Necessary Conditions for the CAPM*

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Received December 23, 1994; revised March 8, 1996

The general restrictions on all economic primitives (i.e., (a) endowments, (b) preferences, and (c) asset return distributions) that yield the CAPM under the expected utility paradigm are provided. These results are then used to derive the class of restrictions on preferences and the distribution of asset returns alone that provides the CAPM. We also show that the conditions that provide the CAPM and derived preferences over mean and variance are equivalent. Consequently, this paper also resolves the question of when mean-variance maximization is consistent with expected utility maximization.

Journal of Economic Literature Classification Numbers: G12, G11, D50.

1. INTRODUCTION

The single-period Capital Asset Pricing Model (CAPM) was originally derived under the assumption that investors have utility functions over only the mean and variance of their end of period consumption [23, 14, 17]. At the time, the only conditions that were known to generate these preferences under the expected utility paradigm were investors with quadratic utility functions or assets with normally distributed returns. Since then, Ross [21] showed that the CAPM can be derived from any two-fund separating distribution and that the normal distribution is a special case of these distributions. However, the complete set of conditions under the expected utility paradigm that are necessary and sufficient for the CAPM

* This research was supported by the SSHRC (Grant 410-94-0725) as well as by a grant from the Bureau of Asset Management, UBC. The author would like to thank Kerry Back, Avi Bick, Jim Brander, Murray Carlson, Kent Daniel, Larry Epstein, Joel Feldman, Rick Green, Burton Hollifield, Vasant Naik, Bryan Routledge, Raman Uppal, an anonymous referee, and the associate editor for their insights, comments, and suggestions. A copy of this paper, in addition to any other working paper by the author, is available on the author’s WWW home page: “http://weber.u.washington.edu/~berk.”
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are not known. In this paper we completely characterize the set of all
economic primitives, that is, endowments, preferences, and asset return
distributions, that yield the CAPM in equilibrium under the expected
utility paradigm.

Both Allingham [1] and Nielsen [18] provide sufficient conditions for
existence of CAPM equilibria. However, these conditions are difficult to
interpret within the expected utility paradigm because both authors assume
that all agents have preferences over mean and variance alone. We show
that if, for any wealth level, agents are assumed to be risk averse and
always prefer more to less then, under the expected utility paradigm, these
conditions are no more general than Ross’ conditions.

A related literature, which has as its goal the answer to the question of
when expected utility maximization is consistent with mean–variance
maximization, has its foundation in Tobin’s [24] original conjecture that
any two-parameter distribution will provide indirect preferences over mean
and variance and Feldstein’s [10] lognormal counter-example. It has since
grown into a research topic unto itself, separate from the literature on the
CAPM.1 We show that the restrictions on primitives that provide the
CAPM in equilibrium are equivalent to the conditions that provide indirect
utility functions over mean and variance alone in equilibrium. Conse-
quently, this paper also provides the conditions under which expected
utility maximization and mean–variance analysis are consistent.

The paper proceeds as follows: Section 2 describes the model and Section 3
derives the main result—the necessary and sufficient conditions that provide
the CAPM. All non-trivial proofs are collected in the Appendix.

2. THE MODEL

Traditionally the CAPM has been derived in a one-period, single con-
sumption good, exchange economy. Each agent \( \kappa \in \mathcal{K} \) has a von Neuman–
Morgenstern utility function, \( E[U'(x_\kappa)] \), over final consumption \( x_\kappa \in X_\kappa \),
where \( X_\kappa \subset L^2 \), the vector space of real-valued random variables whose
variances exist. \( U' : \mathcal{R} \rightarrow \mathcal{R} \) is assumed to be continuous, concave, and dif-
ferentiable with \( E[|U'(z)|] < \infty \) for any \( z \in L^2 \). It is traditional to assume,
in addition, that \( U_\kappa \) is strictly increasing everywhere. However, in the
CAPM literature this assumption is not imposed (e.g., quadratic utility
functions). Therefore, at this point we will make no further assumptions
on \( U_\kappa \).2

1 See [24, 10, 25, 7, 2, 4, 12, 13, 16]. Recently, Löffler [15] has extended this literature
beyond the expected utility paradigm.

2 Strictly speaking, the traditional literature on the CAPM does not even assume concavity
since an agent need not be risk averse in order to be variance averse (see [8, pp. 90, 95; 15]).
At the beginning of the period, agents trade in $I < \infty$ securities that pay off at the end of the period in the consumption good. Each security, $i \in \mathcal{I}$, is uniquely defined by its return $R_i \in R$, where the returns set $R \subseteq L^2$. The riskless asset is always assumed to be an element of $\mathcal{I}$ and is labeled $0$. For convenience we will define $r \equiv R_0$.

A portfolio is a vector $\pi \in \mathcal{R}^I$ with $\sum_{i=0}^I \pi_i = 1$. The return of a portfolio, $\pi$, is just the sum of the return of its constituent securities, $R_\pi = \sum_{i=0}^I \pi_i R_i$. Initially, each agent has wealth $W_\pi$ which is invested in the endowment portfolio $e^\pi$. Since each agent’s endowment is a portfolio, the total endowment is also a portfolio. This portfolio, $z^m = \sum_{i=0}^I W_i e^\pi / \sum_{i=0}^I W_i$, has return $R_m$ (with $R_m = E[R_m]$ and $\sigma_m^2 \equiv \text{var}(R_m)$) and is known as the market portfolio. We assume that the value of the total endowment is non-zero and adopt the price normalization $\sum_{i=0}^I W_i = 1$, that is, the total endowment’s price is 1. Any portfolio $\pi$ is said to be a levered position in the market portfolio if there exists an $a \in \mathcal{R}$ such that, $z = a : m + (1 - a) e_0$, where $e_0$ is an $I$-vector with its first component equal to 1 and the rest 0.

A portfolio $\pi$ is said to be mean–variance efficient in a return set $R$ if it has the minimum variance of all portfolios that have its expected return. We are now ready to define the equilibrium in this economy. The portfolio, $x_\kappa$, that generates agent $\kappa$’s consumption, $x_\kappa = W_\pi \sum_{i=0}^I \pi_i R_i$, is strictly supported by a return set $R$, if for any portfolio $\gamma \in \mathcal{R}^I$ with $z = W_\pi \sum_{i=0}^I \gamma_i R_i$,

$$E[U(z)] > E[U(x_\kappa)] \iff W_\pi > W_\gamma. \quad (1)$$

The set of all consumption portfolios, $\{x_\kappa\}_{\kappa \in \mathcal{X}}$, is said to be market-clearing if

$$\sum_{\kappa \in \mathcal{X}} W_\kappa x_\kappa = \sum_{\kappa \in \mathcal{X}} W_\kappa e^\pi. \quad (2)$$

A set of consumption portfolios and a return set $(\{x_\kappa\}_{\kappa \in \mathcal{X}}, R)$ is an equilibrium, if for every $\kappa \in \mathcal{X}$, $x_\kappa$ is strictly supported by the return set $R$ and markets clear.

Agent $\kappa$’s consumption portfolio $x_\kappa$, is defined to be admissible if the following two properties are satisfied: (i) $U'_\kappa(x_\kappa) > 0$ (i.e., marginal utility

\[\text{NECESSARY CONDITIONS FOR THE CAPM}\]

3 The existence of a riskless asset is assumed in the traditional version of the CAPM. However, like the traditional results in the CAPM literature, the results in this paper do not rely on this assumption. Similar results can be derived for Black’s [3] version of the CAPM.

4 This assumption is standard in the CAPM literature in which it is not generally assumed that the total endowment is positive in every state (e.g., normally distributed returns).
is strictly positive in every state), and (ii) \( x_k \) is an interior point in \( X_k \). If an agent's consumption, \( x_k \), is admissible, then the portfolio, \( \pi_k \), that generates this consumption is also admissible. An equilibrium \( \{(\pi^*)_{k \in K}, R\} \) is an *admissible equilibrium* if the equilibrium set of consumption portfolios \( \{(\pi^*)_{k \in K}\} \) is admissible.

The CAPM is said to hold in an equilibrium, \( \{(\pi^*)_{k \in K}, R\} \), if, and only if, (i) the equilibrium is admissible, (ii) the market portfolio, \( \pi^m \), is mean–variance efficient for \( R \), and (iii) every agent \( k \)'s consumption portfolio, \( \pi^k \), is a levered position in the market portfolio.

Although this definition of a CAPM equilibrium is consistent with other definitions in the literature, past researchers have not defined the equilibrium in precisely this way. For instance, [18, 19, 1] provide existence conditions for CAPM equilibria by deriving conditions for the existence of general equilibria when agents have preferences over mean and variance alone. They therefore implicitly define the model to be any equilibrium that results in an economy in which agents have preferences over mean and variance alone. Since the object of the paper is to derive the set of conditions on primitives that provide the CAPM, we cannot define the model in terms of restrictions on primitives.

We define the CAPM in terms of the distinct characteristics of the equilibrium in which agents have preferences over mean and variance: (i) each asset's expected return is a linear function of the expected return of the market portfolio, and (ii) every agent's portfolio consists of a combination of only the market portfolio and the riskless asset. Admissibility is added because it is traditional in the CAPM literature to make two additional assumptions. First, it is well known that restricting agents to preferences that are increasing in mean and decreasing in variance alone, does not necessarily imply that all agents prefer more to less. Consequently, the traditional literature on the CAPM does not assume that agents' utility functions are strictly increasing everywhere. However, most derivations of the CAPM implicitly assume that, *in equilibrium*, agents prefer more to less (see, for example, [11, p. 96]). Second, since the CAPM pricing result relies on every agent's ability to make marginal trade-offs in equilibrium, no agent can be constrained because she is on the boundary of her choice set.

### 3. THE RESTRICTIONS THAT SUPPORT THE CAPM

Before the complete set of primitives that provide the CAPM can be derived, a parsimonious characterization of the asset span is needed.

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5 Alternatively, one could assume that \( x_k \) is interior to its feasible subset. The latter assumption would be preferred if one has non-negative consumption in mind, because the interior of the set of non-negative random variables with finite variance is empty.
Without loss of generality, assume that a set of \( \{ \delta_j \}_{j=1}^J, \{ \epsilon_i \}_{i \in \mathcal{I}} \) exists such that, for any \( R_i \in \mathcal{R} \),

\[
R_i = \bar{R}_i + \sum_{j=1}^J b_j \delta_j + \epsilon_i
\]

with

\[
\bar{R}_i = E[R_i]
\]

\[
b_j \in \mathcal{R}, \quad j = 1, \ldots, J
\]

\[
E[\epsilon_i \mid R_m] = 0
\]

\[
E[\delta_j] = 0, \quad j = 1, \ldots, J
\]

\[
E[\delta_k \delta_j] = \begin{cases} 1, & k = j, \\ 0, & k \neq j \end{cases}, \quad k, j = 1, \ldots, J
\]

and, in addition, for \( i \in \mathcal{I} \)

\[
\sum_{j=1}^J x_j \epsilon_j = 0.
\]

\( \delta_j \) is referred to as the \( j \)th factor and \( \epsilon_i \) as the residual of the \( i \)th security. The reason that (3)-(9) can always be satisfied is that no restriction is placed on \( J \), the number of factors. By setting the number of factors equal to the number of risky securities (which allows \( \epsilon_i \equiv 0, \forall i \in \mathcal{I} \)), (3)-(9) are trivially satisfied. We will henceforth refer to any \( \{ \delta_j \}_{j=1}^J, \{ \epsilon_i \}_{i \in \mathcal{I}} \) that satisfies (3)-(9) as a factor structure. Any factor structure in which the number of factors, \( J \), is the minimum number of factors required to satisfy (3)-(9) will be referred to as a minimal factor structure.

Now, for a minimal factor structure \( \mathcal{A} = \{ \delta_j \}_{j=1}^J, \{ \epsilon_i \}_{i \in \mathcal{I}} \) and consumption \( y \in L^2 \), define the class of functions of \( y \) that are orthogonal to the factors:

\[
\Gamma_{\mathcal{A}}[y] = \{ \gamma: \mathcal{R} \to \mathcal{R}, \gamma(y) \in L^2, E[\gamma(y) \delta_j] = 0 \quad \forall j = 1, \ldots, J \}.
\]

This class of functions is clearly non-empty. Furthermore, for any given consumption \( y \), the set \( \Gamma_{\mathcal{A}}[y] \) is determined by the span of the factors. In general, since even a minimal factor structure is not uniquely determined, the set \( \Gamma_{\mathcal{A}}[y] \) depends on the choice of factor structure. However, an \( \gamma(y) = k \) is a member of this class.
exception occurs when $y$ is generated by a portfolio that is a levered position in the market portfolio:

**Lemma 3.1.** For any $a, b \in \mathbb{R}$ and any two minimal factor structures $A = \{ \delta_j \}_{j=1}^J$, $\{ \varepsilon_i \}_{i \in \mathcal{X}}$ and $A' = \{ \delta'_j \}_{j=1}^J$, $\{ \varepsilon'_i \}_{i \in \mathcal{X}}$, 

$$\Gamma_d[aR_m + b] = \Gamma_{d'}[aR_m + b].$$

The proof of this lemma is straightforward and is left to the reader. In view of this lemma we will henceforth drop the $A$ subscript on the set $\Gamma_d[y]$ whenever $y$ is generated by a portfolio that is a levered position in the market.

Given a risk free rate $r$, define $\Xi(r)$ to be the set of all market-clearing consumption allocations in which every agent $\kappa$’s consumption, $x_\kappa$, is a levered position in the market portfolio and is admissible, i.e.,

$$\Xi(r) = \left\{ x \mid x_\kappa = C_\kappa R_m + (W_\kappa - C_\kappa) r \quad \forall \kappa \in \mathcal{X}, \quad \sum_{\kappa \in \mathcal{X}} C_\kappa = 1, \text{ } x \text{ is admissible} \right\}.$$

The following proposition and its corollaries are the main contributions of the paper:

**Proposition 3.1** (Necessary and Sufficient Conditions for the CAPM). The CAPM holds in an equilibrium if and only if every agent $\kappa$’s utility function satisfies

$$U'_\kappa(C_\kappa R_m + (W_\kappa - C_\kappa) \rho) = \gamma'_\kappa(C_\kappa R_m + (W_\kappa - C_\kappa) \rho) - \frac{E[\tilde{\gamma}_\kappa(C_\kappa R_m + (W_\kappa - C_\kappa) \rho)]}{\sigma^2_m/[R_m - \rho]} + R_m R_m, \quad (11)$$

where $\{ C_\kappa R_m + (W_\kappa - C_\kappa) \rho \}_{\kappa \in \mathcal{X}, \rho \in \mathbb{R}} \in \Xi(\rho)$, $\rho \in \mathbb{R}$, and $\gamma'_\kappa \in \Gamma[ C_\kappa R_m + (W_\kappa - C_\kappa) \rho]$ for each agent $\kappa \in \mathcal{X}$. Furthermore, the value of $\rho$ that solves the above equations is $r$, the risk free rate in the economy.

**Corollary 3.1.** Assume that $R_m$ has a continuous, strictly increasing, distribution function with support $(D, U) \subseteq \mathbb{R}$. Then Eq. (11) in the above proposition can be replaced with

$$U'_\kappa(z) = A'_\kappa(z) - \frac{E[\tilde{\gamma}_\kappa(C_\kappa R_m + (W_\kappa - C_\kappa) \rho)]}{\sigma^2_m/[R_m - \rho]} + R_m - \frac{z^2 - 2\rho(W_\kappa - C_\kappa)z}{2C_\kappa} \quad (12)$$

for $z \in (C_\kappa D + (W_\kappa - C_\kappa) \rho, C_\kappa U + (W_\kappa - C_\kappa) \rho)$, where $A'_\kappa(z) = \gamma'_\kappa(z)$. 
Proof. Letting \( z = C_r R_m + (W_e - C_e) \rho \) in (11), integrating, and recalling that expected utility functions are unique up to an affine transformation provides (12).

Remark. This proposition provides the conditions in terms of what is traditionally taken, in the CAPM literature, as one of the primitives of the economy, namely, the set of asset returns. In most general equilibrium models the asset return set would not be regarded as a primitive since it requires knowledge of equilibrium prices to compute it. Note, however, that asset returns enter in only three places: \( R_m, I[ \cdot ], \) and \( W_e \). Under the price normalization (i.e., the total endowment has price 1) \( R_m \) is not a function of prices. Since the factors are also not a function of prices (they are a function of the asset span which does not depend on prices), \( I[C_r R_m + (W_e - C_e) \rho] \) also does not depend on prices. Finally, \( W_e \) can be defined without any knowledge of equilibrium prices by valuing each agent’s asset endowment in the return set for which the market portfolio is mean–variance efficient. Consequently, no knowledge of equilibrium prices is required to check the conditions in the proposition.

It follows immediately from Proposition 3.1 that indirect preferences over mean and variance alone are not only sufficient, but also necessary for the CAPM. To see why, note that (11) implies that in a CAPM equilibrium, the non-linear part of each agent’s marginal utility function prices the assets at zero. Consequently, the non-linear part of every agent’s marginal utility function can be deleted without affecting the equilibrium. That is, all agents’ utility functions can be replaced with quadratic functions, so the conditions that provide the CAPM and mean–variance maximization are identical in this economy. The proposition therefore answers the question of when expected utility maximization is consistent with mean–variance maximization.

It has long been recognized (see [9]) that the presence of financial (i.e., zero net supply) assets with non-linear payoffs such as options pose a challenge to the CAPM since the distribution of these assets returns are clearly not normal (or more generally two-fund separating). The question then arises: Under what conditions can a zero net supply asset with an arbitrary return distribution be added to an economy without upsetting an existing CAPM equilibrium? Proposition 3.1 provides the answer to the question. Since the asset span only enters through \( I \), the conditions in the proposition remain unchanged whenever the part of the return of the new asset not spanned by the factors is a residual; i.e., if the return of the new asset is denoted \( R_i \), then the CAPM will continue to hold so long as
\[
E[R_i - R_m - \sum_{j=1}^{n} b_{ij} \delta_j | R_m] = 0.
\]

When the CAPM holds, both the market and agents’ marginal utility functions price assets. Consequently, their projections onto the factors must be the same. The following corollary proves this:
Corollary 3.2. In an equilibrium of the economy, the CAPM holds if, and only if,

\[ \beta_k(x, \omega) = c_k b_m, \quad c_k \in \mathbb{R}, \quad \forall \omega \in \mathcal{X}, \quad x \in \Xi(r), \]  

where the \( j \)th element of \( \beta_j(x, \omega) \in \mathbb{R}^d \) is \( \beta_j(x, \omega) = \text{Cov}(U^y(x, \omega \delta), \delta) \) and \( b_m = (b_{m1}, b_{m2}, \ldots, b_{md}) \) is the market portfolio's sensitivity to the factors (the market betas).

From Proposition 3.1 and its corollaries it is apparent why the CAPM holds under the two well-known conditions. Equation (12) implies that the CAPM can only hold if all agents' utility functions differ from a quadratic function by a term whose derivative is orthogonal to every factor in the economy. The quadratic function itself trivially satisfies this condition. Similarly, when \( J = 1, \) \( \beta_m \) and \( b_m \) are scalars and so (13) does not restrict \( \beta_m \). Thus, any two-fund separating distribution (where one factor is the riskless asset) satisfies the corollary.

Finally, we sketch how the results in this paper can be used to answer a question posed by Ross [22, p. 889]: What joint restrictions on preferences and distributions provide the CAPM? This question was motivated by the observation that quadratic utility and two-fund separating distributions, by themselves, are necessary and sufficient for the CAPM. That is, Ross [21] showed that in the absence of any further assumptions on primitives, the CAPM can only hold under the two-fund separating distributional assumption, and it follows immediately from [5] that in the absence of any further assumptions on primitives, the CAPM can only hold under the quadratic preference assumption. Since assumptions on endowments are ruled out on economic grounds, the only potential set of undiscovered conditions were joint restrictions on preferences and distributions.

Equation (13) implies, for any two factors \( k \) and \( l \), \( E[ U^y(x, \omega)(b_m \delta_l - b_m \delta_k)] = 0 \). Assume all agents have polynomial utility functions of order \( N \). This equation then becomes,

\[ \sum_{n=0}^{N-1} E \left[ a_n^x \left( \theta_x + \beta_x \sum_{j=1}^{J} b_m \delta_j \right)^n \right] (b_m \delta_l - b_m \delta_k) = 0, \]  

where \( \theta_x \) and \( \beta_x \) are defined implicitly. By expanding each term on the left-hand side of the above series, it becomes clear that the only way it will be zero for any endowment allocation (i.e., for every \( b_m \)) and any two factors \( k \) and \( l \) is if for every positive integer \( m \leq N \) and every non-negative integer \( n \) such that \( \sum_{j=1}^{J} n_j = m \),

\[ n \]

7 A complete derivation is available on the author's WWW home page.
These restrictions (i.e., polynomial utility functions of order \( N \) and (15)) therefore define a class of joint restrictions on preferences and distributions that provide the CAPM. Since the only analytic functions with finite power series expansions are the polynomials, every other analytic function must have an infinite power series expansion. It therefore immediately follows that the only way the CAPM can hold when utility functions are assumed to be analytic but not polynomial, is if either \( J = 1 \) or (15) holds with \( N = \infty \). As was pointed out above, the case for \( N = \infty \) (i.e., any analytic function) was solved by Ross [21]. Thus no other joint restrictions on preferences and distributions exist that provide the CAPM.

Intuitively, if agents' utility functions are polynomials of order \( N \) or less, agents will not care about any moment above the \( N \)th. Therefore, to get the CAPM to hold, all central moments from the third to the \( N \)th need to be restricted so that agents are indifferent to them. The above moment restrictions on the distribution of the factors (i.e., (15)) effects this. The case when \( J = 1 \) corresponds to Ross' two-fund separating distributions. When \( J \neq 1 \), and the utility functions are not polynomials, then the CAPM can only hold if all factor moments satisfy (15). An example of such an economy is asset returns that are elliptically distributed. As the reader can verify, every moment of the elliptical distributions satisfies (15). These joint restrictions therefore provide an intuitive link between quadratic utilities on the one hand, and elliptical distributions on the other hand. To get the CAPM, the power series coefficients of agents' utility functions and the central moments of the factor distributions must be restricted. Quadratic utility and elliptical distributions represent the extreme position of using only one of these strategies at a time.

Unfortunately, the complete set of restrictions that provide the CAPM are no more realistic than the previously known set. Even if consumption is restricted to be positive, no finite order polynomial is both strictly increasing and strictly concave in the positive domain. When the class of utility functions is expanded to include any analytic function that is

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8 See [6] or [20]. Note that although these authors assume that all assets are elliptically distributed, all that is actually required is that the factors be elliptically distributed.
everywhere strictly concave and strictly increasing, Ross [21] has shown that the CAPM will only hold if return distributions are two-fund separating. Ross' conditions therefore provide the only realistic way, within the expected utility paradigm, to get the CAPM (or mean–variance preferences).

APPENDIX

A. Proof of Proposition 3.1

Proof. Dybvig and Ingersoll [9] have shown that \( \pi^m \) is mean–variance efficient if and only if

\[
    r = E \left[ R_i \left\{ 1 - \frac{\bar{R}_m - r}{\sigma_m^2} (R_m - \bar{R}_m) \right\} \right] \quad \forall i \in \mathcal{I}. \tag{16}
\]

For any \( \kappa \in \mathcal{X} \) in an admissible equilibrium ([11, p. 66]),

\[
    \frac{1}{E[U'(x_\kappa)]} E[U'(x_\kappa) \cdot R_i] = r \quad \forall i \in \mathcal{I}. \tag{17}
\]

Therefore, when the market portfolio is mean–variance efficient in equilibrium then both (16) and (17) hold so:

\[
    E \left[ \left\{ U'(x_\kappa) - E[U'(x_\kappa) \left( 1 - \frac{\bar{R}_m - r}{\sigma_m^2} (R_m - \bar{R}_m) \right) \right\} \cdot R_i \right] = 0 \quad \forall i \in \mathcal{I}. \tag{18}
\]

Using (3), (6), (9), and that \( J \) is the minimum number of factors, it can be shown that the only way (18) can be satisfied\(^9\) is if

\[
    E \left[ \left\{ U'(x_\kappa) + E[U'(x_\kappa)] \cdot R_i \left( \frac{\bar{R}_m - r}{\sigma_m^2} \right) \cdot \delta_j \right\} \right] = 0 \quad \forall j. \tag{19}
\]

The above derivation shows that the CAPM holds in an equilibrium if and only if (19) holds for every \( \kappa \in \mathcal{X} \) when \( x \in \mathcal{E}(r) \).

We next show that when \( x \in \mathcal{E}(r) \), (11) implies (19) and so the above argument allows us to conclude that the CAPM holds. Setting \( \rho = r \) in (11) provides:

\[
    U'_j(x_\kappa) = \gamma_j(x_\kappa) - \frac{E[U_j(C_x R_m + (W_x - C_x) r)]}{\sigma_m^2 (R_m - r) + R_m} \left[ x_\kappa - r(W_x - C_x) \right] \frac{C_x}{C_x}. \tag{20}
\]

\(^9\) A proof of this fact is available on the author’s WWW home page.
Taking expectations (for every \( \kappa \)),
\[
E[U'_n(x_n)] = E[\gamma_n(x_n)] - \frac{E[\gamma_n(C, R_m + (W_x - C_x)r)]}{\sigma^2_m/(R_m - r) + R_m} \times \left[ \frac{E[x_n] - r(W_x - C_x)}{C_x} \right].
\]
(21)

Using (20) and (21) it is straightforward to verify that (19) is satisfied so the CAPM holds.

Last, we need to show that if the CAPM holds in at least one equilibrium of the economy, then the conditions in the proposition will be satisfied. If the CAPM holds in equilibrium, then (19) holds for every \( \kappa \) when \( x \in \Xi(r) \). This implies that
\[
U'_n(x_n) + E[U'_n(x_n)] \sigma_m^2 R_m (R_m - r) = \gamma_n(x_n),
\]
(22)
where \( \gamma_n \in I[x_n] \). Taking expectations of both sides of (22) and solving for \( E[U'_n(x_n)] \) provides:
\[
E[U'_n(x_n)] = \frac{E[\gamma_n(x_n)]}{1 + R_m (r - R_m) / \sigma_m^2}
= \frac{E[\gamma_n(C, R_m + (W_x - C_x)r)]}{1 + R_m (r - R_m) / \sigma_m^2}.
\]
(23)

Substituting (23) into (22) and writing \( x_n \) out explicitly, we have that for every \( \kappa \),
\[
U'_n(C, R_m + (W_x - C_x)r) = \gamma_n(C, R_m + (W_x - C_x)r)
- \frac{E[\gamma_n(C, R_m + (W_x - C_x)r)]}{\sigma_m^2/(R_m - r) + R_m} R_m,
\]
(24)
which is (11).

B. Proof of Corollary 3.2

Proof: (1) Necessity: By Proposition 3.1, if the CAPM holds in equilibrium then (11) holds. Thus, \( c_n = -\frac{E[\gamma_n(C, R_m + (W_x - C_x)r)]}{\sigma_m^2/(R_m - r) + R_m} \). (2) Sufficiency: Take the allocation in \( \Xi(r) \) for which the projection of every agent’s marginal utility function onto the factors is a constant multiple of the market betas and project agent \( \kappa \)’s marginal utility function at this allocation onto the market
\[
U'_n(x_n) = \alpha^* + c_n R_m + e_n(x_n).
\]
(25)
where,
\[ x_s = C_s R_m + (W_s - C_s) r \]  (26)
\[ E[ \epsilon_s(x_s)] = E[ \epsilon_s(x_s) \delta_j] = 0, \quad \forall j = 1, \ldots, J \]  (27)
\[ c_s = \frac{E[ U'_s(x_s)(R_m - \bar{R}_m)]}{\sigma_m^2} \]  (28)
\[ \alpha^* = E[ U'_s(x_s)] - c_s \bar{R}_m. \]  (29)

Note that (27) follows from the fact that the projection of every agent’s marginal utility function onto the factors is a constant multiple of the market projection onto the factors. Let
\[ \gamma_s(x_s) \equiv \epsilon_s(x_s) + \alpha^*. \]  (30)
so that \( \gamma_s \in \Gamma[ C_s, R_m + (W_s - C_s) r ] \). Using the first-order condition (i.e., (17)) gives \( E[ U'_s(x_s)(R_m - \bar{R}_m)] = E[ U'_s(x_s)(r - \bar{R}_m)] \), and so (28) becomes
\[ c_s = \frac{E[ U'_s(x_s)(r - \bar{R}_m)]}{\sigma_m^2} = -E[ U'_s(x_s)] \frac{\bar{R}_m - r}{\sigma_m^2}. \]  (31)

Substituting (30) and (31) into (25), taking expectations, and solving for \( E[ U'_s(x_s)] \) provides,
\[ E[ U'_s(x_s)] = \frac{E[ \gamma_s(x_s)]}{(1 + \left[ \frac{(R_m - r) R_m}{\sigma_m^2} \right])}. \]  (32)

Substituting (32) into (31) provides another expression for \( c_s \). Using this expression (as well as (30)) in (25) provides,
\[ U'_s(x_s) = \gamma_s(x_s) - \frac{E[ \gamma_s(x_s)]}{[\sigma_m^2/(R_m - r)] + \bar{R}_m} R_m, \]  (33)
which, together with (26), provides (11). By Proposition 3.1 the CAPM holds.

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