \neq 0$) and obtain:

$$[1 + \rho_i + \ldots + (\rho_i)^{nG+1-1}] = \tilde{c}_1 [1 + \rho_i + \ldots + (\rho_i)^{n-1}] + \ldots + \tilde{c}_G [1 + \rho_i + \ldots + (\rho_i)^{n-1}]$$

Note that each element $i$ of $\rho$ needs to satisfy this equation: the matrix $\rho$ can therefore be computed by finding the roots of this polynomial equation. This structure has the convenient feature that we can estimate state dynamics from the yields without any maximization (as is typical in term structure models).
Once $\rho$ has been recovered, we can construct $S^n$ for each maturity $n$. Since

$$P_t = \overline{S}H_t$$

where

$$\overline{S} = f \begin{bmatrix} S^{n_1} \\ S^{n_2} \\ \vdots \\ S^{n_G} \end{bmatrix}$$

is a $K \times K$ matrix, we can write:

$$H_t = \overline{S}^{-1}P_t$$

Therefore, we can also write

$$p_{t,n} = S^n\overline{S}^{-1}P_t + u_{t,n}$$

The matrix of loadings on the “observable factors” $P_t$ is therefore $S^n\overline{S}^{-1}$. These factors can be used to construct a variance ratio test that compares the variance of the component of $p_{t,n}$ predicted (in unrestricted regressions) by the factors $P_t$ to the variance predicted under the model (with coefficients $S^n\overline{S}^{-1}$).

One final consideration is that there will generally be $n_{G+1} - 1$ roots of this polynomial (some of them potentially complex or explosive), while we only seek $K$ parameters. This equation shows that the $Q$ dynamics and the comovements of prices only identify the eigenvalues of $\rho$ up to the set of roots of this polynomial. It does not tell us which roots to choose, as they imply the same covariance among prices (while a full MLE procedure that exploits both information about the $P$ and the $Q$ dynamics will be able to choose among them). Of course, in our baseline case, where we only select the first $K$ prices as factors, we will always have as many roots as parameters ($K$).

We use the following selection procedure for the roots. First, we only consider non-explosive roots. This is motivated by the unambiguous empirical fact that price variances are concave in maturity for all the markets we study, especially at the short end of the curve where our estimation is coming from. If prices rise less than linearly with horizon, the system is best described by stationary dynamics. Second, among the non-explosive roots, we select the $K$ most persistent ones. This ensures that our excess volatility findings will be the most conservative (they will suggest the least excess volatility) of all of the covariance-equivalent roots we could have reported. Finally, following the term structure literature, we choose real roots whenever possible.

## B Model Testing

### B.1 Asymptotic Distribution of Variance Ratio Statistic

To simplify the notation, in this section let us refer to $\rho_v$ as a $K$-vector corresponding to the diagonal of the transition matrix of the factor $\rho$ (so $\rho_v = diag(\rho)$). Let $P_t$ be the vector of
$K$ prices observed with no error, with maturities $n_1, ..., n_K$. Then we can write:

$$P_t = P'H_t$$

with $P$ a matrix that depends only on $\rho_v$ (in the case of forwards, a matrix with row $k$ equal to $\rho$ raised element-wise to the $n_j$ power).

Consider now two additional maturities, $n_s$ and $n_l$, with $n_l > n_s > n_K$. The price of the claim with maturity $n_s$ is used to estimate the vector $\rho_v$. Maturity $n_l$ is then used to test the model. The variance ratio for maturity $l$, therefore, depends on i) the estimate of the projection of $p_{l,t}$ onto the vector of factors $P_t$ (maturities $n_1$ to $n_K$) and ii) the estimate of the projection of $p_{s,t}$ onto $P_t$ which estimates the value of $\rho_v$ used when imposing model restrictions.

The (unrestricted) model at each maturity is

$$p_{t,n_s} = b_{n_s}'P_t + \varepsilon_{t,n_s} \quad (26)$$
$$p_{t,n_l} = b_{n_l}'P_t + \varepsilon_{t,n_l} \quad (27)$$

and, under the restrictions of the null model, we have

$$p_{t,n_s} = f_{n_s}(\rho_v)P_t + \varepsilon_{t,n_s} \quad (28)$$
$$p_{t,n_l} = f_{n_l}(\rho_v)P_t + \varepsilon_{t,n_s} \quad (29)$$

where $f$ is the function that maps $\rho_v$ to the loadings on $P_t$ under the null model’s no-arbitrage restrictions.

We make the following assumptions:

Assumptions 1: $E[\varepsilon_{i,t}|x] = 0$

Assumption 2:

$$Cov(\varepsilon_{1,t}, \varepsilon_{2,s}) = \begin{cases} 
\Sigma & \text{if } s = t \\
0_{2x2} & \text{oth}
\end{cases}, \quad \text{where } \Sigma = \begin{pmatrix} \sigma_1^2 & \sigma_{12} \\ \sigma_{12} & \sigma_2^2 \end{pmatrix}.$$

Under the null we have

$$\sqrt{T} \left( \begin{array}{c} \hat{b}_{n_s} - f_{n_s}(\rho) \\ \hat{b}_{n_l} - f_{n_l}(\rho) \end{array} \right) \sim N(0, \Sigma \otimes V_P^{-1})$$

where $V_P = E[P_tP_t']$.

Our variance ratio test is a non-linear hypothesis test based on parameter estimates $(\hat{b}_{n_s}, \hat{b}_{n_l})$. Because we know the joint asymptotic distribution, we can perform such tests using the delta method. Calling the non-linear function of interest $g(b_{n_s}, b_{n_l})$, then we have

$$\sqrt{T} \left[ g(\hat{b}_{n_s}, \hat{b}_{n_l}) - g(b_{n_s}, b_{n_l}) \right] \rightarrow N(0, \nabla g'[\Sigma \otimes V_x^{-1}] \nabla g).$$

31The derivation that follows therefore does not require consecutive maturities. It also easily extends to the case where principal components of a set of prices is observed without error, and it also extends to the case where cumulative prices are used, and not forwards.
In our case, the function we study depends on the variance of the projection of $p_{t,n}$ on $P_t$. We want to compare $V(p_{t,n} | P_t)$ based on the estimate $\hat{b}_{nt}$ to $V(p_{t,n} | P_t)$ based on $\hat{b}_{ns}$. Under the null model, the ratio of these variances is

$$g(\hat{b}_{ns}, \hat{b}_{nl}) = VR = \frac{\hat{b}_{nl}'V_x\hat{b}_{nl}}{f(\hat{b}_{ns})'V_xf(\hat{b}_{ns})}$$

where $\tilde{f}(b_{ns})$ is the appropriate transformation of short maturity coefficients to long maturity coefficients under the null (i.e., that inverts $\hat{\rho}$ from $\hat{b}_{ns}$, and uses this to construct $f_{nl}(\hat{\rho})$). Calling $\tilde{f}'(b_{ns})$ its derivative, we can write the gradient $\nabla g$ as:

$$\nabla g = \begin{pmatrix} -2VRV_{P}\tilde{f}\tilde{f}'V_{P}\tilde{f} \\ 2V_{P}\hat{b}_{nl}\tilde{f}'V_{P}\tilde{f} \end{pmatrix}$$

### B.2 More General Error Dependence

The assumptions stated above allow for dependence in “noise” in the cross section of yields. But they don’t handle serial dependence, which may be present in this data.

A more general estimation of the variance of $b_{ns}$ is

$$V(b_{ns}) = E[(P'tP_t)^{-1}P't\varepsilon_{ns}P_t'(P'tP_t)^{-1}]$$

$$= E[(P'tP_t)^{-1}(\sum_{s,t}P't,nsP't,ns\varepsilon_{ns,t}\varepsilon_{ns,s})X(X'X)^{-1}]$$

Under our Assumption 2, the sum terms in the variance of $b_{ns}$ are zero if $s \neq t$, which is why in the variance reduces to $V(\varepsilon_{ns,t})(X'X)^{-1}$ under this assumptions.

Let $S$ denote the middle term $S = E[X'\varepsilon_{ns,t}X]$. Instead of calculating it as

$$\hat{S} = \Gamma_0 = \sum_t x_t x_t' \varepsilon_{ns,t}^2$$

we can allow for time series lags to enter into the covariance calculation,

$$\hat{S} = \Gamma_0 + \sum_{j=1}^{J} (\Gamma_j + \Gamma_j')$$

where

$$\Gamma_j = \hat{S} \sum_{t=j+1}^{T} x_t x_{t-j}' \varepsilon_{ns,t} \varepsilon_{ns,t-j}.$$ 

This only delivers the (1,1) block of the variance matrix in equation (30). For the (2,2) block, the above analysis is repeated using $\varepsilon_{ni}$. The (1,2) and (2,1) blocks comes from using
both $\varepsilon_n$ and $\varepsilon_n'$, estimating

$$\text{Cov}(b_n, b_{n'}) = E \left[(X'X)^{-1}X'\varepsilon_n \varepsilon'_n X(X'X)^{-1}\right].$$

C Risk-free Rate Variation, Heteroskedasticity and Other Considerations

In this section we consider in greater detail some additional theoretical consideration that may play a role in our analysis, including the role of interest rate variation, heteroskedasticity, and measurement error.

C.1 Stochastic Risk-free Rates

For many of the asset classes considered in this paper, time variation in the risk-free rate plays a minor role in determining the volatility of prices along the term structure, and is typically ignored in the literature (for example, Ait-Sahalia, Karaman and Mancini (2014) ignore risk-free rate variation when pricing variance swap).

For other asset classes, interest rate variation plays a more important role. Here we show that in exponential-affine models where not only log cash flows $x_t$ but also short-term rates $r_t$ are linear functions of the factors, our test is valid even in the presence of (unmodeled) stochastic interest rates. Consider in particular a cumulative contract that pays all the cash flows at maturity, and has an upfront payment of the price. Then, we can write the price as:

$$p_{t,n} = E_t^Q \left[ e^{x_{t+1} + \ldots + x_{t+n}} \right] = E_t^Q \left[ e^{y_{t+1} + \ldots + y_{t+n}} \right]$$

(31)

where $y_t = x_t - r_{t-1}$. If $y_t$ is a linear function of the factors (for example because $x_t$ and $r_{t-1}$ are driven by the same factors), we can simply see this price as a claim to risk-free-adjusted cash flows $y_t$. Finally, remember that all of our analysis doesn’t require us to actually observe the cash flow (in this case $y_t$): it is enough to know that the price is determined according to an exponential-affine model in some cash flow $y_t$.

The argument also holds when all payments are exchanged at maturity, since in that case

$$p_{t,n} = E_t^Q \left[ \frac{e^{x_{t+1} + \ldots + x_{t+n}}}{e^{r_{t+1} + \ldots + r_{t+n}}} \right]$$

which means that we can construct the price $\tilde{p}_{t,n} = p_{t,n} \delta_{t,n}$, where $\delta_{t,n}$ is the price of a risk-free bond with maturity $n$, and the adjusted price $\tilde{p}_{t,n}$ will have the same form as (31).

C.2 Heteroskedasticity adjustment in exponential-affine models

The exponential-affine model for volatility described in Section 5.2 also allows us to understand the effects of stochastic volatility on the model-predicted factor loadings (remember
that stochastic volatility is inconsequential for the test when modeling volatility in a linear framework).

Below we derive the model-predicted factor loadings in the exponential-affine model for volatility when the conditional variance of the factors is assumed to be proportional to the one-period price (the VIX), capturing the intuition that as the VIX increases, future fluctuations in variance will be more pronounced:

\[ V_t(H_{t+1}) = \Gamma_t \Gamma'_t = \Gamma \Gamma' \sigma_t^2 \]

with:

\[ \sigma_t^2 = a * f_{1,t} \]

for \( a > 0 \) a proportionality constant. In this model, the loadings of log forward prices \( f_{t,n} \) on the factors follow:

\[ b_1 = 1' \rho + \frac{1}{2} a (1' \Gamma \Gamma' 1) 1' \rho \]

\[ b_{n+1} = b_n' \rho + \frac{1}{2} a (b_n' \Gamma \Gamma' b_n') 1' \rho \]

Given that empirically \( a > 0 \) (the volatility of volatility increases when the VIX is high), it immediately follows that the heteroskedasticity adjustment will slow down the decay of factor loadings as the maturity increases. Potentially, this effect can generate higher factor loadings at higher maturities under the null, thus reducing the heteroskedasticity-corrected variance ratios. It is therefore important to quantify the magnitude of this adjustment.

Below we discuss in detail how the adjustment term at each maturity \( n \), \( \frac{1}{2} a (b_n \Sigma \Sigma b_n') 1' \rho \), can be estimated by regressing the conditional variance of \( f_{t,n} \) on \( f_{1,t} \). We can then use these estimated adjustment terms to study how the factor loadings \( b_n \) change once we account for heteroskedasticity. Figure A18 reports the loadings of log cumulative variance swap prices, \( p_{t,n} \), onto the first two prices, in the null model with and without heteroskedasticity adjustment, as well as the unrestricted loadings. The figure shows that quantitatively the heteroskedasticity adjustment has only a minor effect on the loadings on the two factors.

### C.2.1 Derivation and estimation

We consider here the case of forward claims on a cash flow \( \exp\{x_t\} \), where \( x_t \) is linear in the factors. Assume that \( \mathbb{P} \) dynamics follow:

\[ H_{t+1} = c + \rho^P F_t + \Gamma_t \epsilon_{t+1} \]

\[ x_t = \delta_0 + 1' H_t \]

The one-period stochastic discount factor follows:

\[ M_{t,t+1} = \exp(-r_t - \frac{1}{2} \lambda'_t \lambda_t - \lambda'_t \epsilon_{t+1}) \]

The term \( \Gamma_t \) captures stochastic conditional volatility of the factors; we specify the exact assumptions about the dependence of \( \Gamma_t \) on time-t information below.
Figure A18: Variance Swap Loadings (Homoskedastic vs Heteroskedastic Model)

**Note.** The figure plots the loadings of prices of each maturity on the two factors (1-month and 2-month price). Dashed lines indicate loadings in the unrestricted model, solid lines indicate loadings in the restricted model. The thick line reports the coefficients under the homoskedasticity assumption, the thin line adjusts for heteroskedasticity.

For any forward asset on a cash flow $x_t$, with maturity $n + 1$, we have the recursive equation:

$$f_{t,n+1} = E_t[exp\{-r_t - \frac{1}{2}\lambda_t'^2 - \lambda_t'^\epsilon_{t+1} + f_{t+1,n}\}]$$

Now, we conjecture that the forward price is an exponentially-affine function of the factors:

$$f_{t,n+1} = exp\{a_{n+1} + b_{n+1}H_t\}$$

Taking logs:

$$a_{n+1} + b_{n+1}H_t = lnE_t[exp\{-r_t - \frac{1}{2}\lambda_t'^2 - \lambda_t'^\epsilon_{t+1} + a_n + b_nH_{t+1}\}]$$

$$= lnE_t[exp\{-r_t - \frac{1}{2}\lambda_t'^2 - \lambda_t'^\epsilon_{t+1} + a_n + b_n(c + \rho P H_t + \Gamma \epsilon_{t+1})\}]$$

$$= -r_t - \frac{1}{2}\lambda_t'^2 - \frac{1}{2}\lambda_t'^2 + a_n + b_n(c + \rho P H_t) + 1.5V_t((-\lambda_t'^2 + b_n \Gamma \epsilon_{t+1}))$$

$$= -r_t - \frac{1}{2}\lambda_t'^2 + a_n + b_n(c + \rho P H_t) + \frac{1}{2}\lambda_t'^2 + \frac{1}{2}b_n \Gamma \epsilon_{t+1} - b_n \Gamma \lambda_t$$

65
\[= -r_t + a_n + b_n c + b_n \rho^p H_t + \frac{1}{2} b_n \Gamma_t \Gamma_t b_n' - b_n' \Gamma_t \lambda_t \]

For the very first maturity (i.e. \(f_{t,1}\)), we have:

\[a_1 + b_1 H_t = \ln E_t \exp \{-r_t - \frac{1}{2} \lambda_t' \lambda_t - \lambda_t' \epsilon_{t+1} + x_{t+1}\} \]

\[= \ln E_t \exp \{-r_t - \frac{1}{2} \lambda_t' \lambda_t - \lambda_t' \epsilon_{t+1} + \delta_0 + 1'[c + \rho^p H_t + \Gamma_t \epsilon_{t+1}]\} \]

\[= -r_t + \delta_0 + 1'c + 1' \rho^p H_t + \frac{1}{2} 1' \Gamma_t \Gamma_t 1 - 1' \Gamma_t \lambda_t \]

In both expressions for \(n = 1\) and for \(n > 1\), we have the terms \(\Gamma_t \Gamma_t'\) and \(\Gamma \lambda_t\) that are functions of time-\(t\) information. To find an exponentially-affine solution, these terms need to be linear in the factors. Following the term structure literature, we assume that \(\Gamma_t \Gamma_t'\) is linear in \(H_t\) (which makes the term \(b_n \Gamma_t \Gamma_t b_n'\) also linear in \(H_t\)). In particular, we assume that:

\[V_t(H_{t+1}) = \Gamma_t \Gamma_t' = \Gamma_t' \sigma_t^2 \]

with:

\[\sigma_t^2 = a * f_{1,t} \]

for some \(a > 0\).

\(\lambda_t\) is assumed to follow:

\[\lambda_t = \Gamma_t^{-1} \Gamma(\lambda + \Lambda H_t)\]

This makes the term \(\Sigma_t \lambda_t\) also linear in \(H_t\). In addition, if the risk-free rate is \(r_t = a_0 + a_1 H_t\), the term \(a_1\) would also enter the recursion for \(b_n\). In what follows, we ignore risk-free rate variation as it plays a minor role in the pricing of variance swaps.

We can rewrite the expressions under \(Q\), using the same normalizations we have used in our main analysis: \(\rho \equiv \rho^p - \Gamma \lambda\) (the VAR companion matrix under \(Q\)) is diagonal, and \(c^Q \equiv c - \lambda \Gamma = 0\). We can then rewrite the expressions as:

\[a_{n+1} + b_{n+1} H_t = a_n + b_n c + b_n \rho^p H_t + \frac{1}{2} b_n \Gamma \sigma_t^2 b_n' - b_n' \Gamma (\lambda + \Lambda H_t) \]

\[a_1 + b_1 H_t = \delta_0 + 1' c + 1' \rho^p H_t + \frac{1}{2} 1' \Gamma \sigma_t^2 1 - 1' \Gamma (\lambda + \Lambda H_t) \]

or:

\[a_{n+1} + b_{n+1} H_t = a_n + b_n \rho H_t + \frac{1}{2} b_n \Gamma b_n' \sigma_t^2 \]

\[a_1 + b_1 H_t = \delta_0 + 1' \rho H_t + \frac{1}{2} 1' \Gamma \sigma_t^2 1 \]

Now, recall that \(\sigma_t^2 = a * f_{1,t} = a 1' \rho H_t\). The expressions then become:

\[a_{n+1} + b_{n+1} H_t = a_n + b_n \rho H_t + \frac{1}{2} b_n \Gamma b_n'(a 1' \rho H_t) \]
\[ a_1 + b_1 H_t = \delta_0 + 1' \rho H_t + \frac{1}{2} 1' \Gamma \Gamma' 1(a 1' \rho H_t) \]

We can now match coefficients on \( H_t \), and obtain:

\[ b_1 = 1' \rho + \frac{1}{2} 1' \Gamma \Gamma' 1(a 1' \rho) \]

\[ b_{n+1} = b_n \rho + \frac{1}{2} b_n \Gamma b_n' (a 1' \rho) \]

To learn about the magnitude of the coefficient adjustments \( \frac{1}{2} 1' \Gamma \Gamma' 1(a 1' \rho) \) and \( \frac{1}{2} b_n \Gamma b_n' (a 1' \rho) \), we proceed as follows. First, note that the conditional variance of the log cash flow in the model is (up to a constant):

\[ V_t(x_{t+1}) = 1' \Gamma \Gamma' 1(a 1' \rho H_t) = 1' \Gamma \Gamma' 1 a f_{t,1} \]

Therefore, regressing \( V_t(x_{t+1}) \) onto \( f_{t,1} \) would yield an estimate of the term \( 1' \Gamma \Gamma' 1 a \). This would allow us to estimate the heteroskedasticity adjustment for \( b_1 \). Next, consider the conditional variance of the first log price (from the left-hand side of the equations above):

\[ V_t(f_{t+1,1}) = b_1' \Gamma \Gamma b_1 = b_1' \Gamma \Gamma b_1 a f_{t,1} \]

The regression coefficient of \( V_t(f_{t+1,1}) \) onto \( f_{t,1} \) yields an estimate of \( b_1' \Gamma \Gamma b_1 a \), which we can use to adjust the coefficient \( b_2 \) for the effects of conditional volatility. Continuing the recursion, this allows us to compute the adjustment for all maturities.

Two final notes on the implementation. First, the most natural way to implement the conditional variance regression is to regress the monthly realized volatility of each variable \( (x_t, f_{t,1}, f_{t,2}, \text{and so on, computed as the sum of changes in log prices during the month}) \). While we don’t observe high-frequency data on realized volatility \( x_t \) within a month, we can use the realized volatility of \( f_{t,1} \) as a proxy. Second, log realized volatilities for maturities above 12 are very noisy, due to the interpolation-induced errors. We therefore apply the regression coefficients estimated for maturity 12 to all higher maturities. This procedure is conservative because the coefficients of this regression appear to be strongly decreasing with maturity (so after maturity 12 they should be even lower than those observed at maturity 12); in addition, the theory predicts that they should be decreasing as maturity increases, since the overall volatility of forwards should converge to zero as maturities increase.

As a robustness test, we also compute the volatility adjustments by using the the month-to-month squared change in price as left-hand side variable as opposed to the within-month realized volatility. The results are essentially identical.

C.3 Measurement Error

In the theoretical setting of Section 2, the prices derived in Equations (6) and (9) show that the value of a claim at any maturity is representable as an exact, error-free linear function of prices of claims at other points on the term structure.

As Piazzesi (2010) notes, observed prices may not perfectly represent the theoretical expectations of investors, but instead may also include “measurement errors” that arise
from data entry errors, building price series from multiple (and potentially asynchronous) data sources, vendors that interpolate data to fill in missing prices, etc. In the context of US treasury yields, Cochrane and Piazzesi (2005) find evidence indicating that the data indeed contain patterns that are a signature of measurement error.

Measurement error potentially influences our parameter estimates and test statistics. We address this using errors-in-variables methods to conduct tests that are robust to measurement error. The resulting tests produce nearly identical findings to those in Section 3, indicating that measurement error is not responsible for the long-maturity excess volatility that we document.
We use a tilde to represent error-ridden observable prices\footnote{Piazzesi (2010), in Section 6, gives an excellent overview of model specification choices when affine term structures are subject to measurement error. If we assume, as is often the case in affine models, that the first \( K \) prices are perfectly observed and only maturities \( K+1 \) to \( N \) are subject to errors, then our baseline estimator in Section 2 remains consistent, and \( p \)-values of our test retain appropriate size due to our bootstrap standard errors.}

\[
\tilde{f}_{t,j} = f_{t,j} + v_{t,j}, \quad j = 1, \ldots, N.
\]

The variance ratio test in Equation (13) depends crucially on coefficient estimates in restricted on unrestricted projections of the error-free long-maturity price \( f_{t,K+j} \) onto error-free short-maturity prices \( F_{t,1:K} \).\footnote{The specific form of the weighting matrix, which take the value \( \hat{\Sigma}_{1:K} \) Equation (13), appears in both the numerator and denominator and is thus not crucial for consistency of the test.} When the error-free projections are infeasible due to noise in prices, consistency of the variance ratio test faces two obstacles. First, we require a consistent estimate of the unrestricted coefficient based on noisy data \( \tilde{f}_{t,K+j} \) and \( \tilde{F}_{1:K} \) (in analogy with Equation (11)) in order to calculate the numerator of the variance ratio. Second, we need a consistent estimate of the restricted coefficient to construct the denominator of the variance ratio (which relies on \( b \) in Equation (7)) but that is based on the noisy short-end prices \( \tilde{f}_{t,K+1} \) and \( \tilde{F}_{1:K} \).

Instrumental variables (IV) methods are a common means of consistently estimating a regression coefficient when the independent variable is observed with error. For example, suppose the affine model has a single factor and the errors are uncorrelated across maturities. In this case, \( b \) is consistently estimated by an IV regression of \( \tilde{f}_{t,2} \) on \( \tilde{f}_{t,1} \), using any other price \( \tilde{f}_{t,j} \) at maturity \( j > 2 \). Because the errors are uncorrelated across maturities, \( \tilde{f}_{t,j} \) is a valid instrument for the noisy dependent variable \( \tilde{f}_{t,1} \). By the same rationale, the unrestricted long-end projection coefficient can be consistently estimated as well. Given consistent estimates of \( b \) and the long maturity unrestricted coefficient, the variance ratio test will be consistent. The only qualitative difference versus Equation (13) is that the weighting matrix will be replaced with an estimate of short maturity noisy price variance, \( \text{Var}(\tilde{F}_{t,1:K}) \).

In practice, however, it is quite likely that measurement errors are substantiably correlated across maturities, so the strategy of instrumenting with other maturities in the same term structure fails to satisfy the exclusion restriction. It is much less likely that measurement errors would be correlated across different term structures. We therefore use prices from different but related term structures as instruments to help resolve potential inference problems due to errors-in-variables bias.

As a first example, we revisit the two-factor affine model for the terms structure of Apple’s variance claims studied in Figure 4. If prices are measured with error, then we must instrument the regressions of \( \tilde{f}_{t,j}^{\text{Apple}} \) on \( \tilde{F}_{t,1:2}^{\text{Apple}} \) (for \( j = 3, \ldots, 24 \)). As instruments, we use short-end prices of claims to IBM variance, \( \tilde{F}_{t,1:2}^{\text{IBM}} \). This approach is valid under the conditions that true, error-free short prices \( F_{t,1:2}^{\text{Apple}} \) and \( F_{t,1:2}^{\text{IBM}} \) are correlated between the different term structures but the errors \( v_{t,1:2}^{\text{Apple}} \) and \( v_{t,1:2}^{\text{IBM}} \) are not. Indeed, the volatility
of individual stocks tend to exhibit strong cross correlation, but there is no obvious reason to suspect that errors in the measurement of their prices are correlated.

The variance ratio test results for this example are plotted in the left panel of Figure A19. Test statistics based on the IV adjustment are nearly identical to those in the baseline estimation. The same is true if we instrument the variance swap term structure tests using implied volatilities of the S&P 500 options (second panel), and if we instrument the Russian CDS term structure with Brazilian CDS spreads (third panel). In all cases, values of instrumented test statistics are quantitatively similar to those in Section 3, suggesting that our main findings cannot be explained by measurement error.

D Data Details and Asset-specific Modeling Considerations

In this section we show how each asset class considered maps into our linear or loglinear framework.

D.1 Variance Swaps and Related Variance Derivatives

As discussed in the text, the price of a variance swap follows:

\[ p_{t,n} = E_t^Q \left[ \sum_{j=1}^{n} RV_{t+j} \right] \]

We then model \( RV_t \) as a linear function of the factors, which immediately yields:

\[ p_{t,n} = a_n + b_n' H_t \]

(32)

An attractive feature of the simple payoff structure of variance swaps is that dependence of prices on factors, \( b_n' H_t \), is robust to many modifications of the factor model. For example, because the swap price is the expected value of the level of \( RV_t \), having both prices and payoffs linear in the factors no longer requires Gaussianity. Any shock distribution with constant means implies the pricing structure in (32).

One important consideration to keep in mind is that because variances are non-negative, a homoskedastic linear Gaussian model is an imperfect description of \( RV_t \). Stochastic variance is a standard feature in the bond and option pricing literatures, and a number of solutions exist that ensure positive variances. The most common solution is to use a CIR volatility process. In these models, the model innovations remain standard normal, but are multiplied

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34See Kelly, Lustig and Van Nieuwerburgh (2013) and Herskovic et al. (2014).

35In the case of Russian CDS, the standard errors of the instrumented statistics are much larger than our estimates based on OLS, which is likely due to the fact that Russian and Brazilian CDS spreads share a much lower correlation than, for example, Apple and IBM implied volatilities.

36We ignore risk-free rate variation, since its volatility and correlation with the variance swap payoff are small, following Ait-Sahalia, Karaman and Mancini (2014), Egloff, Leippold and Wu (2010), Dew-Becker et al. (2015).
by a volatility that scales with the factors (and hence with the level of volatility). The modified model takes the general form\textsuperscript{37}

\[ H_t = \rho H_{t-1} + \Sigma_{t-1} u_t \]

where \( \Sigma_{t-1} \) is a constant function of \( H_{t-1} \). When the model is specified at a high enough frequency (going to continuous time in the limit), and assuming appropriate Feller conditions for the model parameters (see Dai and Singleton (2002)), the probability of variance going below zero tends to zero.

Note that this stochastic volatility case only affects the scale of the innovation \( u_t \). Therefore, the expected level payoff in is unaffected, hence equation (32) is also unaffected. Different versions of this model are applied by Ait-Sahalia, Karaman and Mancini (2014), Egloff, Leippold and Wu (2010), Dew-Becker et al. (2015).

As discussed in the text, in some of our tests we take ATM implied variance as a proxy for the risk-neutral expected variance. This is motivated by practical considerations (measurement of ATM volatility for many asset classes is more stable than measurement of the synthetic variance swap price, \( VIX^2 \)), and by the theoretical result of Carr and Lee (2009) that to a first-order approximation, ATM implied volatility corresponds in fact to the price of a volatility swap (the claim to realized volatility). In these cases, we use the same model as for variance swaps, but consider \( p_{t,n} = IV^n_t \), where \( IV^n_t \) is the \( n \)-maturity implied variance.

Our variance swap data comes from two industry sources, both described in Dew-Becker et al. (2015). Our implied variance series are obtained from Optionmetrics (equity derivatives) and JP Morgan (currency IV).

D.2 Treasuries

Our development of the exponential-affine model for interest rates follows Hamilton and Wu (2012), who study the class of Gaussian affine term structure models developed by Vasicek (1977), Duffie, Kan et al. (1996), Dai and Singleton (2002), and Duffee (2002), and studied by many others.

In the Gaussian affine term structure model, bonds are claims on short-term interest rates. One-period log risk-free rate \( x_t \) is a linear function of the factors with factor dynamics under the pricing measure described by a VAR, just as in our main set-up. The price of a risk-free bond that pays $1 after \( n \) periods is

\[ P_{t,n} = E^Q \left[ \exp \left( - \sum_{j=1}^{n} x_{t+j} \right) \right]. \]  

We assume that factor shocks are homoskedastic \( \Sigma_t = \Sigma \) following Hamilton and Wu (2012),\textsuperscript{37}

\[ \text{For infinitesimal time intervals, the variance may be constructed to maintain strictly positive variance while retaining the Gaussianity of factor innovations, } u_t. \text{ In discrete time, this heteroskedastic Gaussian process does not perfectly rule out negative variances, but may be constructed to do so with probability arbitrarily close to one.} \]
which implies that the log bond price is

\[ p_{t,n} \equiv \log P_{t,n} = a_n + b_n H_t. \]

The factor loading depends only on the persistence of the factors:

\[ b_n = 1'(I + \rho + \ldots + (\rho)^{n-1}). \]  

(34)

The intercept is an inconsequential constant function of remaining model parameters, and drops out from all variance calculations.

### D.3 Credit Default Swaps

To model CDS spreads, we apply the reduced-form modeling of Duffie and Singleton (1999), in which the price of a defaultable bond is written in terms of a default intensity process \( \lambda_t \) and a process of loss given default \( L_t \). The precise relationship between the price of the bond at time \( t \), \( P_t \), and the processes for \( \lambda_t \) and \( L_t \) does not directly map into our general framework of Section 2.

However, Duffie and Singleton (1999) show that under the assumption of fractional recovery of market value in case of default, the price of a defaultable zero-coupon bond can be written as:

\[ P_{t,n} = E_t^Q \left[ \exp \left( - \int_t^T R_s ds \right) \right] \]

with

\[ R_s = r_s + \lambda_s L_s \]

where \( h_t \) is the default intensity and \( L_t \) the loss given default. The defaultable bond can be modeled as a default-free bond with a default-adjusted interest rate. We assume that:

1. \( r_s \) and \( \lambda_s L_s \) are linear in the factors; 2. underlying factors are homoskedastic; and 3. coupons on the underlying defaultable bonds are small enough (relative to the default-adjusted interest rate) so that the yield of an n-maturity defaultable bond with coupon is close to an n-maturity zero-coupon defaultable bond. We can then write:

\[ p_{t,n} = \log(P_{t,n}) = -ny^n_t = (a^n_r + a^n_{\lambda L}) + (b^n_r + b^n_{\lambda L}) H_t \]

while for the default-free bond (with log yield \( y^F \)) we have:

\[ -ny^F_t = a^n_r + b^n_r H_t \]

To link the bond price to the observed CDS spread, we start from the approximate bond-CDS basis relation, that states

\[ Z^n_t \approx Y^n_t - Y^n_{F,t} \]

i.e. the CDS spread \( Z^n_t \) with maturity \( n \) is approximately equal to the yield of the bond \( Y^n_t \) of that maturity in excess of the corresponding risk-free rate \( Y^n_{F,t} \) with the same maturity.
Given that both $Y_{t}^{n}$ and $Y_{F,t}^{n}$ are close to zero, we can write the yield spread to a first-order approximation as:

$$Y_{t}^{n} - Y_{F,t}^{n} \simeq \log(1 + Y_{t}^{n}) - \log(1 + Y_{F,t}^{n}) = y_{t}^{n} - y_{F,t}^{n}$$

so that:

$$nZ_{t}^{n} \simeq n(y_{t}^{n} - y_{F,t}^{n}) = -a_{\lambda L}^{n} - b_{\lambda L}^{n} F_{t}$$

This representation links the observed CDS spread allows us to focus on the cross-section of CDS spreads stripped of the risk-free rate dynamics, which will highlight the factor structure in default risk.

D.4 Inflation Swaps

Inflation swaps are claims to future inflation where the the buyer commits to pay a prede-termined amount $(1 + p_{t,n})^{n} - 1$ and receives $[I(t + n)/I(t)] - 1$, where $I(t)$ is the price level index. Risk-neutral pricing implies:

$$(1 + p_{t,n})^{n} - 1 = E_{t}^{Q} \left[ \frac{I(t + n)}{I(t)} - 1 \right]$$

Calling $\pi_{t} = \Delta \ln I(t)$, and moving to continuous time, we can write:

$$P_{t,n} = e^{p_{t,n}^{n}} = E_{t}^{Q} \left[ \exp(\int_{t}^{t+n} \pi_{s} ds) \right]$$

Just as in the case of bonds, we will have that log cumulative prices $n \cdot p_{t,n}$ will be linear in the factors:

$$n \cdot p_{t,n} = a_{n} + b_{n} H_{t}$$

D.5 Inflation Expectations

The Survey of Professional Forecasts publishes quarterly forecasts of inflation at different horizons, for the upcoming 4 quarters (starting in 1981). Starting in 1991, the SPF also publishes 10-year forecasts. Calling $\pi_{t}$ the log continuously compounded inflation rate during quarter $t$, we observe forecasts

$$1 + f_{t,1} = E_{t}[\exp(\pi_{t+1})]$$

up to:

$$1 + f_{t,4} = E_{t}[\exp(\pi_{t+4})]$$

In addition, we observe the 10-year (40 quarters) expected inflation rate

$$1 + p_{t,40} = E_{t}[\exp(\pi_{t+1} + ... + \pi_{t+40})]$$

More recently, in 2005, the SPF started reporting also 5-year forecasts; we do not use this additional data here because the sample, at the quarterly frequency, becomes extremely short.
Ignoring Jensen inequality effects within the first year, we construct cumulative forecasts up to quarter 4 as:

\[ 1 + p_{t,1} = 1 + f_{t,1} \]
\[ 1 + p_{t,2} = E_t[exp(\pi_{t+1} + \pi_{t+2})](1 + f_{t,1})(1 + f_{t,2}) \]

and so on. We model \( \pi_t \) as linear in factors, so that forecasts are loglinear in the factors; we therefore perform our estimation using log cumulative forecasts \( \ln(1 + p_{t,1}), \ldots \), and so on. Finally, we use median forecasts throughout the analysis.

E Missing Factors: Empirical Evidence

The following table reports robustness checks varying the number of factors, \( K \). For each term structure, the middle number of factors is the number used in our baseline analysis. We compare these results to tests that include one additional or one less factor. For each choice of \( K \) we report the term structure panel \( R^2 \) along with variance ratios and their bootstrap \( p \)-values at various maturities.
Table A6: Robustness: Varying the Number of Factors

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Variance Swaps Apple IV

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Citigroup IV IBM IV

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S&P 500 NASDAQ IV

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**Inflation Swaps**

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