OPTIMAL INVESTMENT WITH UNDIVERSIFIABLE INCOME RISK

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This paper treats the problem of consumption and portfolio choice in continuous time, with stochastic income that cannot be replicated by trading the available securities. The optimal controls and value functions are characterized in terms of the viscosity solution of the associated Hamilton-Jacobi-Bellman equation, which is shown to exist and is characterized. The optimal policy is then given from the first-order conditions of the Hamilton-Jacobi-Bellman equation.

1. INTRODUCTION

This paper treats a problem of consumption and portfolio choice in continuous time, with stochastic income that cannot be replicated by trading the available securities. Optimal controls and value functions are characterized in terms of the viscosity solution of the associated Hamilton-Jacobi-Bellman (HJB) equation, which is shown to exist and is characterized.

Duffie, Fleming, and Zariphopoulou (1991) and Koo (1991) independently treated the case of stochastic income that is geometric Brownian motion, using a homothetic (power class) utility function. Koo (1991) reduced the problem to one of two ordinary differential equations. Using a different approach, Duffie, Fleming, and Zariphopoulou (1991) showed that this special case allows reduction of the dimension of the state space from 2 to 1 and treated the reduced problem as a "dual" consumption-portfolio problem with "almost-power" utility function. They also proved the existence of a unique $C^2$ value function and computed optimal policies for cases in which they exist.

This paper makes no assumptions that would allow one to reduce the dimension of the state space. We focus principally on the existence of a unique viscosity solution to the HJB equation. The first-order conditions of the solution, given in Duffie, Fleming, and Zariphopoulou (1991), then produce the optimal controls. Typically, these controls could be calculated using numerical schemes that converge to the optimal solution by virtue of the stability properties of viscosity solutions; see, for example, Fitzpatrick and Fleming (1990).

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2. MAIN RESULTS

We consider a market with investments at a constant continuously compounding interest rate \( r > 0 \) and in a risky asset whose price \( P_t \) satisfies

\begin{equation}
    dP_t = bP_t \, dt + \sigma_1 P_t \, dB_t \quad (t \geq 0), \quad P_0 = p
\end{equation}

where \( b \) is the mean rate of return, \( \sigma_1 \) is the volatility, and \( B \) is a standard Brownian motion defined on the underlying probability space \((\Omega, \mathcal{F}, \mathbb{P})\). The market parameters \( b, r, \) and \( \sigma_1 \) are assumed to be constant with \( \sigma_1 \neq 0 \) and \( b > r > 0 \).

The agent’s total current wealth is \( W_t = \pi^0_t + \pi_t \), where \( \pi^0_t \) and \( \pi_t \) are the amounts of wealth invested in bond and stock, respectively. The agent also receives nonnegative stochastic income at a rate of \( Y_t \) that evolves according to the equation

\begin{equation}
    dY_t = \mu(Y_t) \, dt + \sigma(Y_t) \, dB_t \quad (t \geq 0), \quad Y_0 = y
\end{equation}

where \( \mu \) and \( \sigma \) are Lipschitz functions (with Lipschitz coefficient, say, \( K \)) satisfying

\begin{equation}
    \mu(y) > 0, \quad \sigma(y) > 0 \quad (y > 0), \quad \text{and} \quad \mu(0) = \sigma(0) = 0, \quad Y_0 \mapsto Y_t \text{ is concave, for all } t \text{ almost surely.}
\end{equation}

The process \( \tilde{B} \) is a standard Brownian motion defined on \((\Omega, \mathcal{F}, \mathbb{P})\) having constant correlation \( \rho \in (-1, 1) \) with \( B \). For this, we can take \( \tilde{B}_t = \rho B_t + \sqrt{1 - \rho^2} Z_t \), where \((B, Z)\) is a two-dimensional standard Brownian motion. The concavity condition in (2.3) is satisfied, for example, if \( \mu \) and \( \sigma \) are linear, or more generally when \( \mu \) is concave and \( \sigma \) is linear. This follows by comparison theorems for stochastic differential equations. (See, for example, Protter 1990, p. 269.)

The agent’s wealth process \( W \) thus evolves according to the equation

\begin{equation}
    dW_t = \left[ rW_t + (b - r)\pi_t - C_t + Y_t \right] dt + \sigma_1 \pi_t \, dB_t \quad (t \geq 0), \quad W_0 = w
\end{equation}

where \( w \) is the initial wealth endowment.

The control processes are the consumption rate \( C \) and the amount of wealth \( \pi \) invested in stock. To state their properties we introduce the sets:

\[ L_+ = \left\{ \ell \in L : \ell_t \geq 0 \text{ a.s. } (t \geq 0) \text{ and } \int_0^t \ell_s \, ds < +\infty \text{ a.s. } (t \geq 0) \right\}, \]

\[ M = \left\{ \ell \in L : \int_0^t \ell_s^2 \, ds < +\infty \text{ a.s. } (t \geq 0) \right\}, \]

where \( L \) is the space of \( \mathcal{F}_t \)-progressively measurable processes and \( \mathcal{F}_t \) is the augmentation under \( P \) of \( \sigma\{\{B_s, Z_s\} : 0 \leq s \leq t\} \).
The set \( \mathcal{A}(w, y) \) of admissible controls consists of pairs \((C, \pi)\) in \( \mathcal{L}_+ \times \mathcal{M} \) such that \( W_t \geq 0 \) a.e. \((t \geq 0)\), where \( W_t \) is given by the state equation (2.4) using the controls \((C, \pi)\).

The agent’s utility function \( \mathcal{J} : \mathcal{L}_+ \rightarrow \mathbb{R}^+ \) is given by

\[
\mathcal{J}(C) = E\left( \int_0^{+\infty} e^{-\beta t} U(C_t) \, dt \right),
\]

where \( \beta > 0 \) is a discount factor and \( U : \mathbb{R}^+ \rightarrow \mathbb{R}^+ \) is assumed to have the properties

\[
U(c) \leq M(1 + c)^\gamma \quad \text{with} \quad 0 < \gamma < 1 \quad \text{and} \quad M > 0,
\]

\[
U(0) \geq 0, \quad \lim_{c \to 0} U'(c) = +\infty, \quad \lim_{c \to \infty} U'(c) = 0.
\]

The value function \( v : \mathbb{R}^+ \times \mathbb{R}^+ \rightarrow \mathbb{R}^+ \) is given by

\[
v(w, y) = \sup_{(C, \pi) \in \mathcal{A}(w, y)} \mathcal{J}(C).
\]

The goal is to characterize \( v \) as the unique constrained viscosity solution of the HJB equation associated with this stochastic control problem and to use this property to show the convergence of a wide class of numerical schemes for the value function.

We now state the main result:

**Theorem 2.1.** The value function \( v \) is the unique constrained viscosity solution in the class of concave functions of

\[
\beta v = \max_{\pi} \left[ \frac{1}{2} \sigma^2 \pi^2 v_{ww} + \rho \pi \sigma \pi \sigma(y) v_{wy} + (b - r) \pi v_{wy} + \frac{1}{2} \sigma^2(y) v_{yy} \right]
+ \max_{c \geq 0} \left[ -c v_w + U(c) \right] + rv_w + vy_w + \mu(y) v_y.
\]

**3. PROPERTIES OF THE VALUE FUNCTION**

In this section we derive some basic properties of the value function.

**Proposition 3.1.** The value function \( v \) is jointly concave in \((w, y)\), strictly increasing in \( w \) and nondecreasing in \( y \).

**Proof.** The concavity of \( v \) is an immediate consequence of the concavity of the utility function \( U \) and the fact that if \((C^1, \pi^1) \in \mathcal{A}(x_1, y_1)\), \((C^2, \pi^2) \in \mathcal{A}(x_2, y_2)\), and \( \lambda \in (0, 1) \), then

\[
(\lambda C^1 + (1 - \lambda) C^2, \lambda \pi^1 + (1 - \lambda) \pi^2) \in \mathcal{A}(\lambda x_1 + (1 - \lambda) x_2, \lambda y_1 + (1 - \lambda) y_2);
\]

the latter follows from the linear dependence of the dynamics (2.4) with respect to the controls and the state variables and the almost sure concavity of \( Y_0 \mapsto Y_t \) for all \( t \).
That $v$ is nondecreasing in $x$ and $y$ follows from the observation that $\mathcal{A}(x_1, y_1) \subseteq \mathcal{A}(x_2, y_2)$ if $x_1 \leq x_2, y_1 \leq y_2$. The proof that $v$ is strictly monotone in $x$ follows along the lines of the proof in Proposition 2.1 in Zariphopoulou (1992) and therefore is omitted.

In order to obtain a convenient bound on the value function $v$, we next consider a fictitious consumption-investment problem with the same utility as in (2.5), no additional income, and an additional “pseudoasset” with price process $P'$ given by

$$
dP'_t = b'P'_t \, dt + \sigma'_1 P'_t \, dZ_t, \quad (t \geq 0),
$$

$$
P'_0 = p' \quad (p' > 0),
$$

where $b'$ are $\sigma'_1$ are constants to be appropriately chosen in the sequel. We denote by $u$ the value function of this “pseudoproblem.” That is,

$$
u(w) = \sup_{(C, \pi, \pi') \in \mathcal{A}'(w)} \mathcal{J}(C),
$$

where $\mathcal{A}'(w)$ is the set of admissible policies $(C, \pi, \pi')$ in $\mathcal{L}_+ \times M \times M$ such that $W'_t \geq 0$ a.e. $(t \geq 0)$, with $W'_t$ given by

$$
dW'_t = [rW'_t + (b - r)\pi'_t - C_t + (b' - r)\pi'_t] \, dt + \sigma'_1 \pi'_t \, dB_t
$$

$$
+ \sigma'_1 \pi'_t \, dZ_t, \quad (t \geq 0),
$$

$$
W'_0 = w \quad (w \geq 0).
$$

We next define the function $f: [0, +\infty) \rightarrow [0, +\infty)$, representing the initial wealth equivalent of the stochastic income $Y$, by

$$
f(y) = E_y \left( \int_0^{+\infty} e^{-r\xi Y_t} \, dt \right),
$$

where

$$
\xi_t = \exp(-\frac{1}{2}(\theta_1^2 + \theta_2^2)t + \theta_1 B_t + \theta_2 Z_t),
$$

$\theta_1 = (b - r)/\sigma_1$ and $\theta_2 = (b' - r)/\sigma'_1$. We assume that the constants $b'$ and $\sigma'$ can be defined so that

$$
E \left[ \int_0^t (\xi Y_s)^2 \, ds \right] < +\infty, \quad t > 0, \quad r > \eta,
$$

where $\eta = K(1 + \sqrt{1 - \rho^2}/2)$. We claim that $f(y) < +\infty$ for all $y \geq 0$. In fact, applying Itô’s lemma to the process $\xi_t Y_t$ gives

$$
\xi_t Y_t = y + \int_0^t [\xi_s Y_s] + \frac{1}{2}\theta_1^2 \sqrt{1 - \rho^2} \xi_s \sigma(Y_s)] \, ds + \int_0^t \theta_1 \xi_s Y_s \, dB_t
$$

$$
+ \int_0^t [\xi_s \sigma(Y_s) + \theta_2 \xi_s Y_s] \, dZ_t.
$$
Taking expectations and using the fact that the above stochastic integrals are martingales (this follows from (2.3) and (3.51)), we get

\[ E(\xi_Y) \leq y + \eta \int_0^t E(\xi_Y) \, ds, \]

where we used (2.3). Gronwall’s inequality then gives

\[ E(\xi_Y) \leq y \left( 1 + \eta \int_0^t e^{\eta(t-s)} \, ds \right) = ye^{\eta t}. \]

Therefore, for every \( T < +\infty \),

\[ E \int_0^T e^{-r\xi_Y} \, dt < y \int_0^T e^{(\eta - r)t} \, dt = y \frac{1 - e^{(\eta - r)t}}{r - \eta}. \]

Using (3.5) and the monotone convergence theorem we get

\[ (3.6) \quad f(y) = \lim_{T \to \infty} E \int_0^T e^{-r\xi_Y} \, dt < \frac{y}{r - \eta}. \]

The stochastic income rate \( Y \) can be replicated by a self-financing trading strategy involving the riskless asset, the original risky asset with price process \( P \), and the pseudo-asset with price process \( P' \). The associated initial investment required is \( f(y) \). It follows that the stochastic income can be replaced in this pseudoproblem with an extra initial wealth of \( f(y) \). This approach is well known by now; for a standard reference, see Huang and Pagès (1992).

We note that \( f \) is continuous; this follows from the monotone convergence theorem, the definition of \( f \), and the fact that \( Y(w, t) \) is a monotone function of \( y \) for almost every \((w, t)\).

**PROPOSITION 3.2.** For all \((w, y) \in \Omega \),

\[ (3.7) \quad 0 \leq v(w, y) \leq u(w + f(y)). \]

**Proof.** We can look at the original problem as the pseudoproblem described above, with no stochastic income but with total wealth \( w + f(y) \) and the additional trading constraint that no wealth is invested in the pseudoasset with price process \( P' \). Inequality (3.7) then follows since \( u \) is the value function of the unconstrained pseudoproblem.

**PROPOSITION 3.3.** The value function satisfies \( v(w, y) \leq O(w^\gamma + y^\gamma) \) as \( w \to \infty \), \( y \to \infty \).

**Proof.** Using that \( U(c) \leq M(1 + c^\gamma) \) and a modification of Theorem 4.5 in Fleming and Zariphopoulou (1991), we obtain

\[ u(w + f(y)) \leq O(w^\gamma + (f(y))^\gamma). \]

The result follows then from the above inequality combined with (3.6) and (3.7). \( \square \)
4. VISCOSITY SOLUTIONS OF THE HJB EQUATION

In this section we analyze the HJB equation (2.7) and characterize the value function as its solution. The fact that the value function of a stochastic control problem solves the associated HJB equation in the classical sense follows from the dynamic programming principle and an application of Itô’s lemma; this can be done only if the value function is known to be a priori smooth. Conversely, classical verification results (see Fleming and Rishel 1975) yield that if the HJB equation has a unique smooth solution then it coincides with the value function. In the problem at hand, however, it does not follow directly that the value function is smooth. Also, (2.7) is a second-order fully nonlinear and possibly degenerate equation and therefore might not have a unique smooth solution. It is thus imperative to relax the notion of solution of the HJB equation. It turns out that a suitable class of weak solutions are the viscosity solutions; in particular, due to the presence of the state constraint $W_t \geq 0$ a.e. ($t \geq 0$), we will work in the class of constrained viscosity solutions. The characterization of the value function as the unique solution of (2.7) in the above class enables us to get convergence of a wide class of numerical schemes for the value function and the optimal policies. This is desirable given the absence of formulas for the optimal policies due to lack of sufficient regularity of the value function.

The notion of viscosity solutions was introduced by Crandall and Lions (1983) for first-order equations, and by Lions (1983) for second-order equations. For a general overview of the theory of viscosity solutions, we refer to the \textit{User’s Guide} of Crandall, Ishii, and Lions (1990).

Next, we recall the notion of constrained viscosity solutions, introduced by Soner (1986) and Capuzzo-Dolcetta and Lions (1987) for first-order equations. (See also Ishii and Lions 1990 and Katsoulakis 1990.)

To this end, consider a nonlinear second-order partial differential equation of the form

\begin{equation}
F(x, u, u_x, u_{xx}) = 0 \quad \text{in } \Omega,
\end{equation}

where $\Omega$ is an open subset of $\mathbb{R}^2$ and $F: \Omega \times \mathbb{R} \times \mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{R} \to \mathbb{R}$ is continuous and (degenerate) elliptic, meaning that $F(x, t, p, X + Y) \leq F(x, t, p, X)$ if $Y \geq 0$.

**Definition 4.1.** A continuous function $u : \bar{\Omega} \to \mathbb{R}$ is a \textit{constrained viscosity solution} of (4.1) if (i) $u$ is a viscosity subsolution of (4.1) on $\bar{\Omega}$; that is, if for any $\phi \in C^2(\bar{\Omega})$ and any local maximum point $x_0 = (w_0, y_0) \in \bar{\Omega}$ of $u - \phi$,

\[ F(x_0, u(x_0), D\phi(x_0), D^2\phi(x_0)) \leq 0, \]

and (ii) $u$ is a viscosity supersolution of (3.1) in $\Omega$; that is, if for any $\phi \in C^2(\bar{\Omega})$ and any local minimum point $x_0 = (w_0, y_0) \in \Omega$ of $u - \phi$,

\[ F(x_0, u(x_0), D\phi(x_0), D^2\phi_{xx}(x_0)) \geq 0, \]

where $D\phi$ and $D^2\phi$ denote, respectively, the gradient vector and the second derivative matrix of $\phi$. 
We next prove the main result regarding the value function, which was stated in Section 1. For ease of presentation, this is done in two steps; we first characterize the value function as a constrained viscosity solution of the HJB equation (2.7); this is shown in Theorem 4.1. We then establish that the value function is actually the unique concave constrained viscosity solution by presenting, in Theorem 4.2, a comparison result for viscosity solutions of (2.7); this, in turn, implies that (2.7) has a unique solution which, therefore, coincides with the value function.

**Theorem 4.1.** The value function \( v \) is a constrained viscosity solution of (2.7) on \( \bar{\Omega} = [0, +\infty) \times [0, +\infty) \).

The fact that, in general, value functions of (stochastic) control problems and differential games turn out to be viscosity solutions of the associated PDEs follows directly from the dynamic programming principle and the theory of viscosity solutions. (See, for example, Lions 1983, Evans and Souganidis 1984, and Fleming and Souganidis 1989.) The main difficulty for the problem at hand is that neither control, that is, neither consumption rate nor risky investment, is uniformly bounded. In order to overcome this difficulty, we first approximate the value function with a sequence of functions that are viscosity solutions of modified (HJB) equations. We then perform a random time scaling to treat the case of unbounded portfolios, and we work with the normalized (HJB) equation. We finally use the stability properties of viscosity solutions to pass to limits.

**Proof.** We first show that \( v \) is a viscosity supersolution of (2.7) in \( \bar{\Omega} \). Let \( \phi \in C^2(\bar{\Omega}) \) and \( x_0 = (w_0, y_0) \in \Omega \) be a minimum of \( v - \phi \). Without loss of generality we can assume that

\[
(4.2) \quad v(x_0) = \phi(x_0) \quad \text{and} \quad v \geq \phi \quad \text{in} \ \bar{\Omega}.
\]

We need to show that

\[
(4.3) \quad \beta v(x_0) \geq \max_\pi V(\pi, x_0) + rw_0\phi_w(x_0) + y_0\phi_y(x_0) + \mu(y_0)\phi_y(x_0),
\]

where

\[
V(\pi, x_0) = \frac{1}{2}\sigma^2\pi^2 + \rho\pi\sigma_1\sigma(y_0)\phi(w_0(x_0))
+ (b - r)\pi\phi_y(x_0) + \frac{1}{2}\sigma^2(y_0)\phi_{yy}(x_0).
\]

To this end, let \((C, \pi) \in \mathcal{A}(w_0, y_0)\) such that \( C_t = C_0, \pi_t = \pi_0, t \geq 0 \). The dynamic programming principle (Krylov 1980) together with (4.2) yields

\[
(4.4) \quad v(w_0, y_0) \geq E\left[ \int_0^\tau e^{-\beta t}U(C_0) \, dt + e^{-\beta \theta}d(W_\theta, Y_\theta) \right],
\]

where \( W \) is the trajectory of wealth given by (2.4) using the controls \((C_0, \pi_0)\) and starting at \((w_0, y_0)\) and \( \theta = \min(\tau, 1/n) \), with \( n > 0 \), and \( \tau = \inf\{t \geq 0 : W_t = 0\} \).
On the other hand, applying Itô's lemma to \( g(t, X_t) = e^{-\beta t}\phi(X_t) \), where \( X_t = (W_t, Y_t) \), we get

\[
E[e^{-\beta \theta}\phi(X_{\theta})] = \nu(w_0, y_0) + E\int_0^\theta e^{-\beta t}\left[ -\beta \phi(X_t) + V(\pi_0, X_t) \phi_w(X_t)[-C_0 + rW_t + Y_t] + \mu(Y_t)\phi_y(X_t) \right] dt.
\]

Combining the above inequality with (4.4) and using standard estimates from the theory of stochastic differential equations (see Gikhman and Skorohod 1972), we get

\[
E\int_0^\theta [-\beta \nu(X_0) + V(\pi_0, X_0) + U(C_0) + \phi_w(X_0)[r w_0\phi_w(X_0) + y_0\phi_w(X_0) - C_0] + \mu(y_0)\phi_y(X_0)] ds + E\int_0^\theta h(s) ds \leq 0,
\]

where \( h(s) = O(s) \). Dividing both sides by \( E(\theta) \) and passing to the limit as \( n \to \infty \), inequality (4.3) follows.

We next show that \( \nu \) is a viscosity subsolution of (2.7) on \( \bar{\Omega} \). We first approximate \( \nu \) by a sequence of functions \( \nu^N \) defined by

\[
\nu^N(w, y) = \sup_{\mathcal{A}_N(w, y)} E\left[ \int_0^{+\infty} e^{-\beta t}U(C_t) dt \right] \quad ((x, y) \in \bar{\Omega}),
\]

where

\[
\mathcal{A}_N(w, y) = \{(C, \pi) \in A(w, y) : \pi \in \mathcal{M} \text{ and } C \in \mathcal{L}_+ \text{ with } C_t \leq N \text{ a.s., } t \geq 0\}.
\]

We show that \( \nu^N \) is a viscosity subsolution of (2.7) on \( \bar{\Omega} \). Let \( \phi \in C^2(\bar{\Omega}) \) and let us assume that \( \nu - \phi \) has a maximum at a point \( X_0 = (w_0, y_0) \in \bar{\Omega} \). Without loss of generality we may assume that \( \nu(w_0, y_0) = \phi(w_0, y_0) \) and \( \nu(w, y) \leq \phi(w, y), (w, y) \in \bar{\Omega} \). We need to show that

\[
(4.5) \quad \beta \nu^N(w_0, y_0) \leq \max_{0 \leq c \leq N} \left[ -c\phi_w(w_0, y_0) + U(c) + \phi_w(w_0, y_0)[r w_0 + y_0] + \max_\pi \left\{ \frac{1}{2}\sigma_w^2\phi_{ww}(w_0, y_0) + \pi(\rho\sigma_y\sigma(w_0)\phi_{wy}(w_0, y_0) + (b-r)\phi_w(w_0, y_0)) + \frac{1}{2}\sigma_y^2(\phi_{yy}(w_0, y_0)] + \mu(y_0)\phi_y(w_0, y_0). \right\}
\]

In order to show (4.5), we first recall that the value function \( \nu^N \) is a viscosity subsolution on \( \bar{\Omega} \) of the normalized Bellman equation:
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\[
\max_{\pi \in \mathbb{R}, \ c \in [0, \bar{c}]} \left[ \frac{1}{1 + \pi^2} \left( \beta v^N - \left( \frac{1}{2} \sigma^2 \pi^2 v^N_{ww} + \pi (\rho \sigma (y) v^N_{w^2} + (b - r) v^N_w) \right) 
- (-c v^N_w + U(c)) - (r w + y) v^N_w - \mu(y) v^N_y \right) \right] 
= 0.
\]

(For a proof see Lions 1983.) We next look at the following cases.

**Case A.** \( \phi_{w^2}(w_0, y_0) \equiv 0 \) and \( \rho \sigma (y_0) \phi_{w^2}(w_0, y_0) + (b - r) \phi_w(w_0, y_0) \neq 0 \). Then (4.5) is automatically satisfied since \( v^N(w_0, y_0) < +\infty \) and the right-hand side of (4.5) is \( +\infty \).

**Case B.** \( \phi_{w^2}(w_0, y_0) > 0 \) and \( \rho \sigma (y_0) \phi_{w^2}(w_0, y_0) + (b - r) \phi_w(w_0, y_0) = 0 \). This is the same as the situation in Case A.

**Case C.** \( \phi_{w^2}(w_0, y_0) = 0 \) and \( \rho \sigma (y_0) \phi_{w^2}(w_0, y_0) + (b - r) \phi_w(w_0, y_0) = 0 \). Then (4.5) becomes

\[
(4.6) \quad \beta v^N(w_0, y_0) \leq \frac{1}{2} \sigma^2 (y_0) \phi_{y^2}(w_0, y_0) + \max_{0 \leq c \leq \bar{c}} \left[ -c \phi_w(w_0, y_0) + U(c) \right] 
+ \phi_w(w_0, y_0) (r w_0, y_0) + \mu(y_0) \phi_y(w_0, y_0).
\]

We argue by contradiction. Let us assume that (4.6) is not true. Then

\[
(4.7) \quad A = \beta v^N(w_0, y_0) - \left[ \frac{1}{2} \sigma^2 (y_0) \phi_{y^2}(w_0, y_0) + \max_{0 \leq c \leq \bar{c}} \left( -c \phi_w(w_0, y_0) + U(c) \right) 
+ \phi_w(w_0, y_0) (r w_0, y_0) + \mu(y_0) \phi_y(w_0, y_0) \right] 
> 0.
\]

Using the fact that \( v^N - \phi \) has a maximum at \((w_0, y_0)\) and inequality (4.7), the normalized HJB equation yields

\[
\max_{\pi, 0 \leq c \leq \bar{c}} \frac{A}{1 + \pi^2} \leq 0,
\]

which is a contradiction because

\[
\max_{\pi, 0 \leq c \leq \bar{c}} \frac{A}{1 + \pi^2} = A > 0.
\]

**Case D.** \( \phi_{w^2}(w_0, y_0) < 0 \) and \( \rho \sigma (y_0) \phi_{w^2}(w_0, y_0) + (b - r) \phi_w(w_0, y_0) \neq 0 \). Then the maximum with respect to \( \pi \) of

\[
\frac{1}{2} \sigma^2 \pi^2 \phi_{w^2}(w_0, y_0) + \pi [\rho \sigma (y_0) \phi_{w^2}(w_0, y_0) + (b - r) \phi_w(w_0, y_0)]
\]
occurs at a finite point, denoted \( \pi^* \). We argue again by contradiction. Let us assume that (4.5) does not hold, that is,

\[
A = \beta v^N(w_0, y_0) \\
- \left[ \frac{1}{2} \sigma_1^2(\pi^*)^2 \phi_{ww}(w_0, y_0) + \pi^* \rho \phi_{w_y}(w_0, y_0) \right] \\
+ \phi_{ww}(w_0, y_0) + \frac{1}{2} \sigma_1^2 \phi_{yy}(w_0, y_0) \\
- \max_{0 \leq r \leq N} \left[ - c \phi_{w}(w_0, y_0) + U(c) \right] \\
- \phi_{w}(w_0, y_0) \phi_{yy}(w_0, y_0) - \mu(y_0) \phi_y(w_0, y_0) \\
> 0.
\]

From the normalized HJB equation, we get \( \max_{(w, y)} A/(1 + \pi^2) \leq 0 \) which again yields a contradiction.

**Case E.** \( \phi_{ww}(w_0, y_0) < 0 \) and \( \rho \phi_{w}(w_0, y_0) + (b - r) \phi_{w_y}(w_0, y_0) = 0 \). This is the same as the situation in Case C.

In view now of the stability properties of viscosity solutions, in order to conclude this part of the proof, we need only to establish that as \( N \to \infty \), \( v^N \to v \), locally uniformly on \( \bar{\Omega} \).

To this end, we fix \((w, y) \in \bar{\Omega}\) and choose \((C^\varepsilon, \pi^\varepsilon) \in \mathcal{H}(w, y)\) such that

\[
v(w, y) \leq E \left[ \int_0^{+\infty} e^{-\beta t} U(C^\varepsilon_t) \, dt \right] + \varepsilon.
\]

Then \((C^\varepsilon \cap N, \pi^\varepsilon) \in \mathcal{H}^N(w, y)\) and the monotone convergence theorem implies that

\[
\lim_{N \to \infty} E \left[ \int_0^{+\infty} e^{-\beta t} U(C^\varepsilon_t \cap N) \, dt \right] = E \left[ \int_0^{+\infty} e^{-\beta t} U(C^\varepsilon_t) \, dt \right].
\]

Therefore,

\[
v^N(w, y) \leq v(w, y) \leq E \left[ \int_0^{+\infty} e^{-\beta t} U(C^\varepsilon_t \cap N) \, dt \right] + 2\varepsilon \leq v^N(w, y) + 2\varepsilon.
\]

Therefore, \( v^N \to v \) for each \((w, y) \in \bar{\Omega}\). On the other hand, \( v^N \) increases with respect to \( N \) and \( v \) is continuous. Dini's theorem therefore implies that \( v^N \to v \), locally uniformly. \( \square \)

Next we present a comparison result for constrained viscosity solutions of (2.7). Comparison results for a large class of boundary problems were given by Ishii and Lions (1990). The equation on hand, however, does not satisfy some of the assumptions in Ishii and Lions (1990), in view of the fact that the controls are not uniformly bounded. It is
therefore necessary to modify some of the arguments in the proof of Theorem II.2 in Ishii and Lions (1990) in order to take care of these difficulties. Since such modifications were done in a one-dimensional analog of the equation at hand in Zariphopoulou (1992) and Duffie, Fleming, and Zariphopoulou (1991), we will only present the main steps of the proof of Theorem 4.2.

**Theorem 4.2.** Let $u$ be an upper-semicontinuous concave viscosity subsolution of (2.7) on $\bar{\Omega}$ and $v$ a supersolution of (2.7) in $\Omega$ that is bounded from below, uniformly continuous on $\bar{\Omega}$, and locally Lipschitz in $\Omega$, such that $|u(x)| + |v(x)| \leq O(|x|^\gamma)$ for $x$ large. Then $u \leq v$ on $\Omega$.

**Sketch of the proof.** We first observe that (2.7) can be written, with $x = (w, y)$, as

$$\beta v(x) = G(y, D^2v, v_w) + F(v_w) + H(x, Dv),$$

where

$$G(y, D^2v, v_w) = \max \left[ \frac{1}{2} \text{tr} \Sigma(\pi, y) \Sigma(\pi, y) D^2v \right],$$

$$F(v_w) = \max_{\epsilon \leq 0} [-cv_w + U(\epsilon)],$$

$$H(x, Dv) = (rw + y)v_w + \mu(y)v_y,$$

$$\Sigma(\pi, y) = \begin{pmatrix} \sigma_1 \sqrt{1 - \rho^2\pi} & \rho \sigma_1\pi \\ 0 & \sigma(y) \end{pmatrix}.$$
and we look at the maximum of the function $\psi$ defined by

$$
\psi(x, z) = u(x) - v(z) - \phi(x, z).
$$

Then (see Zariphopoulou 1992), the maximum occurs at a (finite) point $(x_0, z_0)$ such that

$$
|x_0 - z_0| \leq \ell(\theta) \delta,
$$

for $\delta$ small and $\ell = \ell(\theta) > 0$ with

$$
\left| \frac{z_0 - x_0}{\delta} - 4\eta \right|^4 \leq \omega_v(k\delta) + \omega_v(\ell(\theta)\delta),
$$

where $k > 0$ and $\omega_v$ is the modulus of continuity of $v$.

Finally (see Zariphopoulou 1992)

$$
\lim_{\theta \to 0} \lim_{\delta \to 0} \theta(w_0 + y_0)^\lambda = 0.
$$

Moreover (see Jensen, Lions, and Souganidis 1988; Ishii and Lions 1990), there are matrices $M$ and $N$ such that

$$
\beta u(x_0) \leq G(y_0, M, \phi_{x,1}) + H(x_0, \phi_x),
$$

$$
\beta v(z_0) \geq G(y_0, N, -\phi_{z,1}) + F(-\phi_{z,1}) + H(z_0, -\phi_z),
$$

and

$$
\begin{pmatrix}
M & 0 \\
0 & N
\end{pmatrix}
\leq
\begin{pmatrix}
\phi_{xx}(x_0, z_0) & \phi_{xz}(x_0, z_0) \\
\phi_{zx}(x_0, z_0) & \phi_{zz}(x_0, z_0)
\end{pmatrix},
$$

where $z_0 = (x_0, y_0)$ and

$$
\phi_{x,1} = \frac{\partial}{\partial w} \phi(x, z), \quad \phi_{z,1} = \frac{\partial}{\partial w} \phi(x, z).
$$

(Note that the above inequalities do not follow immediately from the results in Jensen, Lions, and Souganidis 1988. We have to work first with the upper- and lower-convolution functions $u^\varepsilon$ and $v^\varepsilon$ and then take appropriate limits (as in Ishii-Lions 1990), but we skip these steps for the sake of the length of the proof.)

Using the form of $\phi$, (4.20) becomes

$$
\begin{pmatrix}
M & 0 \\
0 & N
\end{pmatrix}
\leq
\begin{pmatrix}
A + \theta B & -A \\
-A & A
\end{pmatrix},
$$

where $A = D^2[(z - x)/\delta - 4\eta]^4$ and $B = D^2[(w + y)^\lambda]$. 
We next observe that

\[
G(y_0, M, \phi_{x,1}) - G(\bar{y}_0, N, -\phi_{z,1}) \\
\leq \max_{\pi} \left[ \frac{1}{2} \text{tr} \left[ \Sigma(x_0) - \Sigma(z_0) \right] [\Sigma(x_0) - \Sigma(z_0)]^T A \\
+ \frac{1}{2} \text{tr} \Sigma(x_0) \Sigma(x_0)^T (\theta B) + (b - r) \pi(\phi_{x,1} + \phi_{z,1}) \right] \\
\leq \frac{1}{2} [\sigma(y_0) - \sigma(\bar{y}_0)]^2 |A_{22}| + \theta(w_0 + y_0)\lambda^{-2} \\
\cdot \lambda \max_{\pi} \left[ \frac{1}{2}(\lambda - 1)\sigma^2 \pi^2 + \pi(\rho(\lambda - 1)\sigma_1 \sigma(y_0) + (b - r)(w_0 + y_0)) \right] \\
+ \frac{\delta}{2} \theta(w_0, y_0)^{\lambda^{-2}}(1 - \lambda)\sigma^2(y_0) \\
\leq C \left[ \frac{(y_0 - \bar{y}_0)^2}{\delta^2} \left| \frac{z_0 - x_0}{\delta} - 4\eta \right|^2 + \theta(w_0, y_0)^\lambda \right],
\]

for some constant \( C > 0 \), where we used (1.3) repeatedly.

Subtracting (4.19) from (4.18) and using the above inequality together with the fact that \( F \) is a decreasing function gives

\[
\beta[u(x_0) - v(z_0) - \theta(w + y)^\lambda] \leq C \left[ \frac{(y_0 - \bar{y}_0)^2}{\delta^2} \left| \frac{z_0 - x_0}{\delta} - 4\eta \right|^2 + \theta(w_0 + y_0)^\lambda \\
+ \left| \frac{y_0 - \bar{y}_0}{\delta} \right| \left| \frac{z_0 - x_0}{\delta} - 4\eta \right|^3 + \left| \frac{z_0 - x_0}{\delta} - 4\eta \right|^4 \right].
\]

Moreover,

\[
u(\bar{x}) - v(\bar{x}) - \theta(\bar{w} + \bar{y})^\lambda \leq \sup_{\Omega \times \Omega} \phi(x, z) + \omega_\epsilon(\kappa\eta\delta).
\]

We now combine the above inequalities, first sending \( \delta \downarrow 0 \), then \( \theta \downarrow 0 \), and last \( \eta \downarrow 0 \).

Using (4.15), (4.16), and (4.17), we then contradict (4.11).

REFERENCES


