Optimal project termination with an informed agent

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December 10, 2015

Abstract

Economic decision-makers commonly face the problem of determining when to terminate a project whose state, monitored imperfectly, shifts over time. I analyze the optimal provision of dynamic incentives in an agent-assisted version of this problem. A firm must decide when to halt a project which yields a stochastic profit flow in continuous time but eventually becomes unprofitable; the firm is aided by an agent who possesses a superior private monitoring technology, limited liability, and incentives for delayed project termination. I develop techniques for deriving the firm’s optimal contract, which involves occasional inefficient early project termination and a golden parachute that is sensitive to public news and declines with project length. Central to my analysis is a duality result linking payment and termination variables in an optimal contract, closely analogous to the equivalence of price and quantity as control variables in monopoly theory. I analyze both the “price” and “quantity” approaches to optimization, which yield complementary insights into features of the optimal contract. In particular, the latter approach maps the economic problem with an agent onto a single-person decision-making problem exhibiting an artificial wedge between the signal-to-noise ratio of the public news process and the firm’s expected profit flow.

1 Introduction

Economic decision-makers commonly face the problem of determining when to terminate a project whose state, which is monitored imperfectly, deteriorates over time. A classic example is a firm operating a depreciating plant that must eventually be retired. The depreciation can take various forms, such as a decline in output quality or an increase in breakdowns. A key feature of this environment is that the firm cannot directly observe whether the plant has depreciated, but instead sees only a noisy indicator of the current state. For instance, it
might observe the quality of individual units of output; the market price of its output when demand fluctuates exogenously; or the incidence of individual breakdowns when malfunctions arrive randomly over time. Another example is a board of directors deciding when to replace a CEO if firm performance indicates she is no longer a good match for the firm. In all these cases the decision-maker faces an optimal stopping problem of deciding when to terminate the project based on the history of observed performance. An optimal policy must balance the incentive to stop based on current expectations of the project state against the option value of continuing operation and gathering more information.

The single-person problem with a binary state and a single state transition has been well-studied in operations research as the problem of “quickest detection” or “change-point detection”; see, e.g., Peskir and Shiryaev (2006) for a textbook treatment. However, in practice many monitoring tasks in economics are accomplished with the assistance of an agent who brings additional expertise to the problem. For instance, the plant owner discussed earlier might hire a manager to oversee the plant’s operation, who by virtue of experience and proximity maintains direct knowledge of the plant’s state. Similarly, the board of directors may ask the CEO to help make the decision as to when she should be replaced.

In such cases the firm naturally wishes to incorporate the agent’s superior information in deciding when to terminate the project. However, agents attached to lucrative projects often have misaligned incentives and prefer to prolong their involvement as long as possible, for instance due to empire-building concerns, on-the-job perquisites, or job search costs following termination. The presence of such incentives substantially complicates optimal elicitation of the agent’s private information. As the agent must be compensated for revealing news which instigates project termination, the firm may instead simply choose to terminate the project early to economize on payments. In this paper I study the optimal incorporation of an agent’s information into the quickest detection problem when the project’s state is binary and transitions exactly once from good to bad.

The form of an optimal contract is not a priori obvious because the firm must incentivize truthful reporting by agents in each state, a problem which generally requires designing the paths of two continuation utility state variables. On top of this, the agent’s incentive constraint in the bad state is complex due to the possibility of double deviations. If the agent decides to withhold his knowledge of the state change at time \( t \), he need not report it a moment later; instead he has many possible delayed reporting strategies, whose profitability depend on non-local features of the contract. It is therefore not even clear that continuation utilities alone constitute sufficient state variables for the problem.

Despite the complexity of the contracting problem, I find that an optimal termination
policy takes the simple form of a threshold rule in “virtual beliefs.” At each moment the firm asks the agent to report on the current state of the project, but updates its beliefs about whether the state has transitioned as if no agent were present. It then terminates the project the first time either the agent reports that the state has transitioned or the firm’s virtual beliefs drop below a (time- and history-independent) threshold. This threshold is strictly lower than the optimal threshold in the firm’s problem without an agent. The firm is therefore able to partially utilize the agent’s information, fully eliminating late terminations and partially mitigating early terminations that it would have incurred in the agent’s absence. The optimal threshold also satisfies clean comparative statics: it is decreasing in the informativeness of news about the state, increasing in the rate of state transitions, and increasing in the severity of the agent’s incentive misalignment.

Optimal payments take a similarly simple form: all payments are deferred until project termination, at which point the agent receives a lump sum “golden parachute.” If the agent reports a state switch, the optimal golden parachute is exactly the expected discounted stream of project rents he would have received by never reporting the switch. On the other hand, if the project is terminated before a reported state switch he is paid nothing. The dynamics of the optimal golden parachute may be characterized by a one-dimensional HJB equation which determines how fast the parachute drifts down over time and how sensitively the parachute reacts to news about output.

A crucial technical tool in my analysis is a duality result tightly linking payments and termination times in an optimal contract. These two contractual elements function as “dual variables”: either may be eliminated in favor of the other, reducing the analysis to design of a single contractual element. As the two variables control the cost of implementing the contract and the amount of output produced, respectively, this duality is closely analogous to the equivalence between price and quantity as control variables in monopoly theory. I pursue both the “price” and “quantity” approaches to deriving an optimal contract. The price approach is equivalent to a recursive analysis using a particular continuation utility as a state variable, and is therefore the familiar route for practitioners of dynamic contracting. However, this characterization yields an HJB equation with a non-standard free boundary condition at a singularity of the ODE, a technically challenging feature not present in many conventional dynamic contracting problems. The quantity approach, by contrast, turns out to be quite tractable and powerful in my setting. I therefore demonstrate the value of the duality approach to contractual design in an economically interesting setting.
1.1 Related literature

My paper sits closest to the literatures on dynamic contracting under moral hazard and dynamic mechanism design, sharing the continuous-time setting and many analytical tools with the former and the basic information elicitation problem of the latter. I briefly survey each literature and its connection to my model and results.

A large literature studies the optimal provision of dynamic incentives for effort under repeated moral hazard. Spear and Srivastava (1987) formulate the standard infinite-horizon recursive problem in discrete time, while Sannikov (2008) and Williams (2008) study the problem in continuous time when output follows a Brownian motion with drift, affording considerable tractability compared to the discrete-time case. The continuous-time model has been extended in various ways. Williams (2008, 2015) permits general forms of hidden savings which impact the agent’s marginal utility of consumption. Sannikov (2014) assumes actions have long-run impacts on output. Prat and Jovanovic (2013) build in symmetric ex ante uncertainty about the quality of the agent, as in the career concerns literature. Cvitanic et al. (2013) suppose instead that the agent possesses ex ante private information about his marginal cost of effort, a setting which may lead the principal to offer a menu of contracts to separate or screen agents of different types as in mechanism design.

These papers solve the optimal contracting problem using recursive techniques with the agent’s continuation utility as a state variable and eliminate effort from the problem using a first-order condition. Their basic technique is therefore in the same spirit as the price approach of this paper, although with important distinctions. First, in moral hazard models the agent’s continuation utility process is a martingale on-path, which is not true of the relevant continuation utility in my model; so additional representation results are required to analyze my problem recursively. Also, while in moral hazard models the incentive constraint for effort is formulated naturally in terms of the local dynamics of continuation utility, an analogous result for the binding incentive constraint in my model cannot be formulated for general contracts and takes care to establish even for a very restricted class of candidate contracts. Finally, the papers cited focus their efficiency analysis on consumption smoothing and the wedge between induced and first-best effort rather than the incidence of early project termination, which is the key inefficiency in my model.

Another paper in this literature, DeMarzo and Sannikov (2006), simplifies the moral hazard problem by replacing effort with the ability to steal output inefficiently. This is equivalent to assuming marginal cost of effort is constant and less than 1, so efficient effort is always implementable and optimal for the principal. They then tightly characterize the form of an optimal contract in the presence of hidding savings. They show that an optimal
contract takes the form of a constant-limit credit line which the agent can draw down at will, with the caveat that the project is terminated when the credit line is exhausted. This dynamic has close similarities to the optimal payment process in my paper, which features a golden parachute that can be taken by the agent at any time and which when exhausted due to poor project returns leads to termination. Further, their contracting problem boils down to the one-dimensional choice of an optimal credit limit, just as my problem can be reduced to the choosing of a one-dimensional threshold in belief space.

Another literature studies dynamic mechanism design with transfers when agents have private payoff-relevant types which fluctuate over time. Baron and Besanko (1984) and Courty and Li (2000) consider two-period problems with a single agent, while Besanko (1985) and Battaglini (2005) study infinite-horizon settings in discrete time with one agent and special type processes. Pavan et al. (2014) extend these results to an infinite horizon discrete-time setting with many agents and general type processes. Williams (2011) analyzes an infinite-horizon problem in continuous time with one agent whose type evolves as an Ornstein-Uhlenbeck process, yielding additional tractability compared to the general discrete-time problem. In this set of papers payments are eliminated from the principal’s objective function in favor of allocations using an envelope theorem or other argument, yielding a virtual profit function which is maximized to obtain an optimal allocation. Their analysis is therefore analogous to the quantity approach in this paper. However, in my model the agent’s type is payoff-relevant only to the principal, in contrast to the mechanism design convention where type captures payoff-relevant information for the agent. Also unlike the mechanism design literature, I assume that the agent has limited liability, so it is not feasible to charge the agent for the private benefits he receives or to “sell him the firm.” Moreover, in my paper the firm observes exogenous signals and uses them to resolve the balance between inefficient early termination and paying the agent information rents.

Recently, Garrett and Pavan (2012) have studied a contracting problem with both repeated moral hazard and time-varying private information. In their model output is a function of both effort and match value, match value is privately observed by the agent and changes over time, and the agent can be replaced. As in this paper, they are primarily interested in characterizing the firm’s optimal termination policy. Also in common with my model, their principal observes output which is correlated with the current state conditional on effort. Unlike my model, their agent does not intrinsically value association with the project. Moreover, given his lack of limited liability he cannot extract rents from the moral hazard problem. This crucial distinction leads to starkly different optimal termination dynamics, which in particular do not depend on the history of project output and can
eventually become unresponsive to bad news from the agent.

My paper also connects to recent work in the dynamic delegation literature. Guo (2015) studies an environment with public experimentation by an agent who has private time-zero beliefs about the true state and a bias toward experimentation, when no transfers are permitted. In her setting the interests of the firm and the agent are not completely misaligned, because after enough bad news even the agent prefers to stop experimentation. Thus it is possible to elicit information from the agent without transfers. In contrast, in my setup the agent prefers to continue the project in all cases, so payments are necessary to obtain any information from him. Despite the differences in our settings, she derives an optimal policy strikingly similar to mine: the principal delegates the right to stop experimentation to the agent until a virtual belief about the state, which is updated as if the agent weren’t present, reaches a threshold, after which experimentation is cut off forever. Grenadier et al. (2015) consider a principal who elicits an agent’s information about the optimal exercise time of a real option, when the agent has a bias for late exercise exactly equivalent to the flow of benefits specification of my model. Unlike my model, there are no transfers, the agent knows the optimal exercise date at time zero, and the principal can commit only the delegation of exercise authority. Still, they predict an outcome with distinct similarities to the optimal policy in my model: the principal follows the agent’s exercise recommendation until a threshold in the value of the underlying asset is reached, at which point the option is exercised immediately.

2 The model

2.1 The environment

A firm possesses a project with an uncertain lifespan and must decide when to irreversibly scrap the project. The project’s lifespan $\tau^\theta$ is unobserved by the firm, who has prior beliefs that $\tau^\theta$ is exponentially distributed with rate parameter $\alpha$. I let $\theta_t$ denote the state of the project at time $t$, with $\theta_t = G$ if $t < \tau^\theta$ and $\theta_t = B$ otherwise. Then from the perspective of the firm $\theta$ is a hidden Markov process with initial state $\theta_0 = G$ and transition rate $\alpha$ from $G$ (the “good state”) to $B$ (the “bad state”). The project generates average profits $r_\theta$ per unit time when the current state is $\theta$, with $r_G > 0 > r_B$ and $\Delta r \equiv r_G - r_B$ denoting the spread in average profits between the two states. However, variability in instantaneous flow profits prevent the firm from immediately detecting a state transition.

For most of this paper, I study a project which generates observable profits in Brownian
increments. Letting $Y_t$ be cumulative profits up to date $t$, I assume the stochastic process $Y$ has increments

$$dY_t = r_{\theta_t} dt + \sigma dZ_t,$$

where $Z$ is a standard Brownian motion independent of $\tau^\theta$. (It is also possible to analyze variants of the model in which the state of the project impacts the arrival rate of Poisson profit shocks rather than the drift rate of profits. The qualitative conclusions are very similar to the Brownian case.)

Mathematically, I model this setting with a canonical probability space $(\Omega, \Sigma, \mathbb{P})$ sufficiently rich to admit $Z$ and $\tau^\theta$, with $\mathbb{E}$ the expectation operator under $\mathbb{P}$. I let $\mathbb{F}^Y = \{\mathcal{F}^Y_t\}_{t \geq 0}$ denote the natural filtration generated by $Y$; this filtration captures the information available to the firm based on its observation of past project profits. I write $\mathbb{E}_t^Y$ for the conditional expectation under $\mathbb{P}$ wrt $\mathcal{F}^Y_t$. I also let $\mathbb{F} = \{\mathcal{F}_t\}_{t \geq 0}$ denote the natural filtration generated by both $Y$ and $\theta$. Finally, I define $\mathbb{P}^G$ and $\mathbb{P}^B$ to be the equivalent probability measures under which $Z_t^G \equiv \frac{1}{\sigma}(Y_t - r_G t)$ and $Z_t^B \equiv \frac{1}{\sigma}(Y_t - r_B t)$, respectively, are standard Brownian motions, and write $\mathbb{E}_t^G$ and $\mathbb{E}_t^B$ for the conditional expectations under these measures wrt $\mathcal{F}^Y_t$.

The firm is a risk-neutral expected-profit maximizer with discount rate $\rho$. Supposing the firm operates the project until some ($\mathbb{F}^Y$-stopping) time $\tau^Y$, it receives expected profits

$$\Pi = \mathbb{E} \left[ \int_0^{\tau^Y} e^{-\rho t} dY_t \right] = \mathbb{E} \left[ \int_0^{\tau^Y} e^{-\rho t} (\pi_t r_G + (1 - \pi_t) r_B) dt \right],$$

(1)

where $\pi_t = \mathbb{P}^Y_t \{\theta_t = G\} = \mathbb{P}\{\theta_t = G \mid \mathcal{F}^Y_t\}$ are the firm’s posterior beliefs at time $t$ about the state of the project.

The firm may hire an agent to oversee the project and monitor its state. The agent is an expert who costlessly and privately observes the state process $\theta$. (He cannot, however, observe $\tau^\theta$ in advance.) That is, the filtration $\mathbb{F}$ captures the information available to the agent at any time. The agent also faces an incentive problem: he receives flow rents $b > 0$ per unit time while the project operates, regardless of its state.\(^1\) I assume that $r_B + b < 0$, so that it is jointly unprofitable for the firm to run the project in the bad state.\(^2\) The agent

\(^1\)None of the results of this paper would be changed by allowing the agent’s flow rents to vary in different states, so long as they are always positive and weakly higher in the good state. In the more general case only the rents in the bad state will enter into the optimal contract.

\(^2\)It turns out that if $r_B + b \geq 0$ the optimal contract under limited liability is trivial: the firm pays the agent nothing and operates the project ignoring the agent’s information entirely. Intuitively, when $r_B + b > 0$ there is scope for gains from trade by operating the project in the bad state, but because the agent can’t pay the firm none of these gains are realized.
possesses limited liability and has no initial wealth, so cannot be sold the project. He is risk-neutral and possesses the same discount rate $\rho$ as the firm.

### 2.2 Contracts

The firm commits in advance to a contract eliciting reports from the agent over time and specifying a termination policy $\tau$ and a cumulative payment function $\Phi$ as a function of the public history of output and the agent’s reports. By the revelation principle, I restrict attention to contracts which at each moment in time elicit a report $\tilde{\theta}_t \in \{G, B\}$ of the current state from the agent. Equivalently, as the only information to communicate is the time of a state switch, the firm asks the agent to make a single report at the time of a state switch. This simplification leads to a formulation of the contractual setting which is formally very similar to a static mechanism design problem.

**Definition 1.** A revelation contract $C = (\Phi, \tau)$ consists of a family of real-valued stochastic processes $\Phi^t$ and stopping times $\tau^t$ for each $t \in \mathbb{R}_+ \cup \{\infty\}$ such that:

- Each $\Phi^t$ is $\mathbb{F}^Y$-adapted, non-decreasing, right-continuous, and satisfies $\Phi^t_0 = 0$ and $\Phi^t_s = \Phi^t_{\tau^t}$ for $s > \tau^t$,
- Each $\tau^t$ is an $\mathbb{F}^Y$-stopping time,
- For every $t$ and $t' > t$, $\Phi^{t'}_s = \Phi^t_s$ and $1\{\tau^{t'} \geq s\} = 1\{\tau^t \geq s\}$ for $s < t$.

Intuitively, a revelation contract specifies a payment $\Phi^t$ and an allocation $\tau^t$ as a function of the reported type $t$, as in static mechanism design. However, in this setting payments and allocations are dynamic variables that can be conditioned on output history observed both before and after the agent’s report. Definition 1 enforces an appropriate form of dynamic consistency for the setting, by requiring that the contract not “look ahead” and anticipate the timing of a future report when making current payment and termination decisions. The definition additionally requires that a revelation contract be fully “cashed out” immediately upon termination. As both parties have linear utility with the same discount rate and no information arrives after termination, this is without loss of generality. (Definition 1 does not allow contracts to condition on any exogenous randomization. This restriction is without loss of generality, as the firm’s value function turns out to be strictly concave in a natural state variable summarizing contract continuations.)
Definition 2. A revelation contract $(\Phi, \tau)$ is incentive-compatible if
\[
\mathbb{E} \left[ \int_{0}^{\tau^{\theta}} e^{-\rho t} \left( b \, dt + d\Phi^{\theta}_{t} \right) \right] \geq \mathbb{E} \left[ \int_{0}^{\tau^{\sigma}} e^{-\rho t} \left( b \, dt + d\Phi^{\sigma}_{t} \right) \right]
\]
for all $\mathbb{F}$-stopping times $\sigma$.

I refer to an incentive-compatible revelation contract as an IC contract. Under any IC contract, the agent truthfully reports $\tau^{\theta}$. Therefore when there is no confusion, I suppress the superscripts on $\Phi$ and $\tau$ and consider an IC contract to be a pair $C = (\Phi, \tau)$ with $\Phi$ an $\mathbb{F}$-adapted process and $\tau$ an $\mathbb{F}$-stopping time. The firm’s problem is then to maximize
\[
\mathbb{E} \left[ \int_{0}^{\tau} e^{-\rho t} (dY_{t} - d\Phi_{t}) \right]
\]
over all IC contracts $(\Phi, \tau)$. I refer to any contract achieving this maximum as an optimal contract. Implicit in this formulation of the problem is the assumption that the firm requires the agent to operate the project in addition to monitoring it. Therefore the firm cannot terminate the agent without also ceasing operation of the project. I explore alternative assumptions later in the paper.

2.3 Baseline contracts

Before deriving an optimal contract, I describe a pair of contracts which establish a useful baseline for understanding the contractual design problem.

Remark. Let $\Phi_{t} = \frac{b}{\rho} 1 \{ t \geq \tau^{\theta} \}$. Then $(\Phi, \tau^{\theta})$ is an IC contract which is profit-maximizing among all IC contracts implementing efficient project termination.

This contract stops the project whenever the agent reports that the state has changed, and pays the agent a lump-sum transfer of $b/\rho$ upon termination. Intuitively, such a contract is incentive compatible because at each moment the additional flow rents $b \, dt$ from waiting a moment longer to report a state switch and terminate the project are exactly offset by the interest expense $-\rho (b/\rho) \, dt$ of delaying receipt of the contract’s terminal payment. Thus the agent is indifferent between all reporting policies regardless of the true state transition time, and in particular is willing to report truthfully. And it is profit-maximizing in the class of efficient contracts because any IC contract implementing $\tau^{\theta}$ must commit to operating forever if the agent never reports a state switch. Therefore it must pay the agent at least as much to stop as he’d collect by letting the project operate forever, which is precisely $b/\rho$. 

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Thus at one extreme, the firm can fully utilize the information available to the agent provided it pays the agent enough. The following remarks considers the opposite extreme, when the firm disregards the agent’s reports entirely.

Remark. Let $\tau^*$ be the $\mathbb{F}^Y$-stopping time maximizing (1). Then $(0, \tau^*)$ is an IC contract which is profit-maximizing among all IC contracts utilizing only public information.

Evidently, if the firm does not condition payments or project termination on the agent’s reports, incentive compatibility is trivially satisfied. The firm’s problem is then to improve upon $\tau^*$ by extracting some of the agent’s state information, but to economize on transfers to the agent and avoid paying the agent’s entire lifetime rental stream whenever the project is halted. I will show that the firm can do better than either of the baseline contracts described above by striking a balance between efficient termination and the costs of obtaining information from the agent.

3 Dual approaches to contractual design

Over the next several sections I develop two distinct approaches to analyzing the firm’s contracting problem. In this section I show that the firm’s transfer scheme and termination policy are “dual variables”: either may be eliminated in favor of the other, reducing the analysis to design of a single contractual element. This duality is closely analogous to the equivalence between choosing prices or quantities in monopoly theory, a perspective I return to later in the section. I go on to develop the “price approach” of optimizing the transfer scheme fully in Section 4, and then pursue the dual “quantity approach” of designing the termination policy in Section 5. The two approaches provide distinct and complementary insights into properties of the optimal contract, in particular the relationship of the contracting problem with an agent to the firm’s unassisted quickest detection problem.

3.1 Preliminaries

In general an IC contract must satisfy incentive constraints deterring the agent from falsely reporting a state change either too early or too late at each moment in time. In particular, when $\theta_t = G$ the agent’s expected utility from waiting until $\tau^\theta$ to report a state switch must exceed his expected utility from reporting a state switch immediately. And conversely, when $\theta_t = B$ the agent’s expected utility from reporting the switch immediately must exceed the payoff of any delayed reporting strategy. It will be very helpful to characterize incentive compatibility as the combination of these two constraints.
Definition 3. A revelation contract \((\Phi, \tau)\) satisfies IC-G (respectively, IC-B) if

\[
\mathbb{E} \left[ \int_{0}^{\tau_{\theta}} e^{-\rho t} (b dt + d\Phi_{t}) \right] \geq \mathbb{E} \left[ \int_{0}^{\tau_{\sigma}} e^{-\rho t} (b dt + d\Phi_{t}) \right]
\]

for all \(\mathbb{F}\)-stopping times \(\sigma \leq \tau_{\theta}\) (respectively, \(\sigma \geq \tau_{\theta}\)).

Lemma 1. A revelation contract is incentive-compatible iff it satisfies both IC-G and IC-B.

Both the IC-G and IC-B constraints substantively restrict the space of IC contracts. However, I conjecture that the sole binding constraint under an optimal contract is the IC-B constraint. Intuitively, given the agent’s preference to delay project termination absent contractual incentives, the IC-B constraint must bind at least some of the time. But as provisioning incentives is costly for the firm, it should impose them as lightly as possible; in particular, it would be surprising if the firm provisioned such strong incentives for reporting a switch that IC-G were violated and the agent wished to report a state switch prematurely. I therefore solve the relaxed problem of characterizing optimal IC-B contracts, and then verify at the end that such a contract is incentive-compatible and therefore an optimal contract.

A general IC-B contract could specify a complex series of payments to the agent over the course of his employment, as well as a varied set of termination policies depending on the project’s output history and the agent’s past reports. The following pair of lemmas prune away much of this complexity by establishing the optimality of a subset of IC-B contracts obeying parsimonious transfer and termination rules.

Lemma 2 (No late termination). Given any IC-B contract \(C = (\Phi, \tau)\), there exists another IC-B contract \(C' = (\Phi', \tau')\) satisfying \(\tau' = \tau_{\theta} \wedge \tau_{Y}\). Further, the firm’s expected profits under \(C'\) are weakly higher than under \(C\), and strictly higher if \(\tau' < \tau\) with positive probability. Finally, \(\tau' = \tau_{\theta} \wedge \tau_{Y}\) for some \(\mathbb{F}^{Y}\)-stopping time \(\tau_{Y}\).

Lemma 3 (Backloading). Let \(C = (\Phi, \tau)\) be any IC-B contract such that \(\tau = \tau_{\theta} \wedge \tau_{Y}\) for some \(\mathbb{F}^{Y}\)-stopping time \(\tau_{Y}\). Then there exists another IC-B contract \(C' = (\Phi', \tau)\) satisfying \(\Phi'_{t} = F_{t}1\{t \geq \tau\}\) for some \(\mathbb{F}^{Y}\)-adapted process \(F \geq 0\). Further, \(C'\) yields the same expected profits as \(C\).

The first lemma establishes that optimal IC-B contracts never terminate inefficiently late - that is, after the agent reports the state has switched. In principle late termination could be desirable as a way to compensate the agent with rents. However, the assumption that the project is jointly unprofitable in the bad state means that the firm can always
more cheaply compensate the agent with cash at the time of the state switch. Therefore an optimal termination policy always consists of a rule of the form “Terminate the project as soon as the agent reports a state switch or a publicly observable threshold has been reached, whichever comes first”.

Remark. It is not clear a priori that an arbitrary IC contract can be improved upon by terminating as soon as a state switch is reported without violating incentive-compatibility. For the firm might enforce IC-G in a particular contract by monitoring output following a report to ensure that the state has really switched. If such monitoring is removed by truncating the project lifetime, an otherwise IC contract may create incentives to prematurely report a state switch. Lemma 2 therefore illustrates the tractability brought by passing to the relaxed problem, as well as the importance of verifying at the end that an optimal IC-B contract does not violate IC-G.

The second lemma demonstrates that all payments can be backloaded to a single *golden parachute*, denoted $F$, payable upon termination. This is because both parties are risk-neutral and discount the future equally; thus expected profits and incentive-compatibility are unchanged by deferring all promised payments until termination and accruing interest on them at rate $\rho$. And because termination occurs no later than the date of a reported state change, the size of the parachute need not be conditioned on the date of a past report, hence is $\mathbb{F}^Y$-adapted. It is then without loss of generality to study only transfer policies of the form “Pay nothing until project termination, at which point grant the agent a single lump sum golden parachute”.

I therefore restrict attention to backloaded contracts with no late termination. I summarize such contracts by pairs $(F, \tau^Y)$, with $F$ a golden parachute and $\tau^Y$ a (public) termination policy. Under such a contract, an agent’s reporting rule can be thought of as a restricted project termination decision: he can terminate the project immediately at any point by reporting a switch, and can extend the project until $\tau^Y$ by staying silent. Under such a contract, the IC-B constraint can be characterized in a convenient recursive form.

**Lemma 4.** Fix a contract $C = (F, \tau^Y)$. Suppose that for every $t$ and every $\mathbb{F}^Y$-stopping time $\tau' \geq t$,

$$F_t \geq \mathbb{E}_t^B \left[ \int_t^{\tau^Y \wedge \tau'} e^{-\rho(s-t)} b \, ds + e^{-\rho(\tau^Y \wedge \tau' - t)} F_{\tau^Y \wedge \tau'} \right]$$  

whenever $\tau^Y > t$. Then $C$ satisfies IC-B.

Conversely, suppose $C$ satisfies IC-B. Then there exists another IC-B contract $C' =$
(F', τ') such that F' satisfies (2) for every t and τ' ≥ t whenever τ' > t, and further such that F'_{τ'∧θ} = F_{τ∧θ} a.s. Hence C' yields the same expected profits as C.

This lemma formalizes the idea that at each moment in time, an agent in the bad state must prefer collecting the golden parachute now rather than delaying termination and collecting additional flow rents from operating the project. As the agent’s delayed report can be conditioned on the path of output going forward, a general delayed reporting policy is an \( \mathbb{F}^Y \)-stopping time \( \tau' \). Lemma 4 rules out the profitability of any such \( \tau' \) at any time as essentially a necessary and sufficient condition for satisfying IC-B. Without loss of generality, I restrict attention to IC-B contracts satisfying (2) going forward and treat the condition as necessary as well as sufficient without comment.

3.2 Designing the golden parachute

One approach to the optimal contracting problem is to design the golden parachute \( F \), treating the termination policy \( τ^Y \) as an auxiliary variable and eliminating it. To do so, one must determine the optimal termination policy which incentivizes truthful reporting under a given golden parachute \( F \).

It turns out that early termination serves a very limited role in an optimal contract. To incentivize truthful reporting, the firm must commit to limit the agent’s continuation utility (flow rents plus golden parachute) from delaying his report of a state switch. In particular, if the golden parachute ever falls to zero, then any delayed report must also yield at most zero expected utility. Given the agent’s limited liability, immediate project termination is of course the only way to enforce such a promise. Is early termination optimal in any other circumstance? The following lemma answers this question in the negative.

**Lemma 5.** Suppose \( C = (F, τ^Y) \) is an IC-B contract. Then τ' ≤ \( \inf\{t : F_t = 0\} \), and there exists another IC-B contract \( \tilde{C} = (\tilde{F}, τ'') \) such that \( \tilde{F}_t = \min\{F_t, b/ρ\} \) whenever \( τ'' ≥ t \) and \( τ'' = \inf\{t : \tilde{F}_t = 0\} \). Further, \( τ'' ≥ τ' \) and \( \tilde{C} \) yields expected profits at least as high as C.

The first claim of the contract is just a restatement of the conclusion that no contract can operate past the point where the golden parachute hits zero. The lemma then constructs a new contract illustrating two design principles: first, the golden parachute can be capped at \( b/ρ \); and second, if a contract \( (F, τ^Y) \) is ever terminated before \( F \) hits zero, it can be extended in an incentive-compatible way without decreasing profits. The first principle is closely related to the fact that \( (b/ρ, \infty) \) is an IC contract, as I demonstrated in Section 2.3. Essentially, whenever \( F_t \) hits \( b/ρ \), delayed reporting can be deterred indefinitely simply by
holding the fee at that level; in particular, it can be delayed until the next time $F_t$ falls below $b/\rho$. Thus $\min\{F_t, b/\rho\}$ is also incentive compatible. The second principle can be proven by explicitly constructing an IC continuation contract. In particular, set

$$\tilde{F}_t = \max \left\{ \frac{b}{\rho} - \left( \frac{b}{\rho} - \min\{F_{\tau Y}, b/\rho\} \right) e^{\rho(t-\tau Y)}, 0 \right\}$$

for $t \geq \tau Y$. Note that so long as $\tilde{F}_t > 0$, \( \frac{d}{dt} \tilde{F}_t = -b + \rho \tilde{F}_t \), which exactly offsets the flow rents and interest expense of delaying termination by a moment. Hence incentive compatibility is ensured for $t \geq \tau Y$. As for times before $\tau Y$, $\tilde{F}_t$ provides no more expected utility to the agent than a contract that terminates at $\tau Y$ and pays $\tilde{F}_{\tau Y}$. So it provides no additional incentives to delay reporting prior to $\tau Y$. Finally, because this extension of the contract pays no more than $\tilde{F}_{\tau Y}$ to the agent while generating positive revenue as long as the state is good, it must improve profitability relative to termination at $\tau Y$.

Lemma 5 solves the design problem for $\tau Y$ given an incentive-compatible golden parachute $F$. The firm’s problem therefore reduces to the design of the dynamics of the golden parachute, subject to the requirement that the project terminate as soon as the parachute reaches zero. I derive the optimal golden parachute in Section 4.

3.3 Designing the termination policy

The second approach to deriving an optimal contract is to design the termination policy $\tau Y$ while treating the golden parachute $F$ as an auxiliary variable. To proceed this way, one must determine the optimal golden parachute which incentivizes truthful reporting under a given termination policy $\tau Y$.

Consider that one reporting strategy an agent may follow is to never report a state switch, collecting rents until the firm halts the project according to $\tau Y$. The value of any IC-B golden parachute implementing $\tau Y$ must therefore exceed this amount, implying the inequality

$$F_t \geq \mathbb{E}_t^B \left[ \int_{t}^{\tau Y} e^{-\rho(s-t)} b \, ds + e^{-\rho(\tau Y-t)} F_{\tau Y} \right]$$

after all histories. Setting $F_{\tau Y} = 0$ and $F_t$ to saturate this inequality minimizes the golden parachute subject to deterring a deviation to never reporting a state change. The following lemma establishes that such a payment scheme deters all other delayed reporting strategies as well and is therefore optimal within the class of IC-B golden parachutes implementing $\tau Y$.

**Lemma 6.** Every $\mathbb{F}^\tau Y$-stopping time $\tau Y$ is supportable as part of an IC-B contract, and the
The surprising conclusion of Lemma 6 is that the firm faces no real double deviation problem when designing an optimal golden parachute. The parachute in the lemma is derived by pretending that the agent can do just one of two things at the time of a state change - report the change now, or stay silent forever. In general, the agent’s ability to deviate again by reporting a state change at some later date implies additional constraints that might bind at the optimum. In my model, however, delaying a report of a state change for any interval of time leaves the agent in an “on-path” setting identical to a counterfactual agent with the same output history but a later state switch. And under the contract of Lemma 6, that agent’s golden parachute is exactly his expected rents from never reporting. Therefore the true agent’s utility from delaying a report by that interval is identical to his utility from never reporting. This argument establishes the lemma’s claim that no strategy of delayed reporting yields a profitable deviation.

Lemma 6 allows $F$ to be eliminated from the firm’s objective function, reducing the contracting problem to the design of $\tau^Y$ alone. I derive the optimal termination policy in Section 5.

### 3.4 Dual variables and monopoly theory

Lemmas 5 and 6 establish the duality of $F$ and $\tau^Y$ as contractual design variables. Note that $F$ controls the cost of implementing the contract, while $\tau^Y$ regulates the amount of output produced; in addition, the longest IC-B $\tau^Y$ is increasing in $F$. It is therefore tempting to draw an analogy to the price and quantity variables of standard monopoly theory. Indeed, price and quantity are also dual variables in that problem, so this analogy provides useful intuition for understanding the duality result in my model.

Consider a standard monopsony problem, in which a firm faces an upward-sloping supply function for procurement of a good. The firm can be considered to control either price $P$ or quantity purchased $Q$, with the remaining variable eliminated from the firm’s optimization problem. One approach is to optimize profits over $P$, determining $Q$ via the supply curve along the way. The dual approach is to control $Q$, with $P$ eliminated from the problem via the inverse supply curve. In my problem, one can analogously think of Lemma 5 as giving
a generalized supply curve, with Lemma 6 computing the generalized inverse supply curve. The firm then optimizes profits $\Pi(F, \tau^Y)$ either by using the generalized supply curve to eliminate $\tau^Y$ (the quantity variable), and then optimizing $\Pi(F)$ over $F$ (the price variable); or by controlling $\tau^Y$ and optimizing $\Pi(\tau^Y)$, using the generalized inverse supply curve to eliminate $F$.

The analogy may be pushed even further by examining the standard logic of equating gains on marginal units and losses on inframarginal units. In the monopsonist’s problem, the firm finds the point (in either $P$ or $Q$ space) where profits on a marginal unit purchased equal losses due to the increased cost of inframarginal units already purchased. Analogously in my problem, the firm determines when the increased profit from operating the project when $\tau^\theta \in (\tau^Y, \tau^Y + dt)$ is balanced by the increased cost of golden parachutes paid to the agent when $\tau^\theta \leq \tau^Y$. This is essentially the calculation carried out in Sections 4 and 5 using $F$ and $\tau^Y$, respectively, as control variables.

## 4 The price approach

In this section I pursue the approach suggested by Lemma 5 and eliminate $\tau^Y$ from the optimal contracting problem, reducing the contracting problem to the design of $F$ alone. I then use techniques of recursive dynamic contracting to maximize the firm’s profits wrt incentive-compatible $F$ and derive an optimal golden parachute. The main theorem of this section derives a differential equation, the HJB equation, whose solution characterizes the optimal incentive-compatible golden parachute. The associated optimal termination policy is left implicit, but could in principle be calculated by Lemma 5. Throughout this section, I will often summarize a contract by the golden parachute process $F$ alone; whenever $\tau^Y$ is not explicitly stated as an element of the contract, it is understood to be the first time the golden parachute reaches zero.

### 4.1 The golden parachute as state variable

In this subsection I show that the current value of the golden parachute constitutes a sufficient statistic for the state of the contract. In the context of recursive contractual design, this assertion amounts to two statements. First, the firm’s profit function can be written as a discounted sum of flow payments involving only the current value of $F$ at each moment of time. Second, the agent’s IC constraints can be reduced to a set of local restrictions on the dynamics of $F$. With these results in hand, standard techniques of continuous-time dynamic
contracting (see, e.g., Sannikov (2008)) can be used to derive the firm’s value function using $F$ as a state variable.

Recall that the firm’s ex post profits from an IC-B contract $F$ consist of flow payments $r_G$ up to the stopping time $\tau = \tau^\theta \wedge \tau^Y$, followed by a lump-sum payment of $F_\tau$ once $\tau$ is reached. From an ex ante perspective, this payment flow can be analyzed as follows: at each time $t$, there is a probability $e^{-\alpha t}$ that the project has not yet terminated. In this case the firm receives a discounted flow payment of $e^{-\rho t} r_G dt$, and with probability $\alpha dt$ the state changes in the next instant and the firm must pay out $e^{-\rho t} F_t$. The net expected contribution of time $t$’s flows to time-zero profits are therefore $e^{-(\rho + \alpha) t} (r_G - \alpha F_t)$. These flows are then summed up to the public stopping time $\tau^Y$ and appropriately averaged to obtain the firm’s total expected profits from $F$. This result is formalized in the following lemma.

**Lemma 7.** The firm’s expected profits under any contract $F$ when the agent reports truthfully are

$$\mathbb{E}^G \left[ \int_0^{\tau^Y} e^{-(\rho + \alpha) t} (r_G - \alpha F_t) dt \right].$$

Lemma 7 establishes the first prong of the claim that $F_t$ appropriately summarizes the state of the contract at time $t$. Note that the expectation in the profit function of Lemma 7 is wrt to the measure $\mathbb{P}^G$ under which $Z^G$ is a standard Brownian motion. In other words, it assumes a world in which the project is always good. In this formulation, the distortion caused by state switching is entirely captured by the higher discount rate $\rho + \alpha$ applied to flow profits.

As for the second prong, recall that IC-B means stopping immediately following a state switch must provide higher utility to the agent than any delayed reporting policy. In particular, it must pay more, on average, than waiting a moment and then reporting. This latter constraint is purely local, restricting only the instantaneous dynamics of the golden parachute given its current value. In fact, in this model these local constraints collectively imply global IC-B. Intuitively, an agent who deviates from immediate reporting of a state switch to delay momentarily finds himself in a situation identical to an agent with the same public output history and a slightly later state switch who has not (yet) deviated. Therefore if agents have incentives to report a state switch at time $t + dt$, an appropriate local constraint will ensure that agents at time $t$ do as well.

This intuition is difficult to formalize for general golden parachute processes, as I do not rule out the possibility that $F$ evolves very irregularly. However, when $F$ is sufficiently well-behaved incentive compatibility can be characterized quite sharply.
Lemma 8. Fix a contract $C = (F, \tau^Y)$. Suppose there exist $\mathbb{R}^Y$-adapted, right-continuous processes $\gamma$ and $\beta$ such that for each $t$, $F$ satisfies

$$F_t = F_0 + \int_0^t \gamma_s \, ds + \int_0^t \beta_s \, dZ^G_s$$

a.s. Then $(F, \tau^Y)$ is an IC-B contract iff

$$b - \rho F_t + \gamma_t - \frac{\Delta r}{\sigma} \beta_t \leq 0$$

whenever $t < \tau^Y$.

Lemma 8 decomposes the increments $dF_t$ of the golden parachute into two components: an instantaneous average change $\gamma_t \, dt$, and a “surprise” term $\beta_t \, dZ^G_t$ linked to the arrival of information on project output. In a world where the project’s state is known to be good, the surprise term has mean zero and $\gamma_t \, dt$ captures the expected change in the golden parachute in the next instant. The derivation of the incentive constraint deterring late reporting arises from the observation that after the state has switched, $Z^B$ rather than $Z^G$ has zero-mean increments. Further, the increments of the two processes are related via $dZ^G_t = dZ^B_t - \Delta r \sigma \beta_t \, dt$.

It is then straightforward to compute the expected change in utility of an agent who momentarily delays reporting a state change. After the delay, the agent collects additional flow rents $b \, dt$; pays an interest cost $-\rho F_t \, dt$ from collecting the golden parachute a bit later; and receives the expected change in the value of the golden parachute, which from his perspective is $(\gamma_t - \frac{\Delta r}{\sigma} \beta_t) \, dt$. If the sum of these three factors is non-positive everywhere, then the agent never benefits from delaying a report of a state change.

Unfortunately Lemma 8 does not fully characterize the set of IC-B contracts, as it applies only to a particularly well-behaved class of stochastic processes. Indeed, the most involved technical step in the proof of Theorem 1 is establishing the suboptimality of more general golden parachutes.\(^3\) However, Lemma 8 captures the key intuition for the incentive constraint appearing in Theorem 1.

Lemmas 7 and 8 together indicate that the current value of the golden parachute is an appropriate state variable for a recursive formulation of the firm’s contracting problem. The next subsection carries out the recursive derivation of an optimal contract.

Remark. The choice of $F_t$ as a state variable for this problem contrasts with standard con-

\(^3\)Examples of such processes include Ito processes with drift or diffusion terms that are not right-continuous; semimartingales whose BV components are not absolutely continuous or which contain jumps; as well as even more irregular processes which are not semimartingales, such as fractional Brownian motions.
tracting environments, which generally use the agent’s continuation utility as a state variable. One distinctive feature of my model is the presence of asymmetric information, which in general requires tracking a pair of continuation utilities for the agent in each state. Letting \((U_t^G, U_t^B)\) be the agent’s continuation utilities in each state at time \(t\), these variables can be computed explicitly for any IC contract satisfying backloading and no late termination:

\[
U_t^G = \mathbb{E}_t^G \left[ \int_t^\tau e^{-(\rho+\alpha)(s-t)}(b + \alpha F_s) \, ds + e^{-(\rho+\alpha)(\tau-t)} F_{\tau} \right], \quad U_t^B = F_t.
\]

The choice of \(F_t\) as a state variable is therefore equivalent, after appropriate reduction of the contract space, to choosing \(U_t^B\) as a state variable. The key property of my model which further reduces the state space is the fact that IC constraints in the good state don’t bind at the optimum. The analysis therefore amounts to designing the dynamics of \(U_t^B\), i.e. the terminal utility of the contract.

### 4.2 A recursive formulation of the optimal contract

Consider the class of all IC-B contracts promising a golden parachute of \(F_0 \geq 0\) if the agent reports a state switch at time zero, and let \(V(F_0)\) be the maximal profit achievable by the firm among all such contracts. \(V\) is then the firm’s value function. To calculate \(V\), I utilize the accounting identity that \(V(F_0)\) should be equal to the firm’s instantaneous flow profits at time 0 plus continuation profits from following an optimal contract a moment later. Of course, what state is reached a moment later depends on the contract increment \(dF\) chosen at time 0; as \(V\) is the value function \(dF\) must of course be chosen to maximize lifetime profits over all incentive-compatible contract increments. Heuristically, \(V\) must satisfy the “Bellman equation”

\[
V(F_0) = \max_{dF \in \text{IC}(F_0)} \mathbb{E}_0^G \left[ (r_G - \alpha F_0) \, dt + e^{-(\rho+\alpha)dt} V(F_0 + dF) \right].
\]

Supposing the firm chooses \(dF = \gamma dt + \beta dZ_t^G\), its control variables are \((\gamma, \beta)\), which by Lemma 8 must satisfy \(b - \rho F_0 + \gamma - \frac{\Delta r}{\sigma} \beta \leq 0\). Derivation of a proper HJB equation (the continuous-time analog of a Bellman equation) is then accomplished by taking \(dt \to 0\) appropriately.

Ito’s lemma provides the proper machinery to take this limit. It says that, to first order,

\[
V(F_0 + dF) \simeq V(F_0) + \left( \gamma V'(F_0) + \frac{1}{2} \beta^2 V''(F_0) \right) dt + \beta V'(F_0) dZ_t^G.
\]

19
This expression differs from a standard Taylor expansion due to the presence of the second-order term $\frac{1}{2} \beta^2 V''(F_0)$. Intuitively, the extra term captures the impact of the curvature of $V$ on $E^G[V(F_0 + dF)]$, which by Jensen’s inequality differs from $V(F_0 + E^G[dF])$ in the same direction as the sign of $V''$.

Inserting the Ito expansion of $V(F_0 + dF)$ into the “Bellman equation”, the final term $\beta V'(F_0) dZ_g$ has mean zero and vanishes.\(^4\) Expanding $e^{-(\rho + \alpha)dt}$ to first order as $1 - (\rho + \alpha)dt$, eliminating second-order terms, and re-arranging yields the HJB equation

$$(\rho + \alpha)V(F_0) = \max_{(\gamma, \beta) \in IC(F_0)} \left\{ r_G - \alpha F_0 + \gamma V'(F_0) + \frac{1}{2} \beta^2 V''(F_0) \right\}.$$  

This is a second-order differential equation which, combined with appropriate boundary conditions, pins down the firm’s value function. One boundary condition is easy: when $F_0 = 0$ the unique IC contract entails immediate termination, so $V(0) = 0$. The second condition is more subtle, and turns out to be a so-called “free boundary condition”. Note that an upper bound on the value of any contract to the firm is $r_G/(\rho + \alpha)$, as this is what the firm would earn if it could directly observe the state of the project. Conversely, as discussed earlier there is no benefit to offering a golden parachute larger than $b/\rho$. Therefore $V$ is bounded above, and it should reach a maximum for some finite value of the golden parachute. The second boundary condition is then that $V'(F^*) = 0$ for some undetermined value $F^*$ of the golden parachute.

The following theorem provides a formal statement of the conditions under which the heuristic approach just outlined produces a solution to the firm’s contracting problem. It constitutes the main technical result of this section.

**Theorem 1.** Suppose there exists a constant $F^* > 0$ and a $C^2$ function $V: [0, F^*] \to \mathbb{R}$ satisfying $V(0) = 0$, $V'(F^*) = 0$, and the HJB equation

$$(\rho + \alpha)V(F_0) = \max_{(\gamma, \beta) \in IC(F_0)} \left\{ r_G - \alpha F_0 + \gamma V'(F_0) + \frac{1}{2} \beta^2 V''(F_0) \right\}$$

for every $F_0 \in [0, F^*]$, where $IC(F_0) \equiv \{(\gamma, \beta) : b - \rho F_0 + \gamma - \beta \Delta r / \sigma \leq 0\}$. Then:

- $V(F^*)$ is an upper bound on the expected profits of any IC-B contract;
- $F^* < b/\rho$, $V$ is strictly increasing and strictly concave on $[0, F^*]$, and $V(F^*) = \frac{r_G - \alpha F^*}{\rho + \alpha}$;\(^4\)

---

\(^4\)Strictly speaking, stochastic integrals wrt $Z^G$ are martingales only if they satisfy appropriate regularity conditions. Checking that these conditions are met is a standard step in a rigorous verification proof. See the proof of Theorem 1 for details.
There exist unique continuous functions $\gamma^*, \beta^* : \mathbb{R} \rightarrow \mathbb{R}$ such that $(\gamma^*(F_0), \beta^*(F_0))$ maximize the HJB equation on $[0, F^*]$ and $\gamma^*, \beta^*$ are constant on $\mathbb{R} \setminus [0, F^*]$;

$\beta^* \geq 0$, $\beta^*(F^*) = 0$, and $\gamma^*(F^*) < 0$;

There exists a weak solution $F$ to the stochastic differential equation (SDE)

\[ F_t = F^* + \int_0^t \gamma^*(F_s) \, ds + \int_0^t \beta^*(F_s) \, dZ_s^G \]  

for all time;

If $\lim_{F_0 \to F} V'(F_0) V''(F_0)$ exists and is finite, then there exists a unique strong solution $F$ to (3) for all time;

If $F$ is a strong solution to (3), then $\max\{F, 0\}$ is an optimal contract whose expected profits are $V(F^*)$.

Theorem 1 states that solving a particular boundary value problem yields the firm’s value function as well as an optimal contract. Importantly, the value function is actually increasing over a range of golden parachutes. This is because when the parachute reaches zero, the firm is forced to terminate the project and loses all future output from its operation. Thus for sufficiently small values of $F_0$, the revenue from a longer project lifespan outweighs the cost of increasing expected payments to the agent, leading to an upward-sloping value function. The critical point $F^*$ determines when these two factors exactly balance. Because $\gamma^*(F^*) < 0$ and $\beta^*(F^*) = 0$, an optimal contract never promises a golden parachute larger than $F^*$ no matter how good the history of output. Instead, as $F_t$ approaches $F^*$ the volatility of the parachute vanishes and the parachute drifts downward deterministically, holding its value below $F^*$.

The HJB equation also allows one to explore the economics of the firm’s contractual design problem. Given concavity of $V$ as assured by the theorem, the optimal controls $(\gamma^*(F_0), \beta^*(F_0))$ are readily computed as the first-order conditions of the HJB equation given a binding IC constraint:

\[ \beta^*(F_0) = -\frac{\Delta r}{\sigma} V'(F_0), \quad \gamma^*(F_0) = -(b - \rho F_0) + \frac{\Delta r}{\sigma} \beta^*(F_0). \]

Intuitively, the tradeoff the firm faces when designing a golden parachute is whether to provision incentives through downward drift or volatility of the parachute. Both are costly for the firm, because each brings the contract closer to premature termination on average.
An optimal contract therefore provisions incentives as lightly as possible, leading to a binding IC-B constraint at all points along the contract. In other words, \((\gamma^*(F_0), \beta^*(F_0))\) satisfy

\[
b - \rho F_0 + \gamma^*(F_0) - \frac{\Delta r}{\sigma} \beta^*(F_0) = 0.
\]

The optimal mix of incentives is then determined by the slope and curvature of the value function: each unit of downward drift added to the contract incurs a constant marginal cost \(V'(F_0)\), while each unit of volatility added incurs a linearly increasing marginal cost \(-\beta V''(F_0)\). Equating the ratio of marginal costs to the marginal rate of substitution \(\frac{\Delta r}{\sigma}\) between volatility and drift in the incentive constraint determines the optimal mix of incentives.

The fact that an optimal contract provisions incentives as lightly as possible and increases the golden parachute in response to good news verifies an earlier conjecture that the IC constraint for the agent in the good state is slack under an optimal contract. To see this, consider any contract evolving as

\[
dF_t = \gamma_t \, dt + \beta_t \, dZ^G_t,
\]

where \(b - \rho F_t + \gamma_t - \frac{\Delta r}{\sigma} \beta_t = 0\) and \(\beta \geq 0\). Reasoning as in the paragraphs following Lemma 8, one may compute the expected change in the agent’s utility when the state is good and he considers momentarily delaying a (false) report of a state change, which turns out to be \(b - \rho F_t + \gamma_t \geq 0\). Thus it is always at least weakly optimal for an agent in the good state to delay (falsely) reporting a state change a moment. Since delay continues to be optimal as long as the state is good, global incentive compatibility obtains.

Finally, note that Theorem 1 falls short of completely solving the optimal contracting problem, as it does not ensure that a solution to the HJB equation actually exists. The problem is essentially technical, as in principle the firm’s value function may not have the required degree of smoothness to satisfy a standard HJB equation. I defer a thorough examination of the technical issues to Section 4.4. In the next subsection, I take a different approach and exhibit an explicit solution to the HJB equation under a particular parameter restriction. For this class of models, Theorem 1 ensures that such a solution is in fact the firm’s value function.
4.3 A complete solution to a special case

For a particular subspace of model parameters, the boundary value problem posed in Theorem 1 has an elegant closed-form solution. The following proposition states the parameter restriction and describes the form of the value function and the associated optimal controls.

Proposition 1. Suppose \( \rho = \alpha + (\Delta r / \sigma)^2 \). Then there exist unique constants \( a_1 \) and \( a_2 \) such that \( a_1 > 0 > a_2 \) and \((a_1, a_2)\) satisfy

\[
\begin{aligned}
\rho a_1 + 2ba_2 + \alpha &= 0, \\
\frac{1}{4} (\Delta r / \sigma)^2 a_1^2 + ba_1a_2 - r_G a_2 &= 0.
\end{aligned}
\]

Further, \( V(F_0) = a_2 F_0^2 + a_1 F_0 \) satisfies the HJB equation of Theorem 1 on \([0, F^*]\) and the boundary conditions \( V(0) = 0, V'(F^*) = 0, \) and \( V''(F^*) < 0, \) where

\[
F^* \equiv \left( \frac{\Delta r}{\sigma} \right)^2 \left( b + r_G \frac{\rho}{\alpha} - \sqrt{(b + r_G)^2 + r_G^2 \left( \frac{\rho}{\alpha} \right)^2 - 1} \right) \in (0, b/\rho).
\]

The corresponding continuous maximizers of the HJB equation are

\[
\beta^*(F_0) = \frac{\Delta r}{\sigma} (F^* - F_0)
\]

and

\[
\gamma^*(F_0) = -\alpha (F^* - F_0) - (b - \rho F^*).
\]

Further, \( V''' = 0 \), so there exists a unique strong solution to (3).

Proposition 1 solves the firm’s optimal contracting problem when \( \rho = \alpha + (\Delta r / \sigma)^2 \). In this case the firm’s value function is quadratic and the associated controls are linear in the current value of the golden parachute. One implication of this linearity is that the optimal contract’s sensitivity to news increases the closer the contract moves to termination, and conversely vanishes as \( F_0 \to F^* \). This dynamic conforms to a sensible intuition that news should “matter more” for incentives as the contract progresses and it becomes ex ante more likely that the state has transitioned. In Section 5 I explore the issue of optimal sensitivity to news from another angle, when considering how news affects progress toward project termination.

Proposition 1 characterizes all the essential features of the firm’s optimal contract but stops short of deriving the contract explicitly. Given the linearity of the optimal controls,
the optimal contract satisfies the linear SDE
\[ dF_t = -\left(\alpha (F^* - F) + (b - \rho F^*)\right) dt + \frac{\Delta r}{\sigma} (F^* - F) dZ_t^G. \]

Such SDEs are solvable in closed form, yielding an explicit solution for the optimal contract:

**Corollary.** Suppose \( \rho = \alpha + \left(\frac{\Delta r}{\sigma}\right)^2 \). Then letting \( F^* \) be as in Proposition 1, an optimal contract is given by

\[ F_t = F^* - (b - \rho F^*) \exp \left( \left(\frac{3}{2} \alpha - \frac{1}{2} \rho \right) t - \frac{\Delta r}{\sigma} Z_t^G \right) \int_0^t \exp \left( - \left(\frac{3}{2} \alpha - \frac{1}{2} \rho \right) s + \frac{\Delta r}{\sigma} Z_s^G \right) ds \]

for \( t \leq \inf\{t : F_t = 0\} \).

An immediate implication of this corollary is the following: fix a time \( t \) and two states of the world \( \omega_1, \omega_2 \in \Omega \) satisfying \( Y_s(\omega_1) > Y_s(\omega_2) \) for all \( s \in (0, t) \) and \( Y_t(\omega_1) = Y_t(\omega_2) \). Then \( F_t(\omega_1) < F_t(\omega_2) \). In words, \( \omega_1 \) and \( \omega_2 \) encode two different paths of output that reach the same cumulative output by time \( t \). However, output under \( \omega_1 \) outperforms early and lags late compared to \( \omega_2 \); consequently the contract is brought closer to termination by time \( t \) under \( \omega_1 \). Thus in a very strong sense the optimal contract weights news more heavily later in the contract.

### 4.4 Existence of solutions to the HJB equation

This subsection is devoted to discussing technical existence issues of solutions to the HJB equation derived in Theorem 1. It can be safely omitted by readers uninterested in this detail.

To begin with, Theorem 3 in Section 5.3 establishes existence of an optimal IC termination policy \( \tau^* = \inf\{t : \pi_t \leq \pi\} \) for some belief threshold \( \pi \), which by Lemma 6 is implementable with the golden parachute

\[ F_t = \mathbb{E}_t^B \left[ \int_t^{\tau^*} e^{-\rho(s-t)} b ds \right]. \]

An application of the martingale representation theorem reveals that \( F \) is an Ito process satisfying the SDE

\[ dF_t = (\rho F_t - b) dt + \beta_t^1 dZ_t^B \]

for some \( \mathbb{F}^Y \)-adapted, progressively measurable process \( \beta_t^1 \) satisfying \( \mathbb{E}[\int_0^t (\beta_s^1)^2 ds] < \infty \) for all \( t \). In this case a partial converse to Theorem 1 obtains:\(^5\)

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\(^5\)This proposition does not ensure that \( V'' \) is nonvanishing at \( F^* \), which appears to be a requirement
Proposition 2. Let $V$ be the firm’s value function for IC-B contracts. Then $V(0) = 0$. Further suppose $V$ is a $C^2$ function on $[0, F^*]$, where $F^* \equiv \mathbb{E}^B \left[ \int_0^{F^*} e^{-\rho b} dt \right]$. Then $V'(F^*) = 0$ and $V$ satisfies

$$(\rho + \alpha)V(F_0) = \sup_{(\gamma, \beta) \in \text{IC}(F_0)} \left\{ \gamma G - \alpha F_0 + \gamma V'(F_0) + \frac{1}{2}\beta^2 V''(F_0) \right\}$$

for all $F_0 \in [0, F^*]$.

The question of existence then boils down to whether the firm’s value function is a sufficiently smooth function. Provided it is, an appropriate solution to the HJB equation is guaranteed. The usual approach is to apply ODE existence theorems directly to the HJB equation. The following proposition demonstrates the difficulties.

Proposition 3. Let $V : [0, F^*] \to \mathbb{R}$ be a $C^2$ function satisfying $V(0) = 0$ and $V'(F^*) = 0$. Then $V$ satisfies the HJB equation on $[0, F^*]$ iff it is strictly increasing and strictly concave on $[0, F^*)$ and satisfies

$$V''(F_0) = -\frac{1}{2} \left( \frac{\Delta r}{\sigma} \right)^2 \frac{V'(F_0)^2}{(\rho + \alpha)V(F_0) - (\gamma G - \alpha F_0) + (b - \rho F_0)V'(F_0)}$$

for each $F_0 \in [0, F^*)$.

Further, suppose $V$ is such a function. Then $V(F^*) = \frac{\gamma G - \alpha F^*}{\rho + \alpha}$, $F^* < b/\rho$, and $V''(F^*) < 0$.

Proposition 3 demonstrates that finding solutions to the HJB equation boils down to solving a particular nonlinear second-order ODE of the form $V'' = G(F_0, V, V')$. Moreover, the solutions of interest to the optimal contracting problem are singular solutions to this ODE. To see this, consider the numerator and denominator of $G$ as $F_0 \to F^*$. The numerator becomes $-\frac{1}{2} \left( \frac{\Delta r}{\sigma} \right)^2 V'(F^*)^2$, which vanishes by definition of the free boundary condition. As for the denominator, $(\rho + \alpha)V(F^*) - (\gamma G - \alpha F^*) + (b - \rho F_0)V'(F^*) = 0$ given the proposition’s final result that $V(F^*) = \frac{\gamma G - \alpha F^*}{\rho + \alpha}$. This fact follows from the observation that the constant function $V(F_0) = V(F^*)$ is a local solution to the ODE whenever $V(F^*) \neq \frac{\gamma G - \alpha F^*}{\rho + \alpha}$. Given that $G$ is continuously differentiable and thus locally Lipschitz around $(F^*, V(F^*), 0)$, standard local ODE uniqueness results then imply that there can be no strictly increasing solutions to the ODE near $F^*$. Hence it must be that $V(F^*) = \frac{\gamma G - \alpha F^*}{\rho + \alpha}$.

in Theorem 1. In fact this condition is used only to ensure existence of a solution to the SDE (3), and is sufficient but not necessary. When proving the converse I instead use Theorem 3 to ensure existence of an optimal contract satisfying (3) without any extra conditions.
To establish smoothness of $V$ via this approach, then, it is necessary at minimum to show local existence of a solution to the ODE $V'' = G(F_0, V, V')$ around the singularity $(F_0, V(F_0), V'(F_0)) = (F^*, (r_G - \alpha F^*)/(\rho + \alpha), 0)$ for some $F^* > 0$. As far as I am aware no general existence results apply in this setting.

5 The quantity approach

In this section I pursue the dual approach to Section 4, eliminating $F$ from the firm’s problem and optimizing the resulting objective function to derive an optimal termination policy. The main result of this section shows that the optimal incentive-compatible termination policy solves a particular quickest detection problem, closely analogous to the firm’s problem without an agent. I then develop an optimal stopping approach to explicitly solve the quickest detection problem.

5.1 The firm’s objective function

Recall that by Lemma 6, any $\mathbb{F}^Y$-stopping time $\tau^Y$ is implementable by an IC-B contract, with associated profit-maximizing golden parachute

$$F_t = \mathbb{E}^B_t \left[ \int_t^{\tau^Y} e^{-\rho(s-t)} b \, ds \right].$$

The following proposition proves that when $F$ is eliminated from the firm’s profit function, the resulting optimization problem for $\tau^Y$ can be stated elegantly in terms of maximizing an expected discounted flow of virtual profits.

**Proposition 4.** Let $\tau^Y$ be any $\mathbb{F}^Y$-stopping time and $\Pi(\tau^Y)$ be the supremum of profits achievable by IC-B contracts implementing $\tau^Y$. Then

$$\Pi(\tau^Y) = \mathbb{E} \left[ \int_0^{\tau^Y} e^{-\rho t} \left( \pi_t r_G - (1 - \pi_t) b \right) \, dt \right].$$

(4)

Though the formal derivation of (4) is somewhat involved, the intuition behind it is very simple. Consider any contract with termination policy $\tau^Y$ and golden parachute as specified in Lemma 6. Upon a reported state change the firm pays the agent his expected discounted flow of rents, conditional on the state being bad, from allowing the project to operate until $\tau^Y$. The firm’s expected profits under such a contract are therefore identical
to a counterfactual setting in which its project pays expected returns $-b$ rather than $r_B$ when the state is bad. And conditional on the public information available at time $t$, the probability of being in the good state at that time is $\pi_t$, so instantaneous expected returns at any time are $\pi_t r_G - (1 - \pi_t)b$.

Proposition 4 reduces the contracting problem with an agent to solving a particular virtual quickest detection problem, closely related to the firm’s problem without an agent (cf. equation (1)). In this virtual problem the firm learns about the state of the project as if no agent were available but incurs a flow cost of operating the project in the bad state of $-b$ rather than $r_B$. Importantly, this reduction does not imply that the optimal contract implements the same termination policy as in a quickest detection problem with no agent and $r_B$ replaced by $-b$. First, the firm’s termination policy is $\tau = \tau^0 \wedge \tau^Y$, so the presence of an agent prevents operation of the project in the bad state; this is in stark contrast to the problem without an agent, where both early and late terminations occur. Second, the firm learns about the current state more quickly in the virtual problem, as its signal-to-noise ratio (SNR) is $\frac{\Delta r}{\sigma}$ with an agent, versus $\frac{r_G + b}{\sigma}$ in the quickest detection problem with $r_B$ replaced by $b$.

5.2 Threshold termination policies

Given the form of $\Pi(\tau^Y)$ in Proposition 4 and the fact that posterior beliefs drift downward over time, a natural conjecture for an optimal termination policy is a threshold rule in the firm’s posterior beliefs, i.e. $\tau^\pi \equiv \inf\{t : \pi_t \leq \pi\}$ is an optimal policy for some $\pi \in [0, 1]$. To verify this conjecture, it is necessary to establish that $\pi_t$ is a sufficient statistic for the history of the project when projecting the future path of beliefs; in other words, one must rule out the possibility that the time-$t$ conditional distribution of future beliefs depends on $\{Y_s\}_{s \leq t}$ in a more complicated way than through $\pi_t$. The following lemma accomplishes this task by fully characterizing the dynamics of $\pi$:

**Lemma 9.** $\pi$ satisfies the SDE

$$d\pi_t = -\alpha \pi_t dt + \frac{\Delta r}{\sigma} \pi_t (1 - \pi_t) d\bar{Z}_t$$

with initial condition $\pi_0 = 1$, where $\bar{Z}$ is a $\mathbb{P}$-standard Brownian motion adapted to $\mathbb{F}^Y$ with increments

$$d\bar{Z}_t = \frac{1}{\sigma} (dY_t - (\pi_t r_G + (1 - \pi_t) r_B) dt).$$

The SDE describing the evolution of $\pi_t$ is derivable from Bayes’ rule, and consists of
two contributions. First, the known transition rate $\alpha$ from the good to the bad state leads to a deterministic drop in beliefs over time at rate $\alpha \pi_t$. Second, the firm learns about the state via unexpected deviations in observed incremental output. The quadratic dependence of this learning on $\pi$ is related to the fact that beliefs are most dispersed at $\pi = 0.5$, and hence most subject to revision as new information is received; more extreme beliefs require stronger evidence to change.

The process $\mathcal{Z}$ is known as the *innovation process*, and tracks deviations of the project’s output from time-$t$ expectations. The fact that $\mathcal{Z}$ is a standard Brownian motion implies that the distribution of the future path of beliefs conditional on $\mathcal{F}_t^Y$ depends on the public history only through $\pi_t$. Thus $\pi_t$ is indeed a sufficient statistic for the history of the project.

This insight reduces the firm’s contractual design problem to the choice of a single number $\pi \in [0, 1]$. A natural baseline contract is the break-even termination policy, under which $\pi$ satisfies $\pi r_G - (1 - \pi) b = 0$. This termination policy would be optimal if, at the breakeven point, the firm had to either end the project immediately or commit to continuing it forever. However, the firm retains a real option to terminate the project in the future, which increases the value of continuing past the break-even point in the hopes of receiving good news about the project’s state which raises flow returns. An optimal threshold policy should therefore satisfy $\pi < b/(b + r_G)$. The next subsection determines the optimal threshold by calculating the value of the firm’s real option over time.

### 5.3 Solving the quickest detection problem

Let

$$R_t \equiv \sup_{\tau^Y \geq t} \mathbb{E}_t^Y \left[ \int_t^{\tau^Y} e^{-\rho(s-t)}(\pi_s r_G - (1 - \pi_s) b) \, ds \right]$$

be the value of the firm’s real option from continuing the project at time $t$. Given that $\pi_t$ is a sufficient statistic for the project history, this option value can be written $R_t = \tilde{V}(\pi_t)$ for some function $\tilde{V} : [0, 1] \to \mathbb{R}_+$, which I refer to as the firm’s *virtual value function*. It is natural to expect that $\tilde{V}$ is a weakly increasing function which vanishes for sufficiently small posterior beliefs, at which point the firm’s real option is worthless and an optimal contract is immediately terminated. An optimal threshold is then the largest $\pi$ such that $\tilde{V}(\pi) = 0$.

The following theorem derives a differential equation characterizing $\tilde{V}$ and establishes that the procedure just outlined maximizes $\Pi(\tau^Y)$. 

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Theorem 2. Suppose there exists a \( \pi \in (0, 1) \) and a \( C^2 \) function \( \tilde{V} : [\pi, 1] \to \mathbb{R} \) satisfying

\[
\rho \tilde{V}(\pi_0) = \pi_0 r_G - (1 - \pi_0)b - \alpha \pi_0 \tilde{V}'(\pi_0) + \frac{1}{2} \left( \frac{\Delta r}{\sigma} \right)^2 \pi_0^2 (1 - \pi_0)^2 \tilde{V}''(\pi_0)
\]

on \([\pi, 1]\) and boundary conditions \( \tilde{V}(\pi) = \tilde{V}'(\pi) = 0 \). Then \( \pi \leq \frac{b}{b + r_G}, \tilde{V}(\pi_0) > 0 \) for \( \pi_0 \in (\pi, 1] \), and \( \tilde{V} \) may be extended to a \( C^1 \), piecewise \( C^2 \) function on \([0, 1]\) by setting \( \tilde{V}(\pi_0) = 0 \) for \( \pi_0 \in [0, \pi] \). On this extended domain, \( \tilde{V} \) is the firm’s virtual value function.

Further, let \( \tau^* \equiv \inf\{t : \pi_t \leq \pi\} \). Then \( \tau^* \) maximizes \( \Pi(\tau^Y) \) among all \( \mathbb{R}^Y \)-stopping times \( \tau^Y \), and \( \Pi(\tau^*) = \tilde{V}(1) \).

The differential equation stated in Theorem 2 may be derived heuristically in a fashion analogous to the method outlined in Section 4.2 for obtaining the HJB equation of Theorem 1. The boundary condition \( \tilde{V}(\pi) = 0 \) reflects the fact that at \( \pi \) the project is terminated. It turns out that, given any choice of \( \pi \in (0, 1) \), the boundary condition \( \tilde{V}(\pi) = 0 \) along with the requirement that \( \tilde{V} \) be well-behaved (in particular, finite) at \( \pi_0 = 1 \) select a unique solution \( \tilde{V} \) to the ODE in Theorem 2.\(^6\) Further, \( \tilde{V}(\pi_t) \) gives the time—\( t \) continuation profits for the termination policy \( \tau^\pi = \inf\{t : \pi_t \leq \pi\} \).

The final boundary condition, \( \tilde{V}'(\pi) = 0 \), is a so-called smooth pasting condition pinning down the optimal \( \pi \). If \( \tilde{V}'(\pi) < 0 \), then \( \pi \) is too low, because continuation profits eventually become negative prior to termination. The firm could increase profits by terminating sooner in the region of beliefs where continuation profits are strictly negative. On the other hand, if \( \tilde{V}'(\pi) > 0 \), then the threshold is too high. The reasoning here is more subtle due to the kink in the value function at \( \pi \), but it can still be understood heuristically. Consider a firm at the threshold belief who instead of terminating immediately continues an instant \( dt \) further and then reverts to terminating the next time beliefs fall below \( \pi \). The firm’s expected gains from such a deviation are, to first order,

\[
\Delta \Pi \simeq (\pi r_G - (1 - \pi)b) \ dt + \mathbb{E} \left[ \tilde{V} \left( \pi - \alpha \pi \ dt + \frac{\Delta r}{\sigma} \pi (1 - \pi) d\mathbb{Z}_t \right) \right].
\]

Now, assume as a simplification that \( d\mathbb{Z}_t \) is a binary random variable taking values \( \pm \sqrt{dt} \) with equal probability (i.e. having the same mean and variance as a Brownian increment

\(^6\) Technically speaking, the requirement that \( \tilde{V} \) be \( C^2 \) at the right boundary is a transversality condition closely analogous to the growth conditions common in macroeconomics and growth theory. It is imposed for the same reason that the growth condition is required to rule out spurious explosive solutions and select a unique growth path for the economy in those models.
over the same time interval). Then for sufficiently small $dt$

$$\mathbb{E}\left[ \tilde{V} \left( \pi - \alpha \pi dt + \frac{\Delta r}{\sigma} \pi (1 - \pi) dZ_t \right) \right] = \frac{1}{2} \tilde{V} \left( \pi - \alpha \pi dt + \frac{\Delta r}{\sigma} \pi (1 - \pi) \sqrt{dt} \right),$$

or to first order in $\sqrt{dt}$,

$$\mathbb{E}\left[ \tilde{V} \left( \pi - \alpha \pi dt + \frac{\Delta r}{\sigma} \pi (1 - \pi) dZ_t \right) \right] \approx \frac{1}{2} \tilde{V}'(\pi^+) \frac{\Delta r}{\sigma} \pi (1 - \pi) \sqrt{dt}.$$

Thus any first-order gains or losses from the flow of profits over the (sufficiently short) interval $dt$ are dwarfed by the lower-order expected gain in continuation profits. $\tilde{V}'(\pi) = 0$ is therefore a necessary condition for optimality of the threshold $\pi$. Theorem 2 ensures that smooth pasting is also sufficient for optimality.

The next lemma establishes that the function $\tilde{V}$ stipulated in Theorem 2 actually exists by explicitly exhibiting one. The lemma makes use of Tricomi’s confluent hypergeometric function $U(m, n, z)$, which is a solution to Kummer’s differential equation defined and may be written in closed form as

$$U(m, n, z) = \frac{1}{\Gamma(m)} \int_0^\infty e^{-zt} t^{m-1} (1 + t)^{n-m-1} dt$$

when $m, z > 0$.

**Lemma 10.** Let $k \equiv \frac{\Delta r}{\sigma}$ and $\beta \equiv \frac{k^2 + 2\alpha + \sqrt{(k^2 + 2\alpha)^2 + 8k^2 \rho}}{2k^2}$. Then $\beta > 1$ and there exist constants $C > 0$ and $\pi \in (0, 1)$ such that

$$\tilde{V}(\pi_0) = \frac{r_G + b}{\rho + \alpha \pi_0} - \frac{b}{\rho} + C \pi_0^\beta (1 - \pi_0)^{1-\beta} U \left( \beta - 1, 2\beta - \frac{2\alpha}{k^2}, \frac{2\alpha}{k^2}, \frac{\pi_0}{1 - \pi_0} \right)$$

is a $C^2$ function on $[\pi, 1)$ satisfying

$$\rho \tilde{V}(\pi_0) = \pi_0 r_G - (1 - \pi_0) b - \alpha \pi_0 \tilde{V}'(\pi_0) + \frac{1}{2} \left( \frac{\Delta r}{\sigma} \right)^2 \pi_0^2 (1 - \pi_0)^2 \tilde{V}''(\pi_0)$$

and $\tilde{V}(\pi) = \tilde{V}'(\pi) = 0$. Further, $\tilde{V}$ may be extended to a $C^2$ function on $[\pi, 1]$ satisfying the ODE on the extended domain.

This lemma relies on the fortuitous fact that the ODE of Theorem 2 is solvable in closed form in terms of known special functions. As the ODE is second-order, the general solution contains two linearly independent components; however, one of them diverges as $\pi_0 \to 1$,
and so must be discarded. Most of the work of the lemma then goes toward establishing existence of constants $C$ and $\pi$ satisfying the remaining boundary conditions.

The final theorem of this section ties the previous results together and verifies that the stopping time maximizing $\Pi(\tau^Y)$ is actually incentive-compatible.

**Theorem 3.** There exists a $\pi \in (0, b/(b + r_G)]$ such that $\tau^* \equiv \inf\{t : \pi_t \leq \pi\}$ maximizes

$$\mathbb{E}\left[\int_0^{\tau^Y} e^{-\rho t}(\pi_t r_G - (1 - \pi_t)b) \, dt\right]$$

among all $\mathbb{F}^Y$-stopping times $\tau^Y$. Letting $F^*$ be the associated golden parachute defined in Lemma 6, $(F^*, \tau^*)$ is an optimal contract.

To see why $(F^*, \tau^*)$ is incentive compatible, note first that by the martingale representation theorem $F^*_t = \mathbb{E}_t^B\left[\int_t^{\tau^Y} e^{-\rho(s-t)} b \, ds\right]$ evolves as

$$dF^*_t = (\rho F^*_t - b) \, dt + \beta_t \, dZ^B_t = \left(\rho F^*_t - b + \frac{\Delta r}{\sigma} \beta_t\right) \, dt + \beta_t \, dZ^G_t$$

for some $\mathbb{F}^Y$-adapted, progressively measurable process $\beta$. Further, given that $\tau^*$ is a threshold policy in $\pi_t$, time to termination and the size of the golden parachute are increasing in $\pi_t$. The arrival of good news about the state must therefore increase $F^*$, in which case $\beta \geq 0$. This contract is then a member of a class shown to be incentive compatible in the discussion following Theorem 1.

### 5.4 More on existence of solutions to the HJB equation

The results of Section 5.3 prove affirmatively that the firm’s virtual value function $\tilde{V}$ is a $C^2$ function. It is natural to wonder whether this result can be leveraged to establish smoothness of the firm’s value function $V$ in the dual problem, as required by Proposition 2. This subsection explores that possibility. As with Section 4.4, this subsection can be safely skipped by a reader uninterested in technical existence details.

Let $(F^*, \tau^*)$ be defined as in Theorem 3. Because $\pi_t$ is a sufficient statistic for the state of the project, one can write $F^*_t = f(\pi_t)$, where $f : [\pi, 1] \to \mathbb{R}_+$ is a function defined by

$$f(x) \equiv \mathbb{E}^B\left[\int_0^{\tau^*_x} e^{-\rho t} b \, dt\right].$$

Here $\tau^*_x$ is an $\mathbb{F}^Y$-stopping time defined by $\tau^*_x(x) \equiv \inf\{t : p_t^x \leq \pi\}$ for the stochastic process
$p^x$ satisfying $p^x_0 = x$ and

$$dp^x_t = -\alpha p^x_t \, dt + \frac{\Delta r}{\sigma} p^x_t (1 - p^x_t) \left( dY_t - (p^x_t r_G + (1 - p^x_t) r_B) \, dt \right).$$

Informally, $\tau^*_x$ gives the total remaining project lifetime when the current posterior belief is $x \in [0,1]$.

**Lemma 11.** $f$ is a continuous, strictly increasing function on $[\pi,1]$ satisfying $f(\pi) = 0$ and $f(1) = F^*_0$.

Using $f$ it is straightforward to relate $V$, the firm’s value function when designing $F$, and $\tilde{V}$, the firm’s virtual value function when designing $\tau^Y$:

**Lemma 12.**

$$\tilde{V}(x) = xV(f(x)) - (1 - x)f(x)$$

for all $x \in [\pi,1]$.

Essentially, when the firm’s posterior belief is $x$, then either the state is good; in which case the continuation value of the contract is $V(f(x))$; or the state is bad, in which case the continuation value of the contract is the termination fee paid to the agent at that point. An immediate corollary of this theorem is that $V(\cdot)$ is a continuous function, as $f^{-1}$ exists and is continuous by the inverse function theorem.

If it could be established that $f$ were $C^2$, appropriate smoothness of $V$ could be directly verified by differentiating (5) twice and using the fact that $f'$ is nonzero to establish existence of the first and second derivatives of $f^{-1}$. The closest approach to proving such a conjecture is the following result.

**Proposition 5.** Suppose that $g : [\pi,1] \rightarrow \mathbb{R}$ is a $C^2$ function satisfying

$$\rho g(x) = b - \left( \alpha x + \left( \frac{\Delta r}{\sigma} \right)^2 x^2 (1 - x) \right) g'(x) + \frac{1}{2} \left( \frac{\Delta r}{\sigma} \right)^2 x^2 (1 - x)^2 g''(x)$$

on $[\pi,1]$ and the boundary condition $g(\pi) = 0$. Then $g = f$.

Standard ODE existence results guarantee a unique $C^2$ solution to the ODE on $[\pi,1]$ for any choice of $g'(\pi)$. However, the ODE is singular at $x = 1$ due to the vanishing coefficient on $g''$ there, and so it is not guaranteed that any solution is well-behaved, in particular finite, as $x \to 1$. 

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6 Conclusion

I have studied the problem of a firm seeking to dynamically elicit the information of a rent-seeking agent about when to terminate a project with a limited lifespan. I show that the firm’s problem can be analyzed through the lens of classic monopoly theory, as the firm faces a tight price-quantity tradeoff between payments to the agent and output from the project. I find that the firm’s optimal contract can be derived by eliminating either price or quantity from the objective function and optimizing with respect to the remaining variable alone. While the price approach is equivalent to a standard recursive analysis using continuation utility as a state variable, the quantity approach provides a much sharper characterization of the firm’s optimal contract.

In an optimal contract, the firm tracks its posterior beliefs about the project’s state as if no agent were present even though the agent truthfully reports whether the state has switched. It optimally terminates the project the first time either the agent reports the state has changed or its “virtual beliefs” fall below a threshold. This threshold solves a modified version of the unassisted optimal stopping problem, from which I obtain simple and intuitive comparative statics. The threshold is strictly lower than the optimal threshold in the firm’s unassisted optimal stopping problem, and is decreasing in the informativeness of news about the state, increasing in the rate of state transitions, and increasing in the severity of the agent’s incentive misalignment. The benefits of hiring an agent can therefore be succinctly summarized: the agent’s information completely eliminates late termination and partially mitigates early termination universally across all projects and output histories. The optimal payment rule can also be stated briefly: when the agent is terminated for reporting a state change, he receives a lump sum golden parachute equal to his expected lifetime rents from withholding his report forever.

While I have assumed that the firm’s project produces a Brownian output flow, the major conclusions of the paper all hold under alternative Poisson technologies involving either steady output flows and occasional costly “breakdowns” or steady expenditures and occasional profitable “breakthroughs.” Another natural generalization of the model is to agents with imperfect knowledge of the state. This change creates an additional incentive for the agent to delay reporting bad news inducing project termination, as there is a chance his beliefs about the state will improve in the future and he will not have to report the bad news after all. In this setting the agent must be additionally compensated for the value of his real option to wait and see.

Another generalization is to projects with many states. In this setting the agent expects
the project to continue deteriorating over time after a termination threshold is reached, so
that delaying a report produces a growing belief asymmetry which can be profitably exploited
by the agent. (By contrast, with only two states the belief asymmetry is constant over time,
and equal to its initial value at the time of the state switch.) The agent must therefore be
additionally compensated for the wedge between the marginal belief asymmetry at the time
of a state switch and his average belief asymmetry over the lifetime of the contract.

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Appendices

A Proofs of theorems

A.1 Proof of Theorem 1

Fix $F^*$ and $V$ as described in the theorem statement. Lemma 23 in the technical appendix establishes that $V$ is a positive, strictly increasing, strictly concave function on $[0, F^*)$, and that maximizers $\gamma^*, \beta^*$ of the HJB equation exist on $[0, F^*)$ and are continuous except possibly at $F^*$. And by Proposition 3, $V(F^*) = \frac{rG - \alpha F^*}{\rho + \alpha}$ and $F^* < b/\rho$. Finally, as this proposition also implies $V''(F^*) < 0$, the maximizers $(\gamma^*, \beta^*)$ are continuous on $[0, F^*)$.

Extend $V$ to $\mathbb{R}_+$ by setting $V(F_0) = V(F^*)$ for $F_0 > F^*$. Then $V$ is a $C^1$, piecewise $C^2$ function on this extended domain and satisfies

$$(\rho + \alpha)V(F_0) \geq \sup_{(\gamma, \beta) \in \text{IC}(F)} \left\{ rG - \alpha F + \gamma V'(F_0) + \frac{1}{2} \beta^2 V''(F_0) \right\}$$

everywhere, with equality for $F_0 \in [0, F^*)$.

I now carry out a verification argument for IC-B contracts $(\tilde{F}, \tilde{\tau})$ such that $\tilde{F}$ is cadlag and bounded above by $b/\rho$ and $\tilde{\tau} = \inf \{ t : \tilde{F}_t = 0 \}$. Lemma 5 justifies the boundedness restriction and form of the termination policy, while Lemma 22 in the technical appendix justifies the focus on cadlag golden parachutes. Verification within this class therefore suffices to prove that $V$ is an upper bound on the lifetime value of any IC-B contract.

Fix an IC-B contract $(\tilde{F}, \tilde{\tau})$ satisfying the conditions just described. Wlog assume that $\tilde{F} = F^\tilde{\tau}$, as the stopped version of any contract remains IC-B and yields the same profits for the firm in all cases. Define the process $M$ by

$$M_t = \frac{b}{\rho}(1 - e^{-\rho t}) + e^{-\rho t}F_t.$$

By Lemma 21 in the technical appendix, $M$ is a $\mathbb{P}^B$-supermartingale, and as $F$ is cadlag and bounded above by $b/\rho$, so is $M$. $M$ then satisfies the conditions of the Doob-Meyer decomposition theorem (see Theorem 1.4.10 of Karatzas and Shreve (1991) for a statement), hence

$$M = \tilde{F}_0 + X - A$$

for some $\mathbb{F}^Y$-adapted process $A$ which is increasing, right-continuous, and starts at zero; and some $\mathbb{F}^Y$-adapted process $X$ which is a cadlag $\mathbb{P}^B$-martingale starting at zero.
Further, by the martingale representation theorem there exists an \( \mathbb{F}^Y \)-adapted, progressively measurable process \( \beta^M \), satisfying \( \mathbb{P} \left\{ \int_0^t (\beta^M)^2_s \, ds < \infty \right\} = 1 \) for all \( t \), such that

\[
X_t = \int_0^t \beta^M_s \, dZ^B_s
\]

for all \( t \). And as \( Z^B_t = Z^G_t + \Delta \sigma r_t \), \( X \) may be equivalently decomposed as

\[
X_t = \int_0^t \frac{\Delta r}{\sigma} \beta^M_s \, ds + \int_0^t \beta^M_s \, dZ^G_s.
\]

Now write \( \tilde{F}_t \) in terms of \( M_t \) as

\[
\tilde{F}_t = e^{\rho t} M_t - \frac{b}{\rho} (e^{\rho t} - 1),
\]

and use Ito’s lemma to obtain

\[
\tilde{F}_t = \tilde{F}_0 + \int_0^t e^{\rho s} \left( \rho M_s - b + \frac{\Delta r}{\sigma} \beta^M_s \right) \, ds + \int_0^t e^{\rho s} \, dA_s + \int_0^t e^{\rho s} \beta^M_s \, dZ^G_s,
\]

or equivalently,

\[
\tilde{F}_t = \tilde{F}_0 + \int_0^t \left( \rho \tilde{F}_s - b + \frac{\Delta r}{\sigma} \tilde{\beta}_s \right) \, dt - \int_0^t e^{\rho s} \, dA_s + \int_0^t \tilde{\beta}_s \, dZ^G_s,
\]

where \( \tilde{\beta}_t \equiv e^{\rho t} \beta^M_t \). Let \( \tilde{\Delta} \tilde{A}_t \equiv \tilde{A}_t - \lim_{s \uparrow t} \tilde{A}_s \) track the jumps of \( \tilde{A} \), of which there are at most countably many pathwise. I will let \( D_t \equiv \tilde{A}_t - \sum_{0 \leq s \leq t} \Delta \tilde{A}_t \) denote the continuous part of \( A \).

In general it is not assured that \( \tilde{\beta} \) is sufficiently regular to guarantee that \( \int \tilde{\beta} \, dZ^G \) is a \( \mathbb{P}^G \)-martingale. I therefore perform the verification procedure using a localized version of the process. For each \( N = 1, 2, \ldots \), define the \( \mathbb{F}^Y \)-stopping time \( \tau_{t,N} \equiv \inf \left\{ t : \left| \int_0^t \tilde{\beta}_s \, dZ^G_s \right| \geq N \right\} \). For each \( N \), the Ito isometry implies that \( \mathbb{E}^G \left[ \int_0^{t \wedge \tau_{t,N}} \tilde{\beta}_s^2 \, ds \right] \leq N^2 \), and so \( \int_0^{t \wedge \tau_{t,N}} \tilde{\beta}_s \, dZ^G_s \) is a \( \mathbb{P}^G \)-martingale.
Now fix a time $t$. Ito’s lemma says that

\[
e^{-\left(\rho + \alpha\right)(t \land \tilde{\tau} \land \tau^l,N)} V(\tilde{F}_{t \land \tilde{\tau} \land \tau^l,N}) = V(\tilde{F}_0) + \sum_{0 \leq s \leq t \land \tilde{\tau} \land \tau^l,N} e^{-\left(\rho + \alpha\right)s} \Delta V_s
+ \int_0^{t \land \tilde{\tau} \land \tau^l,N} e^{-\left(\rho + \alpha\right)s} \left( -(\rho + \alpha) V(\tilde{F}_s) + \frac{1}{2} \tilde{\beta}^2_s V''(\tilde{F}_s) \right) \, ds
+ \int_0^{t \land \tilde{\tau} \land \tau^l,N} e^{-\left(\rho + \alpha\right)s} V'(\tilde{F}_s) \left( \rho \tilde{F}_t - b + \frac{\Delta r}{\sigma} \tilde{\beta}_s \right) \, ds - dD_s
+ \int_0^{t \land \tilde{\tau} \land \tau^l,N} e^{-\left(\rho + \alpha\right)s} \tilde{\beta}_s V''(\tilde{F}_s) \, dZ^G_s,
\]

where $\Delta V_t \equiv V(\tilde{F}_t) - \lim_{s \uparrow t} V(\tilde{F}_s)$. Note that $\Delta V \leq 0$ given that $V$ is monotone increasing and all jumps in $\tilde{F}$ are downward. As $V'$ is bounded, the final term is a martingale. Taking expectations wrt $\mathbb{E}^G$ leaves

\[
V(\tilde{F}_0) = \mathbb{E}^G \left[ e^{-\left(\rho + \alpha\right)(t \land \tilde{\tau} \land \tau^l,N)} V(\tilde{F}_{t \land \tilde{\tau} \land \tau^l,N}) \right] - \mathbb{E}^G \left[ \sum_{0 \leq s \leq t \land \tilde{\tau} \land \tau^l,N} e^{-\left(\rho + \alpha\right)s} \Delta V_s \right]
- \mathbb{E}^G \left[ \int_0^{t \land \tilde{\tau} \land \tau^l,N} e^{-\left(\rho + \alpha\right)s} \left( -(\rho + \alpha) V(\tilde{F}_s) + \frac{1}{2} \tilde{\beta}^2_s V''(\tilde{F}_s) \right) \, ds \right]
- \mathbb{E}^G \left[ \int_0^{t \land \tilde{\tau} \land \tau^l,N} e^{-\left(\rho + \alpha\right)s} V'(\tilde{F}_s) \left( \rho \tilde{F}_t - b + \frac{\Delta r}{\sigma} \tilde{\beta}_s \right) \, ds - dD_s \right].
\]

As $\Delta V \leq 0$, $D$ is monotone increasing, and $V' \geq 0$, this expression implies the inequality

\[
V(\tilde{F}_0) \geq \mathbb{E}^G \left[ e^{-\left(\rho + \alpha\right)(t \land \tilde{\tau} \land \tau^l,N)} V(\tilde{F}_{t \land \tilde{\tau} \land \tau^l,N}) \right]
- \mathbb{E}^G \left[ \int_0^{t \land \tilde{\tau} \land \tau^l,N} e^{-\left(\rho + \alpha\right)s} \left( -(\rho + \alpha) V(\tilde{F}_s) + \frac{1}{2} \tilde{\beta}^2_s V''(\tilde{F}_s) \right) \, ds \right]
- \mathbb{E}^G \left[ \int_0^{t \land \tilde{\tau} \land \tau^l,N} e^{-\left(\rho + \alpha\right)s} V'(\tilde{F}_s) \left( \rho \tilde{F}_t - b + \frac{\Delta r}{\sigma} \tilde{\beta}_s \right) \, ds \right].
\]

Letting $\tilde{\gamma}$ on $[0, t]$ via $\tilde{\gamma}_s = \frac{\Delta r}{\sigma} \tilde{\beta}_s - (b - \rho \tilde{F}_s)$, the previous inequality may be written

\[
V(\tilde{F}_0) \geq \mathbb{E}^G \left[ e^{-\left(\rho + \alpha\right)(t \land \tilde{\tau} \land \tau^l,N)} V(\tilde{F}_{t \land \tilde{\tau} \land \tau^l,N}) \right]
- \mathbb{E}^G \left[ \int_0^{t \land \tilde{\tau} \land \tau^l,N} e^{-\left(\rho + \alpha\right)s} \left( -(\rho + \alpha) V(\tilde{F}_s) + \tilde{\gamma}_s V'(\tilde{F}_s) + \frac{1}{2} \tilde{\beta}^2_s V''(\tilde{F}_s) \right) \, ds \right].
\]

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Note that \((\tilde{\gamma}_s, \tilde{\beta}_s) \in \text{IC}(\tilde{F}_t)\) for all \(s \in [0, t]\), so the extension of the HJB equation stated at the beginning of the proof implies

\[
V(\tilde{F}_0) \geq \mathbb{E}^G \left[ e^{-(\rho + \alpha)(t \wedge \tilde{\tau} \wedge \tau_{L,N})} V(\tilde{F}_{t \wedge \tilde{\tau} \wedge \tau_{L,N}}) + \int_0^{t \wedge \tilde{\tau} \wedge \tau_{L,N}} e^{-(\rho + \alpha)s} \left( r_G - \alpha \tilde{F}_s \right) ds \right].
\]

Finally, take \(N \to \infty\) and \(t \to \infty\). The interior of the expectation is uniformly bounded over all \(t\) and \(N\), so the bounded converge theorem yields

\[
V(\tilde{F}_0) \geq \mathbb{E}^G \left[ e^{-(\rho + \alpha)\tilde{\tau}} V(\tilde{F}_{\tilde{\tau}}) + \int_0^{\tilde{\tau}} e^{-(\rho + \alpha)s} \left( r_G - \alpha \tilde{F}_s \right) ds \right].
\]

As \(\tilde{F}_{\tilde{\tau}} = 0\) and \(V(0) = 0\), this reduces to

\[
V(\tilde{F}_0) \geq \mathbb{E}^G \left[ \int_0^{\tilde{\tau}} e^{-(\rho + \alpha)s} \left( r_G - \alpha \tilde{F}_s \right) ds \right].
\]

By Lemma 7, the rhs is the expected profit of the contract \(\tilde{F}\). Therefore \(V(\tilde{F}_0)\) is an upper bound on the expected profits of \(\tilde{F}\). As \(F^*\) is a global maximum of \(V\), it follows that \(V(F^*)\) is as well.

The final step of verification is establishing that the optimal controls to the HJB equation trace out a contract whose profits are equal to \(V(F^*)\). Extend \(\gamma^*\) and \(\beta^*\) continuously to all \(F_0\) by setting \(\gamma^*(F_0) = \gamma^*(0)\) and \(\beta^*(F_0) = \beta^*(0)\) for all \(F_0 < 0\), and similarly \(\gamma^*(F_0) = \gamma^*(F^*)\) and \(\beta^*(F_0) = 0\) for all \(F_0 > F^*\).

**Lemma 13.** There exists a weak solution to (3) for all time. If \(\lim_{F_0 \to F^*} V'(F_0)V'''(F_0)\) exists and is finite, then there exists a unique strong solution \(F\) for all time. Any strong solution \(F\) satisfies \(F \leq F^*\) a.s.

**Proof.** Because \(\gamma^*\) and \(\beta^*\) are continuous and bounded, Theorem 5.4.22 of Karatzas and Shreve (1991) ensures existence of a weak solution. To show existence of a unique strong solution, I establish under the additional condition in the lemma statement that \(\gamma^*\) and \(\beta^*\) are both globally Lipschitz continuous, from which Theorem 5.2.9 and 5.2.5 of Karatzas and Shreve (1991) ensure strong existence and uniqueness, respectively.

It is sufficient to show that \(\beta^*\) is differentiable on \([0, F^*]\) with bounded derivative. For then \(\gamma^*\) is differentiable with bounded derivative as well, and the mean value theorem implies that both are Lipschitz continuous on \([0, F^*]\). Lipschitz continuity on the extended domain follows trivially given that \(\gamma^*\) and \(\beta^*\) are constant outside of \([0, F^*]\).
Recall from the proof of Proposition 3 that for $F_0 \in [0, F^*)$, $V$ satisfies 

$$V''(F_0) = -\left(\frac{\Delta r}{\sigma}\right)^2 \frac{V'(F_0)^2}{\Gamma(F_0)},$$

where $\Gamma(F_0) \equiv (\rho + \alpha)V(F_0) - (r_G - \alpha F_0) + (b - \rho F_0)V'(F_0) > 0$. As both $V'$ and $\Gamma$ are continuously differentiable, $V'''(F_0)$ exists and is a continuous function on $[0, F^*)$. Then the derivative of $\beta^*$ exists, is continuous, and is given by

$$\frac{d\beta^*}{dF_0}(F_0) = \frac{\Delta r}{\sigma} \left(-1 + \frac{V'(F_0)}{V''(F_0)^2} V'''(F_0)\right)$$

for $F_0 < F^*$. Recall that by Proposition $V''(F^*) < 0$. Then under the assumption of the lemma statement, the limit of this expression exists and is finite as $F_0 \to F^*$. Finally, the mean value theorem implies that $\frac{d\beta^*}{dF_0}(F^*) = \lim_{F_0 \to F^*} \frac{d\beta^*}{dF_0}(F_0)$. Thus $\beta^*$ is continuously differentiable on the compact interval $[0, F^*)$, meaning it has a bounded derivative on that domain.

Now, fix any strong solution $F$. Suppose that on some $\omega \in \Omega$, $F$ satisfies $F_t(\omega) > F^*$ for some $t > 0$. Let $t' = \sup\{s < t : F_s(\omega) = F^*\}$. Then a.s. $F_t(\omega)$ satisfies the ODE

$$\frac{df}{ds} = \gamma^*(F^*)$$

on $[t', t]$ with boundary conditions $f(t') = F^*$ and $f(t) = F_t(\omega) > F^*$. This is a contradiction of the fact that this ODE has a unique global solution $f$ satisfying $f(s) > f(t)$ for all $s < t$ given $\gamma^*(F^*) < 0$. Thus such paths can occur with at most probability zero. In other words, $F \leq F^*$ a.s. \hfill $\Box$

Assume a strong solution $F$ to (3) exists for all time, and let $\tau^Y = \inf\{t : F_t = 0\}$. I show that the expected profits of $F$ are precisely $V(F^*)$. Because $\beta^*$ is bounded, $\int_0^t \beta^*(F_s)V'(F_s) \, dZ_s^G$ is a martingale for all time, so no regularization is needed for this step of verification. Fix $t$ and use Ito’s lemma to write

$$e^{-(\rho + \alpha)(t \wedge \tau^Y)} V(F_{t \wedge \tau^Y})$$

$$= V(F^*) + \int_0^{t \wedge \tau^Y} e^{-(\rho + \alpha)s} \left(-(\rho + \alpha)V(F_s) + \gamma^*(F_s)V'(F_s) + \frac{1}{2} \beta^*(F_s)^2 V''(F_s)\right) \, ds$$

$$+ \int_0^{t \wedge \tau^Y} e^{-(\rho + \alpha)s} \beta^*(F_s)V'(F_s) \, dZ_s^G.$$
Take expectations wrt $\mathbb{P}^G$ to eliminate the martingale term, leaving
\[
\mathbb{E}^G[e^{-(\rho+\alpha)(t\wedge \tau^Y)}V(F_{t\wedge \tau^Y})] = V(F^*) + \mathbb{E}^G \left[ \int_0^{t\wedge \tau^Y} e^{-(\rho+\alpha)s} \left( -(\rho + \alpha)V(F_s) + \gamma^* F_s \nu'(F_s) + \frac{1}{2} \beta^* F_s^2 \nu''(F_s) \right) ds \right].
\]

Because $F_t \in [0, F^*]$ for $t \leq \tau^Y$ and $(\gamma^*(F_0), \beta^*(F_0))$ maximize the rhs of the HJB equation for $F_0 \in [0, F^*]$, the final term is equal to $-\mathbb{E}^G \left[ \int_0^{t\wedge \tau^Y} e^{-(\rho+\alpha)s}(r_G - \alpha F_s) ds \right]$, or after rearrangement
\[
V(F^*) = \mathbb{E}^G \left[ \int_0^{\tau^Y} e^{-(\rho+\alpha)s}(r_G - \alpha F_s) ds \right] + \mathbb{E}^G[e^{-(\rho+\alpha)(t\wedge \tau^Y)}V(F_{t\wedge \tau^Y})].
\]

Now take $t \to \infty$. The interior of each expectation is uniformly bounded for all time, so the bounded convergence theorem allows limits and expectations to be swapped. As $V(F_{\tau^Y}) = 0$, this leaves
\[
V(F^*) = \mathbb{E}^G \left[ \int_0^{\tau^Y} e^{-(\rho+\alpha)s}(r_G - \alpha F_s) ds \right].
\]

By Lemma 7, the rhs are the expected profits of $F$, which was the desired result.

Finally, note that max\{F, 0\} is an Ito process for $t \leq \tau^Y$ with progressively measurable increments $\gamma, \beta$, where $\gamma_t = \gamma^*(F_t)$ and $\beta_t = \beta^*(F_t) \geq 0$ satisfy $b - \rho F_t + \gamma_t - \frac{\Delta r}{\sigma} \beta_t = 0$. Then Lemma 20 in the technical appendix implies that (max\{F, 0\}, $\tau^Y$) is an IC contract.

### A.2 Proof of Theorem 2

Let $\tilde{V}$ be as stated in the theorem, with $\tau^* \equiv \inf\{t : \pi_t \leq \pi \}$ and $\tau^x \equiv \inf\{t : \pi_t \leq x \}$ for any $x \geq \pi$. Then by Ito’s lemma, for all $t$
\[
e^{-\rho(\tau^*\wedge t-\tau^*)}\tilde{V}(\pi_{\tau^*\wedge t}) = \tilde{V}(x) + \int_{\tau^x}^{\tau^*\wedge t} e^{-\rho(s-\tau^*)} \left( -\rho \tilde{V}(\pi_s) - \alpha \pi_s \tilde{V}'(\pi_s) + \frac{1}{2} \left( \frac{\Delta r}{\sigma} \right)^2 \tilde{V}''(\pi_s) \right) ds \\
+ \int_{\tau^x}^{\tau^*\wedge t} e^{-\rho(s-\tau^*)} \tilde{V}'(\pi_s) dZ_s.
\]

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Now, using the fact that \( \tilde{V} \) satisfies the ODE in the theorem statement on \( s \leq \tau^* \), the previous expression reduces to

\[
e^{-\rho(\tau^* \land t - \tau^*)} \tilde{V}(\pi_{\tau^* \land t}) = \tilde{V}(x) - \int_{\tau^*}^{\tau^* \land t} e^{-\rho(s - \tau^*)} (\pi_s r_G - (1 - \pi_s)b) \, ds + \int_{\tau^*}^{\tau^* \land t} e^{-\rho(s - \tau^*)} \tilde{V}'(\pi_s) \, d\mathbb{Z}.
\]

Because \( \tilde{V}' \) is continuous on a compact domain and therefore bounded, the final term is a martingale. Further, by the strong Markov property of stochastic integrals wrt Brownian motions, its conditional expectation wrt \( \mathcal{F}_{\tau^*} \) vanishes. Taking conditional expectations then yields

\[
\mathbb{E}_{\tau^*}[e^{-\rho(\tau^* \land t - \tau^*)} \tilde{V}(\pi_{\tau^* \land t})] = \tilde{V}(x) - \mathbb{E}_{\tau^*} \left[ \int_{\tau^*}^{\tau^* \land t} e^{-\rho(s - \tau^*)} (\pi_s r_G - (1 - \pi_s)b) \, ds \right].
\]

Now take \( t \to \infty \). As the terms in both expectations are bounded, limits and expectations may be swapped, yielding

\[
\tilde{V}(x) = \mathbb{E}_{\tau^*} \left[ \int_{\tau^*}^{\tau^*} e^{-\rho(s - \tau^*)} (\pi_s r_G - (1 - \pi_s)b) \, ds \right]
\]

given the boundary condition \( \tilde{V}(\pi) = 0 \). In particular, \( \tilde{V}(1) = \Pi(\tau^*) \).

Now, suppose \( \pi > b/(b + r_G) \). Then by the expression just derived \( \tilde{V}(x) > 0 \) for all \( x > \pi \), as the flow benefits of operation are always strictly positive on \( s \leq \tau^x \) and \( \tau^x < \tau^* \). But on the other hand, by the ODE in the theorem statement it follows that

\[
\frac{1}{2} \left( \frac{\Delta r}{\sigma} \right)^2 \pi^2 (1 - \pi)^2 \tilde{V}''(\pi) = -(\pi r_G - (1 - \pi)b) < 0,
\]

meaning \( \tilde{V}''(\pi) < 0 \) and implying that for \( x \) sufficiently close to \( \pi \), \( \tilde{V}(x) < 0 \). Thus \( \pi \leq b/(b + r_G) \).

Next I claim that \( \tilde{V} \) is strictly increasing on \( [\pi, b/(b + r_G)] \). This claim is trivial if \( \pi = b/(b + r_G) \), so assume \( \pi < b/(b + r_G) \) for the following argument. By a similar argument to the previous paragraph, \( \tilde{V}''(\pi) > 0 \); then given the boundary conditions at \( \pi \), \( \tilde{V}'(x) > 0 \) for \( x \) sufficiently close to \( \pi \). Suppose \( \tilde{V}' \) vanishes somewhere on \( [\pi, b/(b + r_G)] \), and let \( x^* = \inf\{x > \pi : \tilde{V}'(x) = 0\} \). Continuity ensures \( \tilde{V}'(x^*) = 0 \), and by assumption \( x^* < b/(b + r_G) \). As \( \tilde{V}'(x) > 0 \) for all \( x \in (\pi, x^*) \), and as \( \tilde{V}(\pi) = 0 \), it must be that \( \tilde{V}(x^*) > 0 \). But then by
the ODE $\tilde{V}''(x^*) > 0$, implying that $\tilde{V}'(x) < 0$ for $x$ below and sufficiently close to $x^*$. This contradicts the definition of $x^*$ as the minimal point at which $\tilde{V}'$ vanishes. Thus no such point can exist, i.e. $\tilde{V}' > 0$ on $[\pi, b/(b + r_G))$, proving the claim.

Now fix $x > b/(b + r_G)$. Using the expression for $\tilde{V}(x)$ derived earlier and the law of iterated expectations, one can show that

$$\tilde{V}(x) = E_{\pi} \left[ \int_{\pi}^{x/(b + r_G)} e^{-\rho(s-\tau^*)} (\pi_s r_G - (1 - \pi_s)b) \, ds \right] + E_{\pi} \left[ e^{-\rho(x/(b + r_G)-\tau^*)} \tilde{V}(b/(b + r_G)) \right].$$

Then as the first term is strictly positive while the second is non-negative, I conclude that $\tilde{V}(x) > 0$ for all $x > b/(b + r_G)$, and hence $\tilde{V}(x) > 0$ for all $x > \pi$.

I have now established enough properties of $\tilde{V}$ to verify the optimality of $\tau^*$. Extend $\tilde{V}$ to $[0,1]$ by setting $\tilde{V}(x) = 0$ for $x < \pi$. Note that on this extended domain $\tilde{V}$ is $C^1$ and piecewise $C^2$, so Ito’s lemma still applies. Further, given $\pi \leq b/(b + r_G)$, $\tilde{V}$ satisfies

$$\rho \tilde{V}(\pi_0) \geq \pi_0 r_G - (1 - \pi_0)b - \alpha \pi_0 \tilde{V}'(\pi_0) + \frac{1}{2} \left( \frac{\Delta r}{\sigma} \right)^2 \pi_0^2 (1 - \pi_0)^2 \tilde{V}''(\pi_0)$$

on $[0,1] \setminus \{b/(b + r_G)\}$, with equality iff $\pi_0 = \pi$. ($\tilde{V}''$ is potentially discontinuous at $\pi_0 = \pi$, so I do not specify an inequality holding at that point. This will not matter for what follows.) Fix any $\mathbb{P}^Y$-stopping time $\tau^Y$ (not necessarily a threshold policy), and use Ito’s lemma as before to write

$$e^{-\rho(\tau^Y \wedge t)} \tilde{V}(\pi_{\tau^Y \wedge t}) = \tilde{V}(1) + \int_0^{\tau^Y \wedge t} e^{-\rho s} \left( -\rho \tilde{V}(\pi_s) - \alpha \pi_s \tilde{V}'(\pi_s) + \frac{1}{2} \left( \frac{\Delta r}{\sigma} \right)^2 \pi_s^2 (1 - \pi_s)^2 \tilde{V}''(\pi_s) \right) \, ds$$

$$+ \int_0^{\tau^Y \wedge t} e^{-\rho s} \tilde{V}'(\pi_s) \, dZ_s.$$
Given that \( \tilde{V} \geq 0 \), the final term is non-negative and we’re left with \( \tilde{V}(1) \geq \Pi(\tau^Y) \). Thus given that \( \Pi(\tau^*) = \tilde{V}(1) \), \( \tau^* \) is an optimal termination policy.

Finally, I verify that \( \tilde{V} \) is the firm’s virtual value function, in the sense that \( \tilde{V}(\pi_t) = R_t \) for all time. To verify this, fix any \( \tau^Y \geq t \) and use a variant of the argument above to conclude that
\[
\tilde{V}(\pi_t) \geq \mathbb{E}_t \left[ \int_t^{\tau^Y} e^{-\rho(s-t)}(\pi_s r - (1 - \pi_s)b) \, ds \right],
\]
with equality when \( \tau^Y = \inf\{s \geq t : \pi_t \leq \pi\} \) given that the two inequalities in the verification argument hold with equality for such a policy. Thus
\[
\tilde{V}(\pi_t) = \sup_{\tau^Y \geq t} \mathbb{E}_t \left[ \int_t^{\tau^Y} e^{-\rho(s-t)}(\pi_s r - (1 - \pi_s)b) \, ds \right]
\]
for all time a.s., as desired.

### A.3 Proof of Theorem 3

Theorem 2 in conjunction with Lemma 10 imply the existence of a threshold \( \pi \in (0, b/(r_G+b)] \) such that \( \tau^* = \inf\{t : \pi_t \leq \pi\} \) optimizes \( \Pi(\tau^Y) \) among all \( \mathbb{F}^Y \)-stopping times \( \tau^Y \). Therefore by Proposition 4 \( (F^*, \tau^*) \) is an optimal IC-B contract. It remains only to establish that \( (F^*, \tau^*) \) is an IC contract.

Define a process \( X \) by
\[
X_t = \mathbb{E}_t^B \left[ \int_0^{\tau^*} e^{-\rho s} b \, ds \right]
\]
for \( t \leq \tau^* \), with \( X_t = X_{\tau^*} \) for \( t > \tau^* \). Then \( X \) is a \( \mathbb{F}^Y \)-adapted square-integrable (indeed bounded) \( \mathbb{F}^B \)-martingale, so by the martingale representation theorem there exists an \( \mathbb{F}^Y \)-adapted, progressively measurable process \( \beta^X \) satisfying
\[
\mathbb{E}^B \left[ \int_0^t (\beta^X_s)^2 \, ds \right] < \infty
\]
for all \( t \) such that
\[
X_t = \int_0^t \beta^X_s \, dZ^B_s
\]
for all time. Next note that
\[
X_t = \frac{b}{\rho} (1 - e^{-\rho t}) + e^{-\rho t} F^*_t
\]
for \( t \leq \tau^Y \), hence by Ito’s lemma
\[
F^*_t = F^* + \int_0^t e^{\rho s}(\rho X_s - b) \, ds + \int_0^t e^{\rho s} \beta^X_s \, dZ^B_s = \int_0^t (\rho F_s - b) \, ds + \int_0^t \beta_s \, dZ^B_s,
\]
for all time.
where $\beta_t \equiv e^{\rho t} \beta_t^X$. Written in terms of $Z^G$, this becomes

$$F_t = F^* + \int_0^t \gamma_s \, ds + \int_0^t \beta_s \, dZ_s^G,$$

where $\gamma_t \equiv \rho F_t - b + \Delta \sigma \beta_t$. So $F$ is an Ito process satisfying $b - \rho F_t + \gamma_t - \Delta \sigma \beta_t = 0$ for all $t \leq \tau^Y$. Additionally, a change of measure combined with the Cauchy-Schwartz inequality and the Ito isometry implies that $E^{G} \left[ \int_0^t \beta_s^2 \, ds \right] < \infty$ for all $t$.

If in addition $\beta \geq 0$, then Lemma 20 in the technical appendix implies that $(F^*, \tau^*)$ is an IC contract. This must be true, because $F_t^*$ is increasing in the current value of $\pi_t$ (see Lemma 11), and $\pi_t$ increases in response to positive news shocks. If the loading on $dZ_t^G$ were ever negative, it would be possible to find paths of output over which $\pi_t$ grows but $F_t^*$ shrinks, a contradiction.
B Proofs of propositions

B.1 Proof of Proposition 1

Divide the first equation of the system in the proposition statement through by $a_1$ and solve for $1/a_1$ to obtain

$$\frac{1}{a_1} = -\frac{1}{\alpha} \left( \rho + 2b \frac{a_2}{a_1} \right).$$

Divide the second equation through by $a_1 a_2$ and insert the equality above to obtain

$$\frac{1}{4} \left( \frac{\Delta r}{\sigma} \right)^2 \frac{a_1}{a_2} + \frac{2br_G a_2}{\alpha} \frac{a_1}{a_1} + b + \frac{r_G \rho}{\alpha} = 0.$$

Multiply through by $a_1/a_2$ to obtain a quadratic equation satisfied by $a_1/a_2$:

$$\frac{1}{4} \left( \frac{\Delta r}{\sigma} \right)^2 \left( \frac{a_1}{a_2} \right)^2 + \left( b + \frac{\rho r_G}{\alpha} \right) \frac{a_1}{a_2} + \frac{2br_G}{\alpha} = 0.$$

Letting $F^* = -a_1/(2a_2)$, $F^*$ is a root of the quadratic polynomial

$$\phi(F) \equiv \frac{\alpha}{2} \left( \frac{\Delta r}{\sigma} \right)^2 F^2 - (ab + \rho r_G)F + br_G.$$

The two roots of $\phi$ are

$$F_{\pm}^* = \frac{\pm \sqrt{(ab + \rho r_G)^2 - 2abr_G \left( \frac{\Delta r}{\sigma} \right)^2}}{\alpha \left( \frac{\Delta r}{\sigma} \right)^2}.$$

Now, by assumption $\left( \frac{\Delta r}{\sigma} \right)^2 = \rho - \alpha$. Therefore

$$(ab + \rho r_G)^2 - 2abr_G \left( \frac{\Delta r}{\sigma} \right)^2 > (ab + \rho r_G)^2 - 2abr_G = (ab)^2 + (\rho r_G)^2 > 0.$$

So the discriminant of the quadratic is strictly positive, and there exist two distinct positive roots of the equation. Note that

$$\phi(b/\rho) = \frac{\alpha}{2} \left( \frac{\Delta r}{\sigma} \right)^2 \frac{b^2}{\rho^2} - (ab + \rho r_G) \frac{b}{\rho} + br_G = \frac{\alpha}{2} (\rho - \alpha) \frac{b^2}{\rho^2} - \frac{\alpha b^2}{\rho} = -\frac{\alpha b^2}{2\rho^2} < 0,$$
hence $F^- < b/\rho < F^+$. Recall also that

$$a_1 = -\frac{\alpha}{\rho - b/F^*}.$$  

Hence there exists one solution to the system of equations, corresponding to $F^+$, with $a_1 > 0$, and another, corresponding to $F^-$, with $a_1 < 0$. As $F^*$ is positive for both solutions, there is one solution satisfying $a_1 > 0 > a_2$, and another satisfying $a_2 > 0 > a_1$. The first solution is the one claimed in the theorem statement, and I will restrict attention to it going forward.

Now, let $V(F) = a_2F^2 + a_1F$. Note that $V$ is strictly increasing and strictly concave on $[0, F^*]$, and that $V(0) = 0$ while $V'(F^*) = 0$ and $V''(F^*) = 2a_2 < 0$. Then inserting $V$ into the rhs of the HJB equation, there are unique maximizers

$$\beta^*(F) = -\frac{\Delta r}{\sigma} \frac{V'(F)}{V''(F)} = \frac{\Delta r}{\sigma} (F^* - F)$$

and

$$\gamma^*(F) = \frac{\Delta r}{\sigma} \beta^*(F) - (b - \rho F)$$

$$= \left( \left( \frac{\Delta r}{\sigma} \right)^2 - \rho \right) (F^* - F) - (b - \rho F^*)$$

$$= -\alpha (F^* - F) - (b - \rho F^*),$$

for all $F \in [0, F^*)$. The continuous extension of $\gamma^*$ and $\beta^*$ to $F = F^*$ constitute the unique continuous maximizers of the HJB equation on the extended domain. The HJB equation then becomes

$$(\alpha + \rho)(a_2F^2 + a_1F) = r_G - \alpha F - (\alpha (F^* - F) + (b - \rho F^*)) (2a_2F + a_1) + \left( \frac{\Delta r}{\sigma} \right)^2 (F^* - F)^2 a_2.$$  

As this equation must hold for all $F$, the two sides of the equation must match term by term. The zeroeth order term is

$$0 = r_G - a_1((\alpha - \rho)F^* + b) + \left( \frac{\Delta r}{\sigma} \right)^2 F^* a_2.$$
Using the fact that $F^* = -a_1/(2a_2)$ and $\rho - \alpha = (\frac{\Delta r}{\sigma})^2$, this may equivalently be written

$$0 = r_G - ba_1 - \frac{1}{4} \left( \frac{\Delta r}{\sigma} \right)^2 \frac{a_1^2}{a_2},$$

which is just a rearrangement of the second equation defining $a_1$ and $a_2$. So the zeroth order terms match.

Meanwhile the first-order terms match if

$$(\alpha + \rho)a_1 = -\alpha - 2a_2((\alpha - \rho)F^* + b) + a_1\alpha - 2a_2F^* \left( \frac{\Delta r}{\sigma} \right)^2.$$

Again using $\rho - \alpha = (\frac{\Delta r}{\sigma})^2$, this reduces to

$$\rho a_1 = -\alpha - 2a_2 b,$$

which is just a rearrangement of the first equation defining $a_1$ and $a_2$. So the first-order terms match.

Finally, the second-order terms match if

$$(\alpha + \rho)a_2 = 2\alpha a_2 + \left( \frac{\Delta r}{\sigma} \right)^2 a_2.$$

This is implied by $\rho = \alpha + (\frac{\Delta r}{\sigma})^2$, so second-order terms match. Thus $V$ satisfies the HJB equation on $[0, F^*]$.

### B.2 Proof of Proposition 3

Suppose first that $V$ satisfies the HJB equation. Then Lemma 23 in the technical appendix establishes that $V$ is strictly increasing and strictly concave on $[0, F^*)$, and that

$$\beta^*(F_0) = -\frac{\Delta r}{\sigma} \frac{V'(F_0)}{V''(F_0)}, \quad \gamma^*(F_0) = \frac{\Delta r}{\sigma} \beta^*(F_0) + \rho F - b$$

are maximizers of the HJB equation on $[0, F^*)$. Substituting these maximizers into the HJB equation yields

$$(\rho + \alpha)V(F_0) = r_G - \alpha F_0 + (b - \rho F_0)V'(F_0) - \frac{1}{2} \left( \frac{\Delta r}{\sigma} \right)^2 \frac{V'(F_0)^2}{V''(F_0)}$$

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for $F_0 \in [0, F^*)$. And because $V''(F_0) > 0$ and $V''(0) < 0$, it must be that $(\rho + \alpha)V(F_0) - (r_G - \alpha F_0) - (b - \rho F_0)V'(F_0) > 0$, so this equation may be solved for $V''(F_0)$ to obtain the ODE in the proposition statement.

Conversely, suppose $V$ is strictly increasing and strictly concave and satisfies the ODE in the proposition statement. Rearranging it yields

$$(\rho + \alpha)V(F_0) = r_G - \alpha F_0 + \left(b - \rho F_0 - \left(\frac{\Delta r}{\sigma}\right)^2 \frac{V'(F_0)}{V''(F_0)}\right) V''(F_0) + \frac{1}{2} \left(\frac{\Delta r \left(V'(F_0)\right)^2}{\Delta r V''(F_0)}\right)^2 V''(F_0),$$

or

$$(\rho + \alpha)V(F_0) = r_G - \alpha F_0 + \gamma^*(F_0)V'(F_0) + \frac{1}{2} \beta^*(F_0)^2 V''(F_0).$$

And the proof of Lemma 23 shows that $(\gamma^*(F_0), \beta^*(F_0))$ maximizes the rhs subject to IC-B given any strictly increasing, strictly concave $V$. So $V$ satisfies the HJB equation.

Now, for $V$ satisfying the conditions of the proposition, write the ODE satisfied by $V$ on $[0, F^*)$ as $V'' = G(f, V, V')$, where

$$G(f, V, V') = -\frac{1}{2} \left(\frac{\Delta r}{\sigma}\right)^2 \frac{(V')^2}{(\rho + \alpha)V - (r_G - \alpha f) + (b - \rho f)V'}$$

is defined on the domain $U = \mathbb{R}^3 \setminus \{(f, V, V') : (\rho + \alpha)V - (r_G - \alpha f) + (b - \rho f)V' = 0\}$, with $U$ an open set. Suppose $V(F^*) \neq \frac{rg_0-\alpha r^*}{\rho+\alpha}$. Then $x_0 = (F^*, V(F^*), 0) \in U$, and as $V$ is $C^2$ by assumption, $V$ satisfies $V'' = G(f, V, V')$ on the entire domain $[0, F^*)$. Let $G$ be uniformly Lipschitz continuous wrt $(V, V')$ for fixed $f$ in $U$. Then by the Picard-Lindelof theorem, there exists a unique solution of $V'' = G(f, V, V')$ locally around $x_0$. Now observe that $\tilde{V}(f) = V(F^*)$ is a solution of the ODE sufficiently close to $x_0$, thus $V(F_0) = V(F^*)$ for $F_0$ sufficiently close to $F^*$. This contradicts the fact that $V$ is strictly increasing on $[0, F^*)$. So $V(F^*) = \frac{rg_0-\alpha r^*}{\rho+\alpha}$.

Finally, define $\Gamma : [0, F^*) \rightarrow \mathbb{R}$ by

$$\Gamma(F_0) \equiv (b - \rho F_0)V'(F_0) + (\rho + \alpha)V(F_0) - (r_G - \alpha F_0).$$

As $V$ is a $C^2$ function, $\Gamma$ is a $C^1$ function. Using $\Gamma$, the ODE satisfied by $V$ on $[0, F^*)$ may be written

$$\Gamma(F_0) = -\frac{1}{2} \left(\frac{\Delta r}{\sigma}\right)^2 \frac{V''(F_0)^2}{V''(F_0)}.$$ 

Given $V'(F_0) > 0$ and $V''(F_0) < 0$ for $F < F^*$, it must be that $\Gamma(F_0) > 0$ for $F < F^*$. And
as $V(F^*) = \frac{r_0 - \alpha F^*}{\rho + \alpha}$ and $V'(F^*) = 0$, $\Gamma(F^*) = 0$. It must therefore be that $\Gamma'(F^*) \leq 0$. As 

$$\Gamma'(F_0) = (b - \rho F_0)V''(F_0) + \alpha(1 + V'(F_0))$$

and $V'(F^*) = 0$, this implies 

$$(b - \rho F^*)V''(F^*) \leq -\alpha.$$ 

As $V''(F^*) \leq 0$, if $F^* \geq b/\rho$ this inequality would be violated. So $F^* < b/\rho$. The inequality would also be violated if $V''(F^*) = 0$, so $V''(F^*) < 0$.

**B.3 Proof of Proposition 4**

For each $\mathbb{F}^Y$-stopping time $\tau^Y$, let $F$ be the golden parachute defined by 

$$F_t = \mathbb{E}_t^F \left[ \int_t^\tau Y e^{-\rho(s-t)}b ds \right].$$

Lemma 6 says that 

$$\Pi(\tau^Y) = \mathbb{E} \left[ \int_0^{\tau^Y \wedge \tau^\theta} e^{-\rho t} r_G dt - e^{-\rho(\tau^\theta \wedge \tau^Y)} F_{\tau^\theta \wedge \tau^Y} \right].$$

**Lemma 14.** $e^{-\rho(\tau^\theta \wedge \tau^Y)} F_{\tau^\theta \wedge \tau^Y} = \mathbb{E}_{\tau^\theta} \left[ \int_{\tau^\theta \wedge \tau^Y} e^{-\rho t} b dt \right]$ a.s.

*Proof.* Note that $1\{\tau^Y \geq \tau^\theta\}$ is measurable wrt $\mathcal{F}_{\tau^\theta}$. Then as 

$$\int_{\tau^\theta \wedge \tau^Y} e^{-\rho t} b dt = 1\{\tau^Y \geq \tau^\theta\} \int_{\tau^\theta}^{\tau^Y} e^{-\rho t} b dt$$

it follows that 

$$\mathbb{E}_{\tau^\theta} \left[ \int_{\tau^\theta \wedge \tau^Y} e^{-\rho t} b dt \right] = 1\{\tau^Y \geq \tau^\theta\} \mathbb{E}_{\tau^\theta} \left[ \int_{\tau^\theta}^{\tau^Y} e^{-\rho t} b dt \right] = 0 = e^{-\rho(\tau^\theta \wedge \tau^Y)} F_{\tau^\theta \wedge \tau^Y}$$

on $\{\tau^Y < \tau^\theta\}$. On the other hand, on $\{\tau^Y \geq \tau^\theta\}$ 

$$\mathbb{E}_{\tau^\theta} \left[ \int_{\tau^\theta \wedge \tau^Y} e^{-\rho t} b dt \right] = \mathbb{E}_{\tau^\theta} \left[ \int_{\tau^\theta}^{\tau^Y} e^{-\rho t} b dt \right].$$
Further, on \( \{ \tau^\theta = s \} \), a standard property of stopping time sigma algebras yields
\[
\mathbb{E}_{\tau^\theta} \left[ \int_{\tau^\theta}^{\tau^Y} e^{-\rho t} b \, dt \right] = \mathbb{E}_s \left[ \int_{\tau^\theta}^{\tau^Y} e^{-\rho t} b \, dt \right] = \mathbb{E}_s \left[ \int_{s}^{\tau^Y} e^{-\rho t} b \, dt \right].
\]
And on \( \{ s \geq \tau^\theta \} \)
\[
\mathbb{E}_s \left[ \int_{s}^{\tau^Y} e^{-\rho t} b \, dt \right] = \mathbb{E}_s \left[ \int_{s}^{\tau^Y} e^{-\rho t} b \, dt \right].
\]
Thus on \( \{ \tau^Y \geq \tau^\theta \} \cap \{ \tau^\theta = s \} \),
\[
\mathbb{E}_{\tau^\theta} \left[ \int_{\tau^Y \land \tau^\theta}^{\tau^Y} e^{-\rho t} b \, dt \right] = \mathbb{E}_s \left[ \int_{s}^{\tau^Y} e^{-\rho t} b \, dt \right] = e^{-\rho s} F_s = e^{-\rho(\tau^\theta \land \tau^Y)} F_{\tau^\theta \land \tau^Y}.
\]
This relationship holds for all \( s \), proving the result.

Therefore by the law of iterated expectations,
\[
\Pi(\tau^Y) = \mathbb{E} \left[ \int_0^{\tau^\theta \land \tau^Y} e^{-\rho t} r_G \, dt - \int_{\tau^\theta \land \tau^Y}^{\tau^Y} e^{-\rho t} b \, dt \right] = \mathbb{E} \left[ \int_0^{\tau^\theta \land \tau^Y} e^{-\rho t} (r_G + b) \, dt - \int_0^{\tau^Y} e^{-\rho t} b \, dt \right].
\]
Another application of the law of iterated expectations yields
\[
\mathbb{E} \left[ \int_0^{\tau^\theta \land \tau^Y} e^{-\rho t} (r_G + b) \, dt \right] = \mathbb{E} \left[ \int_0^{\tau^\theta \land \tau^Y} e^{-\rho t} (r_G + b) \, dt \right] = \mathbb{E} \left[ \int_0^{\tau^\theta \land \tau^Y} e^{-\rho t} (r_G + b) \, dt \right] = \mathbb{E} \left[ \int_0^{\tau^\theta \land \tau^Y} e^{-\rho t} (r_G + b) \, dt \right] = \mathbb{E} \left[ \int_0^{\tau^\theta \land \tau^Y} e^{-\rho t} (r_G + b) \, dt \right].
\]
Thus
\[
\Pi(\tau^Y) = \mathbb{E} \left[ \int_0^{\tau^Y} e^{-\rho t} (\pi t r_G - (1 - \pi t)b) \, dt \right].
\]
B.4 Proof of Proposition 5

Let $g$ be as hypothesized. Fix $x \in [\pi, 1]$ and let $p^x$ be the unique strong solution to the SDE

$$p_t = p_0 + \int_0^t \left( -\alpha p_s - \left( \frac{\Delta r}{\sigma} \right)^2 p_s^2 (1 - p_s) \right) \, ds + \int_0^t \frac{\Delta r}{\sigma} p_s (1 - p_s) \, dZ_s^B$$

with initial condition $p_0 = x$ satisfying $0 < p_t^x < 1$ for all $t > 0$ a.s. (See the proof of Lemma 11 for a proof of existence and uniqueness.) Define $\tau^Y_x \equiv \inf\{ t : p_t^x \leq \pi \}$. By Ito’s lemma

$$g(x) = e^{-\rho \tau^Y_x} g(p_{t \wedge \tau^Y_x}^x)$$

$$- \int_0^{\tau^Y_x} e^{-\rho t} \left( -\rho g(p_t^x) - \left( \frac{\Delta r}{\sigma} \right)^2 \left( p_t^x \right)^2 (1 - p_t^x) \right) g'(p_t^x) \, dt$$

$$+ \frac{1}{2} \left( \frac{\Delta r}{\sigma} \right)^2 \left( p_t^x \right)^2 (1 - p_t^x)^2 g''(p_t^x) \, dt$$

$$- \int_0^{\tau^Y_x} \frac{\Delta r}{\sigma} e^{-\rho t} p_t^x (1 - p_t^x) \, dZ_t^B.$$ 

As $p^x$ is pathwise continuous a.s., $g(p_{t \wedge \tau^Y_x}^x) = g(\pi) = 0$. Hence

$$g(x) = - \int_0^{\tau^Y_x} e^{-\rho t} \left( -\rho g(p_t^x) - \left( \frac{\Delta r}{\sigma} \right)^2 \left( p_t^x \right)^2 (1 - p_t^x) \right) g'(p_t^x) \, dt$$

$$+ \frac{1}{2} \left( \frac{\Delta r}{\sigma} \right)^2 \left( p_t^x \right)^2 (1 - p_t^x)^2 g''(p_t^x) \, dt$$

$$- \int_0^{\tau^Y_x} e^{-\rho t} \frac{\Delta r}{\sigma} p_t^x (1 - p_t^x) \, dZ_t^B.$$ 

Given the ODE satisfied by $g$, this reduces to

$$g(x) = \int_0^{\tau^Y_x} e^{-\rho t} b \, dt - \int_0^{\tau^Y_x} \frac{\Delta r}{\sigma} p_t^x (1 - p_t^x) \, dZ_t^B.$$ 

Now take expectations wrt $P^B$ to eliminate the martingale term, yielding

$$g(x) = E^B \left[ \int_0^{\tau^Y_x} e^{-\rho t} b \, dt \right] = f(x).$$
C \hspace{1em} \text{Proofs of lemmas}

C.1 \hspace{1em} \text{Proof of Lemma 1}

Clearly IC-G and IC-B are implied by incentive-compatibility. For the converse result, suppose a contract $C = (\Phi, \tau)$ satisfies IC-G and IC-B, and fix an arbitrary $\mathcal{F}$-stopping time $\sigma$. Let $\underline{\sigma} \equiv \sigma \wedge \tau^\theta$ and $\overline{\sigma} \equiv \sigma \vee \tau^\theta$. Then by IC-G,

$$E \left[ \int_0^{\tau^\theta} e^{-\rho t} \left( bd\tau + d\Phi_t^\theta \right) \right] \geq E \left[ \int_0^{\overline{\sigma}} e^{-\rho t} (bd\tau + d\Phi_t^\theta) \right]$$

or after subtracting the final term from both sides,

$$E \left[ 1\{\sigma \leq \tau^\theta\} \int_0^{\tau^\theta} e^{-\rho t} (bd\tau + d\Phi_t^\theta) \right] \geq E \left[ 1\{\sigma \leq \tau^\theta\} \int_0^{\underline{\sigma}} e^{-\rho t} (bd\tau + d\Phi_t^\theta) \right].$$

A very similar argument using $\overline{\sigma}$ and the IC-B constraint yields

$$E \left[ 1\{\sigma > \tau^\theta\} \int_0^{\tau^\theta} e^{-\rho t} (bd\tau + d\Phi_t^\theta) \right] \geq E \left[ 1\{\sigma > \tau^\theta\} \int_0^{\overline{\sigma}} e^{-\rho t} (bd\tau + d\Phi_t^\theta) \right].$$

Sum these two inequalities to obtain

$$E \left[ \int_0^{\tau^\theta} e^{-\rho t} (bd\tau + d\Phi_t^\theta) \right] \geq E \left[ \int_0^{\tau^\theta} e^{-\rho t} (bd\tau + d\Phi_t^\theta) \right].$$

As this inequality holds for arbitrary $\sigma$, $C$ is an incentive-compatible contract.
C.2 Proof of Lemma 2

Suppose $C = (\Phi, \tau)$ is an IC-B contract. Define a new contract $\tilde{C} = (\tilde{\Phi}, \tilde{\tau})$ by $\tilde{\tau}_t = \tau_t \land t$ for all $t$, $\tilde{\Phi}_s^t = \Phi_s^t$ for all $t$ and $s < \tilde{\tau}_t$, and

$$\tilde{\Phi}_s^t = \Phi_s^t + \mathbb{E}_t^B \left[ \int_{\tilde{\tau}_t}^{\tau_t} e^{-\rho(s-\tilde{\tau}_t)}(b ds + d\Phi_s^t) \right]$$

for all $t$ and $s \geq \tilde{\tau}_t$.

To compute the agent’s expected utility under $\tilde{C}$, I will need to make a minor technical detour. Let $\tilde{\mathbb{E}}_t^B$ be the conditional expectation under $\mathbb{P}^B$ wrt $\mathcal{F}_t$. (Recall that $\mathbb{E}_t^B$ was defined to be the conditional expectation wrt $\mathcal{F}_t^Y$, which does not capture the full information available to the agent at time $t$.)

**Lemma 15.** For any $\mathbb{F}$-stopping time $\sigma$ and every time $t$,

$$\tilde{\mathbb{E}}_t^B \left[ \int_{\tilde{\tau}_\sigma}^{\tau_t} e^{-\rho(s-\tilde{\tau}_\sigma)}(b ds + d\Phi_s^\sigma) \right] = \mathbb{E}_t^B \left[ \int_{\tilde{\tau}_\sigma}^{\tau_t} e^{-\rho(s-\tilde{\tau}_\sigma)}(b ds + d\Phi_s^\sigma) \right].$$

whenever $\sigma = t$.

**Proof.** Fix any $\mathbb{F}$-stopping time $\sigma$. By a standard property of stopping time sigma algebras,

$$\tilde{\mathbb{E}}_t^B \left[ \int_{\tilde{\tau}_\sigma}^{\tau_t} e^{-\rho(s-\tilde{\tau}_\sigma)}(b ds + d\Phi_s^\sigma) \right] = \mathbb{E}_t^B \left[ \int_{\tilde{\tau}_\sigma}^{\tau_t} e^{-\rho(s-\tilde{\tau}_\sigma)}(b ds + d\Phi_s^\sigma) \right]$$

whenever $\sigma = t$. Now, the interior of the latter expectation is a function of the path of output only; as the marginal distribution over future output under $\mathbb{P}^B$ is the same conditional on $\mathcal{F}_t$ and $\mathcal{F}_t^Y$, it is therefore the case that

$$\tilde{\mathbb{E}}_t^B \left[ \int_{\tilde{\tau}_t}^{\tau_t} e^{-\rho(s-\tilde{\tau}_t)}(b ds + d\Phi_s^t) \right] = \mathbb{E}_t^B \left[ \int_{\tilde{\tau}_t}^{\tau_t} e^{-\rho(s-\tilde{\tau}_t)}(b ds + d\Phi_s^t) \right]$$

for all $t$.

The agent’s expected utility under $\tilde{C}$ and an arbitrary reporting policy $\sigma \geq \tau^\theta$ may therefore be written

$$U^\sigma = \mathbb{E} \left[ \int_{0}^{\tilde{\tau}_\sigma} e^{-\rho t}(b dt + d\Phi_t^\sigma) + e^{-\rho \tilde{\tau}_\sigma} \tilde{\mathbb{E}}_t^B \left[ \int_{\tilde{\tau}_\sigma}^{\tau_t} e^{-\rho(s-\tilde{\tau}_\sigma)}(b ds + d\Phi_s^\sigma) \right] \right].$$
Given $\sigma \geq \tau^\theta$, either $\tilde{\tau}^\sigma = \tau^\theta$ or $\tilde{\tau}^\sigma = \tau^\sigma$. In either case

$$E_{\tilde{\tau}^\sigma}^{B} \left[ \int_{\tilde{\tau}^\sigma}^{\tau^\sigma} e^{-\rho(s-\tilde{\tau}^\sigma)}(b \, ds + d\tilde{\Phi}_s^\sigma) \right] = E_{\tilde{\tau}^\sigma} \left[ \int_{\tilde{\tau}^\sigma}^{\tau^\sigma} e^{-\rho(s-\tilde{\tau}^\sigma)}(b \, ds + d\tilde{\Phi}_s^\sigma) \right],$$

as the conditional measure under $\mathbb{P}$ wrt the information in $\mathcal{F}_t$ is the same as under $\mathbb{P}^{B}$ when $t \geq \tau^\theta$. Then by the law of iterated expectations

$$U^\sigma = E \left[ \int_0^{\tau^\sigma} e^{-\rho t} (b \, dt + d\Phi_t^\sigma) \right].$$

The agent’s expected utility under a given reporting strategy is therefore the same under $\tilde{C}$ and $\tilde{\tilde{C}}$, so the fact that the former contract is IC-B implies the latter is as well.

As $\tilde{C}$ is IC-B, I suppress superscripts and consider $\tilde{\Phi}$ and $\tilde{\tau}$ to be $\mathbb{F}$-adapted. The firm’s expected profits under $\tilde{C}$ are

$$\tilde{\Pi} = E \left[ \int_0^{\tau^\theta \wedge \tau^\sigma} e^{-\rho t} (r_G \, dt - d\Phi_t) - e^{-\rho (\tau^\sigma \wedge \tau^\theta)} \tilde{E}^{B}_{\tau^\sigma \wedge \tau^\theta} \left[ \int_{\tau^\sigma \wedge \tau^\theta}^{\tau^\sigma} e^{-\rho (t-\tau^\sigma \wedge \tau^\theta)}(b \, dt + d\Phi_t) \right] \right].$$

As before, it is true that

$$E^{B}_{\tau^\sigma \wedge \tau^\theta} \left[ \int_{\tau^\sigma \wedge \tau^\theta}^{\tau^\sigma} e^{-\rho (t-\tau^\sigma \wedge \tau^\theta)}(b \, dt + d\Phi_t) \right] = E_{\tau^\sigma \wedge \tau^\theta} \left[ \int_{\tau^\sigma \wedge \tau^\theta}^{\tau^\sigma} e^{-\rho (t-\tau^\sigma \wedge \tau^\theta)}(b \, dt + d\Phi_t) \right],$$

and so by the law of iterated expectations

$$\tilde{\Pi} = E \left[ \int_0^{\tau^\sigma \wedge \tau^\theta} e^{-\rho t} r_G \, dt + \int_{\tau^\sigma \wedge \tau^\theta}^{\tau^\sigma} e^{-\rho t} (-b) \, dt - \int_0^{\tau^\sigma} e^{-\rho t} \, d\Phi_t \right].$$

By comparison, the firm’s profits under $\tilde{C}$ are

$$\Pi = E \left[ \int_0^{\tau^\sigma \wedge \tau^\theta} e^{-\rho t} r_G \, dt + \int_{\tau^\sigma \wedge \tau^\theta}^{\tau^\sigma} e^{-\rho t} r_B \, dt - \int_0^{\tau^\sigma} e^{-\rho t} \, d\Phi_t \right].$$

Recall that $0 > r_B + b$ by assumption. Therefore $\tilde{\Pi} \geq \Pi$, and this inequality is strict if $\tau > \tau^\sigma \wedge \tau^\theta = \tilde{\tau}$ with positive probability.

Finally, note that formally speaking $\tilde{\tau}$ should be written $\tilde{\tau}^\sigma = \tau^\sigma \wedge \tau^\theta$, and $\tau^\sigma \wedge \tau^\theta = \tau^\infty \wedge \tau^\theta$ given that $\{\tau^\infty \leq t\} = \{\tau^\theta \leq t\}$ whenever $t \leq \tau^\theta$. As $\tau^\infty$ is a $\mathbb{F}^{\gamma}$-stopping time, this proves the last claim of the lemma.
C.3 Proof of Lemma 3

Fix an IC-B contract \( C = (\Phi, \tau) \) such that \( \tau^t = \tau^Y \wedge t \) for some \( \mathbb{F}^Y \)-stopping time \( \tau^Y \). Define a new contract \( \tilde{C} = (\tilde{\Phi}, \tilde{\tau}) \) as follows: when \( t < \infty \), set \( \tilde{\Phi}^t_s = 0 \) for \( s < \tau^t \) and

\[
\tilde{\Phi}^t_s = e^{\rho s} \int_0^{\tau^t} e^{\rho u} d\Phi^t_u
\]

for \( s \geq \tau^t \). As \( \tau^t \leq t \) this construction is well-defined. And for \( t = \infty \) set \( \tilde{\Phi}^t = 0 \). The agent’s expected utility under \( \tilde{C} \) and any \( \mathcal{F} \)-stopping time \( \sigma \) is then just

\[
U^\sigma = \mathbb{E} \left[ \int_0^{\tau^\sigma} e^{-\rho t} b dt + 1\{\sigma < \infty\} e^{-\rho \tau^\sigma} \Delta \Phi^\sigma_\tau \right] = \mathbb{E} \left[ \int_0^{\tau^\sigma} e^{-\rho t} b dt + 1\{\sigma < \infty\} \int_0^{\tau^\sigma} e^{-\rho u} d\tilde{\Phi}^u_t \right].
\]

This is weakly lower than the agent’s expected utility from \( \sigma \) under \( C \), and identical when \( \sigma < \infty \). As truthful reporting satifies this condition, the fact that \( C \) is IC-B means \( \tilde{C} \) is as well. A nearly identical calculation shows that the firm’s expected profits under \( C \) and \( \tilde{C} \) are the same.

Now, define a \( \mathbb{F}^Y \)-adapted process \( F \) by \( F_t = \tilde{\Phi}^t_t \). Note that in general \( \tilde{\Phi}^t_s = \tilde{\Phi}^t_{\tau^t} 1\{s \geq \tau^t\} \) for all \( t \) and \( s \). Then in particular \( \tilde{\Phi}^t_s = F_{\tau^t} 1\{s \geq \tau^t\} \) whenever \( \tau^t = t \). Consider the remaining case that \( \tau^t < t \). In this case it must be that \( \tilde{\Phi}^t_{\tau^t} = \tilde{\Phi}^t_{\tau^t} \). For in this state of the world the project is immediately terminated regardless of whether the agent reports a state switch at time \( t \). If the payoffs at that time are not equal, then all agents will make the same report regardless of their type, either all reporting a state switch at \( t \) or else refraining from reporting depending on which yields a higher payoff. This violates IC, so the report at time \( t \) must not impact payments. Therefore \( \tilde{\Phi}^t_s = F_{\tau^t} 1\{s \geq \tau^t\} \) in all cases. This establishes the representation in the theorem statement.

C.4 Proof of Lemma 4

Fix a contract \( (F, \tau^Y) \). IC-B holds iff

\[
\mathbb{E} \left[ \int_0^{\tau^Y \wedge \tau^\sigma} e^{-\rho t} b dt + e^{-\rho (\tau^Y \wedge \tau^\sigma)} F_{\tau^Y \wedge \tau^\sigma} \right] \geq \mathbb{E} \left[ \int_0^{\tau^Y \wedge \sigma} e^{-\rho t} b dt + e^{-\rho (\tau^Y \wedge \sigma)} F_{\tau^Y \wedge \sigma} \right]
\]

for every \( \mathbb{F} \)-stopping time \( \sigma \geq \tau^\sigma \). This inequality may be rearranged to obtain

\[
0 \geq \mathbb{E} \left[ 1\{\tau^Y > \tau^\sigma\} e^{-\rho \tau^\sigma} \left( \int_{\tau^\sigma}^{\tau^Y \wedge \sigma} e^{-\rho (t-\tau^\sigma)} b dt + e^{-\rho (\tau^Y \wedge \sigma - \tau^\sigma)} F_{\tau^Y \wedge \sigma} - F_{\tau^\sigma} \right) \right].
\]

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Then by applying the law of iterated expectations, IC-B is equivalent to the condition that

\[ 0 \geq \mathbb{E} \left[ 1\{\tau^Y > \tau^\theta \} e^{-\rho \tau^\theta} \left( \mathbb{E}_{\tau^\theta} \left[ \int_{\tau^\theta}^{\tau^Y \wedge \sigma} e^{-\rho(t-\tau^\theta)} b \, dt + e^{-\rho(\tau^Y \wedge \sigma - \tau^\theta)} F_{\tau^Y \wedge \sigma} \right] - F_{\tau^\theta} \right) \]  

for every \( \sigma \geq \tau^\theta \). I will rely on this alternate characterization of IC-B throughout this proof.

Now suppose that for each \( t \) and every \( \mathbb{F}^Y \)-stopping time \( \tau^t \geq t \), equation (2) holds whenever \( \tau^Y > t \).

Fix an \( \mathbb{F} \)-stopping time \( \sigma \geq \tau^\theta \) and a time \( t \), and define an \( \mathbb{F}^Y \)-stopping time \( \tau'^t \geq t \) by setting \( \tau'^t = \sigma \) on \( \{\tau^\theta = t\} \) and completing \( \tau'^t \) uniquely to ensure its value is independent of \( \tau^\theta \) conditional on the history of output. Then by assumption

\[ F_t \geq \mathbb{E}_t^B \left[ \int_t^{\tau^Y \wedge \tau'^t} e^{-\rho(s-t)} b \, ds + e^{-\rho(\tau^Y \wedge \tau'^t - t)} F_{\tau^Y \wedge \tau'^t} \right] \]

on \( \{\tau^Y > t\} \). In particular, on \( \{\tau^Y > \tau^\theta = t\} \) this inequality can be written

\[ F_{\tau^\theta} \geq \mathbb{E}_{\tau^\theta} \left[ \int_{\tau^\theta}^{\tau^Y \wedge \sigma} e^{-\rho(s-\tau^\theta)} b \, ds + e^{-\rho(\tau^Y \wedge \sigma - \tau^\theta)} F_{\tau^Y \wedge \sigma} \right]. \]

(Recall that the conditional distribution on paths of output after \( t \) when \( \tau^\theta \leq t \) is \( \mathbb{P}^B \).) This argument holds for each \( t \), therefore the inequality holds on \( \{\tau^Y > \tau^\theta\} \). In other words,

\[ 0 \geq 1\{\tau^Y > \tau^\theta\} \left( \mathbb{E}_{\tau^\theta} \left[ \int_{\tau^\theta}^{\tau^Y \wedge \sigma} e^{-\rho(t-\tau^\theta)} b \, dt + e^{-\rho(\tau^Y \wedge \sigma - \tau^\theta)} F_{\tau^Y \wedge \sigma} \right] - F_{\tau^\theta} \right). \]

Taking expectations establishes that \( F \) satisfies IC-B.

In the other direction, suppose \((F, \tau^Y)\) satisfies IC-B. Define a new golden parachute \( F' \) by setting \( F'_t = F_t \) when \( t \geq \tau^Y \) and

\[ F'_t = \sup_{\tau'^t \geq t} \mathbb{E}_t^B \left[ \int_t^{\tau^Y \wedge \tau'^t} e^{-\rho(s-t)} b \, ds + e^{-\rho(\tau^Y \wedge \tau'^t - t)} F_{\tau^Y \wedge \tau'^t} \right] \]

when \( \tau^Y > t \), with the supremum ranging over all \( \mathbb{F}^Y \)-stopping times. Clearly \( F' \geq F \) by taking \( \tau'^t = t \) on the rhs.

I claim that for every \( t \) and \( \mathbb{F}^Y \)-stopping time \( \tau'^t \geq t \),

\[ F'_t = \sup_{\tau'^t \geq t} \mathbb{E}_t^B \left[ \int_t^{\tau^Y \wedge \tau'^t} e^{-\rho(s-t)} b \, ds + e^{-\rho(\tau^Y \wedge \tau'^t - t)} F'_{\tau^Y \wedge \tau'^t} \right] \]

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when \( \tau^Y > t \), from which it follows that \((F', \tau^Y)\) satisfies IC-B by the converse just proven. Obviously the rhs is weakly greater than the lhs, so it is sufficient to establish that

\[
F'_t \geq \sup_{\tau' \geq t} \mathbb{E}^B_t \left[ \int_t^{\tau^Y \wedge \tau'} e^{-\rho(s-t)} b \, ds + e^{-\rho(\tau^Y \wedge \tau' - t)} F'_{\tau^Y \wedge \tau'} \right].
\]

Fix a time \( t \) such that \( \tau^Y > t \). For each \( \varepsilon > 0 \) there exists an \( \mathbb{F}^Y \)-stopping time \( \tau^\varepsilon \geq t \) such that

\[
\sup_{\tau' \geq t} \mathbb{E}^B_t \left[ \int_t^{\tau^Y \wedge \tau'} e^{-\rho(s-t)} b \, ds + e^{-\rho(\tau^Y \wedge \tau' - t)} F'_{\tau^Y \wedge \tau'} \right] \leq \mathbb{E}^B_t \left[ \int_t^{\tau^Y \wedge \tau^\varepsilon} e^{-\rho(s-t)} b \, ds + e^{-\rho(\tau^Y \wedge \tau^\varepsilon - t)} F'_{\tau^Y \wedge \tau^\varepsilon} \right] + \frac{\varepsilon}{2}.
\]

And by definition of \( F' \), there exists a \( \tilde{\tau}^\varepsilon \geq \tau^Y \wedge \tau^\varepsilon \) such that

\[
F'_{\tau^Y \wedge \tau^\varepsilon} \leq \mathbb{E}^B_{\tau^Y \wedge \tau^\varepsilon} \left[ \int_{\tau^Y \wedge \tau^\varepsilon}^{\tau^Y \wedge \tau^\varepsilon} e^{-\rho(s-\tau^Y \wedge \tau^\varepsilon)} b \, ds + e^{-\rho(\tau^Y \wedge \tau^\varepsilon - t)} F'_{\tau^Y \wedge \tau^\varepsilon} \right] + e^{\rho(\tau^Y \wedge \tau^\varepsilon - t)} \frac{\varepsilon}{2}.
\]

(This stopping time may be constructed by using the definition of \( F' \) on each \( E^s = \{ \tau^Y > \tau^\varepsilon = s \} \) for \( s \geq t \) to choose \( \tilde{\tau}^\varepsilon \) on \( E^s \). The inequality is trivially true when \( \tau^Y \leq \tau^\varepsilon \)). Thus by the law of iterated expectations

\[
\sup_{\tau' \geq t} \mathbb{E}^B_t \left[ \int_t^{\tau^Y \wedge \tau'} e^{-\rho(s-t)} b \, ds + e^{-\rho(\tau^Y \wedge \tau' - t)} F'_{\tau^Y \wedge \tau'} \right] \leq \mathbb{E}^B_t \left[ \int_t^{\tau^Y \wedge \tilde{\tau}^\varepsilon} e^{-\rho(s-t)} b \, ds + e^{-\rho(\tau^Y \wedge \tilde{\tau}^\varepsilon - t)} F'_{\tau^Y \wedge \tilde{\tau}^\varepsilon} \right] + \varepsilon.
\]

And as \( \tilde{\tau}^\varepsilon \geq t \), the definition of \( F' \) implies

\[
\sup_{\tau' \geq t} \mathbb{E}^B_t \left[ \int_t^{\tau^Y \wedge \tau'} e^{-\rho(s-t)} b \, ds + e^{-\rho(\tau^Y \wedge \tau' - t)} F'_{\tau^Y \wedge \tau'} \right] \leq F'_t + \varepsilon.
\]

As this inequality holds for all \( \varepsilon > 0 \), the desired inequality must hold.

Finally, I show that \( F'_{\tau^\theta \wedge \tau^Y} = F_{\tau^\theta \wedge \tau^Y} \) a.s. For each \( \varepsilon > 0 \), construct an \( \mathbb{F} \)-stopping time
\( \sigma^\varepsilon \geq \tau^\theta \) by setting \( \sigma^\varepsilon = \tau^{t,\varepsilon} \) when \( \tau^\theta = t \), where \( \tau^{t,\varepsilon} \geq t \) is an \( \mathbb{F}^Y \)-stopping time satisfying

\[
\mathbb{E}^B_t\left[ \int_t^{\tau^Y \wedge \tau^{t,\varepsilon}} e^{-\rho(s-t)} b \, ds + e^{-\rho(\tau^Y \wedge \tau^{t,\varepsilon} - t)} F'_{\tau^Y \wedge \tau^{t,\varepsilon}} \right] \geq F'_t - \varepsilon
\]

whenever \( \tau^Y > t \). (Such a \( \tau^{t,\varepsilon} \) must exist given earlier results about \( F' \).) Then by construction

\[
\mathbb{E}^B_{\tau^\theta}\left[ \int_{\tau^\theta}^{\tau^Y \wedge \sigma} e^{-\rho(t - \tau^\theta)} b \, dt + e^{-\rho(\tau^Y \wedge \sigma - \tau^\theta)} F_{\tau^Y \wedge \sigma} \right] \geq F'_{\tau^\theta} - \varepsilon
\]

whenever \( \tau^Y > \tau^\theta \). Now as \( (F', \tau^Y) \) satisfies IC-B, the earlier characterization of this property implies

\[
0 \geq \mathbb{E}\left[ 1\{\tau^Y > \tau^\theta\} e^{-\rho \tau^\theta} (F'_{\tau^\theta} - F_{\tau^\theta} - \varepsilon) \right].
\]

This inequality holds for all \( \varepsilon > 0 \), hence

\[
0 \geq \mathbb{E}\left[ 1\{\tau^Y > \tau^\theta\} e^{-\rho \tau^\theta} (F'_{\tau^\theta} - F_{\tau^\theta}) \right].
\]

As \( F' \geq F \), it must be that \( F'_{\tau^\theta} = F_{\tau^\theta} \) a.e. on \( \{\tau^Y > \tau^\theta\} \). And by construction \( F'_{\tau^Y} = F_{\tau^Y} \), hence \( F'_{\tau^Y \wedge \tau^\theta} = F_{\tau^Y \wedge \tau^\theta} \) a.s.

### C.5 Proof of Lemma 5

Fix an IC-B contract \( C = (F, \tau^Y) \). I first claim that \( \tau^Y \leq \inf \{t : F_t = 0\} \). For suppose not; then there exists some time \( t \) and state of the world in which \( F_t = 0, \tau^Y > t, \) and \( \theta_t = B \). In this case the agent’s utility from reporting a state change is 0, while his expected utility from never reporting the change is strictly positive given that he collects strictly positive expected flow rents and receives a non-negative golden parachute. Hence IC-B is violated, a contradiction of our assumption.

I now show that the contract \( C' = (F', \tau^Y) \) with \( F' = \min\{F, b/\rho\} \) satisfies IC-B. Consider first the agent’s incentives under \( C' \) at time \( t \) in any state of the world such that \( \theta_t = B \) and \( F_t \leq b/\rho \). The agent’s utility from reporting the state change is the same under \( F \) and \( F' \), while his expected utility from any delayed reporting strategy is weakly lower under \( F' \) than under \( F \). Thus given that \( C \) satisfies IC-B, so does \( C' \).

Next consider the agent’s incentives at time \( t \) in any state of the world such that \( \theta_t = B \) and \( F_t > b/\rho \). In this case his expected utility under \( F' \) from any delayed reporting strategy is at most \( b/\rho \), as \( F' \) is bounded above by \( b/\rho \). Meanwhile his expected utility from reporting
the state change immediately is exactly $b/\rho$. Thus IC-B is satisfied for $F'$ everywhere. Note that as $C'$ implements the same stopping strategy and pays the agent weakly less than $C$, it must be weakly more profitable for the firm.

Now I construct a new contract $\tilde{C} = (\tilde{F}, \tilde{\tau}^Y)$ as follows. Whenever $t \leq \tau^Y$, set $\tilde{F}_t = F'_t$. For $t > \tau^Y$ set

$$\tilde{F}_t = \max \left\{ \frac{b}{\rho} - \left( \frac{b}{\rho} - F'_Y \right) e^{\rho(t-\tau^Y)}, 0 \right\}.$$ 

For the termination policy, let $\tilde{\tau}^Y = \inf \{ t : \tilde{F}_t = 0 \}$.

Suppose that in some state of the world $\tau^Y \geq \tilde{\tau}^Y$. This means that at $\tilde{F}_{\tilde{\tau}^Y} = F'_{\tilde{\tau}^Y}$, but also by definition $\tilde{F}_{\tilde{\tau}^Y} = 0$, meaning $F'_{\tilde{\tau}^Y} = 0$ and thus $\tau^Y \leq \tilde{\tau}^Y$. Hence $\tau^Y \geq \tilde{\tau}^Y$ implies $\tau^Y = \tilde{\tau}^Y$, meaning $\tau^Y \geq \tau^Y$ in all cases.

Next I check incentive compatibility. First consider times $t$ and states of the world in which $\theta_t = B$ and $\tilde{\tau}^Y > t \geq \tau^Y$. In this case, note that $\tilde{F}_t$ is pathwise differentiable, and $\frac{d}{dt} \tilde{F}_t = \rho \tilde{F}_t - b$. The change in the agent’s utility from waiting a moment $dt$ to report is

$$b dt - \rho \tilde{F}_t dt + \frac{d}{dt} \tilde{F}_t dt = 0.$$ 

As this equality holds everywhere until $\tilde{\tau}^Y$, the agent is indifferent between reporting now and any delayed reporting strategy. Thus IC-B holds in these cases. Next consider states of the world in which $\theta_t = B$ and $\tilde{\tau}^Y > t \geq \tau^Y$. In this case no delayed reporting strategy $\tau'$ such that $\tau^Y \geq \tau'$ is more profitable than immediate reporting given that $C'$ is IC-B. And on the other hand any delayed reporting strategy $\tau'$ yields exactly the same expected utility to the agent as the reporting strategy $\tau' \wedge \tau^Y$, as we’ve seen that the agent’s expected utility is unchanged past that point. Thus IC-B holds in all cases.

Finally, I claim that $\tilde{C}$ is at least as profitable for the firm as $C'$, and therefore as $C$. Because $\tilde{C}$ operates for at least as long as $C'$ in the good state in all cases and pays the agent no more than $C'$ whenever $\tilde{\tau}^Y = \tau^Y$, I need only check the cases in which $\tilde{\tau}^Y > \tau^Y$. But in this case the path of $\tilde{F}_t$ is constructed to be (weakly) declining given $b/\rho \geq F'$, and so the firm pays the agent no more stopping at $\tilde{\tau}^Y$ than he would have stopping at $\tau^Y$; in other words, $\tilde{F}_{\tilde{\tau}^Y} \leq \tilde{F}_{\tau^Y} = F'_{\tau^Y}$. Thus payments under $\tilde{C}$ are weakly lower than under $C'$, proving the claim.

### C.6 Proof of Lemma 6

Fix an $\mathbb{F}^Y$-stopping time $\tau^Y$. One feasible reporting policy for the agent at time $t$ is to never report a state change, i.e. to implement the stopping time $\tau' = \infty$. Then if $F$ is any
IC-B golden parachute implementing $\tau^Y$, the IC-B constraint implies

$$F_t \geq \mathbb{E}_t^B \left[ \int_t^{\tau^Y} e^{-\rho(s-t)} b \, ds + e^{-\rho(\tau^Y-t)} F_{\tau^Y} \right] \geq \mathbb{E}_t^B \left[ \int_t^{\tau^Y} e^{-\rho(s-t)} b \, ds \right].$$

It follows that if the golden parachute

$$F_t^* = \mathbb{E}_t^B \left[ \int_t^{\tau^Y} e^{-\rho(s-t)} b \, ds \right]$$

satisfies IC-B, then it minimizes fees paid to the agent state by state over all IC-B contracts implementing $\tau^Y$, and is therefore profit-maximizing.

Fix a time $t$ such that $\tau^Y > t$ and a delayed reporting policy $\tau' \geq t$, and rewrite the agent’s expected utility under $\tau'$ using the law of iterated expectations as

$$\mathbb{E}_t^B \left[ \int_t^{\tau' \wedge \tau^Y} e^{-\rho(s-t)} b \, ds + e^{-\rho(\tau' \wedge \tau^Y-t)} F_{\tau' \wedge \tau^Y} \right]$$

$$= \mathbb{E}_t^B \left[ \int_t^{\tau' \wedge \tau^Y} e^{-\rho(s-t)} b \, ds + e^{-\rho(\tau' \wedge \tau^Y-t)} \right] \mathbb{E}_{\tau' \wedge \tau^Y}^B \left[ \int_t^{\tau^Y} e^{-\rho(s-t)} b \, ds \right]$$

$$= \mathbb{E}_t^B \left[ \int_t^{\tau' \wedge \tau^Y} e^{-\rho(s-t)} b \, ds + \mathbb{E}_{\tau' \wedge \tau^Y}^B \left[ \int_t^{\tau^Y} e^{-\rho(s-t)} b \, ds \right] \right]$$

$$= \mathbb{E}_t^B \left[ \int_t^{\tau^Y} e^{-\rho(s-t)} b \, ds \right] = F_t^*.$$

Hence every delayed reporting policy $\tau'$ yields the same expected utility under $F^*$ as truthful reporting, i.e. $F^*$ is IC-B.

**C.7 Proof of Lemma 7**

Given truthful reporting, under any contract $F$ the firm collects flow payments $dY$ until $\tau^Y \wedge \tau^\theta$ and pays out a lump sum of $F_{\tau^Y \wedge \tau^\theta}$ at project termination. Its expected profits under $F$ are therefore

$$\Pi = \mathbb{E} \left[ \int_0^{\tau^Y \wedge \tau^\theta} e^{-\rho s} dY_s - e^{-\rho(\tau^Y \wedge \tau^\theta)} F_{\tau^Y \wedge \tau^\theta} \right] = \mathbb{E} \left[ \int_0^{\tau^Y \wedge \tau^\theta} e^{-\rho s} r_G \, ds - e^{-\rho(\tau^Y \wedge \tau^\theta)} F_{\tau^Y \wedge \tau^\theta} \right].$$
Now I write out the expectation wrt uncertainty in the switching time explicitly, using the fact that the conditional measure on output paths up to time $t$ given $\tau^\theta \geq t$ is equal to $\mathbb{P}^G$:

$$\Pi = \int_0^\infty dt \alpha e^{-\alpha t} \mathbb{E}^G \left[ \int_0^{\tau^Y \land t} e^{-\rho s}r_G ds - e^{-\rho(\tau^Y \land t)}F_{\tau^Y \land t} \right].$$

The expression in brackets is bounded, so I use Fubini’s theorem to exchange the order of integration, yielding

$$\Pi = \mathbb{E}^G \left[ \int_0^\infty dt \alpha e^{-\alpha t} \left( \int_0^{\tau^Y \land t} e^{-\rho s}r_G ds - e^{-\rho(\tau^Y \land t)}F_{\tau^Y \land t} \right) \right].$$

(In the last line I have used the fact that $\tilde{F}_{\tau^Y} = 0$.) Using integration by parts, the first term can be evaluated as

$$\int_0^{\tau^Y} dt \alpha e^{-\alpha t} \int_0^t e^{-\rho s}r_G ds = -e^{-\alpha \tau^Y} \int_0^{\tau^Y} e^{-\rho s}r_G ds + \int_0^{\tau^Y} e^{-(\rho + \alpha)t}r_G dt.$$

Hence

$$\Pi = \mathbb{E}^G \left[ \int_0^{\tau^Y} e^{-(\rho + \alpha)t}(r_G - \alpha \tilde{F}_t) dt \right].$$

### C.8 Proof of Lemma 8

I prove this lemma with the aid of Lemma 21 in the technical appendix, which establishes that a contract $(F, \tau^Y)$ satisfies IC-B iff $M_{\tau^Y}$ is a $\mathbb{P}^B$-supermartingale, where

$$M_t = \frac{b}{\rho}(1 - e^{-\rho t}) + e^{-\rho t}F_t.$$

Fix a contract $(F, \tau^Y)$ evolving as the Ito process

$$dF_t = \gamma_t dt + \beta_t dZ^G_t = \left( \gamma_t - \frac{\Delta r}{\sigma} \beta_t \right) dt + \beta_t dZ^B_t$$

for $\mathbb{F}^Y$-adapted, progressively measurable processes $\gamma$ and $\beta$. For every $n = 1, 2, \ldots$, define
\[ \tau^n \equiv \inf \left\{ t : \int_0^t |\gamma_s| \, ds + \int_0^t \beta_s^2 \, ds \geq n \right\} . \] And for every \( t \) and \( m = 1, 2, ..., \) define \( \sigma^{t,m} \equiv \inf \{ s \geq t : F_s + |\gamma_s| + |\beta_s| \geq m \} \). (Right-continuity of \( \mathbb{P}^Y \) under the usual conditions ensures that each \( \sigma^{t,m} \) is an \( \mathbb{F}^Y \)-stopping time.)

Suppose first that \( F \) satisfies

\[ b - \rho F_t + \gamma_t - \frac{\Delta r}{\sigma} \beta_t \leq 0 \]

whenever \( t < \tau^Y \). Fix times \( t \) and \( s > t \), and let \( n \) be large enough that \( \tau^n > t \). Ito’s lemma implies that

\[
\mathbb{E}_t[B]\left[M_{s \wedge \tau^n}^Y\right] = M_t^Y + \mathbb{E}_t[B] \left[ \int_{t \wedge \tau^Y}^{s \wedge \tau^Y \wedge \tau^n} e^{-\rho u} \left( b - \rho F_u + \gamma_u - \frac{\Delta r}{\sigma} \beta_u \right) \, du + \int_{t \wedge \tau^Y}^{s \wedge \tau^Y \wedge \tau^n} e^{-\rho u} \beta_u \, dZ_u \right].
\]

Now, given the regularization imposed by \( \tau^n \), the last term is a \( \mathbb{P}^B \)-martingale whose expectation vanishes. And the remaining term within the expectation is non-positive a.s. by assumption. Hence \( \mathbb{E}_t[B]\left[M_{s \wedge \tau^n}^Y\right] \leq M_t^Y \). This holds for all \( n \) sufficiently large, so also

\[
\lim_{n \to \infty} \inf \mathbb{E}_t[B]\left[M_{s \wedge \tau^n}^Y\right] \leq M_t^Y.
\]

Given that \( M \) is non-negative, Fatou’s lemma allows the limit and expectation on the lhs to be swapped, yielding \( \mathbb{E}_t[B]\left[M_{s \wedge \tau^n}^Y\right] \leq M_t^Y \). So \( M^Y \) is a supermartingale, implying by the previous theorem that \( F \) satisfies IC-B.

Conversely, suppose that \( F \) is an IC-B contract. For this direction, assume additionally that \( \gamma \) and \( \beta \) are pathwise right-continuous. Fix a time \( t < \tau^Y \) and take \( m \) large enough that \( \sigma^{t,m} > t \). (Right-continuity of \( F, \gamma, \) and \( \beta \) ensures that such an \( m \) exists.) Use Lemma 21 in the technical appendix to conclude that for all \( s > t \),

\[
\mathbb{E}_t[B] \left[ \frac{b}{\rho} \left( 1 - e^{-\rho (s \wedge \tau^Y \wedge \sigma^{t,m})} \right) + e^{-\rho (s \wedge \tau^Y \wedge \sigma^{t,m})} F_{s \wedge \tau^Y \wedge \sigma^{t,m}} \right] \leq \frac{b}{\rho} (1 - e^{-\rho t}) + e^{-\rho t} F_t.
\]

Using Ito’s lemma, this is equivalent to

\[
\mathbb{E}_t[B] \left[ \int_t^{s \wedge \tau^Y \wedge \sigma^{t,m}} e^{-\rho u} \left( b - \rho F_u + \gamma_u - \frac{\Delta r}{\sigma} \beta_u \right) \, du + \int_t^{s \wedge \tau^Y \wedge \sigma^{t,m}} e^{-\rho u} \beta_u \, dZ_u \right] \leq 0.
\]

Due to the regularization imposed by \( \sigma^{t,m} \), the final term is a \( \mathbb{P}^B \)-martingale whose expec-
tation vanishes, yielding
\[
\mathbb{E}_t^B \left[ \int_t^s e^{-\rho u} \left( b - \rho F_u + \gamma_u - \frac{\Delta r}{\sigma} \beta_u \right) du \right] \leq 0.
\]

Now, given regularization under \( \sigma^{t,m} \), the integral \( \int_t^s e^{-\rho u} 1\{\tau^Y \land \sigma^{t,m} \geq u\} \left( b + \gamma_u - \frac{\Delta r}{\sigma} \beta_u \right) du \) is uniformly bounded state by state and thus in expectation, and so Fubini's theorem permits swapping the order of integration for this term. Meanwhile the non-negativity of \( F \) allows the swapping of the order of integration in the remaining term by Tonelli's theorem. This procedure yields
\[
\int_t^s e^{-\rho u} \mathbb{E}_t^B 1\{\tau^Y \land \sigma^{t,m} \geq u\} \left( b - \rho F_u + \gamma_u - \frac{\Delta r}{\sigma} \beta_u \right) du \leq 0.
\]

Given \( \tau^Y \land \sigma^{t,m} \geq t \) by assumption, \( 1\{\tau^Y \land \sigma^{t,m} \geq u\} \) must converge pathwise to 1 as \( u \downarrow t \). Hence given pathwise right-continuity of \( \gamma \) and \( \beta \), the expression
\[
e^{-\rho u} 1\{\tau^Y \land \sigma^{t,m} \geq u\} \left( b - \rho F_u + \gamma_u - \frac{\Delta r}{\sigma} \beta_u \right)
\]
is pathwise right-continuous at \( t \). Further, \( 1\{\sigma^{t,m} \geq u\} \left( b - \rho F_u + \gamma_u - \frac{\Delta r}{\sigma} \beta_u \right) \) is uniformly bounded by \( b + (1 + \rho + \frac{\Delta r}{\sigma}) m \) for all \( u \geq t \). Therefore the dominated convergence theorem applies, and
\[
e^{-\rho u} \mathbb{E}_t^B 1\{\tau^Y \land \sigma^{t,m} \geq u\} \left( b - \rho F_u + \gamma_u - \frac{\Delta r}{\sigma} \beta_u \right) \]
is right-continuous at \( t \).

Now I apply the mean value theorem: given right continuity of the integrand at \( t \), for each \( s \) there exists a \( u^*(s) \in (t, s) \) such that
\[
\int_t^s e^{-\rho u} \mathbb{E}_t^B 1\{\tau^Y \land \sigma^{t,m} \geq u\} \left( b - \rho F_u + \gamma_u - \frac{\Delta r}{\sigma} \beta_u \right) du = (s - t) e^{-\rho u^*(s)} \mathbb{E}_t^B 1\{\tau^Y \land \sigma^{t,m} \geq u^*(s)\} \left( b - \rho F_{u^*(s)} + \gamma_{u^*(s)} - \frac{\Delta r}{\sigma} \beta_{u^*(s)} \right).
\]

Inserting this formula into the inequality derived earlier yields
\[
\mathbb{E}_t^B 1\{\tau^Y \land \sigma^{m,t} \geq u^*(s)\} \left( b - \rho F_{u^*(s)} + \gamma_{u^*(s)} - \frac{\Delta r}{\sigma} \beta_{u^*(s)} \right) \leq 0
\]
for \( s \) sufficiently close to \( t \). Now take \( s \downarrow t \), using right-continuity of the expectation and the
fact that \(u^*(s) \to t\) by the squeeze theorem to conclude that

\[
b - \rho F_t + \gamma_t - \frac{\Delta r}{\sigma} \beta_t \leq 0.
\]

### C.9 Proof of Lemma 9

See Propositions 8.10 and 8.11 of Harrison (2013).

### C.10 Proof of Lemma 10

Throughout this proof I freely invoke well-known properties of \(U(m, n, z)\). In particular, I will often invoke properties holding when \(m > 0\) and \(n > m + 2\), as is the case when \(m = \beta - 1\) and \(n = 2\beta - \frac{2\alpha}{k^2}\) given \(\beta > 1 + \frac{2\alpha}{k^2}\).

I begin by deriving a general solution to the ODE

\[
\rho v = xr_G - (1 - x)b - \alpha xv' + \frac{k^2}{2}x^2(1 - x)^2v''.
\]

This is an inhomogeneous second-order linear ODE, whose solution can be found by conjecturing a particular solution and then solving the associated homogeneous equation. A natural conjecture is linear in \(x\); inserting \(v_0(x) = c_1x + c_0\) and matching coefficients reveals that

\[
v_0(x) = \frac{r_G + b}{\rho + \alpha} x - \frac{b}{\rho}
\]

is a particular solution to the ODE. The problem of solving the ODE then reduces to solving the associated homogeneous equation

\[
\rho v_H = -\alpha xv'_H + \frac{k^2}{2}x^2(1 - x)^2v''_H.
\]

Now I make the transformation \(z \equiv \frac{x}{1-x}\), obtaining the transformed ODE

\[
\rho v_H = \left(-\alpha z + k^2 - \alpha - \frac{k^2}{z + 1}\right) z \frac{dv_H}{dz} + \frac{k^2}{2}z^2 \frac{d^2v_H}{dz^2}.
\]

Next I guess that \(w_H(z) \equiv \frac{z^\beta}{1+z}v_H(z)\) satisfies a simpler ODE than \(v_H\) itself for some positive power of \(\beta\). Inserting into the ODE yields

\[
\left(\alpha(\beta - 1)z + \rho + \alpha + (\alpha - k^2)(\beta - 1) - \frac{1}{2}k^2(\beta - 1)(\beta - 2)\right) w_H = (k^2\beta - \alpha - \alpha z) z \frac{dw_H}{dz} + \frac{k^2}{2}z^2 \frac{d^2w_H}{dz}.
\]
The right choice of $\beta$ is therefore a solution to
\[ \rho + \alpha + (\alpha - k^2)(\beta - 1) - \frac{1}{2}k^2(\beta - 1)(\beta - 2) = 0, \]
which is the quadratic
\[ \beta^2 - \left(1 + \frac{2\alpha}{k^2}\right)\beta - 2\frac{\rho}{k^2} = 0. \]
It is straightforward to verify that a unique positive solution to this equation exists. Taking this choice of $\beta$, the ODE for $w_H$ reduces to
\[ \alpha(\beta - 1)w_H = (k^2\beta - \alpha - \alpha z)\frac{dw_H}{dz} + \frac{k^2}{2}z\frac{d^2w_H}{dz^2}. \]
Finally, make the substitution $t \equiv \frac{2\alpha}{k^2}z$. I arrive at the ODE
\[ (\beta - 1)w_H = \left(2\beta - \frac{2\alpha}{k^2} - t\right)\frac{dw_H}{dt} + t\frac{d^2w_H}{dt^2}. \]
This is Kummer’s differential equation, which has general solution
\[ w_H(t) = C_1U(m, n, t) + C_2M(m, n, t), \]
where $U$ and $M$ are Tricomi’s and Kummer’s confluent hypergeometric functions and $m \equiv \beta - 1$ and $n \equiv 2\beta - \frac{2\alpha}{k^2}$ are both strictly positive. Transforming back to the original variables, a general solution to the homogeneous equation is
\[ v_H(x) = x^{m+1}(1-x)^{-m}\left(CU\left(m, n, \frac{2\alpha}{k^2} \frac{x}{1-x}\right) + DM\left(m, n, \frac{2\alpha}{k^2} \frac{x}{1-x}\right)\right). \]
Now, Kummer’s function $M(m, n, t)$ is known to diverge in the limit as $t \to \infty$. As the leading term of $v_H$ also diverges in this limit, no solution with $D \neq 0$ can satisfy the desired regularity conditions at $x = 1$. This leaves solutions to the ODE in the lemma statement of the form
\[ v(x) = \frac{r_G + b}{\rho + \alpha}x - \frac{b}{\rho} + Cx^{m+1}(1-x)^{-m}U\left(m, n, \frac{2\alpha}{k^2} \frac{x}{1-x}\right) \]
as candidates for solving the desired boundary value problem.
Let
\[ V_H(x) \equiv x^\beta(1-x)^{1-\beta}U\left(\beta - 1, 2\beta - \frac{2\alpha}{k^2}, \frac{2\alpha}{k^2} \frac{x}{1-x}\right) \]
for $x \in (0, 1)$. $U(m, n, \cdot)$ is known to be a strictly positive function on $(0, \infty)$ when $m > 0,$
so the function $\Gamma(x) \equiv \log V_H(x)$ is also well-defined on $(0,1)$. As $U(m, n, \cdot)$ is an analytic function on $(0, \infty)$, $V_H$ and $\Gamma$ are $C^2$ functions on $(0,1)$. En route to solving the boundary value problem, I establish several basic facts about $V_H$, $\Gamma$, and their derivatives in the limits $x \to 0$ and $x \to 1$.

**Lemma 16.** The following limiting expressions hold as $x \to 1$:

\[
\begin{align*}
\lim_{x \to 1} V_H(x) &= \left(\frac{k^2}{2\alpha}\right)^{\beta-1}, \\
\lim_{x \to 1} V_H'(x) &= -\frac{\rho}{\alpha} \left(\frac{k^2}{2\alpha}\right)^{\beta-1}, \\
\lim_{x \to 1} \Gamma'(x) &= -\frac{\rho}{\alpha}, \\
\lim_{x \to 1} V_H''(x) &= \left(\frac{k^2}{2\alpha}\right)^{\beta+1} \beta(\beta-1) \left(\beta - \frac{2\alpha}{k^2}\right) \left(\beta - \frac{2\alpha}{k^2} - 1\right).
\end{align*}
\]

**Proof.** Let $m \equiv \beta - 1$, $n \equiv 2\beta - \frac{2\alpha}{k^2}$, $\gamma \equiv \frac{k^2}{2\alpha}$, and $z \equiv \gamma^{-1} \frac{x}{1-x}$. Then I may write $V_H$ as

\[
V_H(x) = x^\gamma z^m U(m, n, z).
\]

Now I invoke the well-known asymptotic expansion of $U$ as $z \to \infty$ when $z \in \mathbb{R}$. The full expansion is $U(m, n, z) \sim z^{-m} 2F_0(m, m-n+1; -1/z)$, where $2F_0$ is the well-known generalized hypergeometric series. To third order this expansion says that

\[
U(m, n, z) = z^{-m} \left(a_0 - a_1 z^{-1} + \frac{1}{2} a_2 z^{-2} + \Phi(z)\right),
\]

where $a_0 = 1$, $a_1 = m(m-n+1)$, $a_2 = m(m+1)(m-n+1)(m-n+2)$, and $\Phi(z) \sim O(z^{-3})$. In other words, $\lim_{z \to \infty} z^N \Phi(z) = 0$ when $N < 3$. As $U$ is analytic, so is $\Phi$, and L’hopital’s rule implies $\Phi'(z) \sim O(z^{-4})$ and $\Phi''(z) \sim O(z^{-5})$.

Now insert this expansion into $V_H$ to obtain

\[
V_H(x) = x^\gamma z^m \left(a_0 - a_1 z^{-1} + \frac{1}{2} a_2 z^{-2} + \Phi(z)\right).
\]

Taking $x \to 1$ implies $\lim_{x \to 1} V_H(x) = a_0 \gamma^m$, which is the first formula in the lemma. Now differentiate wrt $x$, noting that $\frac{dz}{dx} = \frac{\gamma^{-1}}{(1-x)^2} = \gamma^{-1}(1 + \gamma z)^2$. I obtain

\[
V_H'(x) = \gamma^m \left(a_0 - a_1 z^{-1} + \frac{1}{2} a_2 z^{-2} + \Phi(z)\right) + x \gamma^{m-1}(1 + \gamma z)^2 \left(a_1 z^{-2} - a_2 z^{-3} + \Phi'(z)\right).
\]
Thus
\[ \lim_{x \to 1} V'_H(x) = \gamma^m a_0 + \gamma^{m+1} a_1 = \left( \frac{k^2}{2\alpha} \right)^m + \left( \frac{k^2}{2\alpha} \right)^{m+1} m(m - n + 1). \]

Writing this explicitly in terms of model parameters, this is
\[ \lim_{x \to 1} V'_H(x) = \left( \frac{k^2}{2\alpha} \right)^{\beta-1} \left( 1 + \frac{k^2}{2\alpha} (\beta - 1) \right) = \left( \frac{k^2}{2\alpha} \right)^{\beta-1} \left( 1 + \frac{2\alpha}{k^2} \right)^{\beta} - \frac{k^2}{2\alpha} \beta^2. \]

Using the quadratic formula characterizing \( \beta \), this simplifies to
\[ \lim_{x \to 1} V'_H(x) = - \left( \frac{k^2}{2\alpha} \right)^{\beta-1} \frac{\rho}{\alpha}, \]
which is the second formula in the lemma statement. The limiting expression for \( \Gamma'(x) = \frac{V'_H(x)}{V_H(x)} \) may then be obtained by combining previous results.

Finally, the second derivative of \( V_H \) is
\[
V''_H(x) = (x + 1) \gamma^{m-1} (1 + \gamma z)^2 \left( a_1 z^{-2} - a_2 z^{-3} + \Phi'(z) \right) \\
+ 2x \gamma^{m-1} (1 + \gamma z)^3 \left( a_1 z^{-2} - a_2 z^{-3} + \Phi'(z) \right) \\
+ x \gamma^{m-2} (1 + \gamma z)^4 \left( -2a_1 z^{-3} + 3a_2 z^{-4} + \Phi''(z) \right).
\]

The first term converges to \( 2\gamma^{m+1} a_1 \) as \( x \to \infty \). As for the rest, pull out the common factor of \( x\gamma^{m-2}(1 + \gamma z)^3 \) and expand
\[
2\gamma \left( a_1 z^{-2} - a_2 z^{-3} + \Phi'(z) \right) + (1 + \gamma z) \left( -2a_1 z^{-3} + 3a_2 z^{-4} + \Phi''(z) \right).
\]

The result is
\[
(-2a_1 + \gamma a_2) z^{-3} + 3a_2 z^{-4} + 2\gamma \Phi'(z) + (1 + \gamma z) \Phi''(z).
\]

All but the firm term are \( O(z^{-4}) \), which when multiplied by \( (1 + \gamma z)^3 \) die out as \( z \to \infty \). I conclude that
\[
\lim_{x \to 1} V''_H(x) = 2\gamma^{m+1} a_1 + \gamma^{m+1} (-2a_1 + \gamma a_2) = \gamma^{m+2} a_2,
\]
which is the final formula in the lemma statement. \( \square \)

**Lemma 17.** The following limiting expressions hold as \( x \to 0 \):
\[
\lim_{x \to 0} V_H(x) = \infty,
\]
\[
\liminf_{x \to 0} \Gamma'(x) = -\infty.
\]
Proof. Let $m \equiv \beta - 1$, $n \equiv 2\beta - \frac{3\beta}{k'}$, $\gamma \equiv \frac{k^2}{2\alpha}$, and $z \equiv \gamma^{-1} \frac{m}{1-x}$. Then I may write $V_H$ as

$$V_H(x) = (1 - x)\gamma^m z^{m+1}U(m, n, z).$$

A standard property of $U$ is $\lim_{z \to 0} z^{m+1}U(m, n, z) = \infty$ when $m > 0$ and $n > m + 2$, proving the result.

Given $\lim_{x \to 0} V_H(x) = \infty$, clearly also $\lim_{x \to 0} \Gamma(x) = \infty$. Now suppose by way of contradiction that $\liminf_{x \to 0} \Gamma'(x) > -\infty$. In this case there exists an $x > 0$ and an $M < \infty$ such that $\Gamma'(x) \geq -M$ for all $y \in (0, x]$. The fundamental theorem of calculus then implies

$$\Gamma(y) = \Gamma(x) - \int_y^x \Gamma'(t) \, dt \leq \Gamma(x) + M(x - y) \leq \Gamma(x) + Mx.$$

Thus $\Gamma(y)$ is bounded above on $(0, x]$, contradicting $\lim_{x \to 0} \Gamma(x) = \infty$. 

These lemmas provide the tools to demonstrate the existence of constants $\pi \in (0, 1)$ and $C > 0$ such that

$$V(x; C) = \frac{r_G + b}{\rho + \alpha} x - \frac{b}{\rho} + CV_H(x)$$

satisfies $V(\pi; C) = V'(\pi; C) = 0$. These boundary conditions are equivalent to $CV_H(\pi) = -\frac{r_G + b}{\rho + \alpha} x + \frac{b}{\rho}$ and $CV_H'(\pi) = -\frac{r_G + b}{\rho + \alpha}$. Dividing the second equation through by the first, existence of an appropriate $\pi$ is equivalent to existence of a solution to

$$\Gamma'(x) = \phi(x),$$

where $\phi(x) \equiv -\frac{1}{\pi_x} \pi$ and $\pi \equiv \frac{b(\rho + \alpha)}{(r_G + b)\rho} > 0$. If such a solution $\pi$ exists, a corresponding $C$ is $C = -\frac{1}{V_H(\pi)} \frac{r_G + b}{\rho + \alpha}$. And if $\Gamma'(\pi) < 0$, then given strict positivity of $V_H$ it must be that $V_H'(\pi) < 0$ and so $C > 0$.

Suppose first that $\pi \leq 1$. Then $\lim_{x \to \pi} \phi(x) = -\infty$ while $\phi(0) = -\pi^{-1}$. Given the continuity of $\phi$, it must be bounded on $[0, x_0]$ for any $x_0 < \pi$. Thus given $\lim_{x \to 0} \Gamma'(x) = -\infty$, there exists an $x_1 \in (0, \pi)$ such that $\Gamma'(x_1) < \phi(x_1)$. And given that $\lim_{x \to 1} \Gamma'(x)$ is finite and $\Gamma' \text{ is continuous on } (0, 1)$, $\Gamma'$ must be bounded on the closed interval $[x_1, \pi]$. Thus in particular there exists an $x_2 \in (x_1, \pi)$ such that $\Gamma'(x_2) > \phi(x_2)$. Then as $\Gamma'(x) - \phi(x)$ is a continuous function on $[x_1, x_2]$, by the intermediate value theorem there exists an $x^* \in (x_1, x_2)$ such that $\Gamma'(x^*) = \phi(x^*)$. Further, $\Gamma'(x^*) < 0$ given the negativity of $\phi$ on $[0, \pi]$.

Now suppose that $\pi > 1$. In this case $\phi$ is continuous and decreasing on $[0, 1]$, with $\phi(1) = -1/(\pi - 1)$. The argument of the previous paragraph continues to establish existence
of an $x_1 \in (0, 1)$ such that $\Gamma'(x_1) < \phi(x_1)$. Meanwhile $\lim_{x \to 1} \Gamma'(x) = -\rho/\alpha$, while

$$\phi(1) = -\frac{1}{\frac{b(\rho+\alpha)}{(\rho+\alpha)\rho} - 1} < -\frac{1}{\frac{\rho+\alpha}{\rho} - 1} = -\frac{\rho}{\alpha}. $$

Thus $\lim_{x \to 1} \Gamma'(x) > \phi(1)$, and so there exists an $x_2 \in (x_1, 1)$ such that $\Gamma'(x_2) > \phi(x_2)$. The intermediate value theorem then ensures existence of a solution $x^* \in (x_1, x_2)$ to $\Gamma'(x) = \phi(x)$. Given the negativity of $\phi$, it must be that $\Gamma'(x^*) < 0$.

Finally, I must establish that $\tilde{V}$ can be extended to a $C^2$ function on $[\pi, 1]$. Once I have done this, it is automatic that $\tilde{V}$ satisfies the ODE on the extended domain simply by taking limits of each side of the ODE as $\pi_0 \to 1$. I have already shown that $\lim_{x \to 1} \tilde{V}^{(n)}(x)$ all exist and are finite for $n = 0, 1, 2$. Defining $\tilde{V}(1) = \lim_{x \to 1} \tilde{V}(x)$ extends $\tilde{V}$ continuously to $[\pi, 1]$. But I must check that the first two derivatives of the resulting function exist at 1 and are continuous. To do this, I invoke the mean value theorem. For any $x \in (\pi, 1)$ the MVT ensures existence of a $y(x) \in (x, 1)$ such that

$$\frac{\tilde{V}(1) - \tilde{V}(x)}{1 - x} = \tilde{V}'(y(x)).$$

Taking $x \to 1$ implies $y(x) \to 1$ by the squeeze theorem. Then as the limit of $\tilde{V}'$ exists, so does the derivative of $\tilde{V}$ at $x = 1$, and $\tilde{V}'(1) = \lim_{x \to 1} \tilde{V}'(x)$. Thus $\tilde{V}$ is a $C^1$ function on $[\pi, 1]$. Another application of the MVT in exactly the same fashion ensures that $\tilde{V}$ is $C^2$.

### C.11 Proof of Lemma 11

**Lemma 18.** $f$ is a continuous, strictly increasing function on $[\pi, 1]$ satisfying $f(\pi) = 0$ and $f(1) = F_0^*$. By definition, each $p^\pi$ solves the SDE

$$p_t = p_0 + \int_0^t \left( -\alpha p_s - \left( \frac{\Delta r}{\sigma} \right)^2 p_s^2 (1 - p_s) \right) ds + \int_0^t \frac{\Delta r}{\sigma} p_s (1 - p_s) dZ^B_s$$

with initial condition $p_0 = x$. To avoid the possibility that this SDE does not have a unique strong solution, I assume in particular that $p^\pi$ solves the regularized SDE

$$p_t = p_0 + \int_0^t \left( -\alpha \phi(p_s) - \left( \frac{\Delta r}{\sigma} \right)^2 \phi(p_s)^2 (1 - \phi(p_s)) \right) ds + \int_0^t \frac{\Delta r}{\sigma} \phi(p_s) (1 - \phi(p_s)) dZ^B_s,$$
where \( \phi(y) = \min\{\max\{y, 0\}, 1\} \). (I will show in a moment that a solution to the regularized SDE is also a solution to the original one.) The coefficients of this SDE are continuous, bounded, continuously differentiable on \([0, 1]\) (taking one-sided derivatives at the boundaries), and constant outside \([0, 1]\). Hence they are globally Lipschitz continuous and satisfy a quadratic growth condition, so by Theorems 5.2.5 and 5.2.9 of Karatzas and Shreve (1991) there exists a unique strong solution of the SDE for every initial condition \( x \in \mathbb{R} \) for all time.

In particular, strong uniqueness ensures that for each \( x \in (0, 1) \), \( 0 < p^*_{t} < 1 \) for all \( t > 0 \) a.s. For note that \( p = 0 \) is a solution to the SDE as well, and so if \( p_t = 0 \) then \( p_0 = 0 < x \). And similarly \( p_t = p_0 - \alpha t \) is a solution to the SDE for any \( p_0 > 1 \) and \( t \leq (p_0 - 1)/\alpha \). Thus if \( p_t \geq 1 \) for \( t > 0 \) then \( p_0 = p_t + \alpha t > x \). This verifies the earlier claim that a solution to the regularized SDE is also a solution to the original one.

In addition, strong uniqueness ensures that for every \( x' > x \) in \((0, 1)\), \( p^*_{t} > p^*_{I} \) for all \( t \) a.s. For suppose two solutions \( p \) and \( p' \) to the SDE satisfy \( p_t = p'_t \) for some \( t \). Then strong uniqueness ensures that \( p = p' \) a.s. In particular \( p_0 = p'_0 \) a.s., establishing that \( p^*_{t} \neq p'^*_{t} \) for all time a.s. As solutions to the SDE are pathwise continuous and \( p^*_{0} > p'^*_{0} \), this proves the claim. This result establishes that \( \tau^Y_x > \tau^Y_x \) a.s., and so \( f \) is strictly increasing in \( x \).

As for continuity, use the fact that solutions to SDEs with globally Lipschitz coefficients are continuous flows wrt time and initial data, i.e. the map \((x, t) \rightarrow p^I_t(\omega)\) is continuous in \((x, t)\) for a.e. \( \omega \in \Omega \). Fix a state \( \omega \in \Omega \) and for each \( x \) consider the path \( \pi^\omega(\omega) \) with associated stopping time \( \tau^Y_{x}(\omega) \). (Explicit references to \( \omega \) will be suppressed to conserve on notation.) By the reasoning in the previous paragraph, \( \tau^Y_x \) is strictly increasing in \( x \). So fix \( x \) and consider first taking \( x' \uparrow x \). Let \( \lim_{x' \uparrow x} \tau^Y_{x'} = L \) (guaranteed to exist given monotonicity of \( \tau^Y_x \)); then \( \lim_{x' \uparrow x} \pi^Y_{x'} = \pi^Y_L \). And as \( \pi^Y_{x'} = \pi^\omega \) for every \( x' \), this means that \( \pi^Y_L = \pi^\omega \) and therefore that \( \tau^Y_x \leq L \). So by monotonicity \( \lim_{x' \uparrow x} \tau^Y_{x'} = \tau^Y_x \).

Next take \( x' \downarrow x \). Given \( \pi^Y_{x'} = \pi^\omega \), the non-zero quadratic variation of \( \pi^\omega \) and the strong Markov property imply that for every \( \varepsilon > 0 \) there exists a \( t' \in (\tau^Y_x, \tau^Y_x + \varepsilon) \) such that \( \pi^Y_{t'} < \pi^\omega \). Now, \( \pi^Y_{t'} \rightarrow \pi^\omega \) as \( x' \rightarrow x \), so \( \lim_{x' \downarrow x} \tau^Y_{x'} < t' < \tau^Y_x + \varepsilon \) for every \( \varepsilon \). Thus \( \lim_{x' \downarrow x} \tau^Y_{x'} = \tau^Y_x \) pointwise. This establishes that \( \tau^Y_x \) is continuous in \( x \) a.s. The bounded convergence theorem then implies continuity of \( f \).

### C.12 Proof of Lemma 12

By definition,

\[
\tilde{V}(\pi_t) = \mathbb{E}_t^Y \left[ \int_t^{\tau^Y_x} e^{-\rho(s-t)} (\pi_s r_G - (1 - \pi_s) b) \, ds \right]
\]
for all \( t \leq \tau^* \), where \( \tau^* = \inf\{t : \pi_t \leq \pi\} \). Given that \( \pi_t = \mathbb{E}_t^Y[1\{\tau^\theta > t\}] \), this may equivalently be written
\[
\tilde{V}(\pi_t) = \mathbb{E}_t^Y\left[ \int_t^{t\vee(\tau^* \wedge \tau^\theta)} e^{-\rho(s-t)} r_G \, ds - \int_{t\vee(\tau^* \wedge \tau^\theta)}^{\tau^*} e^{-\rho(s-t)} b \, ds \right].
\]

On \( \{\tau^* \geq t\} \) this latter expression may be partitioned as
\[
\mathbb{E}_t^Y\left[ \int_t^{t\vee(\tau^* \wedge \tau^\theta)} e^{-\rho(s-t)} r_G \, ds - \int_{t\vee(\tau^* \wedge \tau^\theta)}^{\tau^*} e^{-\rho(s-t)} b \, ds \right]
= \mathbb{E}_t^Y\left[ 1\{\tau^\theta > t\} \left( \int_t^{\tau^* \wedge \tau^\theta} e^{-\rho(s-t)} r_G \, ds - \int_{\tau^* \wedge \tau^\theta}^{\tau^*} e^{-\rho(s-t)} b \, ds \right) \right]
- \mathbb{E}_t^Y\left[ 1\{t \geq \tau^\theta\} \int_t^{\tau^*} e^{-\rho(s-t)} b \, ds \right].
\]

Now, the definition of the value function \( V \) implies that
\[
V(F^*_t) = \mathbb{E}_t\left[ \int_t^{\tau^* \wedge \tau^\theta} e^{-\rho(s-t)} r_G \, ds - \int_{\tau^* \wedge \tau^\theta}^{\tau^*} e^{-\rho(s-t)} b \, ds \right]
\]
on \( \{\tau^* \wedge \tau^\theta > t\} \), where
\[
F^*_t = \mathbb{E}_t^B\left[ \int_t^{\tau^*} e^{-\rho(s-t)} b \, ds \right].
\]

On the other hand, on \( \{\tau^* \geq t \geq \tau^\theta\} \) it must be that
\[
\mathbb{E}_t\left[ \int_t^{\tau^*} e^{-\rho(s-t)} b \, ds \right] = \mathbb{E}_t^B\left[ \int_t^{\tau^*} e^{-\rho(s-t)} b \, ds \right] = F^*_t.
\]

Therefore by the law of iterated expectations
\[
\tilde{V}(\pi_t) = \mathbb{E}_t^Y\left[ 1\{\tau^\theta > t\} V(F^*_t) \right] - \mathbb{E}_t^Y\left[ 1\{t \geq \tau^\theta\} F^*_t \right]
= \pi_t V(F^*_t) - (1 - \pi_t) F^*_t
= \pi_t V(f(\pi_t)) - (1 - \pi_t) f(\pi_t)
\]
on \( \{\tau^* \geq t\} \). In particular, this holds whenever \( \pi_t = x \) for each \( x \in [\pi, 1] \), proving the lemma.
D  Technical appendix

This section contains technical lemmas used in the proofs of results appearing in the main text.

Lemma 19. Fix a contract $\mathcal{C} = (F, \tau_Y)$. Suppose that for each $t$,

$$\mathbb{E}_t \left[ \int_t^{\tau_Y \wedge \tau^\theta} e^{-\rho(s-t)}b ds + e^{-\rho(\tau_Y \wedge \tau^\theta-t)}F_{\tau_Y \wedge \tau^\theta} \right] \geq F_t$$

whenever $\tau_Y \wedge \tau^\theta > t$. Then $\mathcal{C}$ satisfies IC-G.

Proof. Fix a contract $\mathcal{C} = (F, \tau_Y)$. IC-G holds iff

$$\mathbb{E} \left[ \int_0^{\tau_Y \wedge \tau^\theta} e^{-\rho t}b dt + e^{-\rho(\tau_Y \wedge \tau^\theta)}F_{\tau_Y \wedge \tau^\theta} \right] \geq \mathbb{E} \left[ \int_0^{\tau_Y \wedge \sigma} e^{-\rho t}b dt + e^{-\rho(\tau_Y \wedge \tau^\theta)}F_{\tau_Y \wedge \tau^\theta} \right]$$

for every $\mathbb{F}$-stopping time $\sigma \leq \tau^\theta$. This inequality may be rearranged to obtain

$$\mathbb{E} \left[ 1_{\{\tau_Y \wedge \tau^\theta > \sigma\}} e^{-\rho\sigma} \left( \int_\sigma^{\tau_Y \wedge \tau^\theta} e^{-\rho(t-\sigma)}b dt + e^{-\rho(\tau_Y \wedge \tau^\theta-\sigma)}F_{\tau_Y \wedge \tau^\theta} - F_\sigma \right) \right] \geq 0.$$ 

Then by applying the law of iterated expectations, IC-G is equivalent to the condition that

$$\mathbb{E} \left[ 1_{\{\tau_Y \wedge \tau^\theta > \sigma\}} e^{-\rho\sigma} \left( \mathbb{E}_\sigma \left[ \int_\sigma^{\tau_Y \wedge \tau^\theta} e^{-\rho(t-\sigma)}b dt + e^{-\rho(\tau_Y \wedge \tau^\theta-\sigma)}F_{\tau_Y \wedge \tau^\theta} \right] - F_\sigma \right) \right] \geq 0.$$ 

for every $\sigma \leq \tau^\theta$. Note that on for every time $t$, on the event $\{\sigma = t\}$ the expression inside the expectation is equal to

$$1_{\{\tau_Y \wedge \tau^\theta > t\}} e^{-\rho t} \left( \mathbb{E}_t \left[ \int_t^{\tau_Y \wedge \tau^\theta} e^{-\rho(s-t)}b ds + e^{-\rho(\tau_Y \wedge \tau^\theta-t)}F_{\tau_Y \wedge \tau^\theta} \right] - F_t \right),$$

which by assumption is non-negative. As this holds for every choice of $t$, it must be that

$$1_{\{\tau_Y \wedge \tau^\theta > \sigma\}} e^{-\rho\sigma} \left( \mathbb{E}_\sigma \left[ \int_\sigma^{\tau_Y \wedge \tau^\theta} e^{-\rho(t-\sigma)}b dt + e^{-\rho(\tau_Y \wedge \tau^\theta-\sigma)}F_{\tau_Y \wedge \tau^\theta} \right] - F_\sigma \right) \geq 0$$

for every $\sigma \leq \tau^\theta$. Hence $\mathcal{C}$ satisfies IC-G.
Lemma 20. Fix a contract \( C = (F, \tau^Y) \). Suppose there exist progressively measurable processes \( \gamma \) and \( \beta \) such that for all \( t \), each of the following holds a.e. on \( \{ \tau^Y > t \} \):

- \( F_t = F_0 + \int_0^t \gamma_s \, ds + \int_0^t \beta_s \, dZ^G_s \),
- \( b - \rho F_t + \gamma_t - \frac{\Delta \tau}{\sigma} \beta_t = 0 \),
- \( \mathbb{E}^G \left[ \int_0^t \beta_s^2 \, ds \right] < \infty \),
- \( \beta_t \geq 0 \).

Then \( C \) is an IC contract.

Proof. The proof of Lemma 8 establishes that \( C \) is IC-B, as right-continuity of \( \gamma \) and \( \beta \) are unnecessary for the sufficiency proof of that lemma. I check IC-G using the sufficiency condition of Lemma 19 in the technical appendix. Define a process \( U \) by

\[
U_t = \mathbb{E}_t \left[ \int_t^{\tau^Y \wedge \tau^\theta} e^{-\rho(s-t)} b \, ds + e^{-\rho(\tau^\theta \wedge \tau^Y - t)} F_{\tau^\theta \wedge \tau^Y} \right]
\]

when \( \tau^Y \wedge \tau^\theta > t \), and \( U_t = U_{\tau^Y \wedge \tau^\theta} \) otherwise. Then IC-G obtains if \( U_t \geq F_t \) whenever \( \tau^Y \wedge \tau^\theta > t \).

When \( \tau^Y \wedge \tau^\theta > t \), \( U_t \) may be written

\[
U_t = \int_t^\infty du \, \alpha e^{-\alpha(u-t)} \mathbb{E}^G_t \left[ \int_t^{u \wedge \tau^Y} e^{-\rho(s-t)} b \, ds + e^{-\rho(u \wedge \tau^Y - t)} F_{u \wedge \tau^Y} \right].
\]

Use Ito’s lemma to expand the final term as

\[
e^{-\rho(u \wedge \tau^Y - t)} F_{u \wedge \tau^Y} = F_t + \int_t^{u \wedge \tau^Y} e^{-\rho(s-t)} (\gamma_s - \rho F_s) \, ds + \int_t^{u \wedge \tau^Y} e^{-\rho(s-t)} \beta_s \, dZ^G_s.
\]

Due to the regularity condition imposed on \( \beta \) in the lemma statement, the final term is a \( \mathbb{P}^G \)-martingale. Thus \( U_t \) reduces to

\[
U_t = F_t + \int_t^\infty du \, \alpha e^{-\alpha(u-t)} \mathbb{E}^G_t \left[ \int_t^{u \wedge \tau^Y} e^{-\rho(s-t)} (\gamma_s + b - \rho F_s) \, ds \right].
\]

And for each \( s \in [t, \tau^Y) \), \( \gamma_s + b - \rho F_s = \frac{\Delta \tau}{\sigma} \beta_s \geq 0 \) a.s. So the final term is non-negative, yielding \( U_t \geq F_t \). \( \square \)
Lemma 21. Fix a contract \( C = (F, \tau^Y) \) and define the \( \mathbb{P}^Y \)-adapted process \( M \) by

\[
M_t = \frac{b}{\rho}(1 - e^{-\rho t}) + e^{-\rho t} F_t.
\]

Then \( C \) is an IC-B contract iff \( M_{\tau^Y} \) is a \( \mathbb{P}^B \)-supermartingale.

Proof. Fix any \( t \) and \( \mathbb{P}^Y \)-stopping time \( \tau' \geq t \), and consider states of the world in which \( \tau^Y > t \). Then the definition of \( M \) implies

\[
E^{B}_t[M_{\tau^Y \wedge \tau'}] = E^{B}_t\left[\frac{b}{\rho}(1 - e^{-\rho(\tau^Y \wedge \tau')}) + e^{-\rho(\tau^Y \wedge \tau')} F_{\tau^Y \wedge \tau'}\right] = \frac{b}{\rho}(1 - e^{-\rho t}) + e^{-\rho t} E^{B}_t\left[\frac{b}{\rho}(1 - e^{-\rho(\tau^Y \wedge \tau' - t)}) + e^{-\rho(\tau^Y \wedge \tau' - t)} F_{\tau^Y \wedge \tau'}\right].
\]

Now, suppose \( C \) is an IC-B contract. Then by Lemma 4 the final term is at most \( F_t \), meaning

\[
E^{B}_t[M_{\tau^Y \wedge \tau'}] \leq \frac{b}{\rho}(1 - e^{-\rho t}) + e^{-\rho t} F_t = M_t,
\]

and in particular this holds when \( \tau' = s > t \). Thus \( M^{\tau^Y} \) is a supermartingale.

On the other hand, suppose \( M^{\tau^Y} \) is a \( \mathbb{P}^B \)-supermartingale. Then \( E^{B}_t[M_{\tau^Y \wedge \tau'}] \) is at most \( M_t^{\tau^Y} = M_t \), implying

\[
F_t \geq E^{B}_t\left[\frac{b}{\rho}(1 - e^{-\rho(\tau^Y \wedge \tau' - t)}) + e^{-\rho(\tau^Y \wedge \tau' - t)} F_{\tau^Y \wedge \tau'}\right].
\]

Thus by Lemma 4 \( C \) is an IC-B contract. \( \square \)

Lemma 22. Fix an IC-B contract \( C = (F, \tau^Y) \) satisfying \( \tau^Y = \inf\{t : F_t = 0\} \) and \( F \leq b/\rho \). Then there exists another IC-B contract \( C' = (F', \tau') \) which is cadlag and satisfies \( F' \leq F \) and \( \tau' = \tau^Y \) a.s. Thus \( C' \) yields weakly higher expected profits than \( C \).

Proof. Fix any IC-B contract \( (F, \tau^Y) \) satisfying \( F \leq b/\rho \). I first establish that \( F^{\tau^Y} \) is a \( \mathbb{P}^B \)-supermartingale. Let

\[
M_t = \frac{b}{\rho}(1 - e^{-\rho t}) + e^{-\rho t} F_t.
\]

By Lemma 21, \( M^{\tau^Y} \) is a \( \mathbb{P}^B \)-supermartingale. Thus for all \( t \) such that \( \tau^Y > t \) and all \( s > t \),

\[
E^{B}_t[F^{\tau^Y}_s] = E^{B}_t\left[e^{\rho(s \wedge \tau^Y)} M^{\tau^Y}_s - \frac{b}{\rho}(e^{\rho(s \wedge \tau^Y)} - 1)\right] = E^{B}_t\left[(e^{\rho(s \wedge \tau^Y)} - 1)\left(M^{\tau^Y}_s - \frac{b}{\rho}\right)\right] + E^{B}_t[M^{\tau^Y}_s].
\]
As $F \leq b/\rho$ by assumption, $M \leq b/\rho$ as well. This establishes the bound

$$(e^{\rho(s\wedge \tau')} - 1) \left( M_{s\wedge \tau'}^{\tau'} - \frac{b}{\rho} \right) \leq (e^{\rho t} - 1) \left( M_{s\wedge \tau'}^{\tau'} - \frac{b}{\rho} \right).$$

Hence

$$
\mathbb{E}_t^B[F_{s\wedge \tau'}^{\tau'}] \leq -\frac{b}{\rho} (e^{\rho t} - 1) + e^{\rho t} \mathbb{E}_t^B[M_{s\wedge \tau'}^{\tau'}] \leq -\frac{b}{\rho} (e^{\rho t} - 1) + e^{\rho t} M_{s\wedge \tau'}^{\tau'} = F_{t\wedge \tau'}^{\tau'}.
$$

Thus $F_{t\wedge \tau'}^{\tau'}$ is a $\mathbb{P}^B$-supermartingale.

Now I assume wlog that $F = F_{t\wedge \tau'}^{\tau'}$, if necessary passing to the stopped process, so that $F$ is a $\mathbb{P}^B$-supermartingale. Then by Proposition 1.3.14 of Karatzas and Shreve (1991), $F$ possesses pathwise right limits for all time almost surely, and the process $F'$ constructed by setting $F'_t(\omega) = \lim_{s\uparrow t} F_s(\omega)$ for each $\omega \in \Omega$ is a $\mathbb{P}^B$-supermartingale wrt to the filtration $\{\mathcal{F}_t\}_{t \geq 0}$. As $\mathbb{P}^Y$ is right-continuous under the usual conditions, this means $F'$ is a $\mathbb{P}^B$-supermartingale wrt $\mathbb{P}^Y$. Further, the proposition says that $F'_t \leq F_t$ for all time a.s.

Suppose additionally that $\tau' = \inf\{t : F_t = 0\}$, and define $\tau' = \inf\{t : F'_t = 0\}$. I claim that $\mathcal{C}' = (F', \tau')$ satisfies the conditions of the lemma statement. First note that as $F$ is non-negative, so is $F'$. So $F'$ is a feasible payment process. Let $\tau' = \inf\{t : F'_t = 0\}$. I claim that $\tau' = \tau'$ a.s. Given $F' \leq F$, clearly $\tau' \leq \tau'$. In the other direction, suppose that for some $t$ and in some state of the world $\tau' = t$, so that in particular $\lim_{s\uparrow t} F_s = 0$. Given that $F$ is a $\mathbb{P}^B$-supermartingale, the inequality $F_t \leq \mathbb{E}^B[F_s]$ holds for all $s > t$. As $F$ is uniformly bounded, the bounded convergence theorem then implies that $F_t \leq \mathbb{E}^B[\lim_{s\uparrow t} F_s] = 0$. Thus $\tau' \leq t$, as desired.

From $F' \leq F$ and $\tau' = \tau'$ it follows that $\mathcal{C}'$ pays the agent weakly less than $(F, \tau')$ while operating the project for the same length of time in the good state in all cases. Thus $\mathcal{C}'$ is weakly more profitable for the firm, as claimed. It remains only to show that $\mathcal{C}'$ is IC-B, which I verify using the characterization of Lemma 4.

Whenever $F'_t = F_t$ IC-B is automatic, as the agent’s profits from any delayed reporting strategy are weakly lower under $F'$. So consider any time $t$ such that $F_t > F'_t$. For every $s > t$ and $\mathbb{P}^Y$-stopping time $\tau'' \geq s$, incentive-compatibility of $F$ implies that

$$
F_s \geq \mathbb{E}_s^B \left[ \frac{b}{\rho} (1 - e^{-\rho(\tau' \wedge \tau'') - s}) + e^{-\rho(\tau' \wedge \tau'') - s} F_{\tau' \wedge \tau''}^{\tau'} \right].
$$

Taking expectations wrt $\mathbb{E}^B_t$, multiplying through by $e^{-\rho(s-t)}$, and taking the supremum over
all \(\tau'' \geq s\) gives
\[
\mathbb{E}_t^B[e^{-\rho(s-t)}F_s] \geq \sup_{\tau'' \geq s} \mathbb{E}_t^B\left[\frac{b}{\rho}(e^{-\rho(s-t)} - e^{-\rho((\tau'')^\prime - t)}) + e^{-\rho((\tau'')^\prime - t)}F_{\tau'' \wedge \tau^\prime} \right].
\]
Rearranging yields
\[
\mathbb{E}_t^B\left[\frac{b}{\rho}(1 - e^{-\rho(s-t)}) + e^{-\rho(s-t)}F_s\right] \geq \sup_{\tau'' \geq s} \mathbb{E}_t^B\left[\frac{b}{\rho}(1 - e^{-\rho((\tau'')^\prime - t)}) + e^{-\rho((\tau'')^\prime - t)}F_{\tau'' \wedge \tau^\prime} \right].
\]
Now take \(s \downarrow t\). The rhs is increasing in \(s\), so the limit of the rhs exists and is equal to the supremum taken over all \(\tau' > t\). Meanwhile the interior of the expectation on the lhs is uniformly bounded by \(b/\rho\), so the bounded convergence theorem allows the limit and expectation to be swapped. As \(\lim_{s \downarrow t} F_s = F_t'\) by definition, the resulting inequality is
\[
F_t' \geq \sup_{\tau'' > t} \mathbb{E}_t^B\left[\frac{b}{\rho}(1 - e^{-\rho((\tau'')^\prime - t)}) + e^{-\rho((\tau'')^\prime - t)}F_{\tau'' \wedge \tau' \wedge \tau''} \right].
\]
As \(F \geq F'\) and \(\tau' = \tau'\) a.s., the inequality is preserved when \(F\) is replaced by \(F'\) and \(\tau'\) is replaced by \(\tau'\) on the rhs, yielding
\[
F_t' \geq \sup_{\tau'' > t} \mathbb{E}_t^B\left[\frac{b}{\rho}(1 - e^{-\rho((\tau'')^\prime - t)}) + e^{-\rho((\tau'')^\prime - t)}F_{\tau'' \wedge \tau''} \right].
\]
Finally, the expected payoff of any stopping time \(\tau'' \geq t\) is equal to a weighted average of \(F_t'\) and the expected payoff from some \(\tau'' > t\). As no stopping time \(\tau'' > t\) can improve on \(F_t'\), neither can a stopping time \(\tau'' \geq t\). It follows that \(C'\) satisfies IC-B.

\[\square\]

Remark. The cadlag golden parachute \(F'\) is not necessarily a modification of \(F\) in the technical sense. That is, there may exist times \(t\) for which \(F_t \neq F'_t\) with strictly positive probability. However, Lemma 22 establishes that any such disagreement preserves incentive-compatibility and weakly improves profitability of the contract.

**Lemma 23.** Suppose that for some \(F^* > 0\), \(V : [0,F^*] \to \mathbb{R}\) is a \(C^2\) function satisfying \(V'(0) = 0\), \(V''(F^*) = 0\), and the HJB equation on \([0,F^*]\). Then:

- \(V'(F_0) > 0\) and \(V''(F_0) < 0\) for all \(F_0 \in [0,F^*)\),
- The unique maximizers \((\gamma^*(F_0),\beta^*(F_0))\) of the HJB equation for each \(F_0 \in [0,F^*)\) are
  \[
  \beta^*(F_0) = -\frac{\Delta r}{\sigma} \frac{V'(F_0)}{V''(F_0)}, \quad \gamma^*(F_0) = \beta^*(F_0) \frac{\Delta r}{\sigma} + \rho F - b,
  \]
• \((\gamma^*(F^*), \beta^*(F^*)) = (\rho F^* - b, 0)\) are maximizers of the HJB equation for \(F_0 = F^*\).

Proof. Fix \(F_0 \in [0, F^*]\) and suppose \(V'(F_0) < 0\). Then \(\gamma\) may be taken arbitrarily large and negative without violating IC, so that the rhs of the HJB equation equals \(\infty\). However, the lhs of the HJB equation is finite, a contradiction. So \(V'(F_0) \geq 0\). An analogous argument rules out \(V''(F_0) > 0\) by considering arbitrarily large and positive \(\beta\). So both \(V'\) and \(-V''\) must be weakly positive. Next suppose that \(V'(F_0) > 0\) and \(V''(F_0) = 0\). Then again the rhs of the HJB equation may be made arbitrarily large by taking both \(\gamma\) and \(\beta\) large and positive. So either \(V'(F_0) > 0\) and \(V''(F_0) < 0\), or else \(V'(F_0) = 0\) and \(V''(F_0) \leq 0\). In particular, \(V\) is weakly increasing on \([0, F^*]\).

Suppose that \(V'(F_0) = 0\) and \(V''(F) \leq 0\) for some \(F_0 \in [0, F^*]\). Then \(\beta^*(F_0)V''(F) = 0\) for any maximizer \(\beta^*(F_0)\), in which case the HJB equation implies \((\rho + \alpha) V(F_0) = r_G - \alpha F\). But at \(F^*\) the same argument applies, so that \((\rho + \alpha) V(F^*) = r_G - \alpha F^* < r_G - \alpha F_0 = \rho V(F_0)\). This contradicts the fact that \(V\) is weakly increasing on \([0, F^*]\), so it must be that \(V'(F_0) > 0\) and \(V''(F_0) < 0\) for all \(F_0 \in [0, F^*]\).

The stated maximizers of the HJB equation for \(F_0 \in [0, F^*]\) follow from noting that the IC constraint must bind, else increasing \(\gamma\) a sufficiently small amount would increase the rhs of the HJB equation while preserving the IC constraint given \(V'(F_0) > 0\). Thus \(\gamma = \frac{\Delta r}{\sigma} \beta + \rho F_0 - b\) at the optimum, and substituting this expression into the rhs of the HJB equation yields a concave quadratic in \(\beta\) with unique maximum \(\beta = -\frac{\Delta r}{\sigma} \frac{V'(F_0)}{V''(F_0)}\). And as \(V''(F^*) \leq 0\) by continuity of \(V''\) while \(V'(F^*) = 0\), the HJB equation at \(F^*\) is maximized by \((\gamma, \beta) = (\rho F^* - b, 0) \in IC(F^*)\). \(\Box\)