Dynamic Matching in a Two-Sided Market

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Motivated by the rise of the sharing economy, we consider an intermediary firm’s problem of dynamically matching demand and supply of heterogeneous types over a discrete-time horizon. More specifically, there are two finite disjoint sets of demand and supply types. Associated with each possible matching of a demand type and a supply type is a reward. In each period, demand and supply of various types arrive in random quantities. The firm’s problem is to decide on the optimal matching policy to maximize the total discounted rewards minus costs, given that unmatched demand and supply will incur waiting or holding costs, and will be carried over to the next period with abandonments.

For this general dynamic matching problem, we obtain a set of distribution-free structural results. First, using only matching rewards, we define a partial order between pairs of demand and supply types (which do not necessarily share a common demand or supply type). With this notion of partial order, we show it is optimal to prioritize the matching of the dominating pair over the dominated pair, and to greedily match a perfect pair that dominates all other pairs sharing a common demand or supply type. Second, we impose a reward structure in which types have (unidirectional) “taste” differences. For these horizontally differentiated types, we show that there exists a matching priority hierarchy related to “taste” locations: for any given demand (or supply) type, the closer its distance to a supply (or demand) type, the higher the priority to match the closer pair. Along the priority hierarchy, the optimal matching policy has a match-down-to structure for any pair of demand and supply types: there exist state-dependent thresholds; if the levels of demand and supply are higher than the thresholds, they should be matched down to the thresholds; otherwise, they should not be matched. Third, we impose a reward structure in which types have “quality” differences. For these vertically differentiated types, the optimal matching policy has an even simpler top-down matching structure (in short, “line up, match up”): line up demand types and supply types in descending order of their “quality” from high to low; match them from the top, down to some level. When demand and supply types have the same abandonment rate, the match-down-to levels have monotonicity properties with respect to the system state, and the one-step-ahead heuristic policy has a simplified state-dependent structure. Lastly, we study the deterministic counterpart of the stochastic problem and show that its solution can be obtained by solving a linear program or approximated by another linear program with much fewer decision variables. It is asymptotically optimal to re-solve the linear program successively for the current time and state and apply the solution as a heuristic policy, when the time and the arrivals of demand and supply are scaled up proportionally.
1. Introduction

Consider a firm that manages the process of matching crowdsourced supply with demand over a finite discrete-time horizon with \( T \) periods. There are \( n \) types of demand and \( m \) types of supply, with a reward \( r_{ij} \) generated by matching one unit of type \( i \) demand and one unit of type \( j \) supply. At the beginning of each period, demand and supply of various types arrive in random quantities. The firm’s problem is to decide how to match them and to what extent, so as to maximize the total discounted rewards minus costs, given that unmatched demand and supply will incur unit waiting and holding cost rates \( c \) and \( h \), respectively, and will be carried over to the next period with carry-over rates \( \alpha \in [0, 1] \) and \( \beta \in [0, 1] \), respectively.

1.1. Motivation

That is exactly the essence of the problem faced by many intermediaries in the sharing economy. Operations management deals with the management of the process of matching supply with demand. There is a new form of such process that calls for active management – a sharing economy with crowdsourced supply. The sharing economy once played a prominent role in human history in the form of bartering long before currency was invented. In recent years, this ancient system has been revitalized, thanks to the booming information technology, and it is greatly challenging the traditional industries as well as our old ways of living. With the help of technological platforms, ordinary people are able to share their excess resources and possibly receive monetary payment in return. In the recent rise of the sharing economy, ride-sharing platforms such as Uber and Lyft match a prospective rider with a nearby car driven by some freelance driver. Wal-Mart is considering tapping instore customers to deliver online orders (Barr and Wohl 2013). Amazon allows third-party merchants to share their inventories of identical items with each other and with Amazon itself, and ships to customers from the nearest warehouse.\(^1\) A nonprofit organization, United Network for Organ Sharing (UNOS), allocates donated organs to patients in need of transplantation.

These popular business and nonprofit sharing-economy models are based on what academics often call a two-sided market (Rochet and Tirole 2006). In such a market, an information technology platform is developed, built, and maintained by an intermediary firm to make sharing-economy activities possible. Three parties are involved, namely, an intermediary firm (e.g., Amazon or UNOS), the demand side (e.g., customers ordering a product online, or patients in need of organ

\(^1\)Amazon calls that an “inventory commingling program.” A product ordered from Amazon or a third-party seller may not have originated from the original seller. The program gives Amazon the flexibility to ship products on the basis of their geographic proximity to customers, thus shortening delivery times and reducing shipping costs.
transplantation), and the supply side (e.g., products owned by third-party vendors or Amazon in Amazon’s warehouses, or donated organs). In this structure, the intermediary organization matches demand and supply of heterogeneous types.

From the intermediary’s perspective, matching different types of demand and supply generates distinct rewards (or equivalently, mismatch costs). That is because, in some cases, types on either side of the market have their own idiosyncratic preferences for types on the other side; in other words, types have “taste” differences. For example, consider geographic locations as types in the Amazon inventory-sharing example. It costs less for Amazon to fill an online order with a product in a warehouse closer to the customer. We say that such a market has horizontally differentiated types. In other cases, types on either side of the market have a uniform preference for types on the other side; in other words, types have “quality” differences. For example, consider health status as types in organ sharing. A patient’s life expectancy after a transplant will be longer if the donated organ or the patient is in a healthier condition. Such a market is said to have vertically differentiated types. Even more often, we see a mix of horizontally and vertically differentiated types in a market. For example, in organ sharing, in addition to health status as vertically differentiated attributes, blood and tissue can be considered as horizontally differentiated attributes. That is because positive rewards are generated only between compatible pairs. Table 1 lists examples of the two-sided market structure and the categorization of their supply- and demand-type differentiation.

<table>
<thead>
<tr>
<th>Service</th>
<th>Intermediary</th>
<th>Supply</th>
<th>Demand</th>
<th>Types (Differentiation Category)</th>
</tr>
</thead>
<tbody>
<tr>
<td>ridesharing</td>
<td>Uber, Lyft</td>
<td>drivers</td>
<td>riders</td>
<td>location of riders (horizontal)</td>
</tr>
<tr>
<td>delivery</td>
<td>Wal-Mart</td>
<td>couriers</td>
<td>senders</td>
<td>location of recipients (horizontal)</td>
</tr>
<tr>
<td>e-commerce</td>
<td>Amazon</td>
<td>merchants</td>
<td>customers</td>
<td>location of warehouses and customers (horizontal)</td>
</tr>
<tr>
<td>organ transplant</td>
<td>UNOS</td>
<td>organs</td>
<td>patients</td>
<td>blood and tissue (horizontal); health status (vertical)</td>
</tr>
</tbody>
</table>

Other than dealing with heterogeneous types, the matching of demand and supply by an intermediary in the sharing economy can be extremely difficult for two reasons. First, there are time-varying uncertainties on both the demand and supply sides, which may be out of the control of the intermediary (e.g., third-party merchants on Amazon’s marketplace use their own inventory regulating policies and may be subject to various kinds of time-varying supply shocks). Second, arrived but unmatched demand and supply may leave the market over time (e.g., freelance drivers go home and unmatched organs perish, or patients drop out of the waiting list).

Economic theories use the tool of “price” to match demand with supply. While price does play an important role in many marketplaces, especially at the strategic and tactical levels, day-to-day
operations often require more than price adjustment to achieve efficiency for practitioners. For example, financial benefit is the same for a third-party merchant in the Amazon commingling program regardless of where its package is sent (because consumers pay separately for shipping), and consumers pay the same shipping fee regardless of where the package is sent from (because the cost of shipping for customers depends only on the speed of delivery). The allocation of donated organs in the United States does not involve prices at all. Given that prices are exogenous or irrelevant, intervention at the operational level, by directly matching supply with demand of various types, provides an efficient way for the intermediary organization to allocate the crowdsourced supply across different types of demand. In summary, the intermediary has the task of matching exogenous streams of demand and supply types to maximize total profit or social welfare, taking into account that there will be time-dependent random arrivals of demand and supply in the future and that unmatched demand and supply need to be compensated and may abandon the wait.

1.2. Main Results

We formulate the intermediary firm’s dynamic matching problem as a finite-horizon discrete-time stochastic dynamic program and analyze it for the structural properties of optimal matching policies and good heuristic policies. Specifically, we obtain the following set of structural results. These results are distribution-free, i.e., they hold for any distribution of random demand and supply and thus hold along any sample path.

**General Priority Properties.** We define a relation between two pairs of demand and supply types, using only matching rewards, and show such a relation is a partial order that specifies a dominance relation between two pairs. (Governed by the partial order, two pairs of demand and supply types do not necessarily share a common demand or supply type.) Using this notion of partial order, we provide three structural properties of an optimal matching policy. First, if two given pairs of demand and supply types, sharing a common demand or supply type, can be compared under the partial order, it is optimal to prioritize the matching of the dominating pair over the dominated pair. Second, it is optimal to greedily match a perfect pair of demand and supply.

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2 In ridesharing, though Uber is notorious for its “surge pricing,” the same rate applies to all rides at the same time regardless of their origin and destination; in other words, the rate at any given time is exogenous to geographic locations as “types” of riders and drivers. The pricing part of Uber’s practice is at a higher level than the matching part. A higher price can encourage more drivers and discourage more riders to arrive at the market. Given the price is determined and announced, the matching decisions are made at the operational level after drivers and riders see the price and arrive at the market. Because our structural results are distribution free, they can be useful for the matching decisions in Uber’s business practice as well.
supply types that dominates all other pairs sharing its demand or supply type. Third, as a global priority property, for any two pairs of demand and supply types with one strictly dominating the other but not necessarily sharing a common demand or supply type, it is still optimal to prioritize the matching of the dominating pair over the dominated pair. The proofs involve induction and construction. The partial-order relations, as sufficient conditions for these structural properties, are “necessary” in a robust sense against all possible scenarios of demand and supply realizations.

We proceed to study two special cases of the general reward structure. For these two reward structures, all pairs sharing a common demand or supply type are shown to be comparable under the partial order. As a result of the general priority properties, the optimal matching policy boils down to a match-down-to structure when considering a pair of demand and supply types, along the priority hierarchy (see below). In the optimal policy, if a pair of demand and supply types is not matched as much as possible, all pairs that are strictly dominated by this pair would not be matched at all, according to the global priority property.

**Horizontally Differentiated Types.** We assume that \( n \) demand and \( m \) supply types are arbitrarily located on a circle. The unidirectional “distance” between a demand type and a supply type is the distance one travels clockwise along the circle from the location of the demand type to that of the supply type. The reward for matching the two decreases in their “distance.” Using the general priority properties, we verify that it is optimal to match as much as possible the two that are closest to each other. In other words, a perfect pair is formed when demand and supply are the closest to each other on the clockwise circle. Moreover, there exists a priority hierarchy in matching imperfect pairs. For any given demand (or supply) type, the closer its distance to a supply (or demand) type, the higher the priority to match the closer pair. As a result, the optimal matching policy is characterized by state-dependent, match-down-to levels, with those levels for perfect pairs all equal to zero. That is, the optimal matching policy has a match-down-to structure: along the priority matching hierarchy, for a pair of demand and supply types, there exist state-dependent thresholds; if demand and supply levels are higher than the thresholds, they should be matched down to the thresholds; otherwise, they should not be matched. For the special case of two demand

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3 Suppose demand and supply are characterized by their locations in an Euclidean space. We show that demand and supply from the same location forms a perfect pair. Our result on the greedy matching between a perfect pair suggests that Uber and Amazon should always match a demand with a supply if they are originated from the same geographic region.

4 Unfortunately, these results on priorities, determined by distances, fail to hold if the “distance” is the shortest distance along the circle. See Figure 5 for an illustration. We leave the exact model of capturing the geographic locations in a 2-dimensional Euclidean space as types, coupled with the shortest distance as traveling cost, to future research.
and two supply types (with the “taste” distance generalizable to be a undirected distance), we show that the match-down-to levels depend only on the aggregated imbalance between the total demand and supply across two types; the match-down-to levels in any period are reduced to constant levels if the unmatched demand or supply is lost.

**Vertically Differentiated Types.** Each demand or supply type is associated with a quality, and it generates a higher reward if matched with a supply or demand type of a higher quality. The optimal matching policy has a simple structure, which we call *top-down matching* (in short, line up, match up): line up demand types and supply types in descending order of their “quality” from high to low; match them from the top, down to some level. As a result, the optimal matching policy in any period can be fully determined by a total matching quantity. Moreover, we can take a dynamic perspective on the optimal matching policy: as in the case of horizontally differentiated types, in the top-down matching procedure there are match-down-to levels (or equivalently, some protection levels) for any pair of demand and supply types. Unlike horizontally differentiated types, these match-down-to levels depend only on the levels of demand and supply of those types that are weakly lower than the types in the focal pair. When demand and supply have the same carry-over rate, we show, by verifying the $L^\infty$-concavity of the value functions of a transformed problem, that the optimal total matching quantity (from the aggregate perspective) or the optimal protection levels (from the dynamic perspective) have monotonicity properties with respect to the system state: An increment in the level of a demand or supply type with higher “quality” leads to a higher optimal matching quantity or lower protection levels, and the rate of change is dominated by 1. State dependence of the optimal matching policy can be further simplified if the unmatched demand or supply is lost, or if the demand or supply side has only one type. When the carry-over rates are the same on both sides, we show that the one-step-ahead heuristic policy has a simplified state-dependent structure.

**Bounds and Heuristics.** We consider the deterministic counterpart (i.e., the fluid model) of the stochastic dynamic problem for any period with $t$ amount of remaining time in the horizon and any levels of demand and supply; this can be written as a linear program with $O(n \times m \times t)$ variables. If the shadow prices of its dual problem are known or can be approximated by good proxies (e.g., the typical market prices that would encourage demand and supply to enter the market for a given time period), then the optimal matching decisions of the fluid model for a given period can be obtained or approximated by a linear program with only $n \times m$ variables. We show that the fluid model provides an upper bound on the optimal total surplus of the stochastic model.
It is asymptotically optimal to re-solve the linear program for the current time and state and apply the solution as a heuristic policy, when the time and the arrivals of demand and supply are scaled up proportionally.

2. Literature Review

See Figure 1 for a high-level positioning of our framework in the literature.

The proposed dynamic-matching framework can be viewed as a generalization of two foundations of operations management, i.e., inventory management where the firm orders the supply centrally (Zipkin 2000), and revenue management where the firm regulates the demand side with a fixed supply side (Talluri and van Ryzin 2006), and of a combination of the two, i.e., joint pricing and inventory control (Chen and Simchi-Levi 2012). Unlike in inventory management and revenue management, the supply in the sharing economy is crowdsourced. More specifically, various supply types arrive sequentially with inter-temporal uncertainty, as demand does in a typical revenue-management setting. Therefore, there are decentralized inter-temporal uncertainties on both sides of the market. This generalization is motivated by the recent rise of the sharing economy. It adds complexity beyond existing operations frameworks of stochastic inventory theory and revenue management, and it has not been systematically studied in the operations literature. With these foundations of operations management, the tools, techniques, and insights developed in those more established areas may be transferred to this new situation. For instance, to derive the monotonicity property of the optimal matching policy, we use $L^1$ concavity of the value functions in a transformed system, which has been applied for deriving structural properties for lost-sales inventory models (Zipkin 2008) and for perishable-inventory models (Chen et al. 2014).
More specifically, in connection with inventory management, our framework is closely related to the literature on inventory rationing, which considers a single supply type and multiple demand types, and allows demand from the less valuable types to be rejected (thus lost) in anticipation of future demand from the more valuable types. The seminal papers by Evans (1968) and Veinott (1965) consider periodic-review systems, while the works by Ha (1997a,b) and de Véricourt et al. (2002) study continuous-review models assuming Poisson demands. Inventory rationing considers inter-temporal allocation of supply capacity across various demand classes. The matching decisions in our framework generalize the idea of inventory rationing in the sense that one needs to consider the characteristics of both the demand and supply types, such as marginal matching costs and abandonment rates. For horizontally and vertically differentiated types, we characterize the optimal matching policy in the form of match-down-to levels, analogously to the threshold structure of rationing levels.

Moreover, in its connection with inventory management, our framework is related to a stream of research on production systems with random yield. Pioneered by Henig and Gerchak (1990), this stream considers unreliable production that yields only a random portion of the planned quantity. Follow-up works consider extensions such as random capacity (Ciarallo et al. 1994), random disruptions of the system (Moinzadeh and Aggarwal 1997), multiple products with random yield (Hsu and Bassok 1999) and a mixture of reliable and unreliable suppliers (Dada et al. 2007). Compared with random-yield production models, our framework considers a class of problems with purely random sources of supply, independently of the firm’s decisions; in contrast, the output from a random-yield production system is a random fraction or perturbation of the planned amount.

In its connection with revenue management, our framework is closely related to quantity-based revenue management (see, e.g., Talluri and van Ryzin 2006, Part I), in particular, dynamic capacity allocation models with upgrading. Shumsky and Zhang (2009) consider multiple supply types with fixed capacities (at least within a no-replenishment horizon) and multiple demand types. Demand types that arrive to find that their supply type has been depleted can be upgraded by at most one level. The authors show that the optimal policy has a structure of parallel allocation and then rationing. Yu et al. (2014) extend such single-step upgrading to allow general upgrading and show that a sequential upgrading policy is optimal. Our framework further generalizes these models in the following aspects: First, the supply side is not fixed but has inter-temporal uncertainty. Second, various demand and supply types, of which there could be different numbers, can be matched freely. Third, we do not necessarily impose a specific functional form on matching rewards. Those upgrading models assume vertically differentiated types, which are a special case of our general
reward structure. For this special case, we characterize the optimal matching policy as “line up, match up,” generalizing the optimal policies obtained for the upgrading models.

Driven by real-life applications, economists, computer scientists, and operations researchers have studied a variety of two-sided matching problems (see, e.g., Roth and Sotomayor 1990, Abdulki-diroğlu and Sönmez 2013, for surveys). These problems include the college admissions problem (with the marriage problem as a special case, see, e.g., Teo et al. 2001), kidney exchange, and the assignment problem. We compare our framework with those problems as follows.

First, the college admissions problem is preference-based. It involves parties on the demand and supply sides submitting preferences over options (see, e.g., Ashlagi and Shi 2014), either relative rankings (so-called ordinal mechanism) or preference intensities (so-called cardinal mechanism). As those matching outcomes such as marriage and college admissions can be life-changing, serious efforts in soliciting and submitting preferences are necessary. In contrast, as the sharing economy penetrates into our everyday lives, preference-based matching may not be practical. For instance, when customers order on Amazon, they do not have the option, or may not even bother, to choose which warehouse the product will be shipped from, because they pay a fixed shipping cost, which depends on the speed of delivery (Amazon absorbs the additional cost if a product has to be shipped from far away). It requires the intermediary to associate pairs of demand and supply with rewards, as they arrive, and make matching decisions accordingly. To capture this situation, we assign a “monetary” contribution to a pair of demand and supply types, e.g., a lower reward (higher cost to Amazon) if a product is shipped from a warehouse that is farther away from where the demand is originated. Moreover, the college admission problem tends to have a static or deterministic nature. Supply and demand arrive with submitted preferences, before the matching decisions will be made, as in the classical marriage problem. On the other hand, our framework, as in inventory and revenue management, emphasizes the dynamic and stochastic nature of a class of matching problems caused by the growth of the sharing economy and characterized by inter-temporal uncertainties.

Similarly to the college admissions problem, kidney exchange has compatibility-based preferences such as blood-type and tissue compatibility. (Note that kidney exchange is different from kidney allocation. In the latter, the organs are harvested from cadaveric donors; see below.) Moreover, in a typical situation the patient and donor arrive in pairs, with an incompatible (or less likely, compatible) patient and donor in each pair. Because of the compatibility issue and the fact that patients and donors arrive in pairs, matching decisions are based on cycles, such as two-way exchanges or chains of patient-donor pairs; see, e.g., Roth et al. (2004, 2007), Ashlagi et al. (2012), Ünver (2010) and Anderson et al. (2015). Most relevant to our framework is Ünver (2010), which studies dynamic
kidney exchange with inter-temporal random arrivals of patient-donor pairs, and attempts to maximize the number of matched compatible pairs. However, with arbitrary unbalanced arrivals of demand and supply, our matching policy exhibits a more general form and the goal is to maximize social welfare or profit.

Third, our framework is closely related to the assignment problem, in which resources are assigned to tasks. As in the assignment problem, we assign a reward when a pair of demand (task) and supply (resource) types is matched. The classical static assignment dates back to Dantzig (1963). The study of a class of sequential assignment match processes (SAMP), in which resources are waiting to be assigned to a sequential stream of tasks with random attributes, originated with Derman et al. (1972). Numerous extensions have been pursued with more recent developments, such as Su and Zenios (2005) that apply SAMP to study kidney allocation with incentive compatibility (see the references therein for earlier developments). The assignment problem essentially assumes a stationary supply side (e.g., assuming a frozen waiting list of patients in need of transplantation), whereas we allow time-varying supply side with uncertainty, like the demand side. Spivey and Powell (2004) study a dynamic assignment problem that does allow dynamic arrivals of tasks and resources, but they concentrate on heuristic policies. For a general dynamic problem with time-varying uncertainties on both demand and supply sides that can be viewed as a generalization of the dynamic assignment problem, we focus on deriving structural properties of the optimal policy.

In the operations literature, researchers have also been using the queueing approach or its fluid counterpart to model two-sided matching. Using a set of single- or multi-server queueing systems, Allon et al. (2012) model a service marketplace where customers seek services from agents. The authors show, among other things, that the moderating firm’s intervention to achieve operational efficiency by pooling agents can be detrimental to the overall efficiency of the marketplace. That is because strategic interaction between customers and agents influences market prices. Using a mean field analysis, Arnosti et al. (2014) study a decentralized two-sided matching market and show that limiting the visibility of applicants can significantly improve the social welfare. In contrast, given that prices are fixed or irrelevant for many centralized sharing-economy platforms, we do not model the incentive issues of customers or agents but focus on optimal matching policies at the operational level. On top of the matching decisions, one may further consider pricing problems or incentive compatibility issues. Since our structural results on matching after demand and supply arrive are distribution free, they help with solving those problems and issues as well. With a fluid approach of modeling stochastic systems, Zenios et al. (2000), Su and Zenios (2006) study kidney allocation by exploring the efficiency-equity trade-off, and Akan et al. (2012) study liver allocation by exploring
the efficiency-urgency trade-off. Though focusing on structural properties of the optimal policy by exploring the stochastic dynamic program, we also propose a heuristic policy based on a fluid model, and show it is asymptotically optimal. Using double-sided queues, Zenios (1999) studies the transplant waiting list and Afèche et al. (2014) study trading systems like crossing networks. Su and Zenios (2004) analyze a queueing model with service discipline FCFS or LCFS to examine the role of patient choice in the kidney transplant waiting system. Adan and Weiss (2012) show that the stationary distribution of FCFS matching rates for two infinite multi-type sequences is of product form. These papers deal with performance evaluation under a given matching policy. Indeed, Gurvich and Ward (2014) study the dynamic control of matching queues with the objective of minimizing cumulative holding costs. The authors observe that in principle, the controller may choose to wait until some “inventory” of items builds up to facilitate more profitable matches in the future. Our match-down-to policy for imperfect pairs has a similar spirit. With a more general reward structure and the objective of maximizing rewards minus costs, we show when it is optimal to greedily match a pair and when it is optimal to save some “inventory” for future.

Lastly, our paper is a generalization of Duenyas et al. (1997), which assumes that one unit of a demand type and a supply type arrives deterministically in each period and characterizes the structure of the optimal policy for special cases of two or three demand and supply types. We generalize the model to account for time-varying randomness in demand and supply arrivals and their abandonment over time, and we explore the structure of the optimal matching policy for arbitrary numbers of demand and supply types.

3. The General Model
We use a boldface letter to denote a vector and its light face with subscript \(i\) to denote its \(i\)-th entry. By default, a vector is treated as a row vector. We also use a boldface letter to denote a matrix and its light face with subscript \(ij\) to denote its \((i,j)\)-th entry. We use \(x_{[k,\ell]}\) to denote the sub-vector of a vector \(x\), containing elements from the \(k\)-th entry to the \(\ell\)-th entry. We denote by \(e_k^\ell\) a \(k\)-dimensional unit vector where the \(\ell\)-th entry is 1 and all other entries are 0 and by \(e_{ij}^{n\times m}\) an \(n \times m\)-dimensional matrix where the \((i,j)\)-th entry is 1 and all other entries are 0. We denote by \(1_k\) a \(k\)-dimensional vector of 1’s and denote by \(0_k\) a \(k\)-dimensional vector of 0’s (we may omit the superscript \(k\) if the dimension of the zero vector is clear from the context). \(\mathbb{R}_+ = \{r \mid r \geq 0\}\). \(x \wedge y = \min\{x,y\}\) and \(x \vee y = \max\{x,y\}\). \(x^+ = \max\{x,0\}\) and \(x^- = -\min\{x,0\}\). All proofs can be found either in the main body or the appendix of the paper.

Consider a finite horizon with a total number of \(T\) periods. In practice, even though demand and supply arrive in continuous time, matching decisions are typically not made in real time.
For example, Amazon periodically optimizes the way in which it matches customer orders and its warehouses (see Xu et al. 2009). Uber batches up ride requests for a few seconds so that multiple rides can be considered when making the assignment decision.

At the beginning of each period, \( n \) types of demand and \( m \) types of supply arrive in random quantities. Let \( D \) denote the set of demand types and \( S \) denote the set of supply types. With a slight abuse of notation, we write \( D = \{1, 2, \ldots, n\} \) and \( S = \{1, 2, \ldots, m\} \), noting that \( D \) and \( S \) are disjoint sets. We use \( i \) to index a demand type and \( j \) to index a supply type. The pairs of demand and supply are shown in Figure 2 as a bipartite graph. An arc \((i, j)\) represents a match between type \( i \) demand and type \( j \) supply. For simplicity, we consider a complete bipartite graph in the base model. In other words, any demand type can potentially be matched with any supply type, obviously with different rewards (or equivalently, mismatch costs). (We consider an incomplete bipartite graph in §8.) We denote the complete set of arcs by \( A = \{(i, j) \mid 1 \leq i \leq n, 1 \leq j \leq m\} \).

![Figure 2 Pairs of demand and supply.](image)

The state for a given period comprises the demand and supply levels of various types before matching but after the arrival of random demand \( D \in \mathbb{R}_+^n \) and supply \( S \in \mathbb{R}_+^m \) for that period. We make no assumption about the distributions of random demand and supply of various types; in other words, our model and its results are completely distribution-free. (Thus, we omit the time index of \( D \) and \( S \) for simplicity of notation in the stochastic problem. We call back the time index in §7 when studying the heuristic policy suggested by the deterministic problem.) We denote, as system states, the demand vector by \( x = (x_1, \ldots, x_n) \in \mathbb{R}_+^n \) and the supply vector by \( y = (y_1, \ldots, y_m) \in \mathbb{R}_+^m \), where \( x_i \) and \( y_j \) are the quantity of type \( i \) demand and type \( j \) supply available to be matched. Although we assume throughout the paper that the states and the demand and supply arrivals are continuous quantities (and therefore so are the matching decisions), our
results can be readily replicated if those quantities are discrete. The assumption of continuous quantities is commonly seen in the periodic-review inventory literature, e.g., the celebrated one-period newsvendor problem being a special example.

On observing the state \((x, y) \in \mathbb{R}_+^{n+m}\), the firm decides on the quantity \(q_{ij}\) of type \(i\) demand to be matched with type \(j\) supply, for any \(i \in D\) and \(j \in S\). For conciseness, we write the decision variables of matching quantities in a matrix form as \(Q = (q_{ij}) \in \mathbb{R}_+^{n \times m}\), with \(Q_i\) its \(i\)-th row (as a row vector) and \(Q_j\) its \(j\)-th column (as a column vector). We assume that there is a reward \(r_{ij}\) for matching one unit of type \(i\) demand and one unit of type \(j\) supply for all \(i, j\). Similarly, we can write the rewards in a matrix form as \(R = (r_{ij}) \in \mathbb{R}^{n \times m}\). Thus the total reward from matching is linear in the matching quantities. That is, \(R \circ Q = \sum_{i=1}^{n} \sum_{j=1}^{m} r_{ij} q_{ij}\), where “\(\circ\)” is the Hadamard product, or the entrywise product. The post-matching levels of type \(i\) demand and type \(j\) supply are given by \(u_i = x_i - 1^m Q_i^T = x_i - \sum_{j'=1}^{m} q_{ij'}\) and \(v_j = y_j - 1^n Q_j^T = y_j - \sum_{i'=1}^{n} q_{i'j}\), respectively. That is,

\[
\begin{align*}
    u &= x - 1^m Q^T \\
    v &= y - 1^n Q.
\end{align*}
\]

The post-matching levels cannot be negative; i.e., \(u \geq 0, v \geq 0\).\(^5\)

After the matching is done in each period, each unit of unmatched demand and supply incurs a holding cost \(c\) and \(h\) respectively. The cost for demand could be loss of goodwill or waiting costs. (We consider type-dependent costs in §8.) Consequently, the total holding cost amounts to \(c1^n u^T + h1^m v^T = c \sum_{i=1}^{n} u_i + h \sum_{j=1}^{m} v_j\). The unmatched demand and supply carry over to the next period with carry-over rates \(\alpha\) and \(\beta\), respectively. (We consider type-dependent and random carry-over rates in §8.) In other words, \((1 - \alpha)\) fraction of demand and \((1 - \beta)\) fraction of supply leave the system. Without loss of generality, we assume they leave the system with zero surplus.

The firm’s goal is to determine a matching policy \(Q^* = (q_{ij}^*)\) that achieves the maximum expected total discounted surplus. (Our perspective is the maximizing of social welfare. Alternatively, the formulation can account for a maximizing of profit if \(r_{ij}\) is interpreted as the revenue collected from a matching, and \(c\) and \(h\) are interpreted as the penalty paid to demand and supply for showing up but without a successful match in a period.) Let \(V_t(x, y)\) be the optimal expected total discounted surplus given that it is in period \(t\) and the current state is \((x, y)\). We formulate the finite-horizon problem by using the following stochastic dynamic program:

\[
V_t(x, y) = \max_{Q \in \{Q \geq 0, u \geq 0, v \geq 0\}} H_t(Q, x, y),
\]

\(^5\) For simplicity, without formal definitions, we will take the liberty of using consistent notation for the post-matching levels, with its corresponding matching decision. For example, if a matching decision is denoted by \(Q\), its corresponding post-matching levels will be denoted by \(\bar{u}\) and \(\bar{v}\).
\[ H_t(Q,x,y) = R \circ Q - c1^m u^T - h1^m v^T + \gamma E V_{t+1}(\alpha u + D, \beta v + S), \] (1)

where \( \gamma \leq 1 \) is the discount factor. The boundary conditions are \( V_{T+1}(x,y) = 0 \) for all \((x,y)\), without loss of generality. In other words, at the end of the horizon, all unmatched demand and supply leave the system with zero surplus.

The existence of a solution to the dynamic program (1) is resolved by the following proposition.

**Proposition 1.** The functions \( H_t(Q,x,y) \) and \( V_t(x,y) \) are continuous and concave. Thus, there exists an optimal matching policy \( Q^*_t(x,y) \).

**Proof of Proposition 1.** We prove this result by induction on \( t \). Clearly, \( V_{T+1}(x,y) \equiv 0 \) is continuous and concave in \((x,y)\). We suppose \( V_{t+1}(x,y) \) is continuous and concave in \((x,y)\), and show that so is \( V_t(x,y) \). First, because \( V_{t+1}(x,y) \) is continuous in \((x,y)\), \( H_t(Q,x,y) \) is continuous in \((Q,x,y)\). Moreover, because the set mapping from \((x,y)\) to the set \( R(Q;x,y) = \{ Q \mid Q \geq 0, u = x - 1^m Q^T \geq 0, v = y - 1^n Q \leq 0 \} \) is compact-valued and continuous, by the maximum theorem, \( V_t(x,y) \) is continuous in \((x,y)\). Second, since the composition of a concave function and an affine function is still concave (Simchi-Levi et al. 2014, Proposition 2.1.3(b)), \( V_{t+1}(\alpha u + D, \beta v + S) \) is concave in \((Q,x,y)\) for any given \((D,S)\). Then, \( E(\alpha,\beta)[V_{t+1}(\alpha u + D, \beta v + S)] \) is concave in \((Q,x,y)\). Then it is immediately clear that \( H_t(Q,x,y) \) is jointly concave in \((Q,x,y)\), because all other terms except the last term in (1) are linear in \((Q,x,y)\). Because the set \( R(Q;x,y) \) is a polyhedron defined by a system of linear inequalities, and a fortiori, a convex set, and the concavity is preserved under maximization over a convex set (Simchi-Levi et al. 2014, Proposition 2.1.15(b)), we have \( V_t(x,y) = \max_{Q \in R(Q;x,y)} H_t(Q,x,y) \) is concave.

The existence of an optimal matching policy \( Q^*_t(x,y) \) follows from the continuity of the function \( H_t(Q,x,y) \) and the compactness of \( R(Q;x,y) \) for a given \((x,y)\). \[ \square \]

Although the existence of an optimal matching policy is guaranteed, in general we expect the state-dependent optimal policy to be extremely complex. Next we characterize some of its structural properties.

### 4. Priority Properties of the Optimal Policy

One may expect some intuitive properties of the optimal matching policy, e.g., matching a “perfect” pair in some sense, as much as possible. We provide sufficient conditions for such properties. Since we aim to address a general problem that has random dynamics, the conditions would sufficiently guarantee those properties even for a static problem. Therefore, the conditions we will provide are on the reward matrix and independent of any other system parameters. These conditions will guarantee that certain structural properties will hold for the dynamic problem at any time and with any realized demand and supply.
4.1. Preliminary: A Partial Order

To facilitate discussion, we define a relation \( \succeq \) between arcs as follows and will show it is a partial order. First, we consider neighboring arcs in the bipartite graph (Figure 2) that share a common demand or supply type.

**Definition 1 (Arcs sharing a common node).** We say \( (i,j) \succeq (i,j') \) if (i) \( r_{ij} \geq r_{ij'} \) and (ii)

\[
r_{ij} + r_{i'j'} \geq r_{ij'} + r_{i'j}
\]

for all \( i' \in D \). (When \( i' = i \), condition (D) holds automatically. It is easy to see that \( (i,j) \succeq (i,j') \) holds automatically for \( j' = j \).) Similarly, we say \( (i,j) \succeq (i',j) \) if \( r_{ij} \geq r_{i'j} \) and condition (D) holds for all \( j' \in S \).

Note. \( (i,j) \succeq (i,j') \) if condition (D) holds for all \( i' \in D \).

Condition (D) is reminiscent of Monge sequence. Hoffman (1963) provides a necessary and sufficient condition for a transportation problem to be solvable by a greedy algorithm, in which a permutation (called a Monge sequence) is followed. A Monge sequence regulates all the arcs in the graph, requiring condition (D) to hold only for all those neighboring arcs \((i,j), (i,j')\) and \((i',j)\) whenever \((i,j)\) precedes \((i,j')\) and \((i',j)\) in the sequence. However, Definition 1 concerns some pairs of neighboring arcs but requires condition (D) to hold for all alternative nodes \( i' \) that are different from the common node \( i \). The Monge sequence is proposed for solving a static demand-supply balanced transportation problem. We propose the partial order to provide sufficient conditions for some structural properties in our dynamic demand-supply unbalanced matching problem.

Part (i) of Definition 1 requires no less reward by matching pair \((i,j)\) than pair \((i,j')\). To understand part (ii) of Definition 1, we compare the following two strategies: (1) matching one unit of type \( i \) demand and type \( j \) supply and another unit of type \( i' \) demand and type \( j' \) supply, and (2) matching one unit of type \( i \) demand and type \( j' \) supply and another unit of type \( i' \) demand and type \( j \) supply. It is easy to see that the two strategies have the same post-matching levels of
demand and supply. Condition (D) requires that the former strategy weakly dominate the latter (see Figure 3 for an illustration). In other words, part (ii) of Definition 1 implies that there does not exist \( i' \in \mathcal{D} \) such that the latter strategy leads to a strictly higher reward than the former. As a result, part (ii) of Definition 1 eliminates the optimality of breaking up the pair \((i, j)\) in matching nodes \(i, j\) and \(j'\).

We further define a relation between arcs that do not share any node but can be connected through a sequence of neighboring arcs regulated by the relation \( \succeq \).

**Definition 2 (Arcs without common nodes).** For \( i \neq i' \) and \( j \neq j' \), we say \((i, j) \succeq (i', j')\) if there exists a sequence of arcs \((i_1, j_1), (i_2, j_2), \ldots, (i_k, j_k)\) such that either \( i_k = i_{k+1} \) or \( j_k = j_{k+1} \) for \( k = 1, \ldots, n - 1 \), and \((i, j) = (i_1, j_1) \succeq (i_2, j_2) \succeq \cdots \succeq (i_k, j_k) = (i', j')\).

In addition, the *equivalence* relation \((i, j) \equiv (i', j')\) means that \((i, j) \succeq (i', j')\) and \((i', j') \succeq (i, j)\) hold simultaneously. The equivalence relation \((i, j) \equiv (i', j)\) can be similarly defined.

**Lemma 1.** If \((i, j) \succeq (i', j')\) and \((i, j') \succeq (i', j'')\), then \((i, j) \succeq (i', j'')\).

Consider two arcs \((i, j) \succeq (i', j')\) and \(i \neq i' \), \(j \neq j' \). By Definition 2, there exists a decreasing sequence \((i_1, j_1) \succeq \cdots \succeq (i_n, j_n)\) connecting \((i, j)\) and \((i', j')\). By virtue of Lemma 1, we can assume without loss of generality that there are no three consecutive arcs sharing the same node. In fact, if \(i_k = i_{k+1} = i_{k+2}\), then \((i_k, j_k) \succeq (i_{k+2}, j_{k+2})\) by Lemma 1. Thus we can remove \((i_{k+1}, j_{k+1})\) from the sequence and the remaining arcs still constitute a decreasing sequence connecting \((i, j)\) and \((i', j')\).

Under the assumption that no three consecutive arcs share the same node, if \((i_k, j_k)\) and \((i_{k+1}, j_{k+1})\) share the same demand node \(i_k = i_{k+1}\), then \((i_{k+1}, j_{k+1})\) and \((i_{k+2}, j_{k+2})\) must share the same supply node \(j_{k+1} = j_{k+2}\). So the sequence connecting \((i, j)\) and \((i', j')\) follows a zigzag path in one of the following forms: (i) \((i_1, j_1) \succeq (i_1, j_2) \succeq (i_2, j_2) \succeq \cdots \succeq (i_{k-1}, j_k)\); (ii) \((i_1, j_1) \succeq (i_1, j_2) \succeq (i_2, j_2) \succeq \cdots \succeq (i_k, j_k)\); (iii) \((i_1, j_1) \succeq (i_2, j_1) \succeq (i_2, j_2) \succeq \cdots \succeq (i_{k-1}, j_{k-1})\); (iv) \((i_1, j_1) \succeq (i_2, j_1) \succeq (i_2, j_2) \succeq \cdots \succeq (i_k, j_k)\).

In the following proofs, we can choose just one form of the path and the argument for the other three forms would be analogous.

Next, we show that for \((i, j) \succeq (i', j')\), \(i \neq i' \) and \(j \neq j' \), there exists a decreasing sequence connecting \((i, j)\) and \((i', j')\) that does not contain a cycle.

**Lemma 2.** Consider a decreasing sequence of arcs in the form \((i_1, j_1) \succeq (i_2, j_1) \succeq (i_2, j_2) \succeq \cdots \succeq (i_\ell, j_\ell) \succeq (i_1, j_\ell),\) with \(i_1, \ldots, i_\ell\) and \(j_1, \ldots, j_\ell\) being distinct nodes. Then, \((i_1, j_1) \succeq (i_1, j_\ell)\).

By Lemma 2, if \((i, j) \succeq (i', j')\), then there exists a zigzag path of decreasing arcs that connects \((i, j)\) and \((i', j')\) and does not visit any node on the path more than once. To see this, suppose that the decreasing sequence connecting \((i, j)\) and \((i', j')\) contains a circle, say, in the form of \((i_1, j_1) \succeq \cdots \succeq (i_\ell, j_\ell) \succeq (i_1, j_\ell)\). Then, the sequence connecting \((i, j)\) and \((i', j')\) can be rewritten as \((i_1, j_1) \succeq (i_1, j_\ell) \succeq (i_\ell, j_\ell) \succeq \cdots \succeq (i_\ell, j_1) \succeq (i', j_1)\).
(i_2, j_1) \succeq (i_2, j_2) \succeq \cdots \succeq (i_\ell, j_\ell) \succeq (i_1, j_1). \) By Lemma 2, we can drop the arcs \((i_2, j_1), (i_2, j_2), \ldots, (i_\ell, j_\ell)\) and the remaining arcs still compose a decreasing sequence of arcs connecting \((i, j)\) and \((i', j')\).

Finally, we verify that the relation defined in Definitions 1 and 2 is indeed a partial order.

**Lemma 3.** The relation \(\succeq\) is a partial order. That is, any arcs \(\rho_1, \rho_2\) and \(\rho_3\) satisfy: (i) (Reflexivity) \(\rho_1 \succeq \rho_1\); (ii) (Antisymmetry) if \(\rho_1 \succeq \rho_2\) and \(\rho_2 \succeq \rho_1\), then \(\rho_1 \simeq \rho_2\); (iii) (Transitivity) If \(\rho_1 \succeq \rho_2\) and \(\rho_2 \succeq \rho_3\), then \(\rho_1 \succeq \rho_3\).

### 4.2. Priority Between Neighboring Pairs

The defined partial order between neighboring arcs provides a sufficient condition for one arc to have priority over its neighboring arc in the optimal matching decision. To show this, we need the following lemma.

**Lemma 4.** The following statements hold for all periods.\(^6\)

(i) For any \(x_i > 0\) and any \(\varepsilon \in [0, x_i]\), there exists \((\lambda_1, \ldots, \lambda_m) \geq 0\) such that \(\sum_{j'=1}^{m} \lambda_{j'} \leq \varepsilon\) and

\[
V_t(x - \varepsilon e_i^n + \varepsilon e_i^n, y) - V_t(x, y) \geq -\sum_{j'=1}^{m} \lambda_{j'} (r_{ij'} - r_{ij'}). 
\]

(ii) For any \(y_j > 0\) and any \(\varepsilon \in [0, y_j]\), there exists \((\xi_1, \ldots, \xi_n) \geq 0\) such that \(\sum_{i=1}^{n} \xi_i \leq \varepsilon\) and

\[
V_t(x, y - \varepsilon e_j^n + \varepsilon e_j^n) - V_t(x, y) \geq -\sum_{i=1}^{n} \xi_i (r_{ij} - r_{ij'}). 
\]

*Proof of Lemma 4.* We prove part (i) by induction on \(t\). Part (ii) follows from part (i) by symmetry.

(i) The result holds for \(t = T + 1\). Because \(V_{T+1}(x, y) \equiv 0\), we can simply let \((\lambda_1, \ldots, \lambda_m) = 0\).

Suppose that the desired result holds for period \(t + 1\). Consider period \(t\). For any given \(\hat{x}, \hat{y}\) with \(\hat{x}_i > 0\), let \(\hat{Q} \in \max_{Q \geq 0, u \geq 0, v \geq 0} H_t(Q, \hat{x}, \hat{y})\) and \((\hat{u}, \hat{v})\) be the corresponding post-matching levels. 

For any \(\varepsilon \in [0, \hat{x}_i]\), we construct \((\mu_1, \ldots, \mu_m)\) in a recursive way: as the first step, let \(\mu_1 = \min\{\hat{q}_{i1}, \varepsilon\}\), and then recursively, let \(\mu_j = \min\{\varepsilon - \sum_{j'=1}^{j-1} \mu_{j'}, \hat{q}_{ij}\}\) for \(j = 2, \ldots, m\).

We first prove that \(\varepsilon - \sum_{j'=1}^{j} \mu_{j'} = (\varepsilon - \sum_{j'=1}^{j} \hat{q}_{ij'})^+\) for all \(j\), by induction, which guarantees that \(\mu_j \geq 0\) and \(\mu \equiv \sum_{j=1}^{m} \mu_j \leq \varepsilon\). For \(j = 1\), \(\varepsilon - \mu_1 = \varepsilon - \min\{\hat{q}_{i1}, \varepsilon\} = (\varepsilon - \hat{q}_{i1})^+\). Thus the equation holds for \(j = 1\). Suppose it holds for \(j\). Then for \(j + 1\),

\[
\varepsilon - \sum_{j'=1}^{j+1} \mu_{j'} = (\varepsilon - \sum_{j'=1}^{j} \mu_{j'}) - \mu_{j+1} = (\varepsilon - \sum_{j'=1}^{j} \mu_{j'}) - \min\{\varepsilon - \sum_{j'=1}^{j} \mu_{j'}, \hat{q}_{i,j+1}\}
\]

\[
= (\varepsilon - \sum_{j'=1}^{j} \hat{q}_{ij'})^+ - \min\{(\varepsilon - \sum_{j'=1}^{j} \hat{q}_{ij'})^+, \hat{q}_{i,j+1}\}
\]

\(^6\)If the states are in discrete units, the lowest value that \(\varepsilon\) can take is the single unit. Similar statements and proof can be replicated for discrete states.
\[
((\varepsilon - \sum_{j'=1}^m \hat{q}_{i,j'})^+ - \hat{q}_{i,j+1})^+ = [\varepsilon - \sum_{j'=1}^m \hat{q}_{i,j'} - \hat{q}_{i,j+1}]^+ = [\varepsilon - \sum_{j'=1}^{j+1} \hat{q}_{i,j'}]^+,
\]
which completes the induction.

Hence, we have \(\varepsilon - \sum_{j=1}^m \mu_j = (\varepsilon - \sum_{j=1}^m \hat{q}_{ij})^+\). If \(\sum_{j=1}^m \hat{q}_{ij} < \varepsilon\), then \(\varepsilon - \sum_{j=1}^m \mu_j = \varepsilon - \sum_{j=1}^m \hat{q}_{ij}\), implying that \(\sum_{j=1}^m \hat{q}_{ij} = \sum_{j=1}^m \mu_j = \mu\). If \(\sum_{j=1}^m \hat{q}_{ij} \geq \varepsilon\), then \(\varepsilon - \sum_{j=1}^m \mu_j = 0\), implying that \(\varepsilon = \mu\).

Thus, \(\mu \leq \varepsilon\).

Let \(\bar{Q} = \hat{Q} - \sum_{j'=1}^m \mu_{j'} e_{ij'}^{n \times m} + \sum_{j'=1}^m \nu_{j'} e_{ij'}^{n \times m}\). Since \(\bar{Q}\) is optimal for the state \((\hat{x}, \hat{y})\), a fortiori, \(\bar{Q}\) is feasible for the state \((\hat{x}, \hat{y})\), i.e., \(\bar{Q} \geq 0\), \(1^m \bar{Q}^T \leq \hat{x}\) and \(1^n \bar{Q} \leq \hat{y}\). We show that \(\bar{Q}\) is feasible for the new state \((\hat{x}, \hat{y})\) \(\overset{\text{def}}{=} (\hat{x} - \varepsilon e_i^n + \varepsilon e_{ij'}^n, \hat{y}^T) \geq 0\), where the inequality is due to \(\hat{x}_i > 0\) and \(\varepsilon \in [0, \hat{x}]\).

To this end, it suffices to show that \(\bar{Q} \geq 0\), \(1^m \bar{Q}^T \leq \hat{x}\) and \(1^n \bar{Q} \leq \hat{y}\).

First, for all \(j\), because \(0 \leq \mu_j \leq \hat{q}_{ij}\), we have \(\bar{q}_{ij} = \hat{q}_{ij} - \mu_j \geq 0\). Also, it is clear that \(\bar{q}_{ij}^T = \hat{q}_{ij}^T + \mu_j \geq 0\) for all \(j\). For any \(i'' \neq i, i'\), we have \(\bar{q}_{i''} = \hat{q}_{i''}^T + \mu_j \geq 0\) for all \(j\). Thus, \(\bar{Q} \geq 0\).

Second, we have \(1^m \bar{Q}^T = 1^m \hat{Q}^T - \mu = \sum_{j'=1}^m \hat{q}_{ij'} - \mu = 0 \leq \hat{x}_i\). If \(\sum_{j=1}^m \hat{q}_{ij} \geq \varepsilon\), then \(\varepsilon = \mu\). Thus, \(1^m \bar{Q}^T = 1^m \hat{Q}^T - \mu = 1^m \hat{Q}^T - \varepsilon \leq \hat{x}_i - \varepsilon = \hat{x}_i\). We also have \(1^m \bar{Q}^T = 1^m \hat{Q}^T + \mu \leq 1^m \hat{Q}^T + \varepsilon \leq \hat{x}_i + \varepsilon = \hat{x}_i\). For any \(i'' \neq i, i'\), it is easy to see that \(1^m \bar{Q}^T = 1^m \hat{Q}^T \leq \hat{x}_i\). Therefore, \(1^m \bar{Q}^T \leq \hat{x}\).

Finally, \(1^n \bar{Q} = 1^n \hat{Q} + 1^n (-\sum_{j'=1}^m \mu_{j'} e_{ij'}^{n \times m} + \sum_{j'=1}^m \nu_{j'} e_{ij'}^{n \times m}) = 1^n \hat{Q} + 0 \leq \hat{y} = \hat{y}\).

Now under the new state \((\hat{x}, \hat{y})\) \(\overset{\text{def}}{=} (\hat{x} - \varepsilon e_i^n + \varepsilon e_{ij'}^n, \hat{y})\) and feasible decision \(\hat{Q}\), the post-matching levels are given by: \(\hat{u} = \hat{x} - 1^m \hat{Q}^T = \hat{x} - \varepsilon e_i^n + \varepsilon e_{ij'}^n - 1^m (\hat{Q} - \sum_{j'=1}^m \nu_{j'} e_{ij'}^{n \times m} + \sum_{j'=1}^m \mu_{j'} e_{ij'}^{n \times m})^T = \hat{x} - 1^m \hat{Q}^T - (\varepsilon - \mu) e_i^n + (\varepsilon - \mu) e_{ij'}^n = \hat{u} - (\varepsilon - \mu) e_i^n + (\varepsilon - \mu) e_{ij'}^n\), and \(\hat{v} = 1^n (\hat{Q} - \sum_{j'=1}^m \mu_{j'} e_{ij'}^{n \times m} + \sum_{j'=1}^m \nu_{j'} e_{ij'}^{n \times m}) = 1^n \hat{Q} + 0 = \hat{v}\).

Since \(\bar{Q}\) is feasible for the state \((\hat{x}, \hat{y})\) \(\overset{\text{def}}{=} (\hat{x} - \varepsilon e_i^n + \varepsilon e_{ij'}^n, \hat{y})\), we have

\[
V_i(\hat{x} - \varepsilon e_i^n + \varepsilon e_{ij'}^n, \hat{y}) - V_i(\hat{x}, \hat{y}) \\
\geq H_i(\hat{Q}, \hat{x} - \varepsilon e_i^n + \varepsilon e_{ij'}^n, \hat{y}) - H_i(\hat{Q}, \hat{x}, \hat{y}) \\
= \sum_{j'=1}^m \mu_{j'} (r_{ij'} - r_{ij'}) + \gamma E V_{i+1}(\alpha \hat{u} + D - \alpha (\varepsilon - \mu) e_i^n + \alpha (\varepsilon - \mu) e_{ij'}^n, \beta \hat{v} + S) - \gamma E V_{i+1}(\alpha \hat{u} + D, \beta \hat{v} + S).
\]

(2)

By the induction hypothesis, given state \((\hat{x}, \hat{y})\), for \(0 \leq \alpha (\varepsilon - \mu) \leq \varepsilon \leq \hat{x}_i\), there exists \((\nu_1, \ldots, \nu_m) \geq 0\) for such that \(\sum_{j'=1}^m \nu_{j'} \leq \alpha (\varepsilon - \mu)\) and

\[
V_{i+1}(\hat{x} - \alpha (\varepsilon - \mu) e_i^n + \alpha (\varepsilon - \mu) e_{ij'}^n, \hat{y}) - V_{i+1}(\hat{x}, \hat{y}) \geq - \sum_{j'=1}^m \nu_{j'} (r_{ij'} - r_{ij'}).\n\]
This ensures that there exists \((\zeta_1, \ldots, \zeta_m) = E_{D,S}[(\nu_1, \ldots, \nu_m)(\hat{x}, \hat{y}) = (\alpha \hat{u} + D, \beta \hat{v} + S)] \geq 0\) such that 
\[
\sum_{j=1}^{m} \zeta_j' \leq \alpha(\varepsilon - \mu) \quad \text{and} \quad EV_{t+1}(\alpha \hat{u} + D - \alpha(\varepsilon - \mu)e_i^n + \alpha(\varepsilon - \mu)e_i'^n, \beta \hat{v} + S) - EV_{t+1}(\alpha \hat{u} + D, \beta \hat{v} + S) \geq -\sum_{j' = 1}^{m} \zeta_j'(r_{ij'} - r_{i'j'}). 
\]
Combining (2) and (3), we have 
\[
V_i(\dot{x} - \varepsilon e_i^n + \varepsilon e_i'^n, \dot{y}) - V_i(\dot{x}, \dot{y}) \geq -\sum_{j'=1}^{m} \mu_{j'}(r_{ij'} - r_{i'j'}) - \gamma \sum_{j'=1}^{m} \zeta_j'(r_{ij'} - r_{i'j'}) = -\sum_{j'=1}^{m} (\mu_{j'} + \gamma \zeta_j')(r_{ij'} - r_{i'j'}). 
\]
Let \(\lambda_{j'} = \mu_{j'} + \gamma \zeta_{j'}\) for all \(j' \in S\). Then, \(\sum_{j'=1}^{m} \lambda_{j'} \leq \mu + \gamma \alpha(\varepsilon - \mu) \leq \varepsilon\) because \(\gamma, \alpha \leq 1\). This completes the induction. 

Lemma 4 provides a lower bound in the deviation of the value functions when the state of one type increases by a small amount and that of another type on the same side decreases by the same amount.

**Theorem 1** (When prioritizing \((i, j)\) over \((i, j')\) is optimal). If \((i, j) \succeq (i, j')\), there exists an optimal matching policy \(Q^*\) such that 
\[
\min\{v_j^*, q_{ij'}^*\} = 0,
\]
i.e., either the post-matching level of type \(j\) supply is 0, or the matching quantity between type \(i\) demand and type \(j'\) supply is 0. Symmetrically, if \((i, j) \succeq (i', j)\), there exists an optimal matching policy \(Q^*\) such that 
\[
\min\{u_i^*, q_{ij}^*\} = 0.
\]

**Proof of Theorem 1.** We prove the first statement of the theorem. Then the second statement follows by symmetry. The idea is to prove by contradiction and construction. If at an optimal matching decision, type \(j\) supply has not been exhausted (i.e., \(v_j^* > 0\)) and the matching quantity on arc \((i, j')\) is positive (i.e., \(q_{ij'}^* > 0\)), then we can always construct a feasible matching decision that weakly improves the original one by increasing the matching quantity on arc \((i, j)\) by a small amount and decreasing the matching quantity on arc \((i, j')\) by the same amount.

Suppose that an optimal matching decision \(Q^*\) and its derived post-matching supply level \(v^*\) in a given period \(t\) satisfy \(v_j^* > 0\) and \(q_{ij'}^* > 0\) (otherwise, we have already reached the desired property).

Then, we select \(\varepsilon = \min(v_j^*, q_{ij'}^*) > 0\). As a result, \(v_j^* - \varepsilon \geq 0\) and \(q_{ij'}^* - \varepsilon \geq 0\). Consider a matching decision \(Q^* + \varepsilon e_i'^{n \times m} - \varepsilon e_i'^{n \times m} \geq 0\) which has a post-matching levels \((u_i^*, v^* - \varepsilon e_j^n + \varepsilon e_j'^n) \geq 0\) and hence is feasible. Moreover, the decision \(Q^* + \varepsilon e_i'^{n \times m} - \varepsilon e_i'^{n \times m} \geq 0\) satisfies the desired property, i.e., that either the post-matching level of type \(j\) supply is 0 (if \(v_j^* \leq q_{ij'}^*\)) or the matching quantity
between type $i$ demand and type $j'$ supply is 0 (if $q_{ij'}^* \leq v_j^*$). Lastly, it is sufficient to show that $Q^* + \varepsilon e_{ij}^{n \times m} - \varepsilon e_{ij'}^{n \times m} \succeq 0$ weakly dominates $Q^*$.

We have

$$H_t(Q^* + \varepsilon e_{ij}^{n \times m} - \varepsilon e_{ij'}^{n \times m}, x, y) - H_t(Q^*, x, y)$$

$$= \varepsilon(r_{ij} - r_{ij'}) + \gamma EV_{t+1}(\alpha u^* + D, \beta(v^* - e_{ij}^m + e_{ij'}^m) + S) - \gamma EV_{t+1}(\alpha u^* + D, \beta v + S).$$

By Lemma 4 part (ii), for $\varepsilon \in [0, v_j^*]$, there exists $(\hat{\xi}_1, \ldots, \hat{\xi}_n)_{D,S} \geq 0$ such that $\sum_{k=1}^n \hat{\xi}_k(D,S) \leq \beta \varepsilon$ and

$$V_{t+1}(\alpha u^* + D, \beta(v^* - e_{ij}^m + e_{ij'}^m) + S) - V_{t+1}(\alpha u^* + D, \beta v + S) \geq -\sum_{\hat{i}'=1}^n \hat{\xi}_{\hat{i}'}(D,S) \cdot (r_{\hat{i}'j} - r_{\hat{i}'j'}).$$

Let $(\xi_1, \ldots, \xi_n) = E_{D,S}[(\hat{\xi}_1, \ldots, \hat{\xi}_n)] \geq 0$. Then,

$$E[V_{t+1}(\alpha u^* + D, \beta(v^* - e_{ij}^m + e_{ij'}^m) + S) - V_{t+1}(\alpha u^* + D, \beta v + S)] \geq -\sum_{\hat{i}'=1}^n \xi_{\hat{i}'}(r_{\hat{i}'j} - r_{\hat{i}'j'}).$$

Because $(i,j) \succeq (i,j')$, we have $r_{ij} \geq r_{ij'}$ and $r_{ij} + r_{\hat{i}'j} - r_{ij'} - r_{\hat{i}'j} \geq 0$. Then,

$$H_t(Q^* + \varepsilon e_{ij}^{n \times m} - \varepsilon e_{ij'}^{n \times m}, x, y) - H_t(Q^*, x, y)$$

$$\geq \varepsilon(r_{ij} - r_{ij'}) - \gamma \sum_{\hat{i}'=1}^n \xi_{\hat{i}'}(r_{\hat{i}'j} - r_{\hat{i}'j'}) \geq \frac{1}{\beta} \sum_{\hat{i}'=1}^n \xi_{\hat{i}'}(r_{ij} - r_{ij'}) - \gamma \sum_{\hat{i}'=1}^n \xi_{\hat{i}'}(r_{\hat{i}'j} - r_{\hat{i}'j'})$$

$$\geq \gamma \sum_{\hat{i}'=1}^n \xi_{\hat{i}'}(r_{ij} - r_{ij'}) - \gamma \sum_{\hat{i}'=1}^n \xi_{\hat{i}'}(r_{\hat{i}'j} - r_{\hat{i}'j'}) = \gamma \sum_{\hat{i}'=1}^n \xi_{\hat{i}'}(r_{ij} + r_{\hat{i}'j} - r_{ij'} - r_{\hat{i}'j}) \succeq 0,$$ \hspace{1cm} (4)

where the second inequality is due to $\sum_{k=1}^n \xi_k \leq \beta \varepsilon$ and $r_{ij} \geq r_{ij'}$, the third inequality is due to $\beta, \gamma \leq 1$, and the last inequality is due to $r_{ij} + r_{\hat{i}'j} - r_{ij'} - r_{\hat{i}'j} \geq 0$. This implies that $Q^* + \varepsilon e_{ij}^{n \times m} - \varepsilon e_{ij'}^{n \times m}$ weakly dominates $Q^*$. \hfill \Box

Theorem 1 focuses on comparing two neighboring pairs of demand and supply types $(i,j)$ and $(i,j')$. It provides a sufficient condition under which it is optimal for type $j$ supply to have a higher priority of being matched with type $i$ demand over type $j'$ supply. That is, if $(i,j) \succeq (i,j')$, it is optimal not to match type $i$ demand with type $j'$ supply unless type $j$ supply has been exhausted; i.e., if $q_{ij'}^* > 0$, then $v_j^* = 0$. The dominance relation $(i,j) \succeq (i,j')$ contains two conditions. First, the condition $r_{ij} \geq r_{ij'}$ is naturally to be expected. In fact, for a special case with one demand type and two supply types, we can easily show that the demand-supply pair $(1,1)$ has priority over the pair $(1,2)$ in the optimal matching if and only if $r_{11} \geq r_{12}$. Second, the other condition requires that condition $(D)$ hold for all $i' \in D$ such that $i' \neq i$. If this fails to hold, it is possible that the firm would prefer to match type $i$ demand with type $j'$ supply instead of type $j$ supply.

The dominance relation given in Theorem 1 may not be necessary for some scenarios, as in the following example.
Example 1. Consider $\mathcal{D} = \{1, 2, 3\}$ and $\mathcal{S} = \{1, 2, 3\}$. Let $r_{13} = r_{22} = r_{31} = r_{33} = \varepsilon$, $r_{12} = r_{21} = N$, $r_{11} = \frac{3}{2} N$, $r_{23} = r_{32} = 2 N$, $c = h = \varepsilon$, where $\varepsilon$ is sufficiently small and $N$ is sufficiently large. In the current period, assume $x = y = (1, 1, 0)$. On the one hand, when there is a high chance of type 3 demand or supply arriving in the next period, it is optimal for the firm to save the unit of type 2 demand and the unit of type 2 supply for the future; i.e., $q_{i'j'}^* = q_{i'2}^* = 0$ for all $i' \in \mathcal{D}$, $j' \in \mathcal{S}$. On the other hand, it is optimal to fully match the unit of type 1 demand and the unit of type 1 supply, i.e., $q_{i1}^* = 1$. Thus, it is optimal to prioritize matching type 1 demand and type 1 supply over matching type 1 demand and type 2 supply. However, in this example, $r_{11} + r_{22} < r_{12} + r_{21}$, which violates the condition $(1, 1) \succeq (1, 2)$. $\Box$

Nevertheless, the dominance relation given in Theorem 1 is “necessary” in a robust sense against all possible scenarios. In other words, if the dominance relation $(i, j) \succeq (i, j')$ fails to hold, one can construct a scenario such that the firm would prefer to match type $i$ demand with type $j'$ supply instead of type $j$ supply.

4.3. Perfect Pair

Next we provide a sufficient condition under which demand and supply are matched as much as possible between a pair of demand and supply types in preference to all other possible matching options.

Theorem 2 (When greedy matching is optimal). If $(i, j) \succeq (i, j')$ for all $j' \in \mathcal{S}$ and $(i, j) \succeq (i', j)$ for all $i' \in \mathcal{D}$, then the optimal quantity to match between type $i$ demand and type $j$ supply is $q_{ij}^* = \min \{x_i, y_j\}$.

Proof of Theorem 2. We prove this theorem by induction on $t$. For $t = T + 1$, it is obvious that the result holds. Suppose that the result holds for period $t + 1$, $t \leq T$. We show that it also holds for period $t$.

Now consider period $t \leq T$. Without loss of generality, we can assume that both $x_i$ and $y_j$ are positive in period $t$. Otherwise if $x_i = 0$ or $y_j = 0$, the result clearly holds because the only feasible choice for $q_{ij}$ is zero and thus $q_{ij}^* = 0 = \min \{x_i, y_j\}$.

Fix any $(x, y) > 0$. Suppose that in the optimal matching policy $Q^*$, $q_{ij}^* < \min \{x_i, y_j\}$. It is sufficient to show that there exists $\varepsilon > 0$ such that an alternative matching plan $\bar{Q}$, in which $\bar{q}_{ij} = q_{ij}^* + \varepsilon$, weakly dominates $Q^*$. In other words, the firm can improve weakly by matching $\varepsilon$ more of type $i$ demand and type $j$ supply.

One of the following scenarios must hold for the post-matching quantities $u_i^*$ and $v_j^*$: Case (i) $u_i^* > 0$ and $v_j^* > 0$; Case (ii) $u_i^* = 0$ and $v_j^* > 0$, or $u_i^* > 0$ and $v_j^* = 0$; Case (iii) $u_i^* = 0$ and $v_j^* = 0$. The
Thus, $q_i^* > 0$ and $v_j^* > 0$. We choose $\varepsilon > 0$ such that $u_i^* - \varepsilon > 0$ and $v_j^* - \varepsilon > 0$. Consider an alternative matching plan $\bar{Q} = Q^* + \varepsilon e_{ij}^n$, which is clearly feasible. Then,

$$H_t(\bar{Q}, x, y) - H_t(Q^*, x, y) = r_{ij} \varepsilon + (h + c) \varepsilon + \gamma EV_{t+1}(\alpha u^* + D - \alpha \varepsilon e_i^n, \beta v^* + S - \beta \varepsilon e_j^m) - \gamma EV_{t+1}(\alpha u^* + D, \beta v^* + S). \quad (5)$$

If $t = T$, then $H_t(\bar{Q}, x, y) - H_t(Q^*, x, y) = r_{ij} \varepsilon + (h + c) \varepsilon \geq 0$.

If $t < T$, then by the induction hypothesis, in period $t + 1$, the optimal quantity to match between type $i$ demand and type $j$ supply is $q_{ij}^*(t + 1) = \min \{x_i(t + 1), y_j(t + 1)\}$. Consider the case $\beta \geq \alpha$. It is easy to see that

$$V_{t+1}(\alpha u^* + D - \alpha \varepsilon e_i^n, \beta v^* + S - \beta \varepsilon e_j^m) = V_{t+1}(\alpha u^* + D + (\beta - \alpha) \varepsilon e_i^n, \beta v^* + S - \beta \varepsilon e_j^m) \geq V_{t+1}(\alpha u^* + D, \beta v^* + S - \beta \varepsilon e_j^m), \quad (6)$$

because of the greedy matching of pair $(i, j)$ for the subsequent periods.

Now compare two systems that start in period $t + 1$ with the states $(\alpha u^* + D + (\beta - \alpha) \varepsilon e_i^n, \beta v^* + S)$ and $(\alpha u^* + D, \beta v^* + S)$, respectively. The former system has the option of holding the additional amount $(\beta - \alpha) \varepsilon$ of type $i$ demand and mimicking the optimal matching policy of the latter system from period $t + 1$ to period $T$. In this way, the former system incurs the extra cost $c(\beta - \alpha) \varepsilon \sum_{t=0}^{T-1} \alpha^t \gamma^t \leq c(\beta - \alpha) \varepsilon / (1 - \alpha \gamma) \leq c \varepsilon$. That is,

$$V_{t+1}(\alpha u^* + D + (\beta - \alpha) \varepsilon e_i^n, \beta v^* + S) \geq V_{t+1}(\alpha u^* + D, \beta v^* + S) - c \varepsilon. \quad (7)$$

Then, combining (5), (6) and (7), we have

$$H_t(\bar{Q}, x, y) - H_t(Q^*, x, y) \geq r_{ij} \varepsilon + (h + c) \varepsilon - \gamma \beta \varepsilon r_{ij} + \gamma EV_{t+1}(\alpha u^* + D + (\beta - \alpha) \varepsilon e_i^n, \beta v^* + S) - \gamma EV_{t+1}(\alpha u^* + D, \beta v^* + S) \geq r_{ij} \varepsilon + (h + c) \varepsilon - \gamma \beta \varepsilon r_{ij} - \gamma c \varepsilon \geq 0,$$

which demonstrates that $\bar{Q}$ weakly dominates $Q^*$. Similarly, we can reach the same conclusion if $\alpha > \beta$.

Case (ii): Suppose that $u_i^* > 0$ and $v_j^* = 0$. By Theorem 1, under the conditions that $(i, j) \succeq (i', j)$ for all $i' \in D$, we know that $q_{ij}^* = 0$ for any $i' \in D$ and $i' \neq i$. Then, $0 = v_j^* = y_j - \sum_{i'=1}^n q_{i'j}^* = y_j - q_{ij}^*$. Thus, $q_j^* = y_j \geq \min \{x_i, y_j\}$, implying that $q_{ij}^* = \min \{x_i, y_j\}$ because $q_{ij}^* \leq \min \{x_i, y_j\}$.

Similarly, we can prove $q_{ij}^* = \min \{x_i, y_j\}$ if $u_i^* = 0$ and $v_j^* > 0$. 
Case (iii): $u^*_i = v^*_j = 0$. Assume $q^*_ij < \min \{x_i, y_j\}$. Then, there must exist $j' \neq j$ and $i' \neq i$ such that $q^*_{ij'} > 0$ and $q^*_i,j > 0$. We choose $\varepsilon > 0$ such that $q^*_{ij'} - \varepsilon > 0$ and $q^*_i,j - \varepsilon > 0$ and define $Q$ as $\tilde{Q} = Q^* + \varepsilon(e^n_{ij} + e^n_{ij'} - e^n_{ij} - e^n_{ij'})$. The decision $\tilde{Q}$ is feasible, because $\tilde{Q} \geq 0$ and the post-matching levels of $\tilde{Q}$ are the same as that of $Q^*$. Then, $H_i(Q, x, y) - H_i(Q^*, x, y) = \varepsilon(r_{ij} + r_{ij'} - r_{ij} - r_{ij'}) \geq 0$, implying that $\tilde{Q}$, in which $\tilde{q}_{ij} = q^*_{ij} + \varepsilon$, weakly dominates $Q^*$. Following the same argument, we can always find an optimal decision $\tilde{Q}$ in which $\tilde{q}_{ij} = \min \{x_i, y_j\}$.

Theorem 2 is not a direct consequence of Theorem 1. By directly applying Theorem 1, we can only say that under the conditions in Theorem 2, it is optimal for the firm to prioritize the matching of type $i$ demand and type $j$ supply over any other possibilities. However, it may still be possible that the firm has reserved some type $i$ demand and type $j$ supply without greedily matching them. The proof of Theorem 2 goes beyond Theorem 1 by eliminating such a possibility.

The conditions in Theorem 2, i.e., $(i, j) \succeq (i', j')$ for all $j'$ and $(i, j) \succeq (i', j)$ for all $i'$, say that the pair of type $i$ demand and type $j$ supply dominates all other pairs that share type $i$ demand or type $j$ supply. We say that such a pair forms a perfect pair in the eyes of the intermediary firm. Example 1 also serves as a counterexample illustrating that conditions in Theorem 2 are not necessary for some scenario, though one can say that they are “necessary” in a robust sense against all possible scenarios. The dominance relations in Theorem 2 contain two sets of conditions. The first set, $r_{ij} \geq \max_{i' \in D, j' \in S} \{r_{ij'}, r_{i'j}\}$, says that the matching between type $i$ demand and type $j$ supply generates the highest reward among other possible uses of those resources. As a result, type $i$ demand and type $j$ supply are the most favorable for each other from their own perspective. However, they may not form a perfect pair from the intermediary’s point of view unless another set of conditions is satisfied. The following example illustrates that the condition $r_{ij} \geq \max_{i' \in D, j' \in S} \{r_{ij'}, r_{i'j}\}$ is not enough for the intermediary firm to adopt a greedy match. This is because from a centralized planner’s perspective, the components of a most favorable pair for each other may be separately paired with others to generate an overall higher reward. This example emphasizes the importance of the second set of conditions in the dominance relations – i.e., condition (D) holds for all $i'$ with any given $j'$ and for all $j'$ with any given $i'$ – for guaranteeing that a greedy match between type $i$ demand and type $j$ supply will be optimal.

Example 2. The claim in Theorem 2 may fail without the set of condition (D)’s even for a single-period model. To see this, consider a one-period example with $D = \{1, 2, 3\}$ and $S = \{1, 2, 3\}$. Suppose that $r_{11} = r_{22} = r_{33} = 2N$, $r_{12} = r_{21} = r_{23} = r_{32} = N + \varepsilon$, $r_{13} = r_{31} = \varepsilon$, where $N > \varepsilon > 0$. Here, $r_{22} \geq \max \{r_{21}, r_{23}, r_{12}, r_{32}\}$, i.e., $(2, 2)$ generates the highest reward. In the current period, assume $x = (1, 1, 0)$ and $y = (0, 1, 1)$. If we fully match type 2 demand with type 2 supply, then
type 1 demand has to be matched with type 3 supply given there is only one period, leading to a total reward of \( r_{22} + r_{13} = 2N + \varepsilon \). Alternatively, if we match the type 2 demand with type 3 supply and match the type 1 demand with type 2 supply, the total reward is \( r_{23} + r_{12} = 2(N + \varepsilon) \), which is higher than \( r_{22} + r_{13} \), violating the condition \((2, 2) \succeq (2, 3)\). In this example, we see that although type 2 demand and type 2 supply are the most favorable for each other in terms of generating the highest reward, they are not a perfect pair in the eyes of the centralized planner and matching them leads to a lower total reward. \( \square \)

As an immediate application of Theorem 2, consider demand and supply types that are specified by their locations in an Euclidean space. The reward of matching supply with demand is a fixed amount minus the disutility proportional to the Euclidean distance between the supply location and the demand location. It is easy to verify that a demand type and a supply type from the same location forms a perfect pair, and by Theorem 2, they should be matched as much as possible. To see why they are a perfect pair, we have \( r_{ii} + r_{i_j j'} \geq r_{ij} + r_{i'_j} \) because \( d_{ij} \leq d_{ij'} + d_{i'j} \), where \( d_{ij} \) is the Euclidean distance between the locations of type \( i \) demand and type \( j \) supply. The latter inequality is simply the triangle inequality.

**Corollary 1.** In an Euclidean space with horizontally differentiated types as locations, it is optimal to greedily match the demand and supply from the same location.

Corollary 1 suggests that in those two-sided markets with geographic locations as types, the intermediary firm such as Uber and Amazon should always match a demand with a supply if they are originated from the same geographic region, or practically speaking, if they are sufficiently close to each other.

### 4.4. Global Priority Structure

We now generalize the previous two theorems. In previous results, we considered local priority properties between neighboring arcs. In the generalization, we will show that there exists an optimal matching policy in which for any two pairs of (not necessarily neighboring) arcs with one strictly dominating the other (i.e., for any \((i, j) \succeq (i', j')\) and \((i, j) \not\simeq (i', j')\)), either the dominating arc has been matched as much as possible before all of its strictly dominated neighboring arcs, or the dominated arc has zero matching quantities.

To achieve this, we first show that, given two pairs of not necessarily neighboring arcs with one dominating the other, for any feasible matching policy such that the post-matching levels along the path connecting the two are positive, one can make a weak improvement by increasing the matching quantity on the dominating arc and decreasing the matching quantity on the dominated
arc by the same amount. The improvement is achieved by repeatedly increasing and decreasing the same amount in the matching quantity of two neighboring arcs along the path from the dominated arc to the dominating arc. It is analogous to channeling flows through the tunnel along the path, with the maximum batch size that can be channeled at a time being equal to the capacity of the tunnel. As a result, to channel as much as possible, one may need to send many full batches (equal to the tunnel capacity) plus a partial batch. This is the essence of the proof. See Figure 4 for an illustration.

**Figure 4** Channel “flow” from a dominated arc to a dominating arc.

Note. \((i, j) \succeq (i', j')\). When there is a “tunnel” with positive capacity, i.e., the post-matching levels \(u_{i''} > 0\) and \(v_{j''} > 0\) for all \(i'' \in D \setminus \{i'\}\) and \(j'' \in S \setminus \{j'\}\) such that \((i'', j'') \in P\), one can channel some matching quantity on arc \((i', j')\) to arc \((i, j)\).

**Lemma 5** (When there is a tunnel with positive capacity). Suppose that \((i, j) \succeq (i', j')\) and \(P\) is a decreasing path of arcs connecting \((i, j)\) and \((i', j')\). If under a matching decision \(Q\) the post-matching levels \(u_{i''} > 0\) and \(v_{j''} > 0\) for all \(i'' \in D \setminus \{i'\}\) and \(j'' \in S \setminus \{j'\}\) such that \((i'', j'') \in P\), then the matching decision \(Q + \varepsilon e_{i_i}^{n \times m} - \varepsilon e_{j_j}^{n \times m}\), for \(\varepsilon \leq \min \{u_i, v_j, q_{i'j'}\}\), is also feasible and weakly dominates \(Q\).

**Proof of Lemma 5.** Suppose that there is a decreasing path \((i, j) = (i_1, j_1) \succeq (i_2, j_2) \succeq \cdots \succeq (i_{\ell-1}, j_{\ell-1}) \succeq (i_{\ell}, j_{\ell}) = (i', j')\), with \(u_{i_k} > 0\) and \(v_{j_k} > 0\) for \(k = 1, 2, \ldots, \ell - 1\). (If the decreasing path has a different form, the proof is analogous.) Let \(\delta \overset{\text{def}}{=} \min_{1 \leq k \leq \ell - 1} \{u_{i_k}, v_{j_k}, \varepsilon\} = \min_{2 \leq k \leq \ell - 1} \{u_{i_k}, v_{j_k}, \varepsilon\}\), where the second equality is due to \(\varepsilon \leq \min \{u_i, v_j\}\). By the way we select \(\delta\), for any \(2 \leq k \leq \ell\), both the matching decisions \(\tilde{Q}^{(k)} \overset{\text{def}}{=} Q + \delta e_{i_{k-1}j_{k-1}}^{n \times m} - \delta e_{i_{k-1}j_k}^{n \times m}\) and \(Q^{(k)} \overset{\text{def}}{=} Q + \delta e_{i_{k-1}j_{k-1}}^{n \times m} - \delta e_{i_{k-1}j_k}^{n \times m}\) are feasible. Repeatedly applying (4), which applies to
two neighboring arcs, we have $H_i(\tilde{Q}^{(k)}, x, y) \geq H_i(\tilde{Q}^{(k)}, x, y) \geq H_i(\tilde{Q}^{(k+1)}, x, y)$ for $k = 2, \ldots, \ell$, because $(i_k, j_{k-1}) \succeq (i_k, j_k) \succeq (i_k, j_k)$. Hence, $H_i(Q - \delta e_{i_1j_1} + \delta e_{i_1j_1}, x, y) = H_i(Q^{(\ell+1)}, x, y) \geq H_i(Q^{(\ell+1)}, x, y) \geq H_i(Q, x, y)$. Thus we have shown that

$$H_i(Q - \delta e_{i_1j_1} + \delta e_{i_1j_1}, x, y) \geq H_i(Q, x, y).$$

If $\varepsilon \leq \min_{2 \leq k \leq \ell-1} \{u_{i_k}, v_{j_k}\}$, then $\delta = \varepsilon$, and we have proved the claim.

Consider $\delta < \varepsilon$. There must exist some integer $I \in \mathbb{N}$ such that $\varepsilon - I\delta \leq \min_{2 \leq k \leq \ell-1} \{u_{i_k}, v_{j_k}\} < \varepsilon - (I - 1)\delta$. Note that it must be true that $\varepsilon - I\delta \geq 0$. Otherwise, $\min_{2 \leq k \leq \ell-1} \{u_{i_k}, v_{j_k}\} > \varepsilon - I\delta < \delta$, contradicting our choice of $\delta$. On the other hand, if $\varepsilon - I\delta > \delta$, then $\min_{2 \leq k \leq \ell-1} \{u_{i_k}, v_{j_k}\} \geq \varepsilon - I\delta > \delta$, implying that $\delta = \varepsilon$, in which case the claim has been already proved by (8). Therefore, we assume $0 \leq \varepsilon - I\delta \leq \delta$ in the rest of the proof without loss of generality. Under this condition, we want to show by induction that for $\kappa = 1, \ldots, I$,

$$H_i(Q - \kappa\delta e_{i_1j_1} + \kappa\delta e_{i_1j_1}, x, y) \geq H_i(Q, x, y).$$

We have already proved the above inequality when $\kappa = 1$. Suppose it holds for $\kappa$. We need to show that it also holds for $\kappa + 1 \leq I$.

The updated matching decision $Q - \kappa\delta e_{i_1j_1} + \kappa\delta e_{i_1j_1}$ leads to the post-matching levels $u_{i_1} - \kappa\delta$ and $v_{j_1} - \kappa\delta$ for type $i_1$ demand and type $j_1$ supply, while the post-matching levels for type $i_k$ demand and type $j_k$ demand are unchanged for $k = 2, \ldots, \ell - 1$. We now repeat the same analysis leading to (8), with $Q$ replaced by $Q - \kappa\delta e_{i_1j_1} + \kappa\delta e_{i_1j_1}$ and $\varepsilon$ replaced with $\varepsilon - \kappa\delta$.

Because $\delta < \varepsilon \leq \min \{u_{i_1}, v_{j_1}\}$ by our assumption, we have $\delta = \min_{2 \leq k \leq \ell-1} \{u_{i_k}, v_{j_k}\}$. Consider $\tilde{\delta} \overset{\text{def}}{=} \min_{2 \leq k \leq \ell-1} \{u_{i_1} - \kappa\delta, v_{j_1} - \kappa\delta, u_{i_k}, v_{j_k}, \varepsilon - \kappa\delta\} = \min_{2 \leq k \leq \ell-1} \{u_{i_k}, v_{j_k}, \varepsilon - \kappa\delta\}$. Since $\varepsilon - \kappa\delta \geq \varepsilon - (I - 1)\delta > \min_{2 \leq k \leq \ell-1} \{u_{i_k}, v_{j_k}\}$, we have $\tilde{\delta} = \min_{2 \leq k \leq \ell-1} \{u_{i_k}, v_{j_k}\} = \delta$. It then follows from the same analysis we use to derive (8) that

$$H_i(Q - (\kappa + 1)\delta e_{i_1j_1} + (\kappa + 1)e_{i_1j_1}, x, y) \geq H_i(Q - \kappa\delta e_{i_1j_1} + \kappa\delta e_{i_1j_1}, x, y).$$

Combining (10) with (9), we have $H_i(Q - (\kappa + 1)\delta e_{i_1j_1} + (\kappa + 1)e_{i_1j_1}, x, y) \geq H_i(Q, x, y)$. Thus the induction is completed.

Finally, consider the decision $Q - I\delta e_{i_1j_1} + I\delta e_{i_1j_1}$. It leads to the post-matching levels $u_{i_1} - I\delta$ and $v_{j_1} - I\delta$ for type $i_1$ demand and type $j_1$ supply, while the post-matching levels for type $i_k$ demand and type $j_k$ demand are unchanged for $k = 2, \ldots, \ell - 1$. We again repeat the analysis leading to (8), but with $Q$ replaced by $Q - I\delta e_{i_1j_1} + I\delta e_{i_1j_1}$ and $\varepsilon$ replaced with $\varepsilon - I\delta$. 
Let \( \hat{\delta} = \min_{2 \leq k \leq \ell - 1} \{u_{i_1} - I\delta, v_{j_1} - I\delta, u_{i_k}, v_{j_k}, \varepsilon - I\delta\} = \min_{2 \leq k \leq \ell - 1} \{u_{i_k}, v_{j_k}, \varepsilon - I\delta\} \). Since we have assumed that \( \varepsilon - I\delta \leq \min_{2 \leq k \leq \ell - 1} \{u_{i_k}, v_{j_k}\} \), we have \( \hat{\delta} = \varepsilon - I\delta \). Then, we have

\[
H_i(Q - \varepsilon e_{i\ell}^{n \times m} + \varepsilon e_{i1j1}^{n \times m}, x, y) = H_i(Q - I\delta e_{i\ell}^{n \times m} + I\delta e_{i1j1}^{n \times m} - \hat{\delta} e_{i\ell}^{n \times m} + \hat{\delta} e_{i1j1}^{n \times m}, x, y) \\
\geq H_i(Q - I\delta e_{i\ell}^{n \times m} + I\delta e_{i1j1}^{n \times m}, x, y) \geq H_i(Q, x, y),
\]

where the first inequality is due to the same analysis as we use to derive (8) and the second inequality is exactly (9) with \( \kappa = I \). Thus the claim is proved. \( \Box \)

Next we show that for a feasible matching policy, which may not have a tunnel with a positive capacity connecting two ordered pairs of arcs (in other words, there exists a node on the connecting path which has zero post-matching level), we can always construct another feasible matching policy such that it does have a desired tunnel through which we can channel flows, and the difference between the two policies can be sufficiently small.

**Lemma 6** (When there exists no tunnel with positive capacity). Consider a decreasing path \( \mathcal{P} \) connecting \((i, j)\) and \((i', j')\), a state \((x, y) > 0\), and a feasible matching decision \( Q \) with \( u_i > 0, v_j > 0 \) and \( q_{i'j'} > 0 \). For any \( \varepsilon > 0 \), there exists a feasible decision \( Q^{[\varepsilon]} \) such that \( \|Q - Q^{[\varepsilon]}\| \leq \varepsilon \), and the corresponding post-matching levels \( u^{[\varepsilon]}_{ij} > 0 \) and \( v^{[\varepsilon]}_{j'i'} > 0 \) for any \( i'' \) and \( j'' \) on \( \mathcal{P} \).

**Proof of Lemma 6.** It is without loss of generality by assuming \( \varepsilon \) to be sufficiently small, e.g., \( 0 < \varepsilon \leq \min\{u_i, v_j, q_{i'j'}\} \).

First, we consider the case in which \((i, j)\) and \((i', j')\) share a node, say \( i = i' \), and \( \mathcal{P} \) takes the form \((i, j) \succeq (i', j')\) without loss of generality. Then, we can simply let \( Q^{[\varepsilon]} = Q - \frac{1}{2} \varepsilon e_{i'j'}^{n \times m} + \frac{1}{2} \varepsilon e_{ij}^{n \times m} \). Thus \( \|Q - Q^{[\varepsilon]}\| \leq \varepsilon \). Moreover, since \( \varepsilon \leq \min\{u_i, v_j, q_{i'j'}\} \), \( Q^{[\varepsilon]} \) is feasible with post-matching levels positive for both \((i, j)\) and \((i', j')\).

Now suppose that the lemma holds for a path \( \mathcal{P}_{\ell, \ell} \) that takes the form \((i, j) = (i_1, j_1) \succeq (i_2, j_2) \succeq \cdots \succeq (i_\ell, j_\ell) = (i', j')\) as the induction hypothesis. We will show that it will also hold for the path \( \mathcal{P}_{\ell+1, \ell+1} \) that takes the form \((i, j) = (i_1, j_1) \succeq (i_2, j_2) \succeq \cdots \succeq (i_\ell, j_\ell) \succeq (i_{\ell+1}, j_{\ell+1}) = (i', j')\).

To that end, we consider the following cases.

**Case 1:** \( q_{i_{\ell+1}j_{\ell+1}} > 0 \).

In this case, we can apply the induction hypothesis to construct a feasible decision \( \tilde{Q} \) such that \( \|\tilde{Q} - Q\| \leq \frac{3}{2} \varepsilon \) and \( \tilde{u}_k > 0, \tilde{v}_k > 0 \) for \( k = 1, \ldots, \ell \). Let \( Q^{[\varepsilon]} = \tilde{Q} - \frac{1}{4} \varepsilon e_{i_{\ell+1}\ell+1}^{n \times m} + \frac{1}{4} \varepsilon e_{i_1j_1}^{n \times m} \). By Lemma 5, \( Q^{[\varepsilon]} \) weakly dominates \( \tilde{Q} \). Also, we have \( u^{[\varepsilon]}_{i_1} = \tilde{u}_{i_1} - \frac{1}{4} \varepsilon \geq u_{i_1} - \frac{3}{4} \varepsilon > 0 \) and \( v^{[\varepsilon]}_{j_1} = \tilde{v}_{j_1} - \frac{1}{4} \varepsilon \geq v_{j_1} - \frac{3}{4} \varepsilon > 0 \) if
ε is sufficiently small, $u_k^{[c]} = \bar{u}_k > 0$ for $k = 2, \ldots, \ell - 1$, $v_{jk}^{[c]} = \bar{v}_{jk} > 0$ for $k = 2, \ldots, \ell$, $u_{t_e}^{[c]} = \bar{u}_{t_e} + \frac{1}{2} \varepsilon > 0$ and $v_{t_e}^{[c]} = \bar{v}_{t_e} + \frac{1}{2} \varepsilon > 0$. Moreover, $\|Q^{[c]} - \bar{Q}\| \leq \|Q^{[c]} - \bar{Q}\| + \|\bar{Q} - Q^{[c]}\| \leq \frac{1}{2} \varepsilon + \frac{1}{2} \varepsilon = \varepsilon$.

Case 2: $q_{t_e} = 0$ and $v_{t_e} > 0$.

Let $\bar{Q} = Q - \frac{1}{8} \varepsilon e_{i_{t_e}j_{t_e+1}} + \frac{1}{8} \varepsilon e_{i_{t_e}j_{t_e+1}}$. Then, $\|\bar{Q} - Q\| = \frac{1}{8} \varepsilon$. Now that $q_{t_e} = \frac{1}{8} \varepsilon > 0$, we can apply the induction hypothesis to construct a feasible decision $Q^{[c]}$ such that $\|Q^{[c]} - \bar{Q}\| \leq \frac{1}{8} \varepsilon$ and $u_{t_e}^{[c]} > 0, v_{t_e}^{[c]} > 0$ for $k = 1, \ldots, \ell$. Also, we have

\[
v_{j_{t_e+1}}^{[c]} = y_{j_{t_e+1}} - \sum_{\ell' \in D} q_{\ell'}^{[c]} = y_{j_{t_e+1}} - \sum_{\ell' \in D} \bar{q}_{\ell'}^{[c]} - \sum_{\ell' \in D} \bar{q}_{\ell'}^{[c]} - \sum_{\ell' \in D} \bar{q}_{\ell'}^{[c]} \\
= \frac{1}{8} \varepsilon = v_{j_{t_e+1}} + \frac{1}{8} \varepsilon - \frac{1}{8} \varepsilon = v_{t_e+1} + \frac{1}{8} \varepsilon > 0.
\]

Moreover, $\|Q - Q^{[c]}\| \leq \|Q - \bar{Q}\| + \|\bar{Q} - Q^{[c]}\| \leq \frac{1}{8} \varepsilon < \varepsilon$.

Case 3: $q_{t_e} = 0$ and $v_{t_e} = 0$.

Since $y_{t_e} > 0$ by assumption $\gamma > 0$ and $v_{t_e} = q_{i_{t_e}j_{t_e}} = 0$, there exists $\ell'' \in D \setminus \{i_{t_e}\}$ such that $q_{\ell''} = 0$. Then, for $\varepsilon < \frac{1}{8} \min \{q_{i_{t_e}j_{t_e}}, q_{\ell''_{t_e+1}}\}$, the decision $\bar{Q} = Q + \frac{1}{8} \varepsilon (e_{i_{t_e}j_{t_e}} + e_{i_{t_e}j_{t_e+1}} - e_{i_{t_e}j_{t_e+1}} - e_{\ell''_{t_e+1}})$ is also feasible and weakly dominates $Q$ because $(i_{t_e}, j_{t_e}) \succeq (i_{t_e}, j_{t_e+1})$. (In fact, $H_i(\bar{Q}, x, y) - H_i(Q, x, y) = \frac{1}{8} \varepsilon (r_{i_{t_e}j_{t_e}} + r_{j_{t_e}j_{t_e+1}} - r_{i_{t_e}j_{t_e+1}} - r_{j_{t_e}j_{t_e+1}}) \geq 0$. We see that $\|Q - \bar{Q}\| = \frac{1}{8} \varepsilon$ and $\bar{q}_{i_{t_e}j_{t_e}} = \frac{1}{8} \varepsilon > 0$. By the induction hypothesis, we can find a feasible decision $\bar{Q}$ such that $\bar{u}_k > 0, \bar{v}_k > 0$ and $\|\bar{Q} - Q\| \leq \frac{1}{8} \varepsilon$. Note that $\bar{u}_i \geq \bar{u}_i - \|\bar{Q} - \bar{Q}\| \geq u_i - \frac{1}{8} \varepsilon$ and analogously, $\bar{v}_i \geq v_i - \frac{1}{8} \varepsilon$. If $\varepsilon$ is chosen to be sufficiently small such that $\varepsilon < \min \left\{\frac{1}{8} u_i, \frac{1}{8} v_i, 2 q_{i_{t_e}j_{t_e+1}}\right\}$, then $q_{i_{t_e}j_{t_e+1}} \geq \bar{q}_{i_{t_e}j_{t_e+1}} - \|\bar{Q} - Q\| = q_{i_{t_e}j_{t_e+1}} - \frac{1}{8} \varepsilon - \|\bar{Q} - Q\| \geq q_{i_{t_e}j_{t_e+1}} - \frac{1}{8} \varepsilon$, and further, by Lemma 5, the decision $Q^{[c]} = Q - \frac{1}{8} \varepsilon e_{i_{t_e}j_{t_e+1}} + \frac{1}{8} \varepsilon e_{i_{t_e}j_{t_e+1}}$ is feasible and weakly dominates $\bar{Q}$. It is easy to see that $\|Q^{[c]} - Q\| \leq \|Q^{[c]} - Q\| + \|Q - \bar{Q}\| + \|\bar{Q} - Q\| \leq \varepsilon$. Finally, $u_{t_e}^{[c]} = \bar{u}_{t_e} - \frac{1}{8} \varepsilon \geq u_{t_e} - \frac{1}{8} \varepsilon - \frac{1}{8} \varepsilon = u_{t_e} - \frac{3}{8} \varepsilon > 0$ and analogously, $v_{t_e}^{[c]} \geq v_{t_e} - \frac{3}{8} \varepsilon > 0$; $u_{i_{t_e}}^{[c]} = \bar{u}_{i_{t_e}} + \frac{1}{8} \varepsilon > 0$ for $k = 2, \ldots, \ell - 1$, $v_{j_{t_e}}^{[c]} = \bar{v}_{j_{t_e}} > 0$ for $k = 2, \ldots, \ell$, $u_{t_e}^{[c]} = \bar{u}_{t_e} + \frac{1}{8} \varepsilon > 0$, and $v_{t_e}^{[c]} = \bar{v}_{t_e} + \frac{1}{8} \varepsilon > 0$.

Following analogous arguments, we can show that if the lemma holds for the decreasing path $P_{t_e+1, t_{t_e+1}}$, it will also hold for the path $P_{t_{t_e+1}, t_{t_e+1}}$. This completes the induction and shows that the lemma holds for any decreasing path $P$. □

Before we introduce the global priority structure, we need the following definitions to facilitate the presentation. For any arc $(i, j) \in A$, we define a set of arcs

\[
\mathcal{L}_{ij} = \{(i', j') \mid (i, j) \succeq (i', j'), (i', j') \not\succeq (i, j)\} \cup \{(i, j') \mid (i, j) \succeq (i, j'), (i, j') \not\succeq (i, j)\},
\]
where \((i'', j) \not\in (i, j)\) means that \((i'', j)\) is not equal or equivalent to \((i, j)\). By this definition, \(L_{ij}\) is the set of neighboring arcs that are strictly dominated by \((i, j)\). We also define
\[
w_{ij} = w_{ij}(Q, x, y) \overset{\text{def}}{=} \min\{x_i - \sum_{j': (i,j') \not\in L_{ij}} q_{ij'}, y_j - \sum_{i': (i',j) \not\in L_{ij}} q_{i'j}\}
\]
\[
\overset{\text{def}}{=} \min\{u_i + \sum_{j': (i,j') \in L_{ij}} q_{ij'}, v_j + \sum_{i': (i',j) \in L_{ij}} q_{i'j}\},
\]

**Definition 3.** For a given matching decision \(Q\), we say \(Q\) matches a pair of type \(i\) demand with type \(j\) supply to the maximum extent outside the set of its strictly dominated neighboring arcs, \(L_{ij}\), if \(w_{ij} = 0\).

With the above definition, \(w_{ij} = 0\) is equivalent to
\[
q_{ij} = \min\{x_i - \sum_{j': (i,j') \not\in (L_{ij})} q_{ij'}, y_j - \sum_{i': (i',j') \not\in (L_{ij})} q_{i'j}\},
\]
which says that with the matching quantities over the neighboring arcs in \(A\backslash (L_{ij} \cup \{(i,j)\})\), including arcs either incomparable with \((i,j)\) or that dominate \((i,j)\), subtracted, the remaining type \(i\) demand and type \(j\) supply are matched as much as possible. In other words, \(w_{ij} = 0\) says that either type \(i\) demand or type \(j\) supply is exhausted by the matching quantities over arcs outside the set \(L_{ij}\).

Now we are ready to present the global priority structure, as a generalization of the previous two theorems on local priority properties. Analogously to Lemma 5, the overall idea is to construct a process that improves any given feasible matching policy by "channeling flows" between any two pairs of arcs with a strictly dominance relation. The process leads to an improved policy that either channels flows as much as possible from the dominated arc so that the dominating pair is matched to the maximum extent, or there is no flow left to be channeled from the dominated arc, at which point the dominated arc has zero matching quantities. The difficulty is that for some cases, we may not always find a tunnel with a positive capacity connecting the two arcs through which we can channel flows. For those cases, with the help of Lemma 6, which constructs a "closely enough" feasible policy that does have a tunnel with positive capacity, the constructed process will give us the optimal policy with the desired property either when the process terminates in finite steps or converges in a limit.

**Theorem 3** (Global priority structure in optimal matching). Without loss of generality, assume \(x > 0\) and \(y > 0\) in period \(t\).\(^7\) There exists an optimal decision \(Q^*\) such that for any \((i,j) \succeq (i', j')\) and \((i,j) \not\sim (i', j')\),
\[
\min\{w_{ij}^*, q_{i'j'}^*\} = 0,
\]
\(^7\)If \(x_i = 0\) (or \(y_j = 0\)), we can delete demand node \(i\) (or supply node \(j\)) and all its connected arcs, whose matching quantities are set to be zeros.
i.e., either $Q^*$ matches type $i$ demand with type $j$ supply to the maximum extent outside $L_{ij}$ or $q_{ij}^* = 0$.

**Proof of Theorem 3.** It suffices to show that, for any feasible decision $Q$, there exists another feasible decision $Q^*$ such that $\min \left\{ w_{ij}^*, q_{ij}^* \right\} = 0$ for any $(i, j) \succeq (i', j')$ and $(i, j) \not\succeq (i', j')$. We will show this in three steps.

1. **Divide the set of arcs $A$.**

   We first divide the set of all arcs $A$ into a number of disjoint subsets, which will determine the hierarchy in the optimal matching. Let
   \[
   A_1 \overset{\text{def}}{=} \{(i'', j'') \in A \mid \nexists (i, j) \in A \text{ such that } (i, j) \succeq (i'', j'') \text{ and } (i, j) \not\succeq (i'', j'')\}.
   \]

   Note that $A_1$ must be nonempty. To see this, we can arbitrarily select $(i^1, j^1) \in A$. If $(i^1, j^1) \in A_1$, then $A_1$ is nonempty. Otherwise, we can find $(i^2, j^2) \in A$ such that $(i^2, j^2) \succeq (i^1, j^1)$ and $(i^2, j^2) \not\succeq (i^1, j^1)$. Again, either $(i^2, j^2) \in A_1$, or $(i^3, j^3) \succeq (i^2, j^2)$ and $(i^3, j^3) \not\succeq (i^2, j^2)$ for some $(i^3, j^3) \in A$. If we repeat this process and it stops at some $(i^k, j^k)$ (i.e., there is no more arc $(i^{k+1}, j^{k+1}) \succeq (i^k, j^k)$), then $(i^k, j^k) \in A_1$. Otherwise, we end up with a series of arcs $\{(i^k, j^k)\}_{k=1}^{\infty}$. Since $A$ contains only a finite number of arcs, there exist $k < \ell$ such that $(i^k, j^k) = (i^\ell, j^\ell)$. This implies that $(i^\theta, j^\theta) \succeq (i^k, j^k)$ for $\theta = k, \ldots, \ell$, contradicting our choice of $(i^{k+1}, j^{k+1}) \not\succeq (i^k, j^k)$. Thus we have shown the nonemptiness of $A_1$.

   Recursively, we can define
   \[
   A_k \overset{\text{def}}{=} \left\{(i'', j'') \in A \setminus \bigcup_{\theta=1}^{k-1} A_\theta \mid \nexists (i, j) \in A \setminus \bigcup_{\theta=1}^{k-1} A_\theta \text{ such that } (i, j) \succeq (i'', j'') \text{ and } (i, j) \not\succeq (i'', j'')\right\}.
   \]

   Following the same logic we used to prove the nonemptiness of $A_1$, we can show that $A_k$ is nonempty as long as $A \setminus \bigcup_{\theta=1}^{k-1} A_\theta$ is nonempty. Thus $\{A_\theta\}_{\theta=1}^{k}$ is a strictly increasing sequence of sets and ends at $A$. Since $A$ is a finite set, there exists an integer $K$ such that $A = \bigcup_{\theta=1}^{K} A_\theta$.

   Furthermore, for a feasible decision $Q$ and $J = 1, \ldots, K$, we define
   \[
   M_j(Q) \overset{\text{def}}{=} \sum_{(i'', j'') \in \bigcup_{\theta=1}^{j} A_\theta} q_{i'', j''}
   \]
   as the total matching quantity over all the arcs in $\bigcup_{\theta=1}^{j} A_\theta$.

2. **Transfer matching quantity from a dominated arc to a dominating arc.**

   For $(i, j) \succeq (i', j')$ and $(i, j) \not\succeq (i', j')$, there exists $1 \leq L < N \leq K$ such that $(i, j) \in A_L$ and $(i', j') \in A_N$.
CLAIM 1. Consider a feasible matching decision $Q$. Given a sufficiently small $\delta > 0$, we can construct another feasible decision $\tilde{Q}(Q, \delta)$ that weakly dominates $Q$, such that

$$M_L(\tilde{Q}) \geq M_L(Q) + \min \{w_{ij}, q_{ij}^\nu\} - 2\delta \quad \text{and} \quad M_J(\tilde{Q}) \geq M_J(Q) - \delta \quad \text{for} \quad J = 1, \ldots, L - 1,$$

where

$$w_{ij} \triangleq \min \left\{ x_i - \sum_{j' : (i, j') \notin \mathcal{L}_{ij}} q_{ij'}^\nu, y_j - \sum_{i' : (i', j) \notin \mathcal{L}_{ij}} q_{i'j}^\nu \right\} \equiv \min \left\{ u_i + \sum_{j' : (i, j') \notin \mathcal{L}_{ij}} q_{ij'}^\nu, v_j + \sum_{i' : (i', j) \notin \mathcal{L}_{ij}} q_{i'j}^\nu \right\}.$$

We consider three cases.

Case 1: $u_i = 0$.

If $\sum_{j' : (i, j') \in \mathcal{L}_{ij}} q_{ij'}^\nu \leq v_j$, then $\tilde{Q} \triangleq Q + \sum_{j' : (i, j') \notin \mathcal{L}_{ij}} (q_{ij'}^\nu e_{ij}^{n \times m} - q_{ij'}^\nu e_{ij}^{n \times m})$ is feasible and weakly dominates $Q$ by Lemma 5. From $Q$ to $\tilde{Q}$, the quantity on $(i, j)$ is increased by $\sum_{j' : (i, j') \notin \mathcal{L}_{ij}} q_{ij'}^\nu = u_i + \sum_{j' : (i, j') \notin \mathcal{L}_{ij}} q_{ij'}^\nu$. Since $u_i + \sum_{j' : (i, j') \in \mathcal{L}_{ij}} q_{ij'}^\nu = \sum_{j' : (i, j') \notin \mathcal{L}_{ij}} q_{ij'}^\nu \leq v_j$, the increment of the matching quantity on arc $(i, j)$, from $Q$ to $\tilde{Q}$, is equal to $w_{ij}$. Moreover, the matching quantity on any other arc in $\bigcup_{\delta = 1}^L \mathcal{A}_\delta$, from $Q$ to $\tilde{Q}$, is unchanged.

If $\sum_{j' : (i, j') \in \mathcal{L}_{ij}} q_{ij'}^\nu > v_j$, then select any $\{\lambda_{ij}^\nu\}_{j' : (i, j') \in \mathcal{L}_{ij}}$ that satisfies $0 \leq \lambda_{ij}^\nu \leq q_{ij}^\nu$ for all $(i, j') \in \mathcal{L}_{ij}$ and $\sum_{j' : (i, j') \in \mathcal{L}_{ij}} \lambda_{ij}^\nu = v_j$. The decision $Q^\lambda \triangleq Q + \sum_{j' : (i, j') \in \mathcal{L}_{ij}} (\lambda_{ij}^\nu e_{ij}^{n \times m} - \lambda_{ij}^\nu e_{ij}^{n \times m})$ is feasible and weakly dominates $Q$ by Lemma 5. We see that $v_j = 0$ under the decision $Q^\lambda$.

Now consider nonnegative $\{\mu_{ij}^\nu\}_{(i', j') \in \mathcal{L}_{ij}, (i', j') \in \mathcal{L}_{ij}}$ such that the following conditions are satisfied: (i) $\sum_{j' : (i', j') \in \mathcal{L}_{ij}} \mu_{ij}^\nu \leq q_{ij}^\nu$ for $(i, j') \in \mathcal{L}_{ij}$, (ii) $\sum_{j' : (i', j') \in \mathcal{L}_{ij}} \mu_{ij}^\nu \leq q_{ij}^\nu$ for $(i', j) \in \mathcal{L}_{ij}$, and (iii) $\sum_{j' : (i', j') \in \mathcal{L}_{ij}, (i', j') \in \mathcal{L}_{ij}} \mu_{ij}^\nu = \min \left\{ \sum_{j' : (i', j') \in \mathcal{L}_{ij}} q_{ij}^\nu, \sum_{j' : (i', j') \in \mathcal{L}_{ij}} q_{ij}^\nu \right\}$. We show that such $\{\mu_{ij}^\nu\}_{(i', j') \in \mathcal{L}_{ij}, (i', j') \in \mathcal{L}_{ij}}$ exists. To see this, assume $\sum_{j' : (i', j') \in \mathcal{L}_{ij}} q_{ij}^\nu \leq \sum_{j' : (i', j') \in \mathcal{L}_{ij}} q_{ij}^\nu$. Without loss of generality, we choose $(i_1, j) \in \mathcal{L}_{ij}$. Since $q_{i_1 j} \leq \sum_{j' : (i', j') \in \mathcal{L}_{ij}} q_{ij}^\nu \leq \sum_{j' : (i', j') \in \mathcal{L}_{ij}} q_{ij}^\nu$, there exists $\{\mu_{i_1, j'}\}_{j' : (i_1, j') \in \mathcal{L}_{ij}}$ such that $0 \leq \mu_{i_1, j'} \leq q_{i_1 j}^\nu$ and $\sum_{j' : (i_1, j') \in \mathcal{L}_{ij}} \mu_{i_1, j'} = q_{i_1 j}^\nu$. Then, it remains to find $\{\mu_{i_1, j'}\}_{i \neq i_1, j', (i', j') \in \mathcal{L}_{ij}, (i', j') \in \mathcal{L}_{ij}}$ such that: (i) $\sum_{j' : (i', j') \in \mathcal{L}_{ij}} \mu_{i_1, j'} \leq q_{i_1 j}^\nu - \mu_{i_1, j'}$ for $(i, j') \in \mathcal{L}_{ij}$, (ii) $\sum_{j' : (i', j') \in \mathcal{L}_{ij}} \mu_{i_1, j'} \leq q_{i_1 j}^\nu$ for $i' \neq i_1$ and $(i', j') \in \mathcal{L}_{ij}$, and (iii) $\sum_{j' : (i_1, j') \in \mathcal{L}_{ij}, (i_1, j') \in \mathcal{L}_{ij}} \mu_{i_1, j'} = \min \left\{ \sum_{j' : (i_1, j') \in \mathcal{L}_{ij}} q_{ij}^\nu, \sum_{j' : (i_1, j') \in \mathcal{L}_{ij}} (q_{ij}^\nu - \mu_{i_1, j'}) \right\}$. We see that (i’)-(iii’) are in the same form as (i)-(iii) but have strictly fewer unknown $\mu_{i_1, j'}$’s. Consequently, it can be shown recursively that there indeed exists $\{\mu_{i_1, j'}\}_{(i', j') \in \mathcal{L}_{ij}, (i', j') \in \mathcal{L}_{ij}}$ such that conditions (i)-(iii) hold.

Let $\tilde{Q} \triangleq Q^\lambda + \sum_{j' : (i, j') \in \mathcal{L}_{ij}, (i, j') \in \mathcal{L}_{ij}} \mu_{i_1, j'} (e_{ij}^{n \times m} + e_{ij}^{n \times m} - e_{ij}^{n \times m} - e_{ij}^{n \times m})$. By conditions (i) and (ii) and since adding $(e_{ij}^{n \times m} + e_{ij}^{n \times m} - e_{ij}^{n \times m} - e_{ij}^{n \times m})$ to a matching decision does not change
post-matching levels, $\tilde{Q}$ is feasible; moreover, $\tilde{Q}$ weakly dominates $Q^\lambda$ because $(i,j) \succeq (i'',j)$ and $(i,j) \succeq (i,j'')$. We also have

$$\bar{q}_{ij} - q_{ij} = (q_{ij}^\lambda - q_{ij}) + (\bar{q}_{ij} - q_{ij}^\lambda) = \sum_{i''(i,j'') \in \mathcal{L}_{ij}} \lambda_{i''} + \sum_{i'',j''(i,j'') \in \mathcal{L}_{ij}} \mu_{i'',j''}$$

$$= \sum_{i'',(i,j'') \in \mathcal{L}_{ij}} \lambda_{i''} + \min \left\{ \sum_{i'',(i,j'') \in \mathcal{L}_{ij}} q_{i'',j''}^\lambda, \sum_{j''(i,j'') \in \mathcal{L}_{ij}} q_{ij}^\lambda \right\}$$

$$= \min \left\{ v_j + \sum_{i'',(i,j'') \in \mathcal{L}_{ij}} q_{i'',j''}, \sum_{j''(i,j'') \in \mathcal{L}_{ij}} \lambda_{i''} + \sum_{j''(i,j'') \in \mathcal{L}_{ij}} q_{ij}^\lambda \right\} = w_{ij}.$$  

Note that the fourth equality holds because $\sum_{j''(i,j'') \in \mathcal{L}_{ij}} \lambda_{i''} = v_j$, while the second-to-last equality holds because $q_{ij}^\lambda = q_{ij}^\lambda$ for all $(i'',j) \in \mathcal{L}_{ij}$ and $q_{ij}^\lambda = q_{ij}^\lambda - \lambda_{i''}$ for all $(i,j'') \in \mathcal{L}_{ij}$.

Thus, for case (i) $u_i = 0$, we have constructed $\bar{Q}$ that increases the matching quantity of arc $(i,j)$ by $w_{ij}$ without changing the matching quantity on any other arc in $\bigcup_{\theta = 1}^{L} A_{\theta}$. As a result, $M_L$ is increased by $w_{ij}$ while $M_J$ is unchanged for $J = 1, \ldots, L - 1$. Thus Claim 1 holds.

Case 2: $v_j = 0$.

By symmetry, analogously to Case 1, we can show Claim 1 holds for Case 2.

Case 3: $u_i > 0$ and $v_j > 0$.

We consider three subcases.

Subcase 3.1: $(i,j) \in \mathcal{L}_{ij}$, $q_{ij} > v_j$.

If $\sum_{(i,j'') \in \mathcal{L}_{ij}} q_{ij''} \geq v_j$, we can construct $Q^\lambda \overset{\text{def}}{=} Q + \sum_{j''(i,j'') \in \mathcal{L}_{ij}} (\lambda_{i''} e_{ij}^{nxm} - \lambda_{i''} e_{ij}^{nxm})$ as we have done in Case 1. The decision $Q^\lambda$ is feasible and weakly dominates $Q$. Since $v_j^\lambda = 0$, we can apply Case 2 to $Q^\lambda$ and contract $\bar{Q}$ such that $\bar{q}_{ij} = q_{ij}^\lambda + w_{ij}$ and $\bar{q}_{ij'} = q_{ij'}^\lambda$ for any $(i'',j'') \in \bigcup_{\theta = 1}^{L} A_{\theta}$ and $(i'',j'') \neq (i,j)$. Also, we have $\bar{q}_{ij} - q_{ij} = \bar{q}_{ij} - q_{ij}^\lambda = w_{ij}^\lambda + \sum_{j''(i,j'') \in \mathcal{L}_{ij}} \lambda_{i''}$. It then follows from $w_{ij}^\lambda = \min \left\{ u_i + \sum_{j''(i,j'') \in \mathcal{L}_{ij}} q_{ij''}^\lambda, v_j + \sum_{j''(i,j'') \in \mathcal{L}_{ij}} q_{ij''}^\lambda \right\} = \min \left\{ u_i + \sum_{j''(i,j'') \in \mathcal{L}_{ij}} (q_{ij''} - \lambda_{i''}), \sum_{j''(i,j'') \in \mathcal{L}_{ij}} q_{ij''} \right\}$ and $v_j = \sum_{j''(i,j'') \in \mathcal{L}_{ij}} \lambda_{i''}$ that $\bar{q}_{ij} - q_{ij} = \min \left\{ u_i + \sum_{j''(i,j'') \in \mathcal{L}_{ij}} q_{ij''}, v_j + \sum_{j''(i,j'') \in \mathcal{L}_{ij}} q_{ij''} \right\} = w_{ij}$. Therefore, we have constructed $\bar{Q}$ such that it increases the matching quantity of arc $(i,j)$ by $w_{ij}$, without changing the matching quantity on any other arc in $\bigcup_{\theta = 1}^{L} A_{\theta}$. Thus Claim 1 holds.

Subcase 3.2: $\sum_{(i,j) \in \mathcal{L}_{ij}} q_{ij} \geq u_i$.

By symmetry, analogously to Subcase 3.1, we can show that Claim 1 holds for Subcase 3.2.

Subcase 3.3: $\sum_{(i,j) \in \mathcal{L}_{ij}} q_{ij} < v_j$ and $\sum_{(i,j) \in \mathcal{L}_{ij}} q_{ij} < u_i$. 

If \( \sum_{(i',j') \in C_{ij}} q_{ij'} < v_j \) and \( \sum_{(i'',j') \in C_{ij}} q_{ij''} < u_i \), then \( \tilde{Q} = Q + \sum_{(i'',j') \in C_{ij}} q_{ij''}(e_{ij}^{n \times m} - e_{ij'}^{n \times m}) + \sum_{(i'',j') \in C_{ij}} q_{ij''}(e_{ij'}^{n \times m} - e_{ij''}^{n \times m}) \) is feasible and weakly dominates \( Q \). It is easy to verify that \( \bar{u}_i = u_i - \sum_{(i'',j') \in C_{ij}} q_{ij''} > 0 \) and \( \bar{v}_j = v_j - \sum_{(i',j') \in C_{ij}} q_{ij'} > 0 \).

If \((i, j)\) and \((i', j')\) are two neighboring arcs sharing a node, say \( i = i' \), then from \( Q \) to \( \tilde{Q} \), the matching quantity of arc \((i, j)\) is increased by \( \sum_{(i'',j') \in C_{ij}} q_{ij''} + \sum_{(i'',j') \in C_{ij}} q_{ij''} > q_{ij'} \) without changing the matching quantity on any other \((i'', j'')\) in \( \bigcup_{\theta=1}^{L} A_{\theta} \). Then if \((i, j)\) and \((i', j')\) share a node, we can simply let \( \tilde{Q} = \tilde{Q} \) and thus Claim 1 holds.

Otherwise, by applying Lemma 6, we can construct a feasible decision \( \tilde{Q}^{[\theta]} \) such that \( \|Q - \tilde{Q}^{[\theta]}\| \leq \delta \), \( \bar{u}_i^{[\theta]} > 0 \) and \( \bar{v}_j^{[\theta]} > 0 \) for all nodes \( i'' \), \( j'' \) on a decreasing path \( P \) that connects \((i, j)\) and \((i', j')\). Furthermore, we define \( \tilde{Q} = \tilde{Q}^{[\theta]} + \eta e_{ij}^{n \times m} - \eta e_{ij'}^{n \times m} \), where \( \eta = \min \{ \bar{u}_i^{[\theta]}, \bar{v}_j^{[\theta]}, \bar{q}_{ij'}^{[\theta]} \} \). By the definition of \( \| \cdot \| \), \( \bar{q}_{ij'}^{[\theta]} - (\bar{q}_{ij'}^{[\theta]} - \bar{q}_{ij'}^{[\theta]}) \geq \bar{q}_{ij'}^{[\theta]} - \|Q - \tilde{Q}^{[\theta]}\| \geq \bar{q}_{ij'}^{[\theta]} - \delta = q_{ij'}^{[\theta]} - \delta \).

Also, we have \( x_i = \bar{u}_i - \sum_{j'' \in S} q_{ij''} = \bar{u}_i - \sum_{j'' \in S} \bar{q}_{ij''} + (\sum_{j'' \in S} \bar{q}_{ij''} - \sum_{j'' \in S} q_{ij''}) = \bar{u}_i + (\sum_{j'' \in S} q_{ij''} - \sum_{j'' \in S} q_{ij''}) \geq \bar{u}_i - \|Q - Q^{[\theta]}\| = \bar{u}_i - \delta \). Similarly, \( \bar{v}_j \geq \bar{v}_j - \delta \).

From the construction of \( \tilde{Q} \), \( \bar{u}_i = u_i - \sum_{(i',j') \in C_{ij}} q_{ij'} \) and \( \bar{v}_j = v_j - \sum_{(i',j') \in C_{ij}} q_{ij'} \), we have \( \eta = \min \{ \bar{u}_i^{[\theta]}, \bar{v}_j^{[\theta]}, \bar{q}_{ij'}^{[\theta]} \} \geq \min \{ u_i - \sum_{(i',j') \in C_{ij}} q_{ij'}, v_j - \sum_{(i',j') \in C_{ij}} q_{ij'}, q_{ij'}^{[\theta]} \} - \delta \).

From \( Q \) to \( \tilde{Q} \), the matching quantity of arc \((i, j)\) is increased by \( \sum_{(i',j') \in C_{ij}} q_{ij'} + \sum_{(i'',j') \in C_{ij}} q_{ij''} \) while the matching quantity of any other arc \((i'', j'')\) in \( \bigcup_{\theta=1}^{L} A_{\theta} \) and \((i'', j'') \neq (i, j)\) is unchanged.

From \( \tilde{Q} \) to \( \tilde{Q}^{[\theta]} \), the matching quantity of arc \((i, j)\) may have a decrease, but can be bounded.

The total change to \( M_J \) can be bounded by \( \|Q - Q^{[\theta]}\| \leq \delta \) for all \( J = 1, \ldots, L \).

From \( \tilde{Q}^{[\theta]} \) to \( \tilde{Q} \), the matching quantity of arc \((i, j)\) is increased by \( \eta \). The only arc on which the matching quantity is decreased is \((i', j')\), which is not in \( \bigcup_{\theta=1}^{L} A_{\theta} \).

Thus, from \( Q \) to \( \tilde{Q} \), the matching quantities on \((i, j)\) and in \( \bigcup_{\theta=1}^{L} A_{\theta} \) are increased by at least

\[
\sum_{(i'',j') \in C_{ij}} q_{ij''} + \sum_{(i',j') \in C_{ij}} q_{ij'} - \delta + \eta \\
\geq \sum_{(i'',j') \in C_{ij}} q_{ij''} + \sum_{(i',j') \in C_{ij}} q_{ij'} - \delta + \min \left\{ u_i - \sum_{(i',j') \in C_{ij}} q_{ij'}, v_j - \sum_{(i',j') \in C_{ij}} q_{ij'}, q_{ij'} \right\} - \delta \\
\geq \min \left\{ u_i + \sum_{(i'',j') \in C_{ij}} q_{ij''}, v_j + \sum_{(i',j') \in C_{ij}} q_{ij'}, q_{ij'} \right\} - 2\delta \\
= \min \{ w_{ij}, q_{ij'} \} - 2\delta.
\]

If we choose \( \delta \leq \frac{1}{2} \min \{ w_{ij}, q_{ij'} \} \), the above increment is at least \( \frac{1}{2} \min \{ w_{ij}, q_{ij'} \} \).

In addition, from \( Q \) to \( \tilde{Q} \), \( M_J(Q) \) does not decrease more than \( \delta \) for any \( J = 1, \ldots, L - 1 \).

Combining the above three cases, from \( Q \) we can construct another feasible decision \( \tilde{Q}(Q, \delta) \) that
weakly dominates $Q$, such that $M_L(\tilde{Q}) \geq M_L(Q) + \min\{w_{ij}, q_{ij}'\} - 2\delta$ and $M_J(\tilde{Q}) \geq M_J(Q) - \delta$ for $J = 1, \ldots, L - 1$. Thus, Claim 1 holds for Subcase 3.3.

3. A process that leads to an optimal policy with the desired property.

Next, we consider the following process that starts with $k = 1$ and $Q^{(1)} = Q$. We will show that the process leads to a feasible decision $Q^*$ that weakly dominates $Q$ and satisfies $min\{w_{ij}^*, q_{ij}'^*\} = 0$ for all $(i, j) \geq (i', j')$ and $(i, j) \neq (i', j')$.

In Step $k$ ($k = 1, 2, \ldots$) of the process:

**Step k.1** Find $(i, j)$ and $(i', j')$ being a pair of arcs such that $(i, j) \geq (i', j')$ and $(i, j) \neq (i', j')$. If there are more than one of such pairs, we choose the pair that yields the maximum $min\{w_{ij}^{(k)}, q_{ij}'^{(k)}\}$.

**Step k.2** If $min\{w_{ij}^{(k)}, q_{ij}'^{(k)}\} = 0$, we stop the process and set $Q^* = Q^{(k)}$. Otherwise, let $Q^{(k+1)} = \tilde{Q}(Q^{(k)}, \delta_k)$. We choose the $\delta_k$’s such that $0 < \delta_k \leq \frac{1}{2} \min\{w_{ij}^{(k)}, q_{ij}'^{(k)}\}$ for all $k \geq 1$ and further, $\delta_k \leq 2^{-k-1} \min_{1 \leq \ell \leq k-1} \Delta \ell$ for $k \geq 2$, where $\Delta \ell \overset{\text{def}}{=} M_{L^{(\ell)}}(Q^{(\ell+1)}) - M_{L^{(\ell)}}(Q^{(\ell)})$ and $L^{(\ell)} \in \{1, \ldots, K - 1\}$ satisfies $(i, j) \in A_{L^{(\ell)}}$.

If the above process stops in finite steps, it is easy to verify that we reached a desired $Q^*$, simply because of the termination criterion. In the followings, we assume that the process will run for infinitely many steps.

Suppose that $L_s$ is the smallest number $L \in \{1, \ldots, K - 1\}$ such that an arc $(i, j) \in A_{L_s}$ is chosen an infinite number of times as the dominating arc in the process. That means there is an integer $k^\text{max}$ such that no $(i, j) \in A_L$ will be chosen as the dominating arc in Step $k > k^\text{max}$ if $L < L_s$. We will focus on steps after $k^\text{max}$ in the rest of the proof.

Suppose that an arc $(i, j) \in A_{L_s}$ is chosen in Steps $k_1, k_2, \ldots$ (after Step $k^\text{max}$). Then, in Step $k_1$, $M_{L_s}$ is increased by $\Delta k_1$. While $M_{L_s}$ may be decreased in steps $k_1 + 1, \ldots, k_{l+1} - 1$, the decrements are bounded from below by $2^{-k_1-2} \Delta k_1, 2^{-k_1-3} \Delta k_1, \ldots, 2^{-k_1+1} \Delta k_1$, respectively, by Claim 1 and the way we choose $\delta_k$ in each step. Thus, in Steps $k_1, k_2, \ldots, k_{l+1} - 1$, $M_{L_s}$ is increased by at least $\Delta k_1 - \sum_{l'=1}^{k_{l+1}-k_1-1} 2^{-k_1-1-l'} \Delta k_1 = \Delta k_1 - 2^{-k_1} \sum_{l'=1}^{k_{l+1}-k_1-1} 2^{-1-l'} \Delta k_1 \geq \frac{1}{2} \Delta k_1$. Thus, $M_{L_s}(Q^{(k_1+1)}) \geq M_{L_s}(Q^{(k_1)}) + \frac{1}{2} \sum_{l'=1}^{k_{l+1}-1} \Delta k_{l'}$.

We claim that for all $\varepsilon > 0$, $\Delta k_1 < \varepsilon$ when $I$ is sufficiently large. If this is not true, there exist infinitely many $I$’s such that $\Delta k_1 \geq \varepsilon$. Suppose that $\Delta k_{I(\theta)} \geq \varepsilon$ for $\theta = 1, 2, \ldots$, where $\{I(\theta)\}_{\theta=1}^{\infty}$ is a subsequence of $\{1, 2, \ldots\}$. Then, $M_{L_s}(Q^{(k_{I(\ell)}+1)}) \geq M_{L_s}(Q^{(k_{I(1)})}) + \frac{1}{2} \sum_{l'=1}^{k_{I(\ell)}} \Delta k_{l'} \geq M_{L_s}(Q^{(k_{I(1)})}) + \frac{1}{2} \sum_{\theta=1}^{\infty} \Delta k_{I(\theta)} \geq M_{L_s}(Q^{(k_1)}) + \frac{1}{2} \varepsilon \to \infty$ as $\xi \to \infty$. This contradicts the fact that $M_{L_s}(Q^{(k_{I(\xi)}+1)}) = \sum_{(l', j'')} \min_{l''=1}^{k_{l(\xi)+1}} a_{l''}^{(l'' \to l' \to j'' \to j')} \leq \sum_{(l', j'')} \min_{l''=1}^{k_{I(\xi)+1}} a_{l''}^{(l'' \to l' \to j'' \to j')} < \infty$.

By Claim 1 and our choice of $\delta_k$, $k = M_{L^{(I)}}(Q^{(k+1)}) - M_{L^{(I)}}(Q^{(k)}) \geq \frac{1}{2} \min\{w_{ij}^{(k)}, q_{ij}'^{(k)}\}$ for any $(i, j) \geq (i', j')$. In Step $k_1$, if $(i, j)$ and $(i', j')$ are the chosen pair of arcs, then $\min\{w_{ij}^{(k)}, q_{ij}'^{(k)}\} \leq \frac{1}{2} \Delta k_1$. Since $\Delta k_1 < \varepsilon$, this implies $\Delta k_1 < \varepsilon$.
2\Delta_{k_1}. If (i, j) and (i', j') are not chosen in Step \( k \) but also satisfy \((i, j) \succeq (i', j')\) and \((i, j) \not\succeq (i', j')\), we still have \( \min \{ w_{ij}^{(k_1)}, q_{ij,j'}^{(k_1)} \} \leq 2\Delta_{k_1} \), because the chosen pair of arcs has the maximum value of \( \min \{ w_{ij}^{(k_1)}, q_{ij,j'}^{(k_1)} \} \) among all arcs such that \((i, j) \succeq (i', j')\) and \((i, j) \not\succeq (i', j')\). Thus, for any \((i, j) \succeq (i', j')\) and \((i, j) \not\succeq (i', j')\), we have \( \min \{ w_{ij}^{(k_1)}, q_{ij,j'}^{(k_1)} \} \leq 2\Delta_{k_1} \rightarrow 0 \) as \( I \rightarrow \infty \).

On the other hand, since \( Q^{(k_1)} \) is bounded (for the given state \((x, y)\)), it has a convergent subsequence. Without loss of generality, suppose that \( Q^{(k_1)} \) itself converges to a limit \( Q^* \). We see that \( Q^* \) is a feasible decision because the decision space in each period is a compact set, and weakly dominates \( Q \) because each \( Q^{(k_1)} \) weakly dominates \( Q \) and \( H_l(Q; x, y) \) is a continuous function. For any \((i, j) \succeq (i', j')\) and \((i, j) \not\succeq (i', j')\), \( 0 = \lim_{I \rightarrow \infty} \min \{ w_{ij}^{(k_1)}, q_{ij,j'}^{(k_1)} \} = \min \{ w_{ij}^*, q_{ij,j'}^* \} \). This proves the theorem.

So far, we assume no specific structure on the matching rewards. In the following two sections, we will impose two different, intuitive reward structures in which all neighboring arcs can be compared under the partial order. We will rely on the structural priority properties that have been shown, as a starting point, to sharpen the characterization of the optimal matching policy for these two reward structures.

5. Horizontally Differentiated Types
In this section, we first consider the model with demand and supply types that are horizontally differentiated in the sense that each type has its own heterogeneous “taste.”

5.1. \( n \) Demand and \( m \) Supply Types

We assume that the \( n \) demand and \( m \) supply types are distributed on a circle \( C \) with a circumference \(|C|\) (see Figure 5 for an illustration). All the demand types have distinct locations (if that is not true, we can simply treat two demand types sharing the same location as the same type) and so do the supply types. We arbitrarily select a point on \( C \) as location 0 and define the location \( l \) of any point \( p \in C \) as the distance from location 0 to the point \( p \), traveling clockwise around the circle. For a demand or supply type \( k \in D \cup S \), we denote its location on the circle by \( l(k) \). To travel from type \( i \) demand to type \( j \) supply, we assume that one needs to go clockwise. That can be a reasonable assumption in many logistics applications where vehicles, ships, and even drones\(^8\), travel along a fixed route in one direction. If \( l(i) \leq l(j) \), the travel distance \( d_{ij} \) equals \( l(j) - l(i) \). Otherwise, \( d_{ij} = |C| - l(i) + l(j) \). Note that the choice of location 0 does not affect the (clockwise) distances between the types. The unit reward of matching type \( i \) demand and type \( j \) supply is assumed to be \( r_{ij} = f(d_{ij}) \), where \( f(\cdot) \) is a nonincreasing function.

\(^8\) A Silicon Valley startup, Matternet, is creating networks of delivery drones that fly on largely fixed routes. That means the drones can operate autonomously without collision-avoidance technology.
Note. If the distance is not unidirectional, e.g., the shortest distance along the circle, it may be optimal to break up the pair of type $i$ demand and type $j$ supply that are closest, e.g., to match type $i$ demand with type $j'$ supply and match type $i'$ demand with type $j$ supply.

Consider $i \in \mathcal{D}$ and $j \in \mathcal{S}$. If one does not pass by any demand or supply type in $\mathcal{D} \cup \mathcal{S} \setminus \{i, j\}$ when traveling clockwise from $i$ to $j$, then such types are the most favorable match for each other from their own perspective. In fact, we have $d_{ij'} = d_{ij} + d_{ij'} \geq d_{ij}$ and $d_{i'j} = d_{i'i} + d_{ij} \geq d_{ij}$. Since $f$ is nonincreasing, $r_{ij} \geq \max_{i' \in \mathcal{D}, j' \in \mathcal{S}} \{r_{ij'}, r_{i'j}\}$. For type $i$ demand and type $j$ supply, we define $\widehat{(i,j)}$ as the clockwise circular segment from type $i$ demand to type $j$ supply. Then, $i \in \mathcal{D}$ and $j \in \mathcal{S}$ are the most favorable match for each other if and only if $\widehat{(i,j)}$ does not contain any other types than themselves. In general, the further the (clockwise) unidirectional distance between the tastes of type $i$ demand and type $j$ supply, the less surplus is generated by matching them.

We have the following result showing that each demand or supply type should be matched as much as possible with its most favorable match. In other words, under the given reward structure, a “self-interested” match between the most favorable pair, assuming they can self-select their partner, is also socially optimal. The proof is to verify that the sufficient condition in Theorem 2 is satisfied for any pair of demand and supply types that are closest to each other.

**Theorem 4 (Greedy match of perfect pairs).** Suppose that $\widehat{(i,j)}$ does not contain any other types other than $i$ and $j$ themselves. If $f$ is nonincreasing and convex, $q_{ij}^* = \min \{x_i, y_j\}$.

**Proof of Theorem 4.** As we have argued, $r_{ij} \geq \max_{i' \in \mathcal{D}, j' \in \mathcal{S}} \{r_{ij'}, r_{i'j}\}$. This implies that type $i$ demand and type $j$ supply are the most favorable match for each other. In view of Theorem 2, it remains to be shown that $r_{ij} + r_{i'j'} \geq r_{ij'} + r_{i'j}$ for any $i' \in \mathcal{D}$ and $i' \neq i$ and $j' \in \mathcal{S}$ and $j' \neq j$.

Since the choice of location 0 does not affect the (clockwise) distance between two locations, we assume without loss of generality that $0 \leq l(i) \leq l(j)$. 

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**Figure 5** Horizontally differentiated types with unidirectional distance.

![Diagram of horizontally differentiated types with unidirectional distance](image_url)
First, consider the case with \( l(j) \leq l(i') \) and \( l(j) \leq l(j') \). If \( l(i') \leq l(j') \), then \( l(j) \leq l(i') \leq l(j') \) and thus \( d_{ij} = d(j') - l(i) = l(j') - l(i') + l(i') - l(i) = d_i r_j + d_i'). It follows that \( r_{ij'} = f(d_i r_j) \geq f(d_i r_j + d_i r_j') = f(d_{ij'}) = r_{ij}, \) where the inequality is due to that \( f \) is nonincreasing. Since \( r_{ij} \geq r_{ij'} \), \( r_{ij} + r_{ij'} \geq r_{ij'} + r_{ij} \). If \( l(i') > l(j') \), then \( d_i r_j = |C| - l(i') + l(j') = |C| - d_i r_j', d_{ij'} = d_{ij} + d_{ij'} \) and \( d_{ij'} = |C| - l(i') + l(i) = |C| - |l(i') - l(j')| - |l(j') - l(j)| = |C| - d_i r_j' - d_{ij} \). Because \( f \) is convex and \( d_{ij} = l(j) - l(i) \leq |C| - l(i') + l(i) = |C| - d_i r_j' - d_{ij} \), we have \( r_{ij'} - r_{ij} = f(d_{ij'}) - f(d_{ij}) \geq f(d_{ij} + d_{ij'}) - f(d_{ij}) = f(d_{ij}) + f(d_{ij'}) - f(d_{ij}) = f(d_{ij'}) - f(d_{ij}) = r_{ij'} - r_{ij}. \) Thus, \( r_{ij} + r_{ij'} \geq r_{ij'} + r_{ij} \).

Similarly, we can show that \( r_{ij} + r_{ij'} \geq r_{ij} + r_{ij'} \) also holds, if \( l(i) > l(i') \) and \( l(i) > l(j') \).

Because we will not visit any types in \( \mathcal{D}_i \cup \mathcal{S}_j \setminus \{i, j\} \) when traveling from \( i \) to \( j \) clockwise, the remaining cases to consider are: (i) \( l(i') < l(i) \) and \( l(j) < l(j') \), and (ii) \( l(j') < l(i) \) and \( l(j) < l(i') \).

Case (i) can again be proved by using the convexity of \( f \). We can verify that \( d_{ij} \leq d_{ij'} \), \( d_{ij} = d_{ij} + d_{ij'} \) and \( d_{ij'} = d_{ij} + d_{ij'} \). Then, \( r_{ij'} - r_{ij} = f(d_{ij'}) - f(d_{ij}) = f(d_{ij} + d_{ij'}) - f(d_{ij}) \leq f(d_{ij} + d_{ij'}) - f(d_{ij}) = f(d_{ij'}) - f(d_{ij}) = r_{ij'} - r_{ij}. \) This leads to \( r_{ij} + r_{ij'} = r_{ij'} + r_{ij} \).

For Case (ii), we can verify that \( d_{ij} \leq d_{ij'} \) and \( d_{ij} = d_{ij} + d_{ij} \) and \( d_{ij'} = d_{ij} + d_{ij} \). Thus, \( r_{ij} + r_{ij'} = f(d_{ij'}) + f(d_{ij}) = f(d_{ij'}) + f(d_{ij}) = r_{ij'} + r_{ij}. \) \( \Box \)

If \( f \) takes a linear form, we can further characterize the priorities in the optimal matching other than the most favorable pairs. The economic interpretation of a linear function \( f \) is that the reward \( r_{ij} \) from matching type \( i \) demand with type \( j \) supply is obtained from a base reward \( r_0 \) from matching minus the mismatch cost associated with traveling from location \( i \) to location \( j \), which is proportional to the distance between locations \( i \) and \( j \).

**Theorem 5** (Priority of imperfect pairs). If \( f \) is nonincreasing and linear, \( (i, j) \succeq (i', j') \) if and only if \( (i, j) \subset (i', j'). \) Thus it is optimal to prioritize matching \( (i, j) \) over \( (i', j') \) if \( (i, j) \subset (i', j'). \) In particular, for any given type \( i \) demand, the closer its distance to a type \( j \) supply, the higher the priority in matching the demand-supply pair \( (i, j) \) over matching type \( i \) demand with other supply types, and a symmetric result holds for any given supply type.

**Proof of Theorem 5.** Without loss of generality, we assume that the travel cost from one location to another equals the travel distance. We first prove that \( (i, j) \) has priority over \( (i', j') \) if and only if \( d_{ij} \leq d_{ij'} \). In view of Theorem 1, it is sufficient to show that \( (i, j) \succeq (i', j') \) if \( d_{ij} \leq d_{ij'} \).

Without loss of generality, let \( l(i) = 0. \) Then, \( d_{ij} \leq d_{ij'} \) implies that \( l(j') \geq l(j) \geq 0 \) and \( d_{ij'} = d_{ij} + d_{ij'} \).

If \( l(i') < l(j) \) or \( l(i') > l(j') \), then \( d_{ij} = d_{ij'} - d_{ij'} \). It follows that \( r_{ij} + r_{ij} = r_0 - d_{ij'} + r_0 - d_{ij'} = r_0 - d_{ij} + r_0 - d_{ij} = r_0 + r_0 - d_{ij} - d_{ij} = r_{ij} + r_{ij} \).
If \( l(j) \leq l(i') \leq l(j') \), then \( d_{ij} = |C| - d_{ij'} \) and \( d_{ij'} = d_{jj'} - d_{ij} \). Thus, \( d_{ij} = |C| - d_{jj'} + d_{ij'} \geq d_{ij'} \).

It follows that \( r_{ij'} + r_{ij'} = r_0 - d_{ij'} + r_0 - d_{ij'} = r_0 - d_{ij} - d_{jj'} + r_0 - d_{ij} = r_{ij} - d_{jj'} + r_0 - d_{ij} \leq r_{ij} - d_{jj'} + r_0 - d_{ij'} = r_{ij} - d_{jj'} + r_{ij'} < r_{ij} + r_{ij'} \).

Thus, we have proved that \( r_{ij} + r_{ij'} \geq r_{ij'} + r_{ij} \) for any \( i' \in D \). On the other hand, it is clear that \( r_{ij} \geq r_{ij'} \) if \( d_{ij} \leq d_{ij'} \). This implies that \((i,j) \geq (i',j')\) if \( d_{ij} \leq d_{ij'} \).

Following exactly the same argument, we can show that \((i,j) \geq (i',j')\) if \( d_{ij} \leq d_{ij'} \).

Next, we show that \((i,j) \geq (i',j')\) if and only if the clockwise circular segment \((i,j) \subseteq (i',j')\).

Necessity. Suppose that \((i,j) \subseteq (i',j')\). Then, \( d_{ij} \geq d_{ij} \), implying that \((i,j) \geq (i',j')\). Furthermore, we also have \( d_{ij'} \geq d_{ij'} \), as a result of \((i,j) \subseteq (i',j')\). Thus, \((i,j) \geq (i',j')\). By transitivity, \((i,j) \geq (i',j')\).

Sufficiency. Suppose that \((i,j) \geq (i',j')\). Then, there exists a decreasing path connecting \((i,j)\) and \((i',j')\). Assume without loss of generality that \((i,j) = (i_1,j_1) \geq (i_2,j_2) \geq \cdots \geq (i_t,j_t)\). From \((i_1,j_1) \geq (i_2,j_2)\), we can infer that \( d_{i_1,j_1} \leq d_{i_2,j_2} \), which further implies that \((i_1,j_1) \subseteq (i_2,j_2)\). Repeatedly applying similar arguments as above leads to \((i,j) = (i_1,j_1) \subseteq (i_2,j_2) \subseteq (i_3,j_3) \subseteq \cdots \subseteq (i_t,j_t) = (i',j')\). \(\square\)

By virtue of Theorem 5, the optimal matching decision in a period can be characterized by state-dependent protection levels \(a_{ij}(t,\cdot,\cdot)\) defined in a matching procedure as follows.

To start with, let \( k = 1 \), \((x^1,y^1) = (x,y)\) and \( Q^1 = Q^\ast = 0^{n \times m} \). Also, we represent the set of arcs that have not been matched yet by \( \bar{A} \). Initially, \( \bar{A} = A \).

Step 1. For each arc \((i,j) \in (\{(i'',j'') \in \bar{A} \mid \bar{P}(i',j')\} \subseteq (i',j')\}\) (i.e., \((i,j)\) is undominated in \( \bar{A} \)), we do the following.

Step 1.1 Match type \( i \) demand with type \( j \) supply until their remaining unmatched quantities reach \((x_i^k - y_j^k)^+ + a_{ij}(t,x^k,y^k)\) and \((x_i^k - y_j^k)^- + a_{ij}(t,x^k,y^k)\) respectively. Remove \((i,j)\) from \( \bar{A} \). Set \( q_{ij}^k = q_{ij}^k = x_i^k - (x_i^k - y_j^k)^- - a_{ij}(t,x^k,y^k)\).

Step 1.2. If \( a_{ij}(t,x^k,y^k) > 0 \), then set \( q_{ij}^k = 0 \) and remove \((i',j')\) from \( \bar{A} \) for all \((i',j') \neq (i,j)\) such that \((i,j) \subseteq (i',j')\).

Step 2. Update the state vectors: \( x^{k+1} = x^k - 1^m(Q^k)^T \), \( y^{k+1} = y^k - 1^nQ^k \). Increase \( k \) by 1 and set \( Q^k = 0^{n \times m} \). Go back to Step 1 if \( \bar{A} \) is nonempty, and stop otherwise.

In Step 1 of the above procedure, the post-matching levels of type \( i \) demand and type \( j \) supply (right after the matching in Step 1) will be \((x_i^k - y_j^k)^+ + a_{ij}(t,x^k,y^k)\) and \((x_i^k - y_j^k)^- + a_{ij}(t,x^k,y^k)\) respectively. The level \( a_{ij}(t,x^k,y^k) \) is the amount we would like to protect from matching. When \( k = 1 \), each arc \((i,j)\) chosen in Step 1 is undominated by any \((i',j') \in A\), meaning that type \( i \) demand and type \( j \) supply will be matched as much as possible. Thus, \( a_{ij}(t,x^1,y^1) = 0 \) for all
such \((i, j)\). Another property of \(a_{ij}(t, x^k, y^k)\) is that it depends on \(x_i^k\) and \(y_j^k\) only through their difference, \(x_i^k - y_j^k\). If an arc \((i, j)\) is ever selected in Step 1, the decision \(Q^*\) under the state \((x, y)\) will lead to exactly the same post-matching levels as the decision \(Q^* + \varepsilon e_{ij}^{n \times m}\) under the state \((x + \varepsilon e_i^n, y + \varepsilon e_j^m)\). Since the total current-period rewards are linear in matching quantities, one can easily verify that \(Q^* + \varepsilon e_{ij}^{n \times m}\) will satisfy the first-order optimality conditions under the state \((x + \varepsilon e_i^n, y + \varepsilon e_j^m)\) if \(Q^*\) does so under the state \((x, y)\). Consequently, \(Q^* + \varepsilon e_{ij}^{n \times m}\) is optimal for the state \((x + \varepsilon e_i^n, y + \varepsilon e_j^m)\) and has the same protection level as \(Q^*\).

Next we further characterize the optimal matching policy described in the above procedure for the model with two demand and supply types.

5.2. 2 Demand and 2 Supply Types

We focus on the special case where there are two horizontally differentiated demand and supply types. The reward structure is such that a favorable match (which happens to be a perfect pair) yields a high reward and a mismatch yields a low reward. (See Figure 6 for an illustration, where a favorable match is connected by a thick line and a mismatch is connected by a thin line; this figure is comparable to Figure 7 for the vertically differentiated case.) For notational simplicity, we index one type by \(i\), \(i = 1, 2\), and the other by \(-i\). If we consider two distinct locations on a circle, with one location shared by type \(i\) demand and supply and the other by type \(-i\) demand and supply, then the model in the previous subsection reduces to the current model with two demand and two supply types. Even more generally, we do not require the distance between the two types to be unidirectional.

![Horizontally differentiated types.](image)

**Observation 1.** For two horizontally differentiated types (even with the shortest distance along the circle), \((i, i) \succeq (i, -i)\) and \((i, i) \succeq (-i, i)\) for \(i = 1, 2\).

This is because obviously, we have \(r_{ii} \geq \max\{r_{i,-i}, r_{-i,i}\}\) for \(i = 1, 2\) (as long as \(f\) is a nonincreasing function) even with the shortest distance along the circle. As a result, by verifying the definition of the partial order, it is easy to see that the relations \((1, 1) \succeq (1, 2), (1, 1) \succeq (2, 1), (2, 2) \succeq (2, 1)\)
and \((2,2) \succeq (1,2)\) are guaranteed. By Theorem 2, type \(i\) demand and supply type \(i\), \(i=1,2\), should be matched as much as possible. The matching process in a period will stop if both supply types or both demand types are exhausted after being matched with their perfect match. Otherwise, it remains to determine the matching quantity between type \(i\) demand and type \(-i\) supply if they have positive quantities left.

**Theorem 6** (2-to-2 horizontal model: optimal matching policy). Consider two horizontally differentiated types. Fix an arbitrary period \(t\). For any \((x,y)\), define the type-specific demand and supply imbalance \(\eta_i \equiv x_i - y_i\) for \(i=1,2\), and the aggregate imbalance \(\eta \equiv \eta_1 + \eta_2\). The following matching procedure is optimal: for \(i=1,2\),

(i) Round 1 (Greedy matching for the perfect pair): match type \(i\) demand and supply as much as possible, i.e., \(q_{i,i}^* = \min \{x_i, y_i\}\).

(ii) Round 2 ("Match down to" policy for the imperfect pair): if \(x_i > y_i\) and \(x_{-i} < y_{-i}\), then \(q_{i,i}^* = 0\) and match the imperfect pair of type \(i\) demand and type \(-i\) supply down to post-matching levels \(u_i^* = \eta_i^+ + a_i^*(t, \eta)\) and \(v_{-i}^* = \eta_i^- + a_i^*(t, \eta)\) respectively, where \(a_i^*(t, \eta) = \min\{a_i(t, \eta), \eta_i - \eta_i^+, \eta_i^-\}\) and \(\bar{a}_i(t, \eta)\) is some protection level. Otherwise, \(q_{i,i}^* = 0\).

**Proof of Theorem 6.** By Observation 1 and Theorem 4, it is optimal to match the perfect pair as much as possible; that is done in the first round. Next we consider after round 1 how to match the imperfect pairs \((1, 2)\) and \((2, 1)\). If \(x_1 \geq y_1\) and \(x_2 \geq y_2\) or \(x_1 \leq y_1\) and \(x_2 \leq y_2\), it is obvious that \(q_{12}^* = q_{21}^* = 0\).

Consider the case where \(x_1 > y_1\) and \(x_2 < y_2\). (The same argument applies to the case where \(x_1 < y_1\) and \(x_2 > y_2\).) After round 1, the remaining quantities for type 1 demand and type 2 supply is \(x_1 - y_1 > 0\) and \(y_2 - x_2 > 0\) respectively. There is no remaining unmatched type 2 demand and type 1 supply, and thus \(q_{21}^* = 0\). It remains to determine the optimal matching quantity \(q_{12}^*\), which is equivalent to determining some optimal protection level \(a_1^*(t, x, y)\). To see this, note that \(\eta_i^+\) and \(\eta_i^-\) would be the unmatched type 1 demand and type 2 supply remaining after the imperfect pair \((1, 2)\) has been matched as much as possible, where \(\eta = \eta_1 + \eta_2 = (x_1 - y_1) - (y_2 - x_2)\). When the protection level is \(a_1\), the post-matching levels of type 1 demand and type 2 supply are \(u_i = \eta_i^+ + a_1\) and \(v_i = \eta_i^- + a_1\), respectively. The protection level needs to satisfy the nonnegativity constraint \(a_1 \geq 0\) and ensure the matching quantity \(q_{12} = \eta_1 - u_1 = \eta_1 - \eta_i^+ - a_1 \geq 0\), resulting in \(a_1 \leq \eta_1 - \eta_i^+\).

After Round 1, the cost-to-go function can be written in terms of the protection level \(a_1\) as:

\[
\bar{H}_t(a_1, x, y) = r_{11}y_1 + r_{22}x_2 + r_{12}(\eta_1 - \eta_1^+ - a_1) - c(\eta_1^+ + a_1) - h(\eta^- + a_1) + \gamma E V_{i+1} (\alpha(\eta_1^+ + a_1) + D_1, D_2, S_1, \beta(\eta^- + a_1) + S_2)
\]
\[ r_{11} y_1 + r_{22} x_2 + r_{12} (\eta_1 - \eta^+) - c \eta^+ - h \eta^- + \hat{H}_i(a_1, \eta), \]

where

\[ \hat{H}_i(a_1, \eta) = -(r_{12} + c + h) a_1 + \gamma E V_{i+1} (\alpha (\eta^+ + a_1) + D_1, D_2, S_1, \beta (\eta^- + a_1) + S_2) \quad (11) \]

depends on \((x, y)\) only through \(\eta\). The optimal protection level can be obtained by solving \(\max_{0 \leq a_1 \leq \eta_1 - \eta^+} \hat{H}_i(a_1, \eta)\). As with the proof of Proposition 1, it is easy to show that \(\hat{H}_i(a_1, \eta)\) is concave in \(a_1\). Thus, the optimal protection level \(a_1^*(t, x, y) = a_1^*(t, \eta) = \min \{ \bar{a}_i(t, \eta, \eta_1 - \eta^+) \} \), where \(\bar{a}_i(t, \eta) \in \arg \max_{a_1 \geq 0} \hat{H}_i(a_1, \eta)\). □

Theorem 6 shows the structure of the optimal matching policy for the 2-to-2 horizontal model. In the first round of matching, type \(i\) demand is matched as much as possible with its most favorable match, type \(i\) supply. After that, if we matched the imperfect pair, type \(i\) demand and type \(-i\) supply, to the full extent, then the post-matching levels of type \(i\) demand and type \(-i\) supply would become \(\eta^+\) and \(\eta^-\) respectively. The optimal matching quantity is characterized by the state-dependent protection level \(\bar{a}_i(t, \eta)\): the amount \(\bar{a}_i(t, \eta)\) is protected from being matched between type \(i\) demand and type \(-i\) supply so that they are saved for the arrival of their perfect match in future periods. The match-down-to levels for type \(i\) demand and type \(-i\) supply after the second round of matching are \(\eta^+ + \bar{a}_i(t, \eta)\) and \(\eta^- + \bar{a}_i(t, \eta)\) respectively. The matching of the imperfect pair has a match-down-to structure: If the quantity of type \(i\) demand, \(\eta_i = x_i - y_i\), after the first round of matching is greater than the match-down-to level \(\eta^+ + \bar{a}_i(t, \eta)\) (and simultaneously, that of type \(-i\) supply down to \(\eta^- + \bar{a}_i(t, \eta)\)). Otherwise, type \(i\) demand and type \(-i\) supply will not be matched. This structure is analogous to many threshold-type structures in the inventory literature, e.g., the celebrated base-stock policy. Moreover, the match-down-to levels only depend on the aggregated discrepancy between total demand and supply across two types. In other words, the match-down-to levels depend on the 4-dimensional state \((x, y)\) only through a scalar \(\eta\), which measures the total mismatch between demand and supply. Such state collapse in the optimal control policy has been seen in the inventory literature. For example, a well-known technique for dealing with deterministic lead-times is to collapse the state into the inventory position.

We can obtain a further state collapse in the protection levels for the imperfect matching when the unmatched demand or supply is lost after the matching in each period is done.

**Corollary 2** (2-to-2 horizontal model with lost demand or supply). Suppose that \(x_i > y_i\) and \(x_{-i} < y_{-i}\). If \(\alpha = 0\), there exists a constant \(\hat{v}_{-i}(t)\) such that the optimal matching quantity between
type i demand and type −i supply is \( q^*_i = \eta^\pm_i - \max \{ \hat{v}_{-i}(t) \wedge \eta^\pm_{-i}, \eta^- \} \). If \( \beta = 0 \), there exists \( \hat{u}_i(t) \) such that is \( \hat{q}^*_{-i} = \eta^\pm_{-i} - \max \{ \hat{u}_i(t) \wedge \eta^\pm_{-i}, \eta^\pm_{-i} \} \).

Consider \( x_i > y_i, x_{-i} < y_{-i} \) for some \( i = 1, 2 \); otherwise, the matching decisions for the imperfect pairs are trivial. Note that the post-matching level of type −i supply must be in the interval \([\eta^-, \eta^-_{-i}]\), because \( \eta^- \) is the post-matching level of type −i supply if it is matched with type i demand as much as possible and \( \eta^-_{-i} \) is the quantity of type −i supply immediately before the second round of matching. Thus Corollary 2 says that if all the unmatched demand is lost, then the second-round matching reduces the quantity of type −i supply as close as possible to a threshold \( \hat{v}_{-i}(t) \), which is independent of \( \eta \). Intuitively, because all the unmatched demand is lost and the post-matching level of supply type i has to be 0 after Round 1 of matching, the firm only cares about how much of supply type −i to carry to the next period. This results in a constant protection level for supply type −i for the current period. Similarly, if all the unmatched supply is lost, then the optimal matching policy reduces the quantity of type i demand as close as possible to the threshold \( \hat{u}_i(t) \).

6. Vertically Differentiated Types

In this section, we consider vertically differentiated demand and supply types. Each demand or supply type is associated with a “quality” and generates a higher reward if it is matched with a supply or demand type of a higher quality. Specifically, we impose an additive form on the reward structure: for all \( 1 \leq i \leq n \) and \( 1 \leq j \leq m \), \( r_{ij} = r^d_i + r^s_j \) (see, e.g., Shumsky and Zhang 2009, Yu et al. 2014), where \( r^d_i \) (or \( r^s_j \)) can be understood as the quality of type i demand (or type j supply). Without loss of generality, we index the types such that \( r^d_1 > \cdots > r^d_n \) and \( r^s_1 > \cdots > r^s_m \). (Note that we can treat demand types i and \( i' \) as the same type if \( r^d_i = r^d_{i'} \), and the same applies to the supply types.) In other words, type 1 has the highest quality, and the higher the index, the lower the quality. In contrast to the horizontally differentiated types, it generates a lesser reward by matching types with lower qualities. (See Figure 7, where type 2 demand and supply are connected with a dashed line, as compared to Figure 6 where they are connected with a thick solid line.)

With the additive reward structure, \( r_{ij} + r_{i'j'} = r_{ij'} + r_{i'j} \) for all \( i, i' \in D \) and \( j, j' \in S \). This implies that for two neighboring arcs, \((i, j) \succeq (i', j)\) if and only if \( r^d_i \geq r^d_{i'} \), and \((i, j) \succeq (i', j')\) if and only if \( r^s_j \geq r^s_{j'} \). This observation can easily be generalized as \((i, j) \succeq (i', j')\) if and only if \( i < i' \) and \( j < j' \). It follows directly from Theorem 2 that it is optimal to match type 1 demand and type 1 supply as much as possible. That is summarized as follows.

**Proposition 2.** Consider vertically differentiated types. Then \( q^*_{11} = \min\{x_1, y_1\} \).
It follows from Theorem 1 that the arc \((i, j)\) has priority over \((i', j')\) and \((i', j)\) for all \(j' > j\) and \(i' > i\), which leads to the following result.

**Lemma 7.** Consider vertically differentiated types. In period \(t\) given the state \((x, y)\), there exists an optimal matching decision \(Q^*\) such that if \(q_{ij}^* > 0\) for some \(i \in D, j \in S\), then type \(i'\) demand and type \(j'\) supply have been fully matched (i.e., \(\sum_{k=1}^{m} q_{i'k}^* = x_{i'}\) and \(\sum_{k=1}^{n} q_{kj'}^* = y_{j'}\)) for all \(i' < i\) and \(j' < j\).

Following the logic of Lemma 7, we observe that there is an optimal policy that follows a top-down matching procedure (see Figure 8 for an illustration):

**Observation 2** (Top-down matching). Line up demand types and supply types separately in descending order of their indices. Match from the top, down to some level. The optimal matching decision \(Q\) in a period can be fully determined by a static, total matching quantity \(Q \overset{\text{def}}{=} \sum_{i'=1}^{n} \sum_{j'=1}^{m} q_{i'j'}\).

Once \(Q\) is known, we can recover the detailed matching decision \(Q\) as follows: Starting with
where $r_i$ is matched with some demand in period $t$, is $t$, with some supply in period 1, $\ldots$, $i$, of demand and supply are given by $Q_i$, will not be matched in the period. With the total matching quantity $Q_i$, the post-matching levels of demand and supply are given by $u_{i'} = v_j' = 0$ for $i' < i$ and $j' < j$, $u_i = \tilde{x}_i - Q$, $v_j = \tilde{y}_j - Q$, $u_{i+1} = x_{i+1}, \ldots, x_n$ and $v_{j+1} = y_{j+1}, \ldots, y_m$.

Given a state $(x, y)$ and the total matching quantity $Q$ in period $t$, the unmatched amount of type $i'$ demand is $(\tilde{x}_{i'} - Q) + (\tilde{x}_{i'-1} - Q)$, where the first term is the unmatched amount of demand in types 1, $\ldots$, $i'$, and the second term is the unmatched amount of demand in types 1, $\ldots$, $i' - 1$. Thus a total amount of $x_{i'} - [(\tilde{x}_{i'} - Q) + (\tilde{x}_{i'-1} - Q)]$ of type $i'$ demand is matched with some supply in period $t$. Similarly, an amount $y_j' - [(\tilde{y}_j' - Q) + (\tilde{y}_{j'-1} - Q)]$ of type $j'$ supply is matched with some demand in period $t$. Consequently, the total reward from matching in period $t$ is

$$
\sum_{i' = 1}^{n} r_{i'}^d \{ x_{i'} - [(\tilde{x}_{i'} - Q) + (\tilde{x}_{i'-1} - Q)] \} + \sum_{j' = 1}^{m} r_{j'}^s \{ y_{j'} - [(\tilde{y}_{j'} - Q) + (\tilde{y}_{j'-1} - Q)] \}
$$

+ $\sum_{i' = 1}^{n} r_{i'}^d x_{i'} + \sum_{j' = 1}^{m} r_{j'}^s y_{j'} - \sum_{i' = 1}^{n} (r_{i'}^d - r_{i'+1}^d) (\tilde{x}_{i'} - Q) + \sum_{j' = 1}^{m} (r_{j'}^s - r_{j'+1}^s) (\tilde{y}_{j'} - Q) +

where $r_{s+1}^d = r_{s+1}^s = 0$.

Thus we can rewrite the DP (1) as follows:

$$
V_t(x, y) = \max_{0 \leq Q \leq \min \{ \tilde{x}_n, \tilde{y}_m \}} G_t(Q, x, y),
$$

$G_t(Q, x, y) = \sum_{i' = 1}^{n} r_{i'}^d x_{i'} + \sum_{j' = 1}^{m} r_{j'}^s y_{j'} - \sum_{i' = 1}^{n} (r_{i'}^d - r_{i'+1}^d) (\tilde{x}_{i'} - Q) + \sum_{j' = 1}^{m} (r_{j'}^s - r_{j'+1}^s) (\tilde{y}_{j'} - Q) +

- c(\tilde{x}_n - Q) - h(\tilde{y}_m - Q) + \gamma EV_{t+1}(\alpha u + D, \beta v + S),
$$

(12)

where $u = (0^{i-1}, \tilde{x}_i - Q, x_{i+1}, n]$ and $v = (0^{j-1}, \tilde{y}_j - Q, y_{j+1}, m)$ if $\tilde{x}_{i-1} \leq Q < \tilde{x}_i$ and $\tilde{y}_{j-1} \leq Q < \tilde{y}_j$. 

$i = 1$ and $j = 1$, we match type $i$ demand with type $j$ supply until one of them is fully matched or the total matching quantity reaches $Q$. If type $i$ demand (or type $j$ supply) is fully matched, we increase $i$ (or $j$) by 1. Then we repeat the above steps until the total matching quantity finally reaches $Q$. We call this a dynamic perspective on the actual matching procedure in each period.
Lemma 8. \( G_t(Q, x, y) \) is concave in \( Q \).

By Lemma 8, the optimal matching decisions in a period become a one-dimensional convex optimization problem. Like the horizontally differentiated types, we can also take a dynamic perspective of the optimal matching procedure within each period. By Lemma 7, it is not optimal to consider matching type \( i \) demand with type \( j \) supply unless all types \( 1, \ldots, i \) demand and types \( 1, \ldots, j - 1 \) supply have been fully matched. Moreover, type \( i \) demand is considered to be matched with type \( j \) supply only if \( \bar{x}_i > \bar{y}_{j-1} \) and \( \bar{x}_{i-1} < \bar{y}_j \). If \( \bar{x}_i < \bar{y}_{j-1} \) or \( \bar{x}_{i-1} > \bar{y}_j \), then it contradicts the requirement that all types \( 1, \ldots, i \) demand and types \( 1, \ldots, j - 1 \) supply have been fully matched; this is because even all types \( 1, \ldots, i \) demand are not enough to match all types \( 1, \ldots, j - 1 \) supply, or even all types \( 1, \ldots, j \) supply are not enough to match all types \( 1, \ldots, i \) demand.

Now take the dynamic perspective of the top-down matching procedure and consider the scenario when it gets to the matching of type \( i \) demand and type \( j \) supply. The available amount of type \( i \) demand is \( \bar{x}_i \) and that of type \( j \) supply is \( \bar{y}_j \). If type \( i \) demand and type \( j \) supply were matched as much as possible, after the matching the amount of type \( i \) demand would become \( \bar{x}_i \) and that of type \( j \) supply would become \( \bar{y}_j \). Note that we have \( \bar{x}_i - \bar{x}_i = x_i - (\bar{y}_{j-1} - \bar{x}_{i-1})^+ - (\bar{x}_i - \bar{y}_j)^+ = y_j - (\bar{y}_{j-1} - \bar{x}_{i-1})^+ - (\bar{x}_i - \bar{y}_j)^+ = \bar{y}_j - \bar{y}_{j-1} \), where the second equality is due to \( x_i - y_j = (\bar{y}_{j-1} - \bar{x}_{i-1})^+ + (\bar{x}_i - \bar{y}_j)^+ \) and \( z = z^+ - z^- \), and the third equality is due to \( -(z)^- = z^+ \).

Thus, determining the optimal matching quantity between type \( i \) demand and type \( j \) supply is equivalent to finding the optimal protection level \( a_{ij}^*(t) \) such that the post-matching levels \( u_i^* = \bar{x}_i + a_{ij}^*(t) = (\bar{x}_i - \bar{y}_j)^+ + a_{ij}^*(t) \) and \( v_j^* = \bar{y}_j + a_{ij}^*(t) = (\bar{y}_j - \bar{x}_i)^+ + a_{ij}^*(t) \). The following theorem further sharpens the optimal policy characterization from the dynamic perspective of the optimal matching procedure within each period.

Theorem 7 (Vertical model: dynamic perspective). Consider vertically differentiated types. In the top-down matching procedure, consider matching type \( i \) demand with type \( j \) supply, which is optimal only if all types \( 1, \ldots, i \) demand and types \( 1, \ldots, j - 1 \) supply have been fully matched, and \( \bar{x}_i > \bar{y}_{j-1} \) and \( \bar{x}_{i-1} < \bar{y}_j \). There exists a protection level \( a_{ij}^*(t) \) depending on \( (\bar{x}_i - \bar{y}_j, x_{[i+1,n]}, y_{[j+1,m]}) \) such that it is optimal to match type \( i \) demand with type \( j \) supply until the level of type \( i \) demand reduces to \( (\bar{x}_i - \bar{y}_j)^+ + a_{ij}^*(t) \) if \( \bar{x}_i - x_i > a_{ij}^*(t) \) (or equivalently, the level of type \( j \) supply reduces to \( (\bar{y}_j - \bar{x}_i)^+ + a_{ij}^*(t) \) if \( \bar{y}_j - y_j > a_{ij}^*(t) \)), and otherwise not to match type \( i \) demand with type \( j \) supply.

Proof of Theorem 7. Let \( a_{ij}^*(t, \bar{x}_i - \bar{y}_j, x_{[i+1,n]}, y_{[j+1,m]}) \in \arg\max_{a \geq 0} \left[ -(r_i^d + r_j^s + c + h)\alpha + \gamma EV_{t+1}(D_{[1,i-1]}, a(x_i + a) + D_i, \alpha x_{[i+1,n]} + D_{[i+1,n]}, S_{[1,j-1]}, \beta(y_j + a) + S_j, \beta y_{[j+1,m]} + S_{[j+1,m]}) \right] \).
If \( x_i - \bar{x}_i = x_i - (\bar{y}_{i-1} - \bar{x}_{i-1})^+ - (\bar{x}_i - \bar{y}_j)^+ > a_{ij}^*(t) \), then it is feasible and optimal to match type \( i \) demand with type \( j \) supply until the quantity of type \( i \) demand reduces to \( x_i + a_{ij}^*(t) = (\bar{x}_i - \bar{y}_j)^+ + a_{ij}^*(t) \). Otherwise, it is optimal not to match type \( i \) demand with type \( j \) supply. □

When we come to the decision on matching type \( i \) demand with type \( j \) supply in the top-down procedure, the optimal protection level \( a_{ij}^*(t) \) in determining how much to match the pair can be zero. In fact, when it is nonzero, the top-down matching policy would terminate after this round of matching type \( i \) demand with type \( j \) supply (with the matching quantity possibly being zero).

As a result, all types \( i + 1, \ldots, n \) demand and types \( j + 1, \ldots, m \) of lower quality would not be matched, but would be saved in anticipation of possible future shortage in demand and supply.

One managerial insight from the top-down matching procedure is that higher types tend to be matched in the current period to realize higher reward and lower types with lower “overstocking” costs tend to be saved as safety stock for the future.

### 6.1. Equal Carry-Over Rates

We now consider the case in which demand and supply have the same carry-over rate, i.e., \( \alpha = \beta \), for which we can further demonstrate monotonicity properties of the optimal matching policy with respect to the system state. To proceed, we define \( \bar{D}_i \equiv \sum_{t'=1}^{t_i} D_{t'} \) and \( \bar{S}_j \equiv \sum_{t'=1}^{t_j} S_{t'} \) for \( 1 \leq i \leq n \) and \( 1 \leq j \leq m \). We define \( \bar{U}_k \) as the \( k \times k \) upper triangular matrix with all the entries on or above the diagonal equal to one. Then the state transformation can be written in a matrix form: \( \bar{x} \bar{U}_n = \bar{x} \) and \( \bar{y} \bar{U}_m = \bar{y} \) (or equivalently, \( x = \bar{x} \bar{U}_n^{-1} \) and \( y = \bar{y} \bar{U}_m^{-1} \)). Also, let \( \tilde{V}_t(\bar{x}, \bar{y}) \equiv V_t(\bar{x} \bar{U}_n^{-1}, \bar{y} \bar{U}_m^{-1}) - \bar{x} \bar{U}_n^{-1}(\bar{r}^d)^T - \bar{y} \bar{U}_m^{-1}(\bar{r}^s)^T \).

Since \( \bar{U}_k^{-1} \) is a \( k \times k \) upper-triangular difference matrix that has all diagonal entries equal to 1, \( (l, l+1) \)-th entry equal to \(-1\) for all \( l = 1, 2, k - 1 \) and all other entries equal to 0, we can rewrite the DP (12) in terms of the value functions \( \tilde{V}_t \) and the state variables \( \bar{x} \) and \( \bar{y} \) for \( t = 1, \ldots, T \):

\[
\tilde{V}_t(\bar{x}, \bar{y}) = \max_{0 \leq Q \leq \min(\bar{x}_n, \bar{y}_m)} \bar{G}_t(Q, \bar{x}, \bar{y}),
\]

\[
\bar{G}_t(Q, \bar{x}, \bar{y}) = -(1 - \gamma \alpha) \sum_{j'=1}^n (r_{ij}^d - r_{ij+1}^d)(\bar{x}_{ij'} - Q) + (1 - \gamma \alpha) \sum_{j'=1}^m (r_{ij}^s - r_{ij+1}^s)(\bar{y}_{ij'} - Q) - c(\bar{x}_n - Q) - h(\bar{y}_m - Q) + \gamma \bar{D} \bar{U}_n^{-1}(\bar{r}^d)^T + \gamma \bar{S} \bar{U}_m^{-1}(\bar{r}^s)^T + \gamma E_{t+1}(\alpha(\bar{x} - Q 1^n)^+ + \bar{D}, \alpha(\bar{y} - Q 1^m)^+ + \bar{S}).
\]

**Lemma 9.** For all \( t \), \( \tilde{V}_t(\bar{x}, \bar{y}) \) is decreasing in \( \bar{x}_k \) for \( 1 \leq k < n \) and in \( \bar{y}_k \) for \( 1 \leq k < m \).

To proceed further on the structural properties of the optimal matching policy for vertically differentiated types, we make use of the notion of \( L^2 \)-concavity. A function \( f : \mathbb{R}^n \to \mathbb{R} \) is called \( L^2 \)-convex if \( f(x - \xi 1^n) \) is submodular in \( (x, \xi) \) (see Murota 2003). A function \( g \) is \( L^2 \)-concave if \(-g \) is \( L^2 \)-convex.
Consider vertically differentiated types and suppose the following monotonicity property of the optimal matching policy for the original system.

Since the higher the original state \((Q, \tilde{x}, \tilde{y})\), denoted by \(\hat{x}\), the optimal solution to (13), \(\hat{G}_t(Q, \tilde{x}, \tilde{y})\) is nondecreasing in \((Q, \tilde{x}, \tilde{y})\) for \(t = 1, \ldots, T\). Thus after the first round of matching type 1 demand and supply, there are the same levels of matching decisions for the remaining types must be the same for the two states, and as a result, \(Q_t^*(x, y + \varepsilon e_n^m - \varepsilon e_{n-1}^m) = Q_t^*(x, y + \varepsilon e_n^m - \varepsilon e_{n-1}^m) - Q_t^*(x, y) \leq \varepsilon\).

**Proof of Theorem 8.** Monotonicity of \(Q_t^*(x, y)\). Since \(L^2\)-concavity implies supermodularity, by Lemma 10, \(\tilde{G}_t(Q, \tilde{x}, \tilde{y})\) is \(L^2\)-concave, a fortiori, supermodular in \((Q, \tilde{x}, \tilde{y})\). By Simchi-Levi et al. (2014, Theorem 2.2.8), the optimal solution to (13), denoted by \(\hat{Q}_t(\tilde{x}, \tilde{y})\), is nondecreasing in \((\tilde{x}, \tilde{y})\). Since the higher the original state \((x, y)\), the higher the transformed state \((\tilde{x}, \tilde{y})\), the optimal solution \(Q_t^*(x, y)\), expressed in terms of the original state, is nondecreasing in \((x, y)\).

(i) Since \((x + \varepsilon e_n^m, y + \varepsilon e_n^m)\) is a state that has \(\varepsilon\) more type 1 demand and supply than state \((x, y)\), by Proposition 2, it is optimal to match type 1 demand and supply as much as possible; thus after the first round of matching type 1 demand and supply, there are the same levels of the remaining types for the system with state \((x + \varepsilon e_n^m, y + \varepsilon e_n^m)\) and with state \((x, y)\). Thus the optimal matching decisions for the remaining types must be the same for the two states, and as a result, \(Q_t^*(x + \varepsilon e_n^m, y + \varepsilon e_n^m) = Q_t^*(x, y) + \varepsilon\).

(ii) By the definition of \(L^2\)-concavity, \(\tilde{G}_t(Q - \xi, \tilde{x} - \xi 1^n, \tilde{y} - \xi 1^n)\) is supermodular in \((Q, \tilde{x}, \tilde{y}, \xi)\). Then, for \(Q > \hat{Q}_t(\tilde{x}, \tilde{y}) + \varepsilon\), we have

\[
\tilde{G}_t(Q, \tilde{x} + \varepsilon 1^n, \tilde{y} + \varepsilon 1^n) - \tilde{G}_t(Q, \tilde{x} + \varepsilon 1^n, \tilde{y} + \varepsilon 1^n) \leq \tilde{G}_t(Q - \varepsilon, \tilde{x}, \tilde{y}) - \tilde{G}_t(\hat{Q}_t(\tilde{x}, \tilde{y}), \tilde{x}, \tilde{y}) \leq 0,
\]

where the first inequality is derived by definition of supermodularity and the second inequality is due to the optimality of \(\hat{Q}_t\). This implies that any matching quantity \(Q > \hat{Q}_t(\tilde{x}, \tilde{y}) + \varepsilon\) is no better than \(\hat{Q}_t(\tilde{x}, \tilde{y}) + \varepsilon\) for the state \((\tilde{x} + \varepsilon 1^n, \tilde{y} + \varepsilon 1^n)\). Therefore, \(\hat{Q}_t(\tilde{x} + \varepsilon 1^n, \tilde{y} + \varepsilon 1^n) \leq \hat{Q}_t(\tilde{x}, \tilde{y}) + \varepsilon\).

By the monotonicity of \(\hat{Q}_t(\tilde{x}, \tilde{y})\), \(\hat{Q}_t(\tilde{x} + \varepsilon 1^n, \tilde{y}) \leq \hat{Q}_t(\tilde{x} + \varepsilon 1^n, \tilde{y} + \varepsilon 1^n) \leq \hat{Q}_t(\tilde{x}, \tilde{y}) + \varepsilon\). Expressed in the original state, \(Q_t^*(x + \varepsilon e_n^m, y) \leq Q_t^*(x, y) + \varepsilon\), which proves the last inequality in part (ii).
For any two original states \((x + \varepsilon e_i^n, y)\) and \((x + \varepsilon e_{k+1}^n, y)\), \(k = 1, \ldots, n - 1\), their transformed states can be ordered as \((\tilde{x} + \varepsilon \mathbf{1}_{[k,n]}, \tilde{y}) \geq (\tilde{x} + \varepsilon \mathbf{1}_{[k+1,n]}, \tilde{y})\), where \(\mathbf{1}_{[k,n]}\) is an \(n\)-dimensional vector with the \(k\)-th up to \(n\)-th entry being one and the rest of the entries being all zeros. By the monotonicty of \(\hat{Q}_i(\tilde{x}, \tilde{y})\), \(\hat{Q}_i(\tilde{x} + \varepsilon \mathbf{1}_{[k,n]}, \tilde{y}) \geq \hat{Q}_i(\tilde{x} + \varepsilon \mathbf{1}_{[k+1,n]}, \tilde{y})\). Translated into the original state, \(Q_i^*(x + \varepsilon e_i^n, y) \geq Q_i^*(x + \varepsilon e_{k+1}^n, y)\) and thus, \(Q_i^*(x + \varepsilon e_i^n, y) - Q_i^*(x, y) \leq Q_i^*(x + \varepsilon e_{k+1}^n, y) - Q_i^*(x, y)\). Combining that with \(Q_i^*(x + \varepsilon e_i^n, y) \leq Q_i^*(x, y) + \varepsilon\), we have the desired series of inequalities in part (ii), with the first inequality implied by the monotonicty of \(Q_i^*(x, y)\).

(iii) The series of inequalities can be proved analogously to part (ii). □

Theorem 8 provides a set of first-order monotonicity properties of the optimal total matching quantity with respect the state for vertically differentiated types. First, the higher the levels of demand and supply, the more quantities are optimally matched in the current period. Second, the equation in part (i) is a direct consequence of Proposition 2. If the levels of type 1 demand and type 1 supply are increased by the same amount, this increased amount will be optimally matched between them in the current period. Third, the series of inequalities (i.e., in parts (ii) and (iii)) show that an increment in the level of a demand or supply type with higher “quality” leads to a higher optimal matching quantity, and the rate of increase is dominated by 1. The statement is consistent with the intuition that higher types are more likely to be matched in the current period. We caution that these results are obtained under the assumption of equal carry-over rates; i.e., \(\alpha = \beta\). This is because these monotonicity properties are built upon the \(L^2\)-concavity of the value functions in the transformed system (Lemma 10). Unlike concavity and supermodularity, \(L^2\)-concavity depends on the scaling of the variables (Zipkin 2008). Here, to obtain the monotonicity properties, we require the carry-over levels of demand and supply to have the same scaling. (One may expect similar properties to hold for unequal carry-over rates, which may call for a novel form of concavity. We leave that to future research.)

The following corollary recounts Theorem 8 in terms of the state-dependent protection levels.

**Corollary 3** (Vertical model: monotonicity property of optimal protection level). Suppose \(\alpha = \beta\). The state-dependent protection level \(a^{1*}_{ij}(t, \tilde{x}_i - \tilde{y}_j, x_{[i+1,n]}, y_{[j+1,m]})\) is nonincreasing in \((\tilde{x}_i - \tilde{y}_j)^+, (\tilde{x}_i - \tilde{y}_j)^-, x_{[i+1,n]}\) and \(y_{[j+1,m]}\), with the decreasing rates no more than 1. In particular, \(a^{1*}_{11}(t) \equiv 0\). Moreover, \(a^{1*}_{ij}(t)\) is most sensitive to \(\tilde{x}_i - \tilde{y}_j\) and is more sensitive to \(x_{i'}\) than to \(x_{i'+1}\) and to \(y_{j'}\) than to \(y_{j'+1}\) for \(i + 1 \leq i' \leq n - 1\) and \(j + 1 \leq j' \leq m - 1\).

### 6.1.1. One-Step-Ahead Heuristic
Assuming equal carry-over rates, we consider the one-step-ahead heuristic, which has a much more simplified state-dependent structure and would be...
much easier to be computed than the optimal matching policy. Under this heuristic, the firm simply optimizes the protection levels for the current period \(t\), in anticipation of implementing greedy matching (i.e., without reserving any demand or supply) from period \(t+1\) to the end of the horizon.

We show that the one-step-ahead heuristic policy for period \(t\) is determined by a state-dependent target level \(\hat{b}_{ij}(t, \tilde{y}_m - \tilde{x}_n)\) and two other quantities \(\pi_i + \sum_{i'=i+1}^n x_{i'}\) and \(\pi_j + \sum_{i'=i+1}^n x_{i'}\). Recall that \(\pi_i = x_i - (\tilde{y}_j - \tilde{x}_{i-1})^+\) is the available quantity of type \(i\) demand before we consider the matching of type \(i\) demand with type \(j\) supply, and \(\pi_j = (\tilde{x}_i - \tilde{y}_j)^+\) is the post-matching level of type \(i\) demand immediately after we match type \(i\) demand with type \(j\) supply as much as possible. Then, \(\pi_i + \sum_{i'=i+1}^n x_{i'}\) is the available quantity in all the demand types before we match type \(i\) demand with type \(j\) supply, and \(\pi_j + \sum_{i'=i+1}^n x_{i'}\) is the aggregate post-matching level of all the demand types immediately after we match type \(i\) demand with type \(j\) demand as much as possible. We show the optimal matching policy has the following structure: If \(\hat{b}_{ij}(t, \tilde{y}_m - \tilde{x}_n) \geq \pi_i + \sum_{i'=i+1}^n x_{i'}\), it is optimal not to match type \(i\) demand with type \(j\) supply. If \(\pi_i + \sum_{i'=i+1}^n x_{i'} < \hat{b}_{ij}(t, \tilde{y}_m - \tilde{x}_n) < \pi_i + \sum_{i'=i+1}^n x_{i'}\), it is optimal to reduce type \(i\) demand to the target level \(\hat{b}_{ij}(t, \tilde{y}_m - \tilde{x}_n)\). If \(\hat{b}_{ij}(t, \tilde{y}_m - \tilde{x}_n) < \pi_i + \sum_{i'=i+1}^n x_{i'}\), it is optimal to match type \(i\) demand with type \(j\) supply as much as possible.

**Theorem 9 (One-step-ahead heuristic).** Suppose \(\alpha = \beta\) and consider the one-step-ahead heuristic, which also follows the top-down matching procedure. In this procedure, consider matching type \(i\) demand with type \(j\) supply, which is optimal only if \(\tilde{x}_i > \tilde{y}_j\) and \(\tilde{x}_{i-1} < \tilde{y}_j\). There exists a target level \(\hat{b}_{ij}(t, \tilde{y}_m - \tilde{x}_n)\) such that the optimal protection level for matching type \(i\) demand with type \(j\) supply is \(a^*_{ij}(t, x, y) = [\hat{b}_{ij}(t, \tilde{y}_m - \tilde{x}_n) - (\pi_i + \sum_{i'=i+1}^n x_{i'})]^+\).

We see that the protection level \(a^*_{ij}(t, x, y)\) is determined by the aggregate imbalance between demand and supply levels, \(\tilde{y}_m - \tilde{x}_n\), at the beginning of the period and the aggregate post-matching level \(\pi_i + \sum_{i'=i+1}^n x_{i'}\) of all demand types (i.e., the sum of post-matching levels in all demand types) if type \(i\) demand is matched with type \(j\) supply as much as possible. Note that \(\pi_i + \sum_{i'=i+1}^n x_{i'} + a^*_{ij}(t, x, y)\) is the target level for the aggregate post-matching demand. If \(\hat{b}_{ij}(t, \tilde{y}_m - \tilde{x}_n) > \pi_i + \sum_{i'=i+1}^n x_{i'}\), the target level becomes \(\hat{b}_{ij}(t, \tilde{y}_m - \tilde{x}_n)\). Otherwise, it is \(\pi_i + \sum_{i'=i+1}^n x_{i'}\), which is the outcome of matching type \(i\) demand with type \(j\) supply to the maximum extent. Therefore, Proposition 9 implies that the firm’s target is to bring the aggregate post-matching demand level as close to \(\hat{b}_{ij}(t, \tilde{y}_m - \tilde{x}_n)\) as possible. We also note that the matching of type \(i\) demand with type \(j\) supply is constrained by the quantity \(\pi_i + \sum_{i'=i+1}^n x_{i'}\), which is the aggregate level of types \(i, i+1, \ldots, n\) demand immediately before the matching of type \(i\) demand with type \(j\) supply. If \(\hat{b}_{ij}(t, \tilde{y}_m - \tilde{x}_n)\) is already equal to or above this level before matching type \(i\) demand with type \(j\) supply, then these two types will not be matched with each other.
**Proposition 3.** The target level $b_{ij}(t, \tilde{y}_m - \tilde{x}_n)$ for the aggregate post-matching demand is weakly decreasing in $\tilde{y}_m - \tilde{x}_n$ and the decreasing rate is dominated by 1.

Clearly, greedy matching will be optimal in the last period, i.e., period $T$. The following corollary is immediate.

**Corollary 4.** The optimal matching policy in period $T-1$ has the structure as stated in Theorem 9 and satisfies the properties as stated in Proposition 3.

As an implication, the one-step-ahead heuristic is optimal in the beginning of a two-period model.

### 6.2. Lost Demand or Supply

When $\alpha = 0$, any unmatched demand does not carry over to the next period and will be lost at the end of the current period. Similarly, $\beta = 0$ means that unmatched supply will be lost. By symmetry, we focus on the case in which $\beta = 0$. The case with $\alpha = 0$ has parallel results.

Like (13) in §6.1, we can write the DP (12) in terms of the state vector $(\tilde{x}, \tilde{y})$ when $\beta = 0$:

$$V_t(\tilde{x}, \tilde{y}) = \max_{0 \leq Q \leq \min(\tilde{x}_n, \tilde{y}_m)} G_t(Q, \tilde{x}, \tilde{y}),$$

$$G_t(Q, \tilde{x}, \tilde{y}) = -(1 - \gamma\alpha) \sum_{i' = 1}^{n} (r_{i'}^d - r_{i'+1}^d)(\tilde{x}_{i'} - Q)^+ - (1 - \gamma\alpha) \sum_{j' = 1}^{m} (r_{j'}^s - r_{j'+1}^s)(\tilde{y}_{j'} - Q)^+ - c(\tilde{x}_n - Q)$$

$$- h(\tilde{y}_m - Q) + \gamma \tilde{D}U_n^{-1}(r^d)^T + \gamma \tilde{S}U_m^{-1}(r^s)^T + \gamma E\tilde{V}_{t+1}(\alpha(\tilde{x} - Q1^n)^+ + \tilde{D}, \tilde{S}).$$

We can show that the protection levels have a lower-dimensional state dependency when $\beta = 0$, which can be proved analogously to Theorem 7.

**Proposition 4 (Vertical model: lost supply).** Consider vertically differentiated types. With a stronger assumption $\beta = 0$, Theorem 7 can be strengthened as follows: In considering the matching of type $i$ demand with type $j$ supply, there exists a state-dependent threshold $\theta_{ij}(t, x_{[i+1,n]})$ such that it is optimal to reduce type $i$ demand to $\theta_{ij}(t, x_{[i+1,n]})$ if $\tilde{x}_i - \tilde{y}_j < \theta_{ij}(t, x_{[i+1,n]})$, to match it with type $j$ supply down to the level $\tilde{x}_i - \tilde{y}_j$ if $\tilde{x}_i - \tilde{y}_j \geq \theta_{ij}(t, x_{[i+1,n]})$, and otherwise not to match type $i$ demand and type $j$ supply.

### 6.3. 1 Demand Type and $m$ Supply Types

We consider the model with only 1 demand type and $m$ supply types. Without loss of generality, we can let $r_{i}^d = 0$ and $r_{j}^s = r_{ij}$ for $1 \leq j \leq m$, then $r_{ij} = r_{i}^d + r_{j}^s$ for $i = 1$ and $1 \leq j \leq m$.

The next result immediately follows from Theorem 7.

**Corollary 5 (1-to-$m$ vertical model).** Consider vertically differentiated types. With a stronger assumption of 1 demand type and $m$ supply types, Theorem 7 can be strengthened as follows: In
considering matching type 1 demand with type j supply, there exists a threshold \( \bar{z}_j(t, x_1 - \bar{y}_j, y_{j+1,m}) \) such that it is optimal to reduce the type 1 demand to \( \min \{ \bar{z}_j(t, x_1 - \bar{y}_j, y_{j+1,m}), x_1 - \bar{y}_{j-1} \} \) by matching it with type j supply, and otherwise not to match type 1 demand and type j supply.

In addition, we have the following result on the state collapse of the optimal match-down-to levels.

**Proposition 5.** The optimal match-down-to level \( \bar{z}_j(t, x_1 - \bar{y}_j, y_{j+1,m}) \) becomes a constant independent of \( x_1 - \bar{y}_j \) and \( y_{j+1,m} \) if \( \beta = 0 \) or \( \alpha = \beta \).

We can obtain analogous results for the vertical model with \( n \) demand types and 1 supply type.

7. **Bound and Heuristic: The Deterministic Model**

In this section we study the deterministic counterpart of the stochastic problem. We show that the heuristic suggested by the deterministic model can be computed efficiently and is asymptotically optimal for the stochastic problem.

**7.1. The Deterministic Heuristic**

We consider the deterministic model by ignoring the uncertainty and assume that the mean demand quantity \( \lambda_{it} = ED_{it} \) and mean supply quantity \( \mu_{jt} = ES_{jt} \) arrive in each period.

We have the following linear program formulation of the model.

\[
\begin{align*}
(P) \quad \max \quad & \sum_{i=1}^{T} \gamma^{t-1} \left[ \sum_{i=1}^{n} \sum_{j=1}^{m} r_{ij} q_{ijt} - c \left( \sum_{i=1}^{n} x_{it} - \sum_{i=1}^{n} \sum_{j=1}^{m} q_{ijt} \right) - h \left( \sum_{j=1}^{m} y_{jt} - \sum_{i=1}^{n} \sum_{j=1}^{m} q_{ijt} \right) \right] \\
\text{s.t.} \quad & \sum_{j=1}^{m} q_{ijt} \leq x_{it}, \quad 1 \leq i \leq n, 1 \leq t \leq T, \\
& \sum_{i=1}^{n} q_{ijt} \leq y_{jt}, \quad 1 \leq j \leq m, 1 \leq t \leq T, \\
& x_{i,t+1} = \alpha (x_{it} - \sum_{j=1}^{m} q_{ijt}) + \lambda_{it}, \quad 1 \leq i \leq n, 1 \leq t \leq T-1, \\
& y_{j,t+1} = \beta (y_{jt} - \sum_{i=1}^{n} q_{ijt}) + \mu_{jt}, \quad 1 \leq j \leq m, 1 \leq t \leq T-1, \\
& q_{ijt} \geq 0, \quad 1 \leq i \leq n, 1 \leq j \leq m, 1 \leq t \leq T,
\end{align*}
\]

where \((x_1, y_1)\) is a given initial state.

We can rewrite the formulation (P) as a minimization problem and write its dual as a maximization problem as follows:

\[
\begin{align*}
(D) \quad \max \quad & \sum_{i=1}^{T-1} n \sum_{i=1}^{n} f_{it}^d \lambda_{it} + \sum_{i=1}^{T-1} m \sum_{j=1}^{m} f_{jt}^d \mu_{jt} - \sum_{i=1}^{n} p_{i1}^d x_{i1} - \sum_{j=1}^{m} p_{j1}^s y_{j1} + \alpha \sum_{i=1}^{n} f_{i1}^d x_{i1} + \beta \sum_{j=1}^{m} f_{j1}^s y_{j1}
\end{align*}
\]
\[ p_{it}^d + f_{jlt}^s - \alpha f_{it}^d + \beta f_{jt}^s \geq \gamma^{t-1}(r_{ij} + c + h), \quad 1 \leq i \leq n, 1 \leq j \leq m, 1 \leq t \leq T, \]
\[ p_{it}^d - \alpha f_{it}^d + f_{it}^{d,t-1} = \gamma^{t-1}c, \quad 1 \leq i \leq n, 2 \leq t \leq T, \]
\[ p_{jt}^s - \beta f_{jt}^s + f_{jt}^{s,t-1} = \gamma^{t-1}h, \quad 1 \leq j \leq m, 2 \leq t \leq T, \]
\[ p_{it}^d, p_{jt}^s \geq 0, \quad 1 \leq i \leq n, 1 \leq j \leq m, 1 \leq t \leq T, \]
where \( f_{it}^d \equiv 0 \) and \( f_{jt}^s \equiv 0 \) for \( 1 \leq i \leq n \) and \( 1 \leq j \leq m \).

The complementary slackness (CS) conditions for the above primal-dual pair are given as follows:

\[
(\text{CS}) \quad p_{it}^d (x_{it} - \sum_{j=1}^{m} q_{ijt}) = 0, \quad 1 \leq i \leq n, 1 \leq t \leq T, \\
p_{jt}^s (y_{jt} - \sum_{i=1}^{n} q_{ijt}) = 0, \quad 1 \leq j \leq m, 1 \leq t \leq T, \\
q_{ijt} [p_{it}^d + p_{jt}^s - \alpha f_{it}^d - \beta f_{jt}^s - \gamma^{t-1}(r_{ij} + c + h)] = 0, \quad 1 \leq i \leq n, 1 \leq j \leq m, 1 \leq t \leq T.
\]

By the strong duality, we immediately have the following result, which is standard, see, e.g., Sethi and Thompson (2005, section 8.2.2).

**Proposition 6** (Solution to the deterministic model). A matching decision \( \{Q_t\}_{t=1,\ldots,T} \) is optimal if and only if there exist dual variables \( \{p_{it}^d, p_{jt}^s\}_{1 \leq i \leq n, 1 \leq t \leq T}, \{f_{it}^d, f_{jt}^s\}_{1 \leq j \leq m, 1 \leq t \leq T-1} \), together with \( \{Q_t\}_{t=1,\ldots,T} \), such that the constraints of problems (P) and (D) and conditions (CS) are satisfied.

The multipliers \( f_{it}^d, f_{jt}^s, 1 \leq i \leq n, 1 \leq j \leq m, 1 \leq t \leq T \), correspond to the state transition equations in problem (P) on the demand and supply side respectively, which can be interpreted as the shadow prices of changing the system states. If these shadow prices are known or can be approximated by good proxies, e.g., the typical market prices that encourage demand and supply to enter the market for a given time period, then the optimal matching decisions for that period can be obtained or approximated by the following linear program for that period. For any given \( t \), consider the following subproblem:

\[
(\text{s-p}) \quad \max_{q_{ijt}} \sum_{i=1}^{n} \sum_{j=1}^{m} \left( \gamma^{t-1}r_{ij} + \alpha f_{it}^d + \beta f_{jt}^s \right) q_{ijt} - \gamma^{t-1}c \left( \sum_{i=1}^{n} x_{it} - \sum_{i=1}^{n} \sum_{j=1}^{m} q_{ijt} \right) - \gamma^{t-1}h \left( \sum_{j=1}^{m} y_{jt} - \sum_{i=1}^{n} \sum_{j=1}^{m} q_{ijt} \right) \\
\text{s.t.} \quad \sum_{j=1}^{m} q_{ijt} \leq x_{it}, \quad 1 \leq i \leq n, \\
\sum_{i=1}^{n} q_{ijt} \leq y_{jt}, \quad 1 \leq j \leq m, \\
q_{ijt} \geq 0, \quad 1 \leq i \leq n, 1 \leq j \leq m.
\]

The following corollary verifies that a necessary condition for an optimal matching decision \( Q_t \) in period \( t \) is that it solves the single-period matching problem (s-p).
Corollary 6. The optimal matching decision \( \{Q_t\}_{1 \leq t \leq T} \) solves the subproblem \((s-p)\) for \(1 \leq t \leq T\).

Intuitively, one would expect that the uncertainty in demand and supply in the stochastic model would result in lower expected surpluses. The following proposition formalizes that idea.

Proposition 7 (Deterministic upper bound). The deterministic model provides an upper bound on the optimal total surplus of the stochastic model.

7.2. Asymptotic Optimality

Next we show that the heuristic policy suggested by the deterministic problem is asymptotically optimal. Consider a series of stochastic systems indexed by \( k = 1, 2, \ldots \), with 1 representing the original system. We scale the time in system \( k \) so that its clock is \( k \) times faster than the original system in any given period \( t \). Thus, instead of having random demand \( D_{it} \) for type \( i \in D \) and random supply \( S_{jt} \) for type \( j \in S \) arriving in a period, system \( k \) will have an amount of \( \sum_{t=1}^{k} D^\ell_{it} \) for type \( i \in D \) and \( \sum_{t=1}^{k} S^\ell_{jt} \) for type \( j \in S \), where the \( D^\ell_{it} \)'s are i.i.d. and have the same distribution as \( D_{it} \), the \( S^\ell_{jt} \)'s are i.i.d. and have the same distribution as \( S_{jt} \). We further scale down the demand and supply in a way such that every \( k \) units is counted as 1 unit. Then, in system \( k \), we have the one-period demand as \( \bar{D}^k = \frac{1}{k}(\sum_{t=1}^{k} D^\ell_{it}, \ldots, \sum_{t=1}^{k} D^\ell_{mt}) \) and supply as \( \bar{S}^k = \frac{1}{k}(\sum_{t=1}^{k} S^\ell_{1t}, \ldots, \sum_{t=1}^{k} S^\ell_{mt}) \).

Note that \( E\bar{D}^k = ED_t = \lambda_t \) and \( E\bar{S}^k = ES_t = \mu_t \). Let \( V_t^k(x,y) \) be the value function in system \( k \), and \( V_t^{\text{det}}(x,y) \) the value function for the deterministic model. The functions \( H_t^k(Q,x,y) \) are defined in the same way as we defined \( H_t(Q,x,y) \). We have the following result that relates the stochastic systems with the deterministic model.

Proposition 8. Suppose that demand and supply of all types are bounded; i.e., there exist \( U^d \in \mathbb{R}^n \) and \( U^s \in \mathbb{R}^m \) such that \( D_t \leq U^d \) and \( S_t \leq U^s \) almost surely for \( t = 1, \ldots, T \). Then, \( \lim_{k \to \infty} V_t^k(x,y) = V_t^{\text{det}}(x,y) \) for \( t = 1, 2, \ldots, T + 1 \). Furthermore, for any compact set \( B \in \mathbb{R}^{n+m} \), the convergence is uniform over \((x,y) \in B\).

While the assumption of bounded demand and supply simplifies the proof and is reasonable for any practical purposes, it is not necessary. In the following proposition, we drop that restriction.

Proposition 9. Suppose that \( ED_t < \infty \) and \( ES_t < \infty \) for \( t = 1, \ldots, T \). Then, \( \lim_{k \to \infty} V_t^k(x,y) = V_t^{\text{det}}(x,y) \) for \( t = 1, 2, \ldots, T + 1 \). Furthermore, for any compact set \( B \in \mathbb{R}^{n+m} \), the convergence is uniform over \((x,y) \in B\).

We define a feasible decision \( Q \) under state \((x,y) \) in period \( t \) as asymptotically optimal for the stochastic system if, for any \( \varepsilon > 0 \), there exists \( K > 0 \) such that \( H_t^k(Q,x,y) \geq H_t^k(Q^{k*},x,y) - \varepsilon = V_t^k(x,y) - \varepsilon \) if \( k \geq K \), where \( Q^{k*} \) is an optimal policy for system \( k \).
Theorem 10 (Asymptotic optimality of the deterministic heuristic). The optimal policy $Q_t(x, y)$ for the deterministic model, solved from the linear program (P) with $x_1 = x$ and $y_1 = y$, is asymptotically optimal for the stochastic system.

8. Extensions

We discuss a few extensions of the base model.

**Time-Dependent Parameters.** All of our results can be readily extended to allow the following parameters to be time-varying: holding costs $c$ and $h$, carry-over rates $\alpha$ and $\beta$, and discount factor $\gamma$. For the extension of time-dependent reward parameters, we need to generalize condition (D) in Definition 1 of the relation $(i, j) \succeq (i', j')$ to the following condition: for any time $0 \leq t_1 \leq t_2 \leq T$, $r_{ij}(t_1) + r_{i'j'}(t_2) \geq r_{ij'}(t_1) + r_{i'j}(t_2)$. The generalized condition eliminates the optimality of saving type $j$ supply for future, which ensures the priority of matching pair $(i, j)$ over pair $(i, j')$ for the current period. All our results remain valid as long as the partial order is generalized with temporal consideration to prevent the current priority from being reversed in the future.

**Type-Dependent Parameters.** It is likely that different types of demand or supply have heterogeneous holding-cost rates and carry-over rates. This may be less of an issue for horizontally differentiated types of geographic locations, e.g., the backlog cost to Amazon or Uber’s unsatisfied customers is independent of the customers’ geographic locations. The results in Theorem 2 can be preserved if we relax the type-homogeneous-parameter assumption but assume that $c_i \geq c_{i'}$, $\alpha_i \leq \alpha_{i'}$ for all $i' \in D$ and $h_j \geq h_{j'}$, $\beta_j \leq \beta_{j'}$ for all $j' \in S$. In other words, it is still optimal for type $i$ demand and type $j$ supply to be matched as much as possible, as long as that they have the highest holding-cost rates and the lowest carry-over rates on each side. If, on the contrary, type $i'$ demand has a higher holding-cost rate or a lower carry-over rate than type $i$ demand, then the firm may want to prioritize matching arc $(i', j)$ over arc $(i, j)$ to reduce the total holding cost or increase the utilization of type $i'$ demand in the current period. Consequently, $q_{ij}^* = \min\{x_i, y_j\}$ may no longer hold. Similarly, Theorem 1 remains true; i.e., the arc $(i, j)$ has priority over $(i, j')$ provided that $h_j \geq h_{j'}$, $\beta_j \leq \beta_{j'}$, in addition to the condition in Theorem 1 (i.e., $(i, j) \succeq (i, j')$). Theorem 3 can be extended as long as the holding-cost rates and carry-over rates are ordered in alignment with the priority structure determined by the rewards. For example, for the model with vertically differentiated types in §6, the results carry over under the assumptions $c_1 \geq \cdots \geq c_n$, $h_1 \geq \cdots \geq h_m$, $\alpha_1 \leq \cdots \leq \alpha_n$ and $\beta_1 \leq \cdots \leq \beta_m$. It is intuitive that the top-down matching procedure is still optimal when the holding-cost rates and carry-over rates are ordered such that the lower the index, the higher the holding cost rate and the lower the carry-over rate. This is because it gives all the incentive needed to prioritize the matching of higher-quality types for the current period.
Random Abandonment. After the matching in each period, unmatched demand and supply may abandon the wait with uncertainty. Our results will go through for random carry-over rates as long as they are realized the same for all types on each side. Moreover, for type-dependent random carry-over rates, Theorems 1 and 2 and results in §6 will continue to hold as long as the random carry-over rates are ordered in the way as just described, and do not have overlap in their supports.

Forbidden Arcs. In the base model, we assume that matching between any pair of type $i$ demand and type $j$ supply is allowed. In reality, however, it is possible that a certain demand-supply pair $(i, j)$ is undesirable or incompatible, and thus the matching between type $i$ demand and type $j$ supply is not an option for the firm. We consider a set of forbidden arcs $F \subset A = \{(i, j) | 1 \leq i \leq n, 1 \leq j \leq m\}$. The set $F^c \defeq A \setminus F$ is the permissible set of arcs along which matching is allowed. We can extend our partial-order definition over the permissible set $F^c$. As long as the rewards satisfy the assumptions over the set $F^c$, all of our results continue to hold, even under the restriction of forbidden arcs.

Forced Maxing Out. In practice, regulations, contracts, or social concerns may prevent the intermediary from deliberately saving demand and supply for future without maxing out the matching of them within any given period. Our priority results remain valid when reserving demand and supply for future is not allowed, i.e., when all quantities of various types on either the demand or the supply side have to be exhausted, with the unmatched demand or supply carried over to the next period. As a matter of fact, we can completely characterize the optimal matching policy under the restriction of forced maxing out, for those special cases in which all neighboring arcs can be compared under the partial order (e.g., the horizontal and vertical models). The optimal policy is simply to match demand with supply in a greedy fashion according to the priority hierarchy: We start from the undominated arcs (i.e., arcs in $A_1$ as defined in the proof of Theorem 3) and match demand with supply as much as possible along these arcs. After removing the demand and supply types that have been exhausted and their associated arcs, we again match the new undominated arcs (i.e., arcs in $A_2$) to the maximum extent. Repeating this process, we will have either all the demand or all the supply are matched for the current period.

Controlled Supply or Demand Process. In the inventory-sharing example mentioned in the introduction, Amazon may implement certain inventory policy to regulate the stream of its own supply, in addition to the exogenous supply streams regulated by third-party merchants. For any given controlled supply process by Amazon itself, our structural results continue to hold, because those results are distribution-free. On top of the optimal dynamic matching decisions given a controlled supply process and exogenous supplies from third parties, Amazon may optimize the
inventory policy for its own supply stream. Similarly, Uber can optimize its pricing decisions on top of the matching policies which are only applied after the demand and supply have accepted the price and entered the market.

9. Conclusion and Future Research

The proposed framework generalizes many classical problems. For example, we generalize inventory rationing problems and dynamic capacity allocation models with upgrading, by allowing for multiple exogenous supply streams and arbitrary substitution (e.g., a demand for a high-quality product can be substituted with a low-quality product, with monetary compensation). We generalize the transportation problems and the assignment problems by allowing for unbalanced demand and supply streams with inter-temporal uncertainties. Our distribution-free results may shed light on inventory models with stochastic lead-times. For any arbitrarily given inventory policy for a multi-product firm, the obtained general priority properties and match-down-to structures can guide the firm in fulfilling multi-class demand in anticipation of stochastic lead-times, on top of which the firm may search for good heuristic inventory policies.

The proposed framework lays out a foundation for further research in the area of dynamic matching at the operational level. We list several possible directions.

**Performance Bounds for Heuristic Policies.** The heuristic policy suggested by the deterministic problem can perform well if the system is on a large scale, but if it is not, there is no guarantee whatsoever. Moreover, a numerical study, no matter how extensive it is, only makes a posterior statement about the cost-effectiveness of a heuristic policy; it is never clear whether the observations from the numerical study will carry over to other cases. Hence, it is highly desirable to be able to propose an easily computable heuristic policy and derive good performance bounds. If worst-case performance bounds that apply to all model parameters are hard to obtain, parameter-dependent performance bounds would still be desirable. In light of previous efforts in the area of inventory management (see Levi 2010 for a survey and Hu and Yang 2014 for recent development) and the connection of our dynamic matching framework to inventory management, we expect that this direction may be promising.

**Dynamic Pricing in Matching.** On some sharing-economy platforms, prices can be set in real time to respond to market conditions. For those settings, one can consider the dynamic pricing problem in which “price” serves as the tool to use for matching. A matching between a pair of demand and supply may materialize only if the demand has a higher willingness-to-pay than the price and the supply has a lower willingness-to-sell than the price. The intermediary can get a cut
of the selling price of the crowdsourced supply. The intermediary’s problem can be to maximize profitability over time.

**Competition among Sharing Platforms.** In practice there can be more than one intermediary platform competing for both supply and demand. For example, in the car-sharing economy, Uber competes with Lyft. One can consider a model related to the previous research direction, in which upon arrival, a unit of demand or supply checks the prices in both platforms and chooses to be matched to a unit of demand or supply that results in a larger surplus. If not matched immediately, the unit may stay in both platforms with possible abandonment over time. One specific objective for a study of this problem could be to investigate the existence and uniqueness of the dynamic-pricing equilibrium strategy by sharing platforms and to analyze its properties.

**Appendix. Proofs.**

**Proof of Lemma 1.** \((i, j) \succeq (i', j')\) implies that \(r_{ij} + r_{i'j'} \geq r_{ij'} + r_{i'j}\) for all \(i' \in D\) and \(r_{ij} \geq r_{ij'}\). \((i, j') \succeq (i, j'')\) implies that \(r_{ij'} + r_{ij''} \geq r_{ij'} + r_{ij'}\) for all \(i' \in D\) and \(r_{ij'} \geq r_{ij''}\). Adding up the two inequalities leads to \(r_{ij} + r_{i'j'} \geq r_{ij} + r_{i'j''}\) for all \(i' \in D\). Moreover, \(r_{ij} \geq r_{ij''}\). It follows from Definition 1 that \((i, j) \succeq (i, j'')\).  

**Proof of Lemma 2.** Since for \(k = 1, \ldots, l - 1\), \((i_k, j_k) \succeq (i_{k+1}, j_{k+1})\), we have \(r_{i_kj_k} + r_{i_{k+1}j'_k} \geq r_{i_kj'_k} + r_{i_{k+1}j_k}\) for all \(j'_k\). Likewise, for \(k = 1, \ldots, l - 1\), \((i_{k+1}, j_k) \succeq (i_{k+1}, j_{k+1})\) implies that \(r_{i_{k+1}j_k} + r_{i'_{k+1}j_k+1} \geq r_{i_{k+1}j_{k+1}} + r_{i'_{k+1}j_k}\) for all \(i'_{k+1}\), and \((i_1, j_1) \succeq (i_1, j_{l-1})\) implies that \(r_{i_1j_1} + r_{i_1j'_{l-1}} \geq r_{i_1j_{l-1}} + r_{i_1j\ell}\) for all \(j'_{l-1}\). Sum up all these inequalities for any given \(i'\) and \(j'\),

\[
\sum_{k=1}^{l-1} (r_{i_kj_k} + r_{i_{k+1}j'_k}) + \sum_{k=1}^{l-1} (r_{i_kj_{k+1}} + r_{i'_{k+1}j_k+1}) + r_{i_1j_1} + r_{i_1j_{l-1}} \geq \sum_{k=1}^{l-1} (r_{i_kj'_k} + r_{i_{k+1}j_k}) + \sum_{k=1}^{l-1} (r_{i_kj_{k+1}} + r_{i'_{k+1}j_k+1}) + r_{i_1j'_{l-1}} + r_{i_1j\ell}.
\]

By rearranging terms, the left-hand-side of the above inequality can be rewritten as \(\sum_{k=1}^{l} r_{i_kj_k} + \sum_{k=1}^{l} r_{i_{k+1}j'_k}\), and the right-hand-side can be rewritten as \(\sum_{k=1}^{l} r_{i_kj'_k} + \sum_{k=1}^{l} r_{i_{k+1}j_k}\). Then, it is easy to see that the inequality becomes \(r_{i_1j_1} + r_{i_1j_{l-1}} \geq r_{i_1j_{l-1}} + r_{i_1j\ell}\), for any \(i'\), after canceling out the common terms on both sides. Moreover, \((i_1, j_1) \succeq (i_2, j_1) \succeq (i_2, j_2) \succeq \cdots \succeq (i_\ell, j_{l-1}) \succeq (i_{\ell-1}, j_{l-1})\) implies that \(r_{i_1j_1} \geq r_{i_2j_1} \geq r_{i_2j_2} \geq \cdots \geq r_{i_{\ell}j_{l-1}} \geq r_{i_{\ell}j_{l-1}}\) and thus \(r_{i_1j_1} \geq r_{i_{\ell}j_{l-1}}\). Therefore, \((i_1, j_1) \succeq (i_{\ell-1}, j_{l-1})\).  

**Proof of Lemma 3.** (Reflexivity) This follows directly from Definition 1.

(Antisymmetry) This follows from the definition of the equivalence relation \(\simeq\).

(Transitivity) \(\rho_1 \succeq \rho_2\) implies that there exists a decreasing sequence of arcs connecting \(\rho_1\) and \(\rho_2\), containing no cycles. Likewise, \(\rho_2 \succeq \rho_3\) implies that there exists a decreasing sequence of arcs connecting \(\rho_2\) and
\(\rho_3\), containing no cycles. Combining these two sequences, we have a decreasing sequence of arcs connecting \(\rho_1\) and \(\rho_3\).

If \(\rho_1\) and \(\rho_3\) share no nodes, then by Definition 2, \(\rho_1 \geq \rho_3\).

Now consider the case in which \(\rho_1\) and \(\rho_3\) have a common node. Let us assume without loss of generality that \(\rho_1 = (i_1, j_1)\) and \(\rho_3 = (i_1, j_3)\). If \(\rho_2\) also shares the same node with them, i.e., node \(i_1\), then by Lemma 1, \(\rho_1 \geq \rho_3\).

It remains to check the case in which \(\rho_2\) is not adjacent to the common node \(i_1\) shared by \(\rho_1\) and \(\rho_3\). Note that there exists a decreasing zigzag path of arcs connecting \(\rho_2\) and \(\rho_3\) that does not visit any node on the path more than once. Consider the arc \(\rho\) immediately before \(\rho_3\). The arc \(\rho\) is adjacent to either node \(j_3\) or node \(i_1\). In the former case, by Definition 2, the relation \(\rho_1 \geq \rho\) holds, because \(\rho_1\) and \(\rho\) share no common nodes and there is a decreasing sequence of arcs connecting \(\rho_1\) and \(\rho\) (consider the decreasing sequence connecting \(\rho_1\) and \(\rho_3\) with \(\rho_3\) removed). Together with the arc \(\rho_3\), the zigzag path connecting \(\rho_1\) and \(\rho\) forms a cycle. By Lemma 2, \(\rho_1 \geq \rho_3\).

If the latter case, i.e., \(\rho\) is adjacent to \(i_1\), then \(\rho = (i_1, j)\) for some \(j\) and there is an arc \(\rho' = (i, j)\) immediately before \(\rho\) in the decreasing path that connects \(\rho_2\) and \(\rho_3\), with \(i \neq i_1\). Since there clearly exists a decreasing sequence connecting \(\rho_1\) and \(\rho'\), which do not share a common node, we have \(\rho_1 \geq \rho'\). Then, together with the arc \(\rho\), the zigzag path connecting \(\rho_1\) and \(\rho'\) forms a cycle. Again by Lemma 2, \(\rho_1 \geq \rho\). Since \(\rho \geq \rho_3\) and all of \(\rho_1, \rho\) and \(\rho_3\) share the same node \(i_1\), it follows from Lemma 1 that \(\rho_1 \geq \rho_3\). \(\square\)

**Proof of Corollary 2.** Consider the case in which \(x_1 \geq y_1\) and \(x_2 \leq y_2\). If \(\alpha = 0\), in the proof of Theorem 6, (11) reduces to \(\tilde{H}_t(a_1, \eta) = -(r_{12} + c + h)a_1 + \gamma EV_{t+1}(D_1, D_2, S_1, \beta(\eta^- + a_1) + S_2)\). To optimize the protection level \(a_1\) is equivalent to optimizing the post-matching level \(v_2 = \eta^- + a_1\) of supply type 2. Let \(\hat{v}_2(t) \in \arg\max_{v_2 \geq 0} -(r_{12} + c + h)v_2 + \gamma EV_{t+1}(D_1, D_2, S_1, \beta v_2 + S_2)\), which is independent of \(\eta\). Since \(0 \leq a_1 \leq \eta_1 - \eta^+\), \(\eta^- \leq v_2 = \eta^- + a_1 \leq \eta_2^-\). Then, \(v^*_2 = \max\{\hat{v}_2(t) \land \eta_2^-, \eta^-, \}, a^*_1(t, \eta) = v^*_2 - \eta^- = [\hat{v}_2(t) \land \eta_2^- - \eta^-]^+\) and \(q^*_1 = \eta_2^- - v^*_2 = \eta_2^- - \max\{\hat{v}_2(t) \land \eta_2^-, \eta^-\}\). Analogously, we can show the desired result for \(\beta = 0\). \(\square\)

**Proof of Lemma 7.** By Theorem 3, we can find an optimal \(Q^*\) such that \(\min\{w^*_{ij}, q^*_{ij}\} = 0\) if \((i', j') \succeq (i, j)\). We show that \(Q^*\) satisfies the desired property.

Suppose on the contrary that \(q^*_{ij} > 0\) and type \(j' < j\) supply is not fully matched (i.e., \(v^*_{j'} > 0\)). Then, \(w^*_{ij} = 0\) by our choice of \(Q^*\). But \(w^*_{ij} = \min\{u^*_i + \sum_{(i', j') \in E_{ij}} q^*_{ij}, v^*_j + \sum_{(i', j') \in E_{ij}} q^*_{i'j'}\} \geq \min\{q^*_{ij}, v^*_j\} > 0\), where the first inequality holds because \((i, j) \in L_{ij}\). Thus, \(v^*_j\) must be equal to zero.

Similarly, we can show that \(u^*_i = 0\) for any \(i' < i\). \(\square\)
Proof of Lemma 8. It is easy to see that $G_t$ is concave in $Q$ within the interior of the ranges $\hat{x}_{i-1} \leq Q < \hat{x}_i$ and $\hat{y}_{j-1} \leq Q < \hat{y}_j$.

Without loss of generality, we assume that $\hat{x}_i \in (\hat{y}_{j-1}, \hat{y}_j)$. We show that $G_t$ is concave in the neighborhood of a breakpoint $a = \hat{x}_i$. To this end, it suffices to show that $G_t(a + \varepsilon, x, y) - G_t(a, x, y) \leq G_t(a - \varepsilon, x, y) - G_t(a, x, y)$, where $0 < \varepsilon < \min\{\hat{x}_i - \hat{y}_{j-1}, \hat{y}_j - \hat{x}_i\}$.

One the one hand, we have

$$G_t(a, x, y) - G_t(a - \varepsilon, x, y)$$

$$= (r_i^d + r_i^s + c + h)\varepsilon$$

$$+ \gamma EV_{i+1}(D_{[i,j]}, \alpha x_{i+1,n} + D_{[i+1,n]}, S_{[1,j-1]}, \beta(\hat{y}_j - \hat{x}_i) + S_j, \beta y_{[j+1,m]} + S_{[j+1,m]})$$

$$- \gamma EV_{i+1}(D_{[i,i]}, \alpha x_{i+1,n} + D_{[i+1,n]}, S_{[1,j-1]}, \beta(\hat{y}_j - \hat{x}_i) + S_j, \beta y_{[j+1,m]} + S_{[j+1,m]})$$

$$\geq (r_i^d + r_i^s + c + h)\varepsilon - \gamma (r_i^d - r_{i+1}^d)\varepsilon$$

where the inequality follows from Lemma 4 and the fact that $-\sum_{j'=1}^m \lambda_{j'}(r_{ij'} - r_{ij'}) = -\sum_{j'=1}^m \lambda_{j'}(r_i^d - r_{i+1}^d)$$

$$\geq -(r_i^d - r_{i+1}^d)\varepsilon$$

If $\alpha < 0$, then $r_i^d \geq r_{i+1}^d$, $(r_i^d + r_i^s + c + h)\varepsilon - \gamma (r_i^d - r_{i+1}^d)\varepsilon \geq (r_i^d + r_i^s + c + h)\varepsilon$. Then, by the concavity of $V_{i+1}()$ (see Proposition 1), it follows that $G_t(a + \varepsilon, x, y) - G_t(a - \varepsilon, x, y) \leq G_t(a, x, y) - G_t(a, x, y)$.

Proof of Lemma 9. By Lemma 4, there exists $(\lambda_1, \ldots, \lambda_m) \geq 0$ such that $\sum_{j'=1}^m \lambda_{j'} \leq \varepsilon$ and $V_t(x, y) - V_t(x + \varepsilon e_i^m - \varepsilon e_i^0, y) = V_t((x + \varepsilon e_i^m - \varepsilon e_i^0) - \varepsilon e_i^0, y) - V_t(x + \varepsilon e_i^0, y) \geq -\sum_{j'=1}^m \lambda_{j'}(r_{ij'} - r_{ij'})$$

$$= -\sum_{j'=1}^m \lambda_{j'}(r_i^d - r_{i+1}^d)$$. If $i < i'$, then $r_i^d > r_{i+1}^d$ and $V_t(x, y) - V_t(x + \varepsilon e_i^0, y) \geq -\sum_{j'=1}^m \lambda_{j'}(r_i^d - r_{i+1}^d)$$

$$\geq -(r_i^d - r_{i+1}^d)\varepsilon$$. Then, for $1 \leq k \leq n$, we have

$$\tilde{V}_t(\tilde{x} + \varepsilon e_k^m, \tilde{y}) - \tilde{V}_t(\tilde{x}, \tilde{y}) = V_t((\tilde{x} + \varepsilon e_k^m)U_n^{-1}, \tilde{y}U_n^{-1}) - V_t(\tilde{x}U_n^{-1}, \tilde{y}U_n^{-1}) - (\tilde{x} + \varepsilon e_k^m)U_n^{-1}(r_i^d)^T - \tilde{x}U_n^{-1}(r_i^d)^T$$
Thus, we have shown that \( \hat{V}_t(\hat{x}, \hat{y}) \) is decreasing in \( \hat{x}_k \) for \( 1 \leq k < n \). Similarly, we can show that \( \hat{V}_t(\hat{x}, \hat{y}) \) is decreasing in \( \hat{y}_k \) for \( 1 \leq k < m \). \( \square \)

**Proof of Lemma 10.** The proof is by induction on \( t \). Clearly, \( \hat{V}_{t+1}(\hat{x}, \hat{y}) = 0 \) is \( L^3 \)-concave in \((\hat{x}, \hat{y})\). We suppose that \( \hat{V}_{t+1}(\hat{x}, \hat{y}) \) is \( L^3 \)-concave in \((\hat{x}, \hat{y})\). Then by definition of \( L^3 \)-concavity and submodularity, for any given \( \hat{D} \) and \( \hat{S} \), \( \hat{V}_{t+1}(\alpha \hat{x} + \hat{D}, \alpha \hat{y} + \hat{S}) \) is \( L^3 \)-concave in \((\hat{x}, \hat{y})\). Now consider period \( t \). Since \( Q \leq \min\{\hat{x}_n, \hat{y}_m\} \) and \( \alpha = \beta \),

\[
\hat{V}_{t+1}(\alpha(\hat{x} - Q1^n)^+ + \hat{D}, \alpha(\hat{y} - Q1^m)^+ + \hat{S})
= \hat{V}_{t+1}(\alpha(\hat{x}_{1:n-1} - Q1^{n-1})^+ + \hat{D}_{1:n-1}, \alpha(\hat{x}_n - Q) + \hat{D}_n, \alpha(\hat{y}_{1:m-1} - Q1^{m-1})^+ + \hat{S}_{1:m-1}, \alpha(\hat{y}_m - Q) + \hat{S}_m),
\]

which is \( L^3 \)-concave in \((Q, \hat{x}, \hat{y})\) by applying Chen et al. (2014, Lemma 4) and noting the monotonicity proved in Lemma 9. By Simchi-Levi et al. (2014, Proposition 2.3.4(c)), \( E_{\hat{D}, \hat{S}}[\hat{V}_{t+1}(\alpha(\hat{x} - Q1^n)^+ + \hat{D}, \alpha(\hat{y} - Q1^m)^+ + \hat{S})] \) is \( L^3 \)-concave in \((Q, \hat{x}, \hat{y})\), thus the last term in (13) is \( L^3 \)-concave in \((Q, \hat{x}, \hat{y})\). The first two terms in (13) are \( L^3 \)-concave in \((Q, \hat{x}, \hat{y})\), because \(- (\hat{x}_i - Q)^+ \) is supermodular in \((Q, \hat{x}_i)\), \(- (\hat{y}_j - Q)^+ \) is supermodular in \((Q, \hat{y}_j)\) and \( L^3 \)-concavity is preserved under any nonnegative linear combination. Since the other terms are linear, \( \bar{G}_t(Q, \hat{x}, \hat{y}) \) is \( L^3 \)-concave in \((Q, \hat{x}, \hat{y})\). By Simchi-Levi et al. (2014, Proposition 2.3.4(e)), \( \hat{V}_t(\hat{x}, \hat{y}) \) is \( L^3 \)-concave in \((\hat{x}, \hat{y})\). This completes the induction. \( \square \)

**Proof of Corollary 3.** In view of Theorem 7, the protection level \( a^*_j(t) \) plays a role in deciding how much to match type \( i \) demand and type \( j \) supply, which happens only when \( \hat{x}_i > \hat{y}_{j-1} \) and \( \hat{x}_{i-1} < \hat{y}_j \) and types \( 1, 2, \ldots, i - 1 \) demand and types \( 1, 2, \ldots, j - 1 \) supply have been fully matched. The optimal total quantity to be matched between demand types \( 1, \ldots, i \) and supply types \( 1, \ldots, j \) is given by \( q^*_j \in \arg\max_{0 \leq q \leq \min(\hat{x}_i, \hat{y}_j)} \bar{G}_t(q, \hat{x}, \hat{y}) \). By the concavity of \( \bar{G}_t(q, \hat{x}, \hat{y}) \) with respect to \( q \), \( q^*_j = \min\{q^*, \hat{x}_i, \hat{y}_j\} \) where \( q^* \in \arg\max_{q \geq 0} \bar{G}_t(q, \hat{x}, \hat{y}) \). Since \( q^* \geq \max\{\hat{x}_{i-1}, \hat{y}_{j-1}\} \) (because types \( 1, 2, \ldots, i - 1 \) demand and types \( 1, 2, \ldots, j - 1 \) supply have been fully matched) and \( \hat{x}_i > \hat{y}_{j-1} \) and \( \hat{x}_{i-1} < \hat{y}_j \), we have \( q^*_j = \min\{q^*, \hat{x}_i, \hat{y}_j\} \geq \max\{\hat{x}_{i-1}, \hat{y}_{j-1}\} \). Thus, from (13), we see that \( q^*_j \) depends only on \((\hat{x}_{[i,n]}, \hat{y}_{[j,m]}))\).

Without loss of generality, consider the case \( \hat{x}_i \geq \hat{y}_j \). In this case, given the protection level \( a^*_j(t) \), the quantity of type \( i \) demand after matching with type \( j \) supply can be expressed as \( \hat{x}_i - \hat{y}_j + a^*_j(t) \). On the other hand, it can also be expressed as \( \hat{x}_i - q^*_j \). Equating these two expressions leads to \( a^*_j(t) = \hat{y}_j - q^*_j \). By \( L^2 \)-concavity of \( \bar{G}_t \), analogous to the proof of Theorem 8, we can show that \( q^*_j \) is nondecreasing in \( \hat{x}_i \) and \( \hat{y}_j \), for \( i \leq i' \leq n \).
and \( j \leq j' \leq m \) with the increasing rate less than or equal to 1. Since \( a^*_k(t) \) depends only on \( \eta_{ij} = \tilde{x}_i - \tilde{y}_j \), \( \hat{x}_{[i+1,n]} \) and \( \hat{y}_{[j+1,m]} \), it is nonincreasing in \( \eta_{ij} = \tilde{x}_i - \tilde{y}_j \), and the decreasing rates are dominated by 1. Moreover, analogous to the proof of Theorem 8, in terms of the original state \((x, y)\), for \( i + 1 \leq i' \leq n - 1 \), \(-1 \leq a^*_{i'}(t, \eta_{i'j} + \varepsilon, x_{[i+1,n]}, y_{[j+1,m]}) - a^*_i(t, \eta_{ij}, x_{[i+1,n]} + \varepsilon e_{i-1}, y_{[j+1,m]}) \leq a^*_j(t, \eta_{ij}, x_{[i+1,n]} + \varepsilon e_{i-1}, y_{[j+1,m]}) - a^*_{i'}(t, \eta_{i'j}, x_{[i+1,n]}, y_{[j+1,m]}) \leq a^*_j(t, \eta_{ij}, x_{[i+1,n]} + \varepsilon e_{i-1}, y_{[j+1,m]}) - a^*_{i'}(t, \eta_{i'j}, x_{[i+1,n]}, y_{[j+1,m]}) \leq 0 \). By symmetry, we have the results with respect to the supply levels. \( \square \)

**Proof of Theorem 9.** As part of the one-step-ahead heuristic, the greedy matching policy is implemented from period \( t + 1 \). We first prove two lemmas on the greedy matching policy. With vertically differentiated types, the greedy matching policy also follows the top-down structure but does not reserve demand or supply. Let \( V^\varepsilon(x, y) \) be the expected total discounted surplus under the greedy matching policy from the current period \( t \) to the end of the horizon, given the current state \((x, y)\). The transformed state \((\tilde{x}, \tilde{y})\) is defined as in the previous subsection.

**Lemma 11.** For \( \varepsilon > 0 \) and \( \delta_k \geq 0 \), \( k = 1, \ldots, j \) such that \( \sum_{k=1}^j \delta_k = \varepsilon \), the difference \( V^\varepsilon(x, y - \sum_{k=1}^j \delta_k e_k + \varepsilon e_j) - V^\varepsilon(x, y) \) depends only on \( \delta_k, k = 1, \ldots, j \), \( \tilde{x}_n \), and \( x_{[1,j-1]} \). Symmetrically, \( V^\varepsilon(x - \sum_{k=1}^i \delta_k e_k + \varepsilon e_i, y) - V^\varepsilon(x, y) \) depends only on \( \delta_k, k = 1, \ldots, i \), \( \tilde{y}_m \), and \( y_{[i+1,n]} \), for \( \varepsilon > 0 \), \( \delta_k \geq 0 \) and \( \sum_{k=1}^i \delta_k = \varepsilon \).

**Proof of Lemma 11.** We will focus on the difference \( V^\varepsilon(x, y - \sum_{k=1}^j \delta_k e_k + \varepsilon e_j) - V^\varepsilon(x, y) \) and the other difference satisfies the desired property by symmetry. If we define \( \delta_{j+1} = \cdots = \delta_m = 0 \) and \( \delta = (\delta_1, \ldots, \delta_m) \), the difference can be rewritten as \( V^\varepsilon(x, y - \delta + \varepsilon e_j) - V^\varepsilon(x, y) \).

We prove the lemma by induction. Suppose the desired property holds for \( t + 1 \).

If \( \tilde{x}_n < \tilde{y}_{j-1} \), there exists \( 1 \leq j' \leq j - 1 \) such that \( \tilde{y}_{j'-1} \leq \tilde{x}_n < \tilde{y}_{j'} \). Under the greedy matching policy, all the demand and types \( 1, \ldots, j' - 1 \) supply is matched, a quantity \( \tilde{x}_n - \tilde{y}_{j'-1} \) in type \( j' \) supply is matched, and types \( j' + 1, \ldots, m \) supply will not be matched. This leads to the post-matching levels \( Y = (0_{[1,j'-1]}, \tilde{y}_{j'} - \tilde{x}_n, y_{[j'+1,m]}) \) for the supply types. Then \( V^\varepsilon(x, y) = r^x x^T + r^y_{[1,j'-1]} y^T_{[1,j'-1]} + r^y_{j'} (\tilde{x}_n - \tilde{y}_{j'-1}) + \gamma E V^\varepsilon_{t+1} (D, \alpha Y + S) \).

On the other hand, under the state \((x, y - \delta + \varepsilon e_j)\), all the demand will again be fully matched, and the total amounts of demand and supply do not change compared to the state \((x, y)\). There exists \( j' \leq j'' \leq j - 1 \) such that types \( 1, \ldots, j'' - 1 \) supply are fully matched and types \( j'' + 1, \ldots, m \) supply are not matched. This leads to the post-matching levels \( Y' = (0_{[1,j''-1]}, \tilde{y}_{j''} - \tilde{y}_{j''-1} - \tilde{x}_n, y_{[j''+1,m]} - \delta_{[j''+1,m]}) + \varepsilon e_j \), where \( \delta_k \) is defined as \( \sum_{k=1}^k \delta_k \) for \( 1 \leq k \leq m \).
Let
\[
\Delta = \begin{cases}
(0_{[1,j')} \gamma \tilde{y}_{j'} - \tilde{x}_n, \chi_{[1,j')} - \delta_{[1,j')})^2 + r_{j'} \gamma (\tilde{x}_n - \tilde{y}_{j'} + \tilde{\delta}_{j'}) \gamma + \epsilon V_{t+1} (D, \alpha Y + S) & \text{if } j' > j', \\
(0_{[1,j+1]} \gamma \tilde{\delta}_{j'} + \delta_{[j+1,m]}) & \text{if } j' = j'.
\end{cases}
\]

It is easy to verify that \( \Delta \geq 0, \sum_{k=1}^{m} \Delta_k = \sum_{k=1}^{m} \delta_k = \epsilon, \Delta_k = 0 \) for \( k = j + 1, \ldots, m \), and \( \Delta \) depends only on \( \epsilon, \delta \) and \((\tilde{x}_n, y_{[1,j-1]}). In addition, \( Y' = Y - \Delta + \epsilon e_j \). Then \( V_{t+1} (D, \alpha Y + S) - V_{t+1} (D, \alpha Y - \alpha \Delta + \alpha \epsilon e_j + S) \)
depends only on \( \delta, \tilde{x}_n, \tilde{D}_n \) and \( y_{[1,j-1]. \) (Note that \( Y_{[1,j-1]} \) is uniquely determined by \( y_{[1,j-1]. \) Then the difference
\[
V_{t}^\gamma (x, y) - \epsilon e_j - V_{t}^\gamma (x, y)
\]
\[
\Delta = \begin{cases}
(0_{[1,j')} \gamma \tilde{y}_{j'} - \tilde{x}_n, \chi_{[1,j')} - \delta_{[1,j')})^2 + r_{j'} \gamma (\tilde{x}_n - \tilde{y}_{j'} + \tilde{\delta}_{j'}) \gamma + \epsilon V_{t+1} (D, \alpha Y + S) & \text{if } j' > j', \\
(0_{[1,j+1]} \gamma \tilde{\delta}_{j'} + \delta_{[j+1,m]}) & \text{if } j' = j'.
\end{cases}
\]

depends only on \( \delta, \tilde{x}_n \) and \( y_{[1,j-1]. \)

If \( \tilde{x}_n \geq \tilde{y}_{j-1} \), the greedy matching policy leads to the same post-matching levels under the two states \((x, y)\)
and \((x, y - \delta + \epsilon e_j)\). We see that \( V_{t}^\gamma (x, y - \delta + \epsilon e_j) - V_{t}^\gamma (x, y) = \sum_{k=1}^{m} \delta_k (r_{j'} - r_{j'+1}) \), which is independent of \((x, y)\).

Combining the above analysis, we see that the difference \( V_{t}^\gamma (x, y - \delta + \epsilon e_j) - V_{t}^\gamma (x, y) \) depends only on \( \delta_k, k = 1, \ldots, j, \tilde{x}_n \) and \( y_{[1,j-1]. \) \( \square \)

**Lemma 12.** The difference \( V_{t}^\gamma (x + e_i, y + \epsilon e_j) - V_{t}^\gamma (x, y) \) depends only on \( \epsilon \) and \((x_{[1,i-1]}, y_{[1,j-1]}, \tilde{x}_n, \tilde{y}_m). \)

**Proof of Lemma 12.** First, consider the case \( \tilde{x}_n \leq \tilde{y}_{j-1} \). If \( \tilde{x}_n \geq \tilde{y}_{j-1} \), then \( V_{t}^\gamma (x + e_i, y + \epsilon e_j) - V_{t}^\gamma (x, y) = (r_{i} \gamma + r_{j}) \gamma, \) which is independent of \((x, y)\).

If \( \tilde{y}_{j-1} \leq \tilde{x}_n \leq \tilde{y}_{j'} \) for some \( 1 \leq j' \leq j - 1 \), Under state \((x, y)\), the post-matching levels for the supply types are \( v = (0_{[1,j')}, y_{[1,j')} - \tilde{x}_n, y_{[j'+1,m]}) \). The additional amount, \( \epsilon \), of type \( i \) demand and the extra quantities \( \delta_{j', \min} = \min \{ \epsilon, \tilde{y}_{j'} - \tilde{x}_n \}, \delta_{j'+1, \min} = \min \{ [\epsilon - (\tilde{y}_{j'} - \tilde{x}_n)]^+, y_{j'+1} \}, \ldots, \delta_{j-1, \min} = \min \{ [\epsilon - (\tilde{y}_{j-2} - \tilde{x}_n)]^+, y_{j-1} \}, \delta_j = [\epsilon - (\tilde{y}_{j-1} - \tilde{x}_n)]^+ \) are matched for supply types \( j', j' + 1, \ldots, j - 1, j \), respectively, under the new state \((x + \epsilon e_i, y + \epsilon e_j)\). Let \( \delta = (0, 0, 0, 0, 0) \in \mathbb{R}_m^n. \) Note that \( \delta \) is a function of \((\tilde{x}_n, y_{[1,j-1]}). \) Then we have
\[
V_{t}^\gamma (x + \epsilon e_i, y + \epsilon e_j) - V_{t}^\gamma (x, y) = r_{i} \gamma + \sum_{k=1}^{m} r_{k} \delta_k + \gamma E [V_{t+1}(D, \alpha (v - \delta + \epsilon e_j) + S) - V_{t+1}(D, \alpha v + S)]
\]
Since \( V_{t+1}(D, \alpha v + S) \) depends only on \( \varepsilon, \delta, \hat{D}_n \) and \( v_{[1,j-1]} \) by Lemma 11, the difference \( V^g_t(x + \varepsilon e_i^m, y + \varepsilon e_j^m) - V^g_t(x, y) \) depends only on \( \varepsilon, \hat{x}_n \) and \( y_{[1,j-1]} \), because \( \delta \) is defined in terms of \( \varepsilon, \hat{x}_n \) and \( y_{[1,j-1]} \).

Now consider the case \( \hat{x}_n > \hat{y}_m \). By symmetry, we can show that \( V^g_t(x + \varepsilon e_i^m, y + \varepsilon e_j^m) - V^g_t(x, y) \) depends only on \( \varepsilon, \hat{y}_m \) and \( x_{[1,i-1]} \). Combining those two cases, the difference depends only on \( \varepsilon \) and \( \langle x_{[1,i-1]}, y_{[1,j-1]}, \hat{x}_n, \hat{y}_m \rangle \). □

For the one-step-ahead heuristic, in period \( t \), the firm follows the greedy matching from period \( t + 1 \) to the end of the horizon. The firm faces the following optimization problem for the protection level \( a \) when matching type \( i \) demand with type \( j \) supply.

\[
\max_{a \geq 0} f_t(a, x, y) \overset{\text{def}}{=} -\left( r_i^t + r_j^t + c + h \right) a + \gamma V^g_{t+1}(D_{[1,i-1]}, \alpha(x_i^t + a) + D_i, \alpha x_{[i+1,n]} + D_{[i+1,n]}, S_{[1,j-1]}, \alpha(y_j^t + a + \varepsilon) + S_j, \alpha y_{[j+1,m]} + S_{[j+1,m]}),
\]

where \( x_i = (\hat{x}_i - \hat{y}_j)^+ \) and \( y_j = (\hat{y}_j - \hat{x}_i)^+ \).

Since \( V^g_{t+1} \) is concave, \( f_t(a, x, y) \) is concave in \( a \) and the optimal protection level \( a^* \in \text{sup} \{ a \geq 0 | \lim_{\varepsilon \downarrow 0} \{ f(a + \varepsilon, x, y) - f(a, x, y) \} / \varepsilon \geq 0 \} \). We have

\[
f_t(a + \varepsilon, x, y) - f_t(a, x, y) = -\left( r_i^t + r_j^t + c + h \right) \varepsilon + \gamma V^g_{t+1}(D_{[1,i-1]}, \alpha(x_i^t + a + \varepsilon) + D_i, \alpha x_{[i+1,n]} + D_{[i+1,n]}, S_{[1,j-1]}, \alpha(y_j^t + a + \varepsilon) + S_j, \alpha y_{[j+1,m]} + S_{[j+1,m]}),
\]

where the second equality follows from Lemma 12. To see this, let \( (X, Y) = (D_{[1,i-1]}, \alpha(x_i^t + a) + D_i, \alpha x_{[i+1,n]} + D_{[i+1,n]}, S_{[1,j-1]}, \alpha(y_j^t + a + \varepsilon) + S_j, \alpha y_{[j+1,m]} + S_{[j+1,m]}), \) and \( (X', Y') = (D_{[1,i-1]}, \alpha(x_i^t + \sum_{t'=i+1}^n x_{t'} + a) + D_i, D_{[i+1,n]}, S_{[1,j-1]}, \alpha(y_j^t + \sum_{j'=j+1}^m y_{j'} + a) + S_j, S_{[j+1,m]}),\) we see that \( \hat{X}_{[1,i-1]} = \hat{X}'_{[1,i-1]}, \hat{x}_n = \hat{X}'_n, \hat{Y}_{[1,j-1]} = \hat{Y}'_{[1,j-1]}, \hat{y}_m = \hat{Y}'_m \). By Lemma 12, we have \( E[V^g_{t+1}(X + \alpha \varepsilon, Y + \alpha \varepsilon) - V^g_{t+1}(X, Y)] = E[V^g_{t+1}(X' + \alpha \varepsilon, Y' + \alpha \varepsilon) - V^g_{t+1}(X', Y')], \) which ensures the second equality in (14).
Let us define
\[
g_t(a, x, y) \overset{\text{def}}{=} -(r_i^d + r_j^s + c + h)a + \gamma EV_{t+1}^g(D_{[i, i-1]}, \alpha(x_i) + \sum_{i'=i+1}^n x_i' + a) + D_i, D_{[i+1, n]},
\]
\[
S_{[1, j-1]}, \alpha(y_j) + \sum_{j'=j+1}^m y_{j'} + a) + S_j, S_{[j+1, m]}.
\]

By the above analysis, we have \(g_t(a, \varepsilon, x, y) - g_t(a, x, y) = f_t(a, \varepsilon, x, y) - f_t(a, x, y)\). Thus it is equivalent to maximize \(g_t(a, x, y)\). By substituting \(b = x_i + \sum_{i'=i+1}^n x_i' + a\), we have
\[
g_t(a, x, y) = g_t(b, x, y) = -(r_i^d + r_j^s + c + h)(b - x_i - \sum_{i'=i+1}^n x_i') + \gamma EV_{t+1}^g(D_{[i, i-1]}, \alpha b + D_i, D_{[i+1, n]}, S_{[1, j-1]}, \alpha(b + \check{y}_m - \check{x}_n) + S_j, S_{[j+1, m]}).
\]

Note that the above expression depends on \((x, y)\) only through \(\check{y}_m - \check{x}_n\). By optimizing \(g_t\) in terms of \(b\) over \(b \geq 0\), we have an optimizer \(\hat{b}_i(t, \check{y}_m - \check{x}_n)\) that depends on \(\check{y}_m - \check{x}_n\) and solves \(\max_{a \geq 0} g_t(b, x, y) = g_t(b, \check{y}_m - \check{x}_n)\). Since \(a \geq 0\), we require \(b \geq x_i + \sum_{i'=i+1}^n x_i'\). Thus \(b_i'(t, \check{y}_m - \check{x}_n) = \hat{b}_i(t, \check{y}_m - \check{x}_n) \vee \big( x_i + \sum_{i'=i+1}^n x_i' \big)\) maximizes \(g_t(b, \check{y}_m - \check{x}_n)\) for \(b \geq (x_i + \sum_{i'=i+1}^n x_i')\) and \(a^* = b_i'(t, \check{y}_m - \check{x}_n) - (x_i + \sum_{i'=i+1}^n x_i') = [\hat{b}_i(t, \check{y}_m - \check{x}_n) - (x_i + \sum_{i'=i+1}^n x_i')]^+\) solves \(\max_{a \geq 0} f_t(a, x, y)\). □

**Proof of Proposition 3.** Let us define \(\hat{V}_t^g(\check{x}, \check{y}) \overset{\text{def}}{=} V_t^g(x, y) - r^d x^d - r^s x^s\) (in the same way as we defined \(\hat{V}_t\)). Now consider a system (M) with the same parameters except that we increase both \(c\) and \(h\) by the same amount \(M\). Let \(\hat{V}_t^M\) denote the value function of this system (M). If \(M\) is sufficiently large, the optimal policy will reduce to the greedy matching policy. Starting from the same initial state, the original system and the new system (M) will have exactly the same trajectory of state because they share the same policy. However, system (M) incurs extra holding and waiting costs. Following this logic, we can infer that \(\hat{V}_t^M(\check{x}, \check{y}) = \hat{V}_t^M(\check{x}, \check{y}) - M \sum_{i'=t}^\tau \gamma^{\tau-t} E[\alpha^{\tau-t}(\check{x}_n - \check{y}_m) + \sum_{i'=t+1}^\tau \alpha^{\tau-t} (\check{D}_{n_i} + \check{S}_{n_i})^+] + M \sum_{i'=t}^\tau \gamma^{\tau-t} E[\alpha^{\tau-t}(\check{x}_n - \check{y}_m) + \sum_{i'=t+1}^\tau \alpha^{\tau-t} (\check{D}_{n_i} + \check{S}_{n_i})^-]^-\), where \(\check{D}_{n_i}\) and \(\check{S}_{n_i}\) are the total supply and demand in period \(\tau\). Since \(\hat{V}_t^M(\check{x}, \check{y})\) is \(L^3\)-concave in \((\check{x}, \check{y})\) by Lemma 10, so is \(\hat{V}_t^g(\check{x}, \check{y})\). (Note that the functions \(- (z_1 - z_2 + C)^+\) and \(- (z_1 - z_2 + C)^-\) are \(L^2\)-concave in \((z_1, z_2)\) for any constant \(C\).) Then in period \(t\), the total matching quantity \(Q_t^g\) in the one-step-ahead heuristic under the practice of greedy matching from period \(t+1\) will satisfy the same set of properties as in Theorem 8. In terms of the aggregate post-matching level for demand, \(\hat{b}_i(t, \check{y}_m - \check{x}_n)\) is weakly decreasing in \((\check{y}_m - \check{x}_n)\) and the decreasing rate is dominated by 1. (One can also derive this property from maximizing (15) over \(b \geq 0\), with \(V_t^g\) rewritten in terms of \(\hat{V}_t^g\).) □
Proof of Proposition 5. The result for $\beta = 0$ follows directly from Proposition 4. For $\alpha = \beta$, the result can be proved by applying the same approach as in Yu et al. (2014), thus we omit the lengthy details. □

Proof of Corollary 6. The dual problem of problem (s-p) is given by

\[
\begin{align*}
(s-d) \quad \max & \quad p^d_{it} - \sum_{i=1}^{n} p^d_{it} x_{it} - \sum_{j=1}^{m} p^s_{jt} y_{jt} \\
\text{s.t.} & \quad p^d_{it} + p^s_{jt} \geq \gamma^{-1}(r_{ij} + c + h) + \alpha f^d_{it} + \beta f^s_{jt}, \quad 1 \leq i \leq n, 1 \leq j \leq m, \\
& \quad p^d_{it}, p^s_{jt} \geq 0, \quad 1 \leq i \leq n, 1 \leq j \leq m.
\end{align*}
\]

The complementary slackness conditions are the same as in (CS) for period $t$. We see that the primal and dual optimal solutions to problems (P) and (D) for period $t$ are primal- and dual-feasible for problems (s-p) and (s-d). Moreover, they satisfy the complementary slackness conditions of the subproblem. Thus, the optimal solution to (P) for period $t$ is optimal for the subproblem (s-p). □

Proof of Proposition 7. Let $\Omega$ be the set of all sample paths of demand and supply realizations over the finite horizon, $\omega \in \Omega$ be a sample path, and $p(\omega)$ be the density at $\omega$. We rewrite the stochastic model in the following form of a stochastic program.

\[
\begin{align*}
\text{(P1) max } & \int_{\Omega} p(\omega) \sum_{t=1}^{T} \left[ \sum_{i=1}^{n} \sum_{j=1}^{m} r_{ij} q_{ijt}(\omega) - c \sum_{i=1}^{n} \sum_{j=1}^{m} \left[ x_{it}(\omega) - \sum_{i=1}^{n} q_{ijt}(\omega) \right] - h \sum_{j=1}^{m} \left[ y_{jt}(\omega) - \sum_{i=1}^{n} q_{ijt}(\omega) \right] \right] d\omega \\
\text{s.t. } & \quad x_{i,t+1}(\omega) = \alpha \left[ x_{it}(\omega) - \sum_{j=1}^{m} q_{ijt}(\omega) \right] + D_{it}(\omega), \text{ for all } 1 \leq i \leq n, 1 \leq t \leq T - 1 \text{ and } \omega \in \Omega, \\
& \quad y_{j,t+1}(\omega) = \beta \left[ y_{jt}(\omega) - \sum_{i=1}^{n} q_{ijt}(\omega) \right] + S_{jt}(\omega), \text{ for all } 1 \leq j \leq m, 1 \leq t \leq T - 1 \text{ and } \omega \in \Omega, \\
& \quad \sum_{i=1}^{n} q_{ijt}(\omega) \leq y_{jt}(\omega) \text{ for all } 1 \leq j \leq m, 1 \leq t \leq T \text{ and } \omega \in \Omega, \\
& \quad \sum_{j=1}^{m} q_{ijt}(\omega) \leq x_{it}(\omega) \text{ for all } 1 \leq i \leq n, 1 \leq t \leq T \text{ and } \omega \in \Omega, \\
& \quad q_{ijt}(\omega) \geq 0 \text{ for all } 1 \leq i \leq n, 1 \leq j \leq m, 1 \leq t \leq T \text{ and } \omega \in \Omega.
\end{align*}
\]

We denote by $Q_{ijt}^*(\omega)$, $\omega \in \Omega$, the optimal matching strategy, and by $x_{it}^*(\omega)$ and $y_{jt}^*(\omega)$ the associated state trajectory. Let $\bar{x}_{it}$, $\bar{y}_{jt}$ and $\bar{q}_{ijt}$ be the expectation of $x_{it}^*(\omega)$, $y_{jt}^*(\omega)$ and $q_{ijt}^*(\omega)$ over $\Omega$, respectively. Because all sample paths satisfy the constraints of problem (P1), as expectations, $(\bar{q}_{ijt}, \bar{x}_{it}, \bar{y}_{jt})$ is feasible for the deterministic problem (P), with the corresponding objective value equal to the optimal value of the stochastic problem (P1). Therefore, the deterministic problem (P) has a larger optimal value than the stochastic problem (P1). □
Proof of Proposition 8. We prove this proposition by induction on \( t \). The result is clearly true for \( t = T + 1 \) because \( V^k_{T+1}(x,y) \equiv 0 \). Suppose that for any compact set \( B \in \mathbb{R}^{m+n} \), \( V^k_{t+1}(x,y) \rightarrow V^\text{det}_{t+1}(x,y) \) uniformly for \((x,y) \in B\) as \( k \rightarrow \infty \). Now consider an arbitrary compact set \( B_0 \in \mathbb{R}^{m+n} \). Since demand and supply are bounded, there exists a compact set \( B_1 \) such that \((\alpha u + \tilde{D}^k_{t+1} + \beta v + \tilde{S}^k_{t+1}) \in B_1 \) and \((\alpha u + \lambda_{t+1} + \beta v + \mu_{t+1}) \in B_1 \) if \((x,y) \in B_0 \), where \( u = x - 1^m Q^T \) and \( v = y - 1^n Q \). By the induction hypothesis, for \( \forall \varepsilon > 0 \), there exists \( K \) such that \( |V^k_{t+1}(\alpha u + \tilde{D}^k_{t+1} + \beta v + \tilde{S}^k_{t+1}) - V^\text{det}_{t+1}(\alpha u + \tilde{D}^k_{t+1} + \beta v + \tilde{S}^k_{t+1})| < \frac{\varepsilon}{2} \) for \( k \geq K \), over all \((x,y) \in B_0 \). We have

\[
\begin{align*}
&|EV^\text{det}_{t+1}(\alpha u + \tilde{D}^k_{t+1} + \beta v + \tilde{S}^k_{t+1}) - EV^\text{det}_{t+1}(\alpha u + \lambda_{t+1} + \beta v + \mu_{t+1})| \\
&\leq E\left(|V^\text{det}_{t+1}(\alpha u + \tilde{D}^k_{t+1} + \beta v + \tilde{S}^k_{t+1}) - V^\text{det}_{t+1}(\alpha u + \lambda_{t+1} + \beta v + \mu_{t+1})| \mathbb{1}\{(\|\tilde{D}^k_{t+1} + \tilde{S}^k_{t+1}\| - (\lambda_{t+1} + \mu_{t+1})\| \leq \delta\}\}\right) \\
&+ E\left(|V^\text{det}_{t+1}(\alpha u + \tilde{D}^k_{t+1} + \beta v + \tilde{S}^k_{t+1}) - V^\text{det}_{t+1}(\alpha u + \lambda_{t+1} + \beta v + \mu_{t+1})| \mathbb{1}\{(\|\tilde{D}^k_{t+1} + \tilde{S}^k_{t+1}\| - (\lambda_{t+1} + \mu_{t+1})\| > \delta\}\}\right) \\
&\leq \frac{\varepsilon}{4} + 2M \cdot E\left(\mathbb{1}\{(\|\tilde{D}^k_{t+1} + \tilde{S}^k_{t+1}\| - (\lambda_{t+1} + \mu_{t+1})\| \geq \delta\}\}\right) \\
&= \frac{\varepsilon}{4} + 2M \cdot \Pr(\|\tilde{D}^k_{t+1} + \tilde{S}^k_{t+1}\| - (\lambda_{t+1} + \mu_{t+1})\| \geq \delta).
\end{align*}
\]  

By the weak law of large numbers, \( \Pr(\|\tilde{D}^k_{t+1} + \tilde{S}^k_{t+1}\| - (\lambda_{t+1} + \mu_{t+1})\| \geq \delta) \rightarrow 0 \) as \( k \rightarrow \infty \). Then, we can find a sufficiently large \( K_1 \geq K \) such that \( \Pr(\|\tilde{D}^k_{t+1} + \tilde{S}^k_{t+1}\| - (\lambda_{t+1} + \mu_{t+1})\| \geq \delta) < \varepsilon/(8M) \) when \( k \geq K_1 \). Combining this with (16), we have

\[
|EV^\text{det}_{t+1}(\alpha u + \tilde{D}^k_{t+1} + \beta v + \tilde{S}^k_{t+1}) - EV^\text{det}_{t+1}(\alpha u + \lambda_{t+1} + \beta v + \mu_{t+1})| < \frac{\varepsilon}{2}
\]

when \( k \geq K_1 \). Thus, \( |EV^\text{det}_{t+1}(\alpha u + \tilde{D}^k_{t+1} + \beta v + \tilde{S}^k_{t+1}) - EV^\text{det}_{t+1}(\alpha u + \lambda_{t+1} + \beta v + \mu_{t+1})| \leq \varepsilon \).

Proof of Proposition 9. The proof is again by induction on \( t \). It is clear that it holds for \( t = T + 1 \). Suppose it holds for \( t + 1 \).

As in the proof of Proposition 8, we first bound the difference \( |EV^\text{det}_{t+1}(\alpha u + \tilde{D}^k_{t+1} + \beta v + \tilde{S}^k_{t+1}) - EV^\text{det}_{t+1}(\alpha u + \lambda_{t+1} + \beta v + \mu_{t+1})| < \frac{\varepsilon}{2} \).
Consider positive vectors \( \mathbf{U}^d = (U^d, \ldots, U^d) \in \mathbb{R}^n \) and \( \mathbf{U}^s = (U^s, \ldots, U^s) \in \mathbb{R}^m \). We define \( \bar{\mathbf{D}}_{t+1}^k(U^d) \) as the truncated demand and supply. Clearly, the truncated demand and supply is weakly less than the original demand and supply. Further, we let \( \lambda_{t+1}(U^d) = \lambda_{t+1} \land \mathbf{U}^d \) and \( \mu_{t+1}(U^s) = \mu_{t+1} \land \mathbf{U}^s \). By the strong law of large numbers, \( \bar{\mathbf{D}}_{t+1}^k(U^d) = \bar{\mathbf{D}}_{t+1}^k \land \mathbf{U}^d \to \lambda_{t+1} \land \mathbf{U}^d = \lambda(U^d) \) almost surely and \( \bar{\mathbf{S}}_{t+1}^k(U^s) \to \mu_{t+1}(U^s) \) almost surely as \( k \to \infty \). It is also clear that \( \lambda_{t+1}(U^d) = \lambda_{t+1} \) when \( U^d \) is sufficiently large (larger than \( \max_1 \leq n \lambda_{t+1} \)) and \( \mu_{t+1}(U^s) = \mu_{t+1} \) when \( U^s \) is sufficiently large. Let \( B_0 \) and \( B_1 \) be compact sets defined as in Proposition 8.

Consider the difference \( V_{t+1}^k(x, y) - V_{t+1}^k(x', y') \) with \( (x, y) \geq (x', y') \). Note that an additional unit of demand contributes at most \( \max_{i \in D, j \in S} r_{ij} + \sum_{t=1}^T (\beta \gamma)^{t-1} h \) (reward from matching and possible saving in holding cost for reducing a unit of supply) to the total surplus, and an addition unit of supply contributes at most \( \max_{i \in D} \sum_{t=1}^T (\beta \gamma)^{t-1} c \) to the total surplus. We have \( V_{t+1}^k(x, y) - V_{t+1}^k(x', y') \leq (\max_{i \in D, j \in S} r_{ij} + \sum_{t=1}^T (\beta \gamma)^{t-1} h) \sum_{i=1}^n (x_i - x'_i) + (\max_{i \in D} \sum_{t=1}^T (\beta \gamma)^{t-1} c) \sum_{j=1}^m (y_j - y'_j) \). On the other hand, in the worst case, an additional unit of demand (or supply) incurs the extra waiting (or holding) cost \( \sum_{t=1}^T (\beta \gamma)^{t-1} c \) (or \( \sum_{t=1}^T (\beta \gamma)^{t-1} h \)). This implies that \( V_{t+1}^k(x, y) - V_{t+1}^k(x', y') \geq - (\sum_{t=1}^T (\beta \gamma)^{t-1} c) \sum_{i=1}^n (x_i - x'_i) - (\sum_{t=1}^T (\beta \gamma)^{t-1} h) \sum_{j=1}^m (y_j - y'_j) \). From the above arguments, there exists constant \( C > 0 \) such that

\[
|V_{t+1}^k(x, y) - V_{t+1}^k(x', y')| \leq C \sum_{i=1}^n (x_i - x'_i) + \sum_{j=1}^m (y_j - y'_j).
\]

Then,

\[
E[V_{t+1}^k(\alpha \mathbf{u} + \bar{\mathbf{D}}_{t+1}^k, \beta \mathbf{v} + \bar{\mathbf{S}}_{t+1}^k) - V_{t+1}^k(\alpha \mathbf{u} + \bar{\mathbf{D}}_{t+1}^k(U^d), \beta \mathbf{v} + \bar{\mathbf{S}}_{t+1}^k(U^s))] \leq C \cdot E \left[ \sum_{i=1}^n \left( \bar{D}_{t+1}^k - \bar{D}_{t+1}^k(U^d) \right) D_{i,t+1}^k + \sum_{j=1}^m \left( \bar{S}_{j,t+1}^k - \bar{S}_{j,t+1}^k(U^s) \right) S_{j,t+1}^k \right] = C \sum_{i=1}^n E[D_{i,t+1}^k] - D_{i,t+1}^k(U^d) + C \sum_{j=1}^m E[S_{j,t+1}^k] - S_{j,t+1}^k(U^s)
\]

For any convex function \( f(D, U) \), we have \( \bar{D}_{t+1}^k(U^d) - U^d \) is dominated by \( D_{i,t+1}^k - U^d \) in convex order. Since the function \( x_1(x) \) is convex, we have \( E[(D_{i,t+1}^k - U^d) 1\{D_{i,t+1}^k - U^d > 0\}] \leq E[(D_{i,t+1}^k - U^d) 1\{D_{i,t+1}^k - U^d > 0\}]. \) Likewise, \( E[(S_{j,t+1}^k - U^s) 1\{S_{j,t+1}^k - U^s > 0\}] \leq E[(S_{j,t+1}^k - U^s) 1\{S_{j,t+1}^k - U^s > 0\}]. \) By the dominated convergence theorem, we know that \( \lim_{k \to \infty} E[(D_{i,t+1}^k - U^d) 1\{D_{i,t+1}^k - U^d > 0\}] = E[(D_{i,t+1}^k - U^d) 1\{D_{i,t+1}^k - U^d > 0\}] = 0 \) and analogously, \( \lim_{k \to \infty} E[(S_{j,t+1}^k - U^s) 1\{S_{j,t+1}^k - U^s > 0\}] = 0. \) This implies that \( E[V_{t+1}^k(\alpha \mathbf{u} + \bar{\mathbf{D}}_{t+1}^k, \beta \mathbf{v} + \bar{\mathbf{S}}_{t+1}^k) - V_{t+1}^k(\alpha \mathbf{u} + \bar{\mathbf{D}}_{t+1}^k(U^d), \beta \mathbf{v} + \bar{\mathbf{S}}_{t+1}^k(U^s))] \) converges to zero uniformly as \( k \to \infty. \)

Consequently, for \( \forall \varepsilon > 0 \), there exists sufficiently large \( U \) (independent of \( k \)) such that when \( U^d > U \) and \( U^s > U \), \( E[V_{t+1}^k(\alpha \mathbf{u} + \bar{\mathbf{D}}_{t+1}^k, \beta \mathbf{v} + \bar{\mathbf{S}}_{t+1}^k) - V_{t+1}^k(\alpha \mathbf{u} + \bar{\mathbf{D}}_{t+1}^k(U^d), \beta \mathbf{v} + \bar{\mathbf{S}}_{t+1}^k(U^s))] < \varepsilon. \)
We choose $U^d > U \land (\max_{1 \leq i \leq n} \lambda_{i,t+1})$ and $U^s > U \land (\max_{1 \leq j \leq m} \mu_{j,t+1})$. Then, $\lambda_{t+1}(U^d) = \lambda_{t+1}$ and $\mu_{t+1}(U^s) = \mu_{t+1}$. Applying the analysis in Proposition 8 for bounded demand (noting that now $\hat{D}_{t+1}(U^d)$ and $\hat{S}_{t+1}(U^s)$ are bounded), for the chosen $U^d$ and $U^s$, there exists $K$ (dependent on the choice of $U^d$ and $U^s$) such that $|V^k_{t+1}(\alpha u + \hat{D}_{t+1}(U^d), \beta v + \hat{S}_{t+1}(U^s)) - V^\text{det}_{t+1}(\alpha u + \lambda_{t+1}(U^d), \beta v + \mu_{t+1}(U^s))| < \frac{\varepsilon}{2}$ when $k \geq K$.

Then,

$$|EV^k_{t+1}(\alpha u + \hat{D}_{t+1}, \beta v + \hat{S}_{t+1}) - EV^\text{det}_{t+1}(\alpha u + \lambda_{t+1}, \beta v + \mu_{t+1})|$$
$$= |EV^k_{t+1}(\alpha u + \hat{D}_{t+1}, \beta v + \hat{S}_{t+1}) - EV^\text{det}_{t+1}(\alpha u + \lambda_{t+1}(U^d), \beta v + \mu_{t+1}(U^s))|$$
$$\leq E|V^k_{t+1}(\alpha u + \hat{D}_{t+1}, \beta v + \hat{S}_{t+1}) - V^k_{t+1}(\alpha u + \hat{D}_{t+1}(U^d), \beta v + \hat{S}_{t+1}(U^s))|$$
$$+ E|V^k_{t+1}(\alpha u + \hat{D}_{t+1}(U^d), \beta v + \hat{S}_{t+1}(U^s)) - V^\text{det}_{t+1}(\alpha u + \lambda_{t+1}(U^d), \beta v + \mu_{t+1}(U^s))| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Then it follows that $|H_t^b(Q, x, y) - H_t^\text{det}(Q, x, y)| < \varepsilon$ when $k \geq K$, for any $(x, y) \in B_0$ and feasible $Q$. The rest of the proof is the same as in Proposition 8. □

**Proof of Theorem 10.** From the proof of Propositions 8 and 9, for any $\varepsilon > 0$, there exists $K > 0$ such that $|H_t^b(Q, x, y) - H_t^\text{det}(Q, x, y)| < \frac{\varepsilon}{2}$ if $k \geq K$, for any feasible $Q$. Then, $H_t^b(\hat{Q}, x, y) \geq H_t^\text{det}(\hat{Q}, x, y) - \varepsilon \geq H_t^b(Q^*, x, y) - \varepsilon$ if $k \geq K$, where the second inequality follows from the optimality of $\hat{Q}$. □

**References**


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