Computing equilibria of GEI by relocalization on a Grassmann manifold

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Abstract

An algorithm is described to compute equilibria of the general economic model with incomplete asset markets, that is, of GEI. The algorithm is based on the existence of a route of zeros of a homotopy whose domain includes the price simplex and a Grassmann Manifold. This route is followed, in effect, by localizing and following diffeomorphic pieces in Euclidean space, and by relocalizing as is necessary.

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1. Introduction

The authors have been involved with two projects, Brown et al. (1996a,b), directed toward the computation of equilibria for general economic models with incomplete asset markets, hereafter denoted GEI. These papers offer the first avenues for computing equilibria for GEI beyond trivial size. The essence of the difficulty of proofs of existence of equilibria as well as the computation thereof is the discontinuity of excess demand, introduced by possible change in dimension of
the asset returns matrix, as the price varies. This feature engendered the celebrated introduction of the Grassmann manifold to GEI; see Duffie and Shafer (1985), and the surveys by Geanakoplos (1990) and Magill and Shafer (1991). Our projects (Brown et al., 1996a,b) offer methods to prove the existence of equilibria as well as computing equilibria for GEI. Brown et al. (1996a), which develops a fixed point problem on the Grassmann manifold as a special case, is cast in a setting more general than the economic problem. Our purpose here is to complete and clarify the connection of Brown et al. (1996a) to the computation of equilibria for GEI. In particular, for generic endowments of one agent and assets we show how to compute an equilibrium of GEI.

The methods of Brown et al. (1996a,b) and, indeed, those herein, use homotopies and path following. This is not surprising since all general purpose, theoretically sound algorithms for computing economic equilibria have, to date, had an underlying homotopy and path-following interpretation; see, for example, Scarf and Hansen (1973), Eaves (1972), and Smale (1976); further homotopies have been suggested for the computation of GEI; see Geanakoplos and Shafer (1990). The avenue of Brown et al. (1996b), and that used herein, underlies current proofs of existence of equilibria for GEI in the present literature and is based on the Grassmann manifold, whereas the approach of Brown et al. (1996b) does not use the Grassmann manifold. The methods of Brown et al. (1996a,b) have relative advantages, namely Brown et al. (1996a) might be more robust, but Brown et al. (1996b) operates in a smaller dimension search space. These methods and others are being developed for computer implementation and comparison.

We borrow basic terminology and results in differential topology from Guillemin and Pollack (1974) and Hirsch (1976).

2. The GEI model

The version of GEI we employ has agents 0, 1, ..., n, time periods 0 and 1, one state 0 in time 0, states 1, ..., s in time 1 and assets 1, ..., t. There are m state-dependent goods yielding a total of \( M = (s + 1)m \) goods.

Let \( R^m \) denote the m-dimensional Euclidean space of \( z = (z_1, \ldots, z_m) \) with \( z_i \) in \( R \). By \( z \) positive or \( z > 0 \) we denote \( z_i > 0 \) for all \( i = 1, \ldots, m \). Let \( R^m_+ = \{ z \in R^m : z \geq 0 \} \), \( R^m_+ = \{ z \in R^m : z > 0 \} \), and \( R^M = (R^m)^{s+1} \). In particular, for \( z \) in \( R^M \) we have \( z = (z_0, \ldots, z_s) \) with each \( z_i \) in \( R^m \) and \( z_{ij} \) in \( R \). Let \( \| \cdot \| \) indicate the Euclidean norm.

Let the \( M \)-vector \( p = (p_0, p_1, \ldots, p_s) \) represent the prices for the \( M \) goods. Let \( P = \{ p \in R^M : p > 0, e^T p = 1 \} \) be the (open) price simplex, where \( e = (1, \ldots, 1) \). We regard \( P \) as an \((M - 1)\)-manifold without boundary.

Let the \( M \)-vector \( z = (z_0, \ldots, z_s) \) represent the consumption bundle of an agent \( i = 0, \ldots, n \) and let the \( M \)-vector \( w^i = (w^i_0, \ldots, w^i_s) \) represent the endowment bundle for agent \( i = 0, \ldots, n \).
We adopt the box notation. For the $M$-vectors $p = (p_0, \ldots, p_s)$ and $z = (z_0, \ldots, z_s)$ we define $p \Box z$ to be the $s$-vector $(p^T z_1, \ldots, p^T z_s)$. We extend the box notation to $M \times t$ matrices $z$.

The $t$ assets of GEI are described by the $M \times t$ matrix

\[
\begin{pmatrix}
A_0 \\
A_1 \\
A_2 \\
\vdots \\
A_s
\end{pmatrix},
\]

where each $A_i$ is $m \times t$ and $A_0 \equiv 0$. We assume $t \leq s$.

The asset returns matrix $p \Box A$ is defined to be the $s \times t$ matrix

\[
\begin{pmatrix}
p_1^T A_1 \\
p_2^T A_2 \\
\vdots \\
p_s^T A_s
\end{pmatrix}
\]

where each $p_i^T A_i$ is $1 \times t$. Let $G \equiv G^s_t$ be the set of all $t$-dimensional planes in $R^s$ through the origin. $G$ is a compact, boundaryless, differential manifold of dimension $(s-t)t$ known as the Grassmann manifold. Given $\gamma$ in $G$ we shall be concerned if the column span of $p \Box A$, denoted $S(p \Box A)$, is a subset of $\gamma$, i.e.

\[S(p \Box A) \subseteq \gamma.\]

Let $u^i : R^M \rightarrow R^1$ be the utility function for agent $i = 0, \ldots, n$. We make common smoothness, monotonicity, curvature, and boundary condition assumptions on the utility functions, namely the function $u^i$ is smooth ($C^\infty$), the derivative $\nabla u^i$ is positive on $R^M_+$, $\nabla u^i(z)y = 0$ with $y \neq 0$ implies $y^T \nabla^2 u^i(z)y < 0$ for $z$ in $R^M_+$, and the closure of $\{y \in R^M : u^i(y) \geq u^i(z)\}$ is a subset of $R^M_+$ for $z$ in $R^M$.

We define the unconstrained excess demand $Z_0 : P \rightarrow R^M$ for agent $0$ by

\[
Z_0(p) = z^0 - w^0, \\
z^0 = \operatorname{argmax}_{z \geq 0} u^0(z)
\]

subject to $p^T z \leq p^T w^0$.

We define the constrained excess demand $Z^i : P \times G \rightarrow R^M$ for agents $i$ in $\nu \equiv \{1, \ldots, n\}$ by

\[
Z^i(p, \gamma) = z^i - w^i, \\
z^i = \operatorname{argmax}_{z \geq 0} u^i(z)
\]

subject to $p^T z \leq p^T w^i$, $p \Box (z - w^i) \in \gamma$. 

The constrained excess demand \( Z_v : P \times G \to R^M \) is defined by
\[
Z_v(p, \gamma) = \sum_{i \in \nu} Z_i(p, \gamma).
\]
The excess demand \( Z_0 + Z_v \) is defined by
\[
(p, \gamma) \mapsto Z_0(p) + Z_v(p, \gamma).
\]
From our assumptions on the utility functions \( u^i \) the excess demands \( Z_0 \) and \( Z_v \) are well-defined, smooth, bounded below and Walrasian. By Walrasian we mean, for example,
\[
p^T Z_v(p, \gamma) = 0 \quad \text{for all} \quad p \in P \quad \text{and} \quad \gamma \in G.
\]
Furthermore, \( Z_0 \) is coercive on \( P \), i.e. \( p \to \infty \) in \( P \) implies \( Z_0(p) \to \infty \) in \( R^M \). By \( p \to \infty \) in \( P \) we mean that \( p \) leaves every compact subset of \( P \).

A pseudo-equilibrium of GEI is defined to be a pair \((p, \gamma)\) in \( P \times G \) where both the excess demand is zero and the span of the asset returns is a subset of the plane \( \gamma \), i.e. both
(a) \( Z_0(p) + Z_v(p, \gamma) = 0 \) and
(b) \( S(p \square A) \subseteq \gamma \).

If, in fact, we also have
\[
S(p \square A) = \gamma,
\]
then a pseudo-equilibrium is defined to be an equilibrium; in this case we have solved \( Z_0(p) + Z_v(p, S(p \square A)) = 0 \). As we shall see, and as is known, for generic \((w^0, A)\), pseudo-equilibria are equilibria.

If \((p, \gamma)\) is a pseudo-equilibrium, then observe that since
\[
S(p \square Z_v(p, \gamma)) \subseteq \gamma \quad \text{and} \quad Z_0(p) = -Z_v(p, \gamma),
\]
we have
\[
p \square Z_0(p) \in \gamma.
\]
That is, agent 0 was released from the net expenditure constraint \( p \square (z - w^0) \in \gamma \), but at a pseudo-equilibrium this constraint is regained. The purpose of this device, referred to as Cass's trick, is to obtain an excess demand which is coercive.

We have bypassed an economically compelling description of an equilibrium as well as the 'no arbitrage' transformation. Indeed, we have proceeded directly to a formulation of a pseudo-equilibrium which is suited to our computation. A full discussion of the economic motivation for GEI and an equilibrium, as we have stated it, can be found in the surveys by Geanakoplos (1990) and Magill and Shafer (1991).

3. The homotopy \( H \)

We set about defining the homotopy \( H \) to be used in the computation of a pseudo-equilibrium for GEI. Our homotopy \( H = (E, F) \) is in two parts. \( E \) is related to the excess demand and \( F \) is related to the span of the asset returns.
For a subset $\gamma$ of $R^m$ let $\gamma^\perp$ denote the orthogonal set, i.e.

$$\gamma^\perp = \{ z \in R^m ; z^T y = 0, \forall y \in \gamma \}.$$

For a vector $v$ in $R^m$ we define $v^\perp \equiv \{0\}^\perp$. If $\gamma$ is a subspace, then $\gamma^\perp$ is the orthogonal complement. Let $Q = e^\perp$ in $R^M$, where $e = (1, \ldots, 1)$.

As usual our homotopy parameter $\theta$ lies in the unit interval $I = [0, 1]$. Our homotopy $H = (E, F)$ has a domain $P \times G \times I$ and a range $Q \times R^m$, where

$$E : P \times G \times I \to Q \quad \text{and} \quad F : P \times G \times I \to R^m.$$

We introduce a rotation that transforms excess demand into a fixed $(M - 1)$-dimensional set, namely $Q$. For each price $p$ in $P$ we define the linear bijection $T_p : p^\perp \to Q$ by

$$T_p(z) = z - e^T z \frac{\sqrt{M} p + \| p \| e}{\sqrt{M} + \| p \| M}.$$

The following lemma indicates that $T_p$ rotates $p^\perp$ to $Q$.

**Lemma 3.1.** Let us fix $p$ in $P$. For any $z$ and $y$ in $p^\perp$ we have.

(a) $T_p(z)$ is in $Q$;

(b) $T_p(z) = T_p(y)$ implies $z = y$;

(c) $\| z \| = \| T_p(z) \|$;

(d) $z^T y = 0$ implies $T_p(z)^T T_p(y) = 0$.

**Proof.** The algebra is straightforward. \(\square\)

We define the parametric excess demand $E : P \times G \times I \to Q$ by

$$E(p, \gamma, \theta) = T_p(Z(p, \gamma, \theta)) \quad \text{and} \quad Z(p, \gamma, \theta) = Z_0(p) + \theta Z_v(p, \gamma).$$

where $Z_0$ and $Z_v$ are the unconstrained and constrained excess demands.

From Lemma 3.1 we have $\| E \| = \| Z \|$ so $E(p, \gamma, \theta) = 0$ if and only if $Z(p, \gamma, \theta) = 0$. The reader should notice that if $Z(p, \gamma, 1) = 0$, then the excess demand $Z_0(p) + Z_v(p, \gamma)$ is zero.

The next result shows that zeros of $E$ cannot tend to infinity in $P$. We observe that $Z(p, \gamma, \theta)$ is coercive on $P$, because $Z_0$ is coercive and since $Z_0$ and $Z_v$ are bounded below on $P$.

**Lemma 3.2.** There is a compact subset $C$ of $P$ such that $E(p, \gamma, \theta) = 0$ implies $p$ is in $C$.

**Proof.** Otherwise for zeros $(p, \gamma, \theta)$ of $E$ we could have $p \to \infty$ in $P$. But $E(p, \gamma, \theta) = 0$ implies $Z(p, \gamma, \theta) = 0$. However, $p \to \infty$ in $P$ implies $Z(p, \gamma, \theta) \to \infty$ in $R^M$, a contradiction to $Z(p, \gamma, \theta) = 0$. \(\square\)
For $\gamma$ in $G$ let $\delta = \gamma^\perp$ be the orthogonal complement of $\gamma$ in $R^s$. Let $\Pi_\delta : R^s \to \delta \subseteq R^s$ be the projection from $R^s$ to $\delta$. The reader should note that $S(p \boxdot A) \subseteq \gamma$ if and only if $\Pi_\delta(p \boxdot A) \equiv (\Pi_\delta(p \boxdot A^1), \ldots, \Pi_\delta(p \boxdot A^e))$ is zero in $(R^s)^e \equiv R^{se}$. Our use of projections in this section follows that in Hirsch et al. (1990).

We define the parametric asset returns constraint $F : P \times G \times I \to R^{se}$ by

$$F(p, \gamma, \theta) \equiv \Pi_\delta'(p \boxdot A).$$

Finally, we define the homotopy $H$ by

$$H(p, \gamma, \theta) = (E(p, \gamma, \theta), F(p, \gamma, \theta)).$$

Our aim is to compute a zero $(p, \gamma, 1)$ of $H$, i.e.

$$E(p, \gamma, 1) = 0, \quad F(p, \gamma, 1) = 0.$$ 

That is,

$$Z(p, \gamma, 1) = 0, \quad \Pi_\delta'(p \boxdot A) = 0$$

and, thereby, obtain a pseudo-equilibrium of GEI.

To compute a zero $(p', \gamma', 1)$ of $H$ our plan is to begin with the unique zero $(p^0, \gamma^0, 0) = (p, \gamma, 0)$ of $H$ with $\theta = 0$ and then follow the route $W$ of $H^{-1}(0)$ in $P \times G \times I$ to the boundary $\theta = 1$ and thereby compute the desired zero $(p', \gamma', 1)$ of $H$. See Fig. 1.

There are four matters to be established before the computation is compelling: (i) that there is a unique starting point in $P \times G \times 0$; (ii) that there is a route $W$ to follow in $P \times G \times I$, and (iii) that the route $W$ terminates in $P \times G \times 1$, and that $W$ can be followed. We address each of these in order.

**4. A unique start $(p^0, \gamma^0, 0)$**

Computation begins in the bottom of $P \times G \times I$ with a point $(p, \gamma, \theta) = (p^0, \gamma^0, 0)$ in $H^{-1}(0)$. We want to show that such a point is unique. That is to
say, we want to show that with generic endowments $w^0$ of agent 0 and assets $A$ the solution $(p, \gamma)$ of

$$Z_0(p) = 0, \quad \Pi_{\delta^i}(p \Box A) = 0$$

is unique.

If we examine the first-order optimality conditions in the utility optimization for calculating $Z_0$ then we see that solving $Z_0(p) = 0$ is to solve

$$z^0 = w^0, \quad \nabla u^0(z^0) = \lambda p^T, \quad e^T p = 1, \quad \lambda > 0.$$ 

Thus we must have

$$\frac{\nabla u^0(w^0)^T}{\nabla u^0(w^0)e}.$$

In particular, given a positive $w^0$, the price $p^0$ is unique.

It remains to solve $\Pi_{\delta^i}(p^0 \Box A) = 0$ for $\gamma^0$ in $G$, where $\delta^\perp = \gamma^0$, i.e. to find a $\gamma^0$ in $G$ with $S(p^0 \Box A) \subseteq \gamma^0$. Clearly, if the rank of $p^0 \Box A$ is $t$, then $\gamma^0$ is unique.

For fixed $p = p^0$ and for $i = 1, \ldots, t$ we consider the solution of the following system in the variable $\xi$ in $R^{t-1}$:

$$(p \Box A^i)\xi - p \Box A^i = 0,$$

where $A^i$ is the $i$th column of $A$ and $A^\perp$ is the matrix $A$ with column $i$ deleted. The range space of the system has dimension $s$ and the derivative with respect to $A$ has rank $s$. Thus by a parametric version of Sard’s Theorem (see Guillemin and Pollack, 1974, p. 68), for generic $A$ the dimension of the solution space of $\xi$ is of dimension $(s - 1) - s = -1$. That is, for generic $A$ the solution set of the system is empty. Because the set $i = 1, \ldots, t$ is finite, for generic $A$ the system $p \Box A$ has rank $t$ for $p = p^0$.

Thus for generic endowments $w^0$ for agent 0 and assets $A$, a point $(p^0, \gamma^0, 0)$ in both $H^{-1}(0)$ and $P \times G \times 0$ is unique.

5. An atlas for $G$

To show that $H^{-1}(0)$ is a one-manifold and to follow the route of the one-manifold, we need an atlas for the differential manifold $G$. To follow the route of $H^{-1}(0)$ beginning at $(p^0, \gamma^0, 0)$ in $P \times G \times I$ we localize and sequentially relocalize to a neighborhood of the Grassmannian $G$ and transfer the computation to Euclidean space.

Let $\| \cdot \|$ be the maximum norm for a vector, i.e. $\| x \| = \max_i | x_i |$, with $x_i$ in $R$. Let us select $r$ in the range $1 < r < +\infty$ and define $X$ by

$$X = \{ x \in R^{(s-1)r}; \| x \| < r + 1 \}. $$
For the vector $x$ in $X$ we define $\text{wrap}(x)$ to be the $((s - t) \times t)$-matrix
\[
\begin{bmatrix}
x_1 & \ldots & x_t \\
x_{t+1} & \ldots & x_{2t} \\
\vdots & & \\
x_{(s-t)t-t+1} & \ldots & x_{(s-t)t}
\end{bmatrix}.
\]

Let $B$ be the set of all subsets of $\sigma = \{1, \ldots, s\}$ of size $t$. We refer to $\beta$ in $B$ as a localization index; this will index a chart. Given $\beta$ in $B$ let $\alpha = \sigma \setminus \beta$ be the complement of $\beta$ in $\sigma$. We take the order of the elements in both $\beta$ and $\alpha$ as the natural order.

For $\beta$ in $B$ let $\Pi_B : R^s \rightarrow R^t$ select the $t$ rows indexed by $\beta$ and similarly let $\Pi_{\alpha} : R^s \rightarrow R^{s-t}$ select the $s - t$ rows indexed by $\alpha$; we also apply both $\Pi_{\beta}$ and $\Pi_{\alpha}$ to the matrices. For $\beta$ in $B$ and $x$ in $X$ we define the $(s \times t)$-matrix $\beta(x)$ by
\[
\Pi_{\beta} \beta(x) = 1,
\]
\[
\Pi_{\alpha} \beta(x) = \text{wrap}(x).
\]
That is, the rows of $\beta(x)$ indexed by $\beta$ form the identity matrix and the rows of $\beta(x)$ indexed by $\alpha$ is the $((s - t) \times t)$-matrix $\text{wrap}(x)$.

**Example.** For $s = 4$, $t = 2$, $\beta = \{1, 3\}$, and $x = (x_1, x_2, x_3, x_4)$ we have
\[
\text{wrap}(x) = \begin{bmatrix} x_1 & x_3 \\ x_3 & x_4 \end{bmatrix}, \quad \beta(x) = \begin{bmatrix} 1 & 0 \\ x_1 & x_2 \\ 0 & 1 \\ x_3 & x_4 \end{bmatrix}.
\]

Similarly, define the $s \times (s - t)$ matrix $\alpha(x)$ by
\[
\Pi_{\beta} \alpha(x) = -\text{wrap}(x)^T,
\]
\[
\Pi_{\alpha} \alpha(x) = 1.
\]

We observe that the columns of $(\alpha(x), \beta(x))$ span $R^s$ and that the columns of $\alpha(x)$ are orthogonal to the columns of $\beta(x)$.

Let $S\beta(x)$ and $S\alpha(x)$ be the vector space spanned by the columns of $\beta(x)$ and $\alpha(x)$, respectively. A vector $y$ in $S\alpha(x)$ is determined by the $\alpha$ coordinates, i.e.
\[
y = \alpha(x) \Pi_{\alpha} y.
\]
We observe that $S\beta(x) = S\beta(y)$ implies $x = y$. The projection $\Pi_{\alpha} : S\alpha(x) \rightarrow R^{s-t}$ is a linear bijection.

We define the multifunction $\Lambda : G \ni B$ by $\Lambda(\gamma) = \{ \beta \in B : \gamma = S\beta(x), |x| \leq 1 \}$. Given the vectors spanning $\gamma$, an element $\beta$ in $\Lambda(\gamma)$ can be effectively computed, see Eaves (1993). The multifunction $\Lambda$ is called a locator and its purpose is to ‘center’ a chart at $\gamma$. For $\beta$ in $B$ we define the open subset $U_\beta$ of $G$ by
\[
U_\beta = \{S\beta(x) : x \in X\}.
\]
The $U_\beta$ for $\beta$ in $B$ yield a finite open cover of the manifold $G$. 
The collection of \( \mathcal{U} = \{S\beta, U_{\beta}\}: \beta \in B \) forms a finite atlas for the manifold \( G \), see Brown et al. (1996a). In particular, for \( \beta \) in \( B \),
\[
U_{\beta} \subseteq G
\]
\[
S\beta \uparrow
\]
\[
X \subseteq R^{(s-n)}
\]
is a diffeomorphism.

For \( \beta \) in \( B \) and \( x \) in \( X \), the \( s \times s \) matrix that projects \( R^s \) to \( S\beta(x) \) is given by
\[
\beta(x)\left( \beta(x)^T \beta(x) \right)^{-1}\beta(x)^T,
\]
and similarly for \( \alpha \).

6. The localization \( H_\beta \) of \( H \)

To justify the properties of \( H^{-1}(0) \) and to follow a route in \( H^{-1}(0) \) we sequentially localize to Euclidean space. We select \( \beta \) in \( B \) and set \( \alpha = \sigma \setminus \beta \). We define the projection \( \Pi_{\alpha}': R^{st} \to R^{(s-n)t} \) by
\[
\Pi_{\alpha}'(y) = (\Pi_{\alpha}(y_1), \ldots, \Pi_{\alpha}(y_s)),
\]
where \( y = (y_1, \ldots, y_s) \) and each \( y_i \) is in \( R^s \). For each \( x \) in \( X \) we observe that \( \Pi_{\alpha}' \) is a linear bijection from \( S\alpha(x)' = (S\alpha(x))' \subseteq R^{nt} \) to \( R^{(s-n)t} \). In particular, for \( y \) in \( S\alpha(x)' \) we have \( y = 0 \) if and only if \( \Pi_{\alpha}'(y) = 0 \). Let \( \iota \) represent the identity map on each of \( P \), \( Q \), and \( I \).

For \( \beta \) in \( B \) we shall understand and follow a route of zeros of \( H \) of
\[
H: P \times U_{\beta} \times I \to Q \times R^{st}
\]
by understanding and following paths of zeros of
\[
H_\beta: P \times X \times I \to Q \times R^{(s-n)t},
\]
where \( H_\beta \equiv (E_\beta, F_\beta) \) is defined by
\[
E_\beta(p, x, \theta) \equiv E(p, S\beta(x), \theta) \quad \text{and} \quad F_\beta(p, x, \theta) \equiv \Pi_{\alpha}'F(p, S\beta(x), \theta)
\]
See Fig. 2.

\[
\begin{array}{c}
P \times U_{\beta} \times I \xrightarrow{H = (E, F)} Q \times R^{st} \\
(t, S\beta, 1) \xrightarrow{H_\beta = (E_\beta, F_\beta)} (t, \Pi_{\alpha}) \\
P \times X \times I \xrightarrow{H_\beta = (E_\beta, F_\beta)} Q \times R^{(s-n)t}
\end{array}
\]

Fig. 2. Localized homotopy \( H_\beta \).
We observe that \((t, S, \delta, \nu)\) is a diffeomorphism from \(P \times X \times I\) to \(P \times U_\beta \times I\) and that 
\[ F(p, S\beta(x), \theta) = \Pi^t_s(p \square A) = 0, \]
where \(\delta = S\alpha(x)\) if and only if 
\[ \Pi^t_s F(p, S\beta(x), \theta) = 0. \]
Thus the zeros of \(H_\beta^t\) on \(P \times X \times I\) are diffeomorphic by \((t, S, \delta, \nu)\) to the zeros of \(H\) on \(P \times U_\beta \times I\). The advantage of following the zeros of \(H_\beta^t\) over that of \(H\) are that the domain and range of \(H_\beta^t\) are Euclidean and the dimensions are 'right'.

7. The regularity of zero for \(H\)

For generic endowments \(w^0\) of agent 0 and generic assets \(A\) we show that 0 is a regular value of \(H\) and \(\partial H\), i.e. that \(H\) and \(\partial H\) are transverse to zero. It is sufficient to show that the derivative of \(H_\beta^t\) with respect to \(w^0\) and \(A\) is onto, or equivalently, that the derivative of \(E_\beta\) with respect to \(w^0\) has rank \(M - 1\) and the derivative of \(F_\beta\) with respect to the assets \(A\) has rank \((s - t)t\). Again we use a parametric version of Sard's Theorem; see Guillemin and Pollack (1974, p. 68).

The derivative of \(Z_0\) with respect to \(w^0\) is of the form 
\[ qp^T - I, \]
where \(p^Tq = 1\). Hence the rank of \(\nabla_{w^0} Z_0(p)\) is \(M - 1\). Because \(p^T Z_0(p) = 0\) we have \(p^T \nabla_{w^0} Z_0(p) = 0\) and we see that the columns of \(\nabla_{w^0} Z_0(p)\) lie in the set \(p\perp\). As the transformation \(T_p\) rotates \(p\perp\) to \(Q\) we see that \(\nabla_{w^0} E_\beta(p, S\beta(x), \theta)\) also has rank \(M - 1\).

Let \(\alpha = \alpha \setminus \beta\) and let \(\delta = S\alpha(x)\) for \(x\) in \(X\). For positive prices the derivative of \(p \square A\) with respect to \(A\) has rank \(st\). Thus the derivative of \(F_\beta = \Pi^t_s(p \square A)\) with respect to \(A\) at \((p, S\beta(x), \theta)\) has rank \((s - t)t\). Thus for generic \((w^0, A)\) zero is a regular value of \(H_\beta\) and \(\partial H_\beta\). Because there are only finitely many \(\beta\) in \(B\) we see that for generic \((w^0, A)\) zero is a regular value of every \(H_\beta\) and \(\partial H_\beta\) for \(\beta\) in \(B\). It follows that zero is a regular value of \(H\) and \(\partial H\); this point is discussed further in the next section.

8. The route \(W\) of zeros

We have shown for generic \((w^0, A)\) that zero is a regular value of \(H_\beta\) and \(\partial H_\beta\), and consequently, of \(H\) and \(\partial H\). A connected component of a one-manifold which is diffeomorphic to a convex set in \(R^1\) is defined to be a route. Assuming that \((w^0, A)\) is generic, we show that there is a unique route \(W\) of \(H^{-1}(0)\) which begins at \((p^0, \gamma^0, 0)\) and ends in \(P \times G \times 1\) and thereby yields a pseudo-equilibrium. The set \(H^{-1}(0)\) is assembled from diffeomorphic pieces \(H_\beta^{-1}(0)\) and the route \(W\) is assembled from certain diffeomorphic components \(W_\beta\) in \(H_\beta^{-1}(0)\). See Fig. 3.
Because $H_\beta: P \times X \times I \to Q \times R^{(r-1)M}$ drops dimension of 1 from domain to range and zero is a regular value, we see that $H_\beta^{-1}(0)$ is a closed smooth one-manifold in $P \times X \times I$. In addition, the one-manifold $H_\beta^{-1}(0)$ is neat in $P \times X \times I$, i.e. the boundary of $H_\beta^{-1}(0)$ is where $H_\beta^{-1}(0)$ meets the boundary of $P \times X \times I$ and such meetings are transverse.

Recall that the $(\iota, S\beta, \iota)$ are diffeomorphisms from $P \times X \times I$ to $P \times U_\beta \times I$ that carry $H_{\beta n}(0)$ to $H^{-1}(0) \cap (P \times U_\beta \times I)$. We transfer the one-manifolds $H_{\beta n}^{-1}(0)$ in $P \times X \times I$ to diffeomorphic copies in $P \times G \times I$ and overlay them. As the $U_\beta$ for $\beta$ in $B$ form an open cover of $G$, we obtain

$$H^{-1}(0) = \bigcup_{\beta \in B} (\iota, S\beta, \iota)H_{\beta n}^{-1}(0).$$

It is essential to understand that

$$(\iota, S\beta, \iota)H_{\beta n}^{-1}(0) \quad \text{and} \quad (\iota, S\beta', \iota)H_{\beta'}^{-1}(0)$$

are identical on $P \times (U_\beta \cap U_{\beta'}) \times I$ for every $\beta$ and $\beta'$ in $B$.

Recall that all prices $p$ of $P$, where $H(p, \gamma, \theta) = 0$ for some $(\gamma, \theta)$ in $G \times I$, lie in a compact subset of $P$. Moving our knowledge of the $H_{\beta n}^{-1}(0)$ to $H^{-1}(0)$ we see that $H^{-1}(0)$ is a compact one-manifold neat in $P \times G \times I$, i.e. the boundary of $H^{-1}(0)$ is where the boundary of $H^{-1}(0)$ meets the boundary $P \times G \times \{0, 1\}$ of $P \times G \times I$, and such meetings are transverse.

Because there is a single point $(p^0, \gamma^0, 0)$ in both $H^{-1}(0)$ and $P \times G \times 0$ for generic $(w^0, A)$ we see that there is a single component of $H^{-1}(0)$ which meets $P \times G \times 0$; let us denote this route as $W$. Beginning at $(p^0, \gamma^0, 0)$ as we move along the route $W$ we cannot return to $P \times G \times 0$, we do not leave $P$, and because $G$ is compact and boundaryless we can only arrive at a point in $P \times G \times 1$. Let us denote this point as $(p^1, \gamma^1, 1)$. We have now proved that for generic endowments $w^0$ of agent 0 and assets $A$ there is a unique route of zeros of $H$ reaching from $P \times G \times 0$ to $P \times G \times 1$. Any point in both $H^{-1}(0)$ and $P \times G \times 1$ yields a pseudo-equilibrium. Furthermore, we see that routes of zeros of $H$ pair all pseudo-equilibria except one, namely $(p^1, \gamma^1, 1)$, which is paired to the starting point $(p^0, \gamma^0, 0)$. Thus for generic $(w^0, A)$ we see that GEI has an odd number of
9. The algorithm

In this section we describe the algorithm for computing an equilibrium for GEI, i.e. for solving $H(p, \gamma, \theta) = 0$ with $\theta = 1$. Our discussion is direct since many aspects of the algorithm are developed in Brown et al. (1996a). Basically, the algorithm follows localized diffeomorphic pieces in Euclidean space and thereby follows the route $W$ of $H^{-1}(0)$ in $P \times G \times I$ beginning in $P \times G \times 0$.

We require that zero is a regular value of $H$ and $\partial H$. This is usually managed in practice by first assuming as much. Then, if the computation does not go well, $(w^0, A)$ is perturbed by something like white noise. We use the notation $[i]$ to indicate the iteration $i$.

We set $p[0] = p^0$ and $\gamma[0] = \gamma^0$ by

$$p[0] = \nabla u^0(w^0)^T / \nabla u^0(w^0) e \quad \text{and} \quad \gamma[0] = S(p[0] \square A),$$

as in Section 4. We apply the locator (see Section 5) to obtain the initial localization index $\beta[0]$ in $\Lambda(\gamma[0])$, and $x[0]$ with $S\beta[0] (x[0]) = \gamma[0]$ and $|x[0]| \leq 1$. The initial position in $P \times X \times I$ is $(p[0], x[0], \theta[0])$ with $\theta[0] = 0$. We have then solved

$$H_{\beta[0]}(p[0], x[0], \theta[0]) = 0.$$

We set the direction in $P \times X \times I$ to be $(\bar{p}[0], \bar{x}[0], \bar{\theta}[0]) = (0, 0, +1)$ for initialization purposes. Now the plan is to follow the route $W_{\beta[0]}$ of zeros of $H_{\beta[0]}$ in $P \times X \times I$ beginning at $(p[0], x[0], \theta[0])$. The route is followed until either $\theta = 1$ is achieved or until the route starts to exit $X$, i.e. $|x| \geq r$.

Let us assume that we are following the route $W_{\beta[j]}$ of zeros of $H_{\beta[j]}$ in $P \times X \times I$. If we reach $\theta = 1$, then we terminate because we have solved $H_{\beta[j]}(p, x, 1) = 0$, i.e. we have solved $H(p, \gamma, 1) = 0$, where $\gamma = S\beta[j](x[j])$. That is, we have a pseudo-equilibrium $(p, \gamma)$ to GEI. However, if in following $W_{\beta[j]}$ the $|x|$ equals $r$ before $\theta = 1$, then we apply the locator to compute $\beta[j+1]$ and a point $x'$ with $|x'| \leq 1$ so that

$$S\beta[j+1](x') = S\beta[j](x).$$

We have relocalized to follow the route $W_{\beta[j+1]}$ of zeros of $H_{\beta[j+1]}$. In this manner the route-following continues and eventually $\theta = 1$. The union of the $(t, S\beta[j], t)W_{\beta[j]}$ over $j$ yields the route $W$ of zeros of $H$ beginning at $(p[0], \gamma[0], 0)$ in $P \times G \times I$. 

pseudo-equilibria. Briefly, for generic $(w^0, A)$ we show that pseudo-equilibria are, in fact, equilibria. By letting the generic $(w^0, A)$ tend to the given $(w^0, A)$ we see that any GEI has a pseudo-equilibrium.
Now we give a more detailed statement of the algorithm for, in effect, following the route \( W \) of \( H^{-1}(0) \). The state of the algorithm is recorded by three state variables:

- the localization index: \( \beta[j] \);
- the position in \( P \times X \times I: (p[i], x[i], \theta[i]) \); and
- the direction in \( P \times X \times I: (\bar{p}[i], \bar{x}[i], \bar{\theta}[i]) \),

where

\[
H_{\beta[j]}( p[i], x[i], \theta[i]) = 0
\]

and \( (\bar{p}[i], \bar{x}[i], \bar{\theta}[i]) \) was the direction traveled to arrive at the position.

We assume that the algorithm has been running and the three state variables have been maintained. The algorithm has two main subroutines, namely

(i) prediction-corrector, and
(ii) relocalization.

If \( \theta[i] = 1 \), then \( p' = p[i] \) and \( \gamma' = S\beta[j](x[i]) \) is a pseudo-equilibrium, and we terminate. Now let us assume \( \theta[i] < 1 \) and we examine \( x[i] \). If \( |x[i]| < r \), then we execute a predictor-corrector iteration. If \( |x[i]| \geq r \), then we relocalize. Recall that we need not be concerned about leaving the price space \( P \).

**Predictor-corrector iteration.** Given the index \( \beta[j] \), position \( (p[i], x[i], \theta[i]) \), and direction \( (\bar{p}[i], \bar{x}[i], \bar{\theta}[i]) \), with \( \theta[i] < 1 \) and \( |x[i]| < r \), we first compute the \((M + (s - t)t) \times (M + (s - t)t + 1)\) derivative

\[
D = \nabla H_{\beta[j]}( p[i], x[i], \theta[i])
\]

and solve

\[
D(\bar{p}, \bar{x}, \bar{\theta}) = 0, \quad |(\bar{p}, \bar{x}, \bar{\theta})| = 1, \quad e^T\bar{p} = 0
\]

for the next direction \( (\bar{p}[i + 1], \bar{x}[i + 1], \bar{\theta}[i + 1]) = (\bar{p}, \bar{x}, \bar{\theta}) \), where the sign is chosen to make an acute angle with the current direction, i.e.

\[
(\bar{p}[i], \bar{x}[i], \bar{\theta}[i])^T(\bar{p}, \bar{x}, \bar{\theta}) > 0,
\]

in order to maintain the correct direction of movement. We set

\[
(\bar{p}, \bar{x}, \bar{\theta}) = (p[i], x[i], \theta[i]) + \tau(\bar{p}, \bar{x}, \bar{\theta})
\]

in \( P \times X \times I \) for some positive \( \tau \). The predictor step has been executed. For the corrector step, we apply Newton's method in the space

\[
(\bar{p}, \bar{x}, \bar{\theta}) + (\bar{p}, \bar{x}, \bar{\theta})^{-1},
\]

beginning at the point \( (\bar{p}, \bar{x}, \bar{\theta}) \) to solve the equation

\[
H_{\beta[j]}( p, x, \theta) = 0
\]
for the new position \((p[i + 1], x[i + 1], \theta[i + 1]) \equiv (p, x, \theta)\). The three state vectors have been updated and the predictor–corrector iteration is now complete.

Adjustment of \(\tau\) in the predictor–corrector iteration is to be expected; we are balancing step size against losing the route. For more information on route-following by differential methods see Allgower and Georg (1992), Davidenko (1953), Kellogg et al. (1976), Watson et al. (1987), and Zangwill and Garcia (1981).

**Relocalization.** Given the index \(\beta[j]\), position \((p[i], x[i], \theta[i])\), and direction \((\bar{p}[i], \bar{x}[i], \bar{\theta}[i])\), with \(\theta[i] < 1\) and \(|x[i]| \geq r\), we apply the locator \(\Lambda\) (see Section 5) to obtain an index

\[
\beta[j + 1] \in \Lambda(S\beta[j](x[i]))
\]

and \(x[i + 1]\) with \(|x[i + 1]| \leq 1\) and

\[
S\beta[j](x[i + 1]) = S\beta[j](x[i]).
\]

We set \(p[i + 1] = p[i]\) and \(\theta[i + 1] = \theta[i]\) and we have our new position. For the direction, we set \(\bar{p}[i + 1] = \bar{p}[i]\) and \(\bar{\theta}[i + 1] = \bar{\theta}[i]\) and it remains only to compute \(\bar{x}[i + 1]\). If \((\bar{p}[i + 1], \bar{\theta}[i + 1])\) is not (near) zero, then we do not need \(\bar{x}[i + 1]\), i.e. we can simply set \(\bar{x}[i + 1] = 0\) and \((\bar{p}[i + 1], \bar{x}[i + 1], \bar{\theta}[i + 1])\) suffices for the direction. If, however, \((\bar{p}[i + 1], \bar{\theta}[i + 1])\) is (near) zero, then we need \(\bar{x}[i + 1]\) for the direction (this seems generically unlikely). In this case, \(\bar{x}[i + 1]\) is selected so that the derivatives of

\[
S\beta[j + 1](x[i + 1] - \epsilon \bar{x}[i + 1])
\]

with respect to \(\epsilon\) are (approximately) equal for \(\epsilon = 0\). See Brown et al. (1993a) for more details. The three state vectors have been updated and the relocalization is now complete.

Assuming the predictor–corrector method can move down the routes \(W_{\beta[j]}\) at some minimum arc-length per unit time, due to the functioning of the locator, the algorithm terminates after finite time with some localization index \(\beta[j]\), where the route \(W_{\beta[j]}\) meets \(P \times X \times 1\) at a point, say \((p^1, x^1, 1)\), and thereby yields a pseudo-equilibrium \((p^1, \gamma^1) = (\bar{p}[j], S\beta[j](x^1))\) of GEI.

In Section 11 we show that all such pseudo-equilibria are, in fact, equilibria. Thus the algorithm computes an equilibrium. For more details on relocalization, see Brown et al. (1996a). A version of the algorithm is executed in the next section.

To compute an equilibrium of a GEI economy with our algorithm, the sufficient properties are that the excess demands \(Z_0\) and \(Z_\nu\) are (Walrasian and) continuously differentiable, \(Z_0(p) = 0\) has a unique solution in \(P\), the \((p, \gamma, \theta)\) in \(P \times G \times I\) solving \(Z_0(p) + \theta Z_\nu(p, \gamma) = 0\) yield \(p\)'s lying in a compact subset of \(P\), and zero is regular.
10. Sample computation

With the algorithm an equilibrium is computed for a model GEl economy. To avoid computing derivatives we execute an approximate version of the algorithm. The prediction direction is taken as the difference of the current and last position, and a secant method is used for the corrector step. The state of the algorithm is a pair of positions \((p[i-1], x[i-1], \theta[i-1])\) and \((p[i], x[i], \theta[i])\).

For the GEl economy we have three agents 0, 1 and 2, four states 0, 1, 2 and 3, two goods 1 and 2, and two assets 1 and 2. Thus \(n = 2\), \(s = 3\), \(m = 2\), and \(t = 2\). Agents 1 and 2 are identical, i.e. have identical endowments \(w^1 = w^2\) and utilities \(u^1 = u^2\). The endowments and assets are given by

\[
w^0 = (20, 20; 5, 10; 10, 10; 15, 10),
\]
\[
w^1 = (10, 10; 25, 20; 20, 20; 15, 20).
\]
\[
A^1 = (0, 0; 10; 0; 1, 0; 0), \quad A^2 = (0, 0; 2, 1; 1, 0; 2, 1).
\]

The utilities are given by

\[
u'(z) = -\sum_{j=0}^{s} c_j \left( b - \prod_{k=1}^{m} (z_{ik})^{d_k} \right)^2
\]

for \(i = 0\) and 1, where \(b = 57\), \(c = (1, \frac{1}{3}, \frac{1}{3}, \frac{1}{3})\), \(d^0 = (\frac{3}{4}, \frac{3}{4})\), and \(d^1 = (\frac{3}{4}, \frac{1}{4})\). The parameter \(r > 1\) of the algorithm is set (arbitrarily) at 1.4.

The computation begins with the initial localization index \(\beta[0] = (1, 3)\), initial position:

\[
p[0] = (0.1085, 0.3256, 0.0799, 0.1199, 0.0460, 0.1379, 0.0331, 0.1491),
\]
\[
x[0] = (0.6710, -0.2311), \quad \theta[0] = 0,
\]

and initial direction \((\tilde{p}, \tilde{x}, \tilde{\theta}) = (0, 0, 1)\). From here the path \(H^{-1}_{\tilde{\beta}[0]}(0)\) is followed in \(P \times X \times I\) to the positions

\[
p[54] = (0.2605, 0.2443, 0.0844, 0.0802, 0.0923, 0.0805, 0.0790, 0.0789),
\]
\[
x[54] = (-0.0431, 1.2140), \quad \theta[54] = 0.5713,
\]
\[
p[55] = (0.2612, 0.2439, 0.0847, 0.0800, 0.0926, 0.0803, 0.0788, 0.0786),
\]
\[
x[55] = (-0.0732, 1.2527), \quad \theta[55] = 0.5763.
\]

This is the first point for which \(|x| \geq r\); in particular, \(x_2[55] = 1.2527\). The new localization index is \(\beta[56] = (1, 2)\). The same positions in the new localization are

\[
p[56] = (0.2605, 0.2443, 0.0844, 0.0802, 0.0923, 0.0805, 0.0790, 0.0789),
\]
\[ x[56] = (0.0355, 0.8237), \quad \theta[56] = 0.5713, \]
\[ p[57] = (0.2612, 0.2439, 0.0847, 0.0800, 0.0926, 0.0803, 0.0788, 0.0786), \]
\[ x[57] = (0.0584, 0.7983), \quad \theta[57] = 0.5763. \]

From here the path \( H_{\beta[56]}^{-1}(0) \) is followed in \( P \times X \times I \) to the position
\[ p[80] = (0.3025, 0.2211, 0.1010, 0.0704, 0.0966, 0.0671, 0.0808, 0.0612), \]
\[ x[80] = (0.6155, 0.1833), \quad \theta[80] = 1, \]
without a relocalization, in particular \( \beta[56] = \beta[80] \). The computed pseudo-equilibrium price \( p^1 \) and plane \( \gamma^1 \) is given by \( p^1 = p[80] \) and \( \gamma^1 = S\beta[80](x[80]) \).

The price \( p^1 \) is, in fact, an equilibrium price. Figs. 4 and 5 show the progress of \((p[i], \theta[i])\) in \( P \times I \) and \((x[i], \beta[i])\) in \( X \times I \).
11. Equilibria under generic \((w^0, A)\)

Under generic endowments \(w^0\) for agent 0 and assets \(A\) we show, for completeness, that pseudo-equilibria are, in fact, equilibria. These results follow those of Duffle and Shafer (1985), Geanakoplos and Shafer (1990), and Husseini et al. (1990), for example.

If \((p, \gamma)\) in \(P \times G\) is a pseudo-equilibrium and not an equilibrium, then the following system

\[
\begin{align*}
(a.1) \quad & T_p(Z_0(p) + Z_p(p, \gamma)) = 0, \\
(a.2) \quad & \Pi_\delta(p \square A) = 0, \\
(a.3) \quad & (p \square A)\xi = 0, \\
(a.4) \quad & \xi^\top \xi = 1,
\end{align*}
\]

has a solution \((p, \gamma, \xi)\), where \(\delta = \gamma^\perp\). Equivalently, for some \(i = 1, \ldots, t\) and \(\beta\) in \(B\) the following system

\[
\begin{align*}
(b.1) \quad & T_p(Z_0(p) + Z_p(p, S\beta(x))) = 0, \\
(b.2) \quad & \Pi_\alpha^i \Pi_\delta^i(p \square A^\setminus i) = 0, \\
(b.3) \quad & (p \square A^\setminus i)\zeta - p \square A^i = 0,
\end{align*}
\]

has a solution \((p, x, \zeta)\), where \(A^i\) is the \(i\)th column of \(A\), \(A^\setminus i\) is the matrix \(A\) with column \(i\) deleted, \(\alpha = \sigma \setminus \beta\), and \(\delta = S\alpha(x)\). The domain and range space dimensions as well as ranks of derivatives for the system \((b)\) are indicated in Table 1.

Thus, for example, the range space of equation \((b.2)\) is \((s - r)(t - 1)\) and the derivative of \((b.2)\) with respect to \(A^\setminus i\) has rank \((s - r)(t - 1)\). A blank indicates that the derivative has not been computed or is not needed. The derivatives computed are straightforward or have already been derived. Examining the table and applying a parametric version of Sard's Theorem, see Guillemin and Pollack (1974), we see that zero is a regular value of the system \((b)\) for generic \((w^0, A)\). Thus the solution set \((p, x, \zeta)\) has dimension \((M - 1) + (s - r)t + (t - 1) - (M - 1) - (s - r)(t - 1) - s = -1\). Because the set of \(i = 1, \ldots, t\) and \(\beta\) in \(B\) are finite, we see that for generic \((w^0, A)\) the system \((a)\) has no solution. That is to say, for generic \((w^0, A)\) all pseudo-equilibria are equilibria.

| Table 1 |
|---|---|---|---|---|---|
| \(w^0\) | \(A^\setminus i\) | \(A^i\) | \(p\) | \(x\) | \(\zeta\) |
| \(b.1\) | \(M - 1\) | 0 | 0 | 0 | \(M - 1\) |
| \(b.2\) | 0 | \((s - r)(t - 1)\) | 0 | 0 | \((s - r)(t - 1)\) |
| \(b.3\) | 0 | \(s\) | 0 | 0 | \(s\) |
| \(M\) | \(M\) | \(M \times (t - 1)\) | \(M - 1\) | \((s - r)t\) | \(t - 1\) |
12. Summary

The results of the paper are summarized in the following theorem. Recall the homotopy $H: P \times G \times I \to Q \times R^m$ defined in Section 3, namely $E = (E, F)$, where

$$E(p, \gamma, \theta) = T_p(Z_0(p) + \theta Z_\gamma(p, \gamma)) \quad \text{and} \quad F(p, \gamma, \theta) = \Pi_\delta'(p \boxempty A),$$

where $\delta = \gamma \perp$.

**Theorem.** For generic $(w^0, A)$ zero is a regular value of $H$ and $\partial H$, and hence the zeros of $H$ form a one-manifold neat in $P \times G \times I$. The accessible point $(p^0, \gamma^0, 0)$ is the only zero of $H$ in $P \times G \times 0$. The route $W$ of zeros of $H$ which contains the point $(p^0, \gamma^0, 0)$ meets $P \times G \times 1$ at a point $(p^1, \gamma^1, 1)$. Beginning with $(p^0, \gamma^0, 0)$ the route $W$ can be followed by the predictor-corrector and relocalization methods to compute $(p^1, \gamma^1, 1)$. All zeros of $H$ in $P \times G \times 1$ are equilibria, and vice versa. All equilibria, except $(p^1, \gamma^1, 1)$, are paired by routes of zeros of $H$, and, consequently, there are an odd number of equilibria. \(\Box\)

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