Learning in Repeated Auctions with Budgets: Regret Minimization and Equilibrium

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Abstract. In online advertising markets, advertisers often purchase ad placements through bidding in repeated auctions based on realized viewer information. We study how budget-constrained advertisers may compete in such sequential auctions in the presence of uncertainty about future bidding opportunities and competition. We formulate this problem as a sequential game of incomplete information, in which bidders know neither their own valuation distribution nor the budgets and valuation distributions of their competitors. We introduce a family of practical bidding strategies we refer to as adaptive pacing strategies, in which advertisers adjust their bids according to the sample path of expenditures they exhibit, and analyze the performance of these strategies in different competitive settings. We establish the asymptotic optimality of these strategies when competitors’ bids are independent and identically distributed over auctions, but also when competing bids are arbitrary. When all the bidders adopt these strategies, we establish the convergence of the induced dynamics and characterize a regime (well motivated in the context of online advertising markets) under which these strategies constitute an approximate Nash equilibrium in dynamic strategies: the benefit from unilaterally deviating to other strategies, including ones with access to complete information, becomes negligible as the number of auctions and competitors grows large. This establishes a connection between regret minimization and market stability, by which advertisers can essentially follow approximate equilibrium bidding strategies that also ensure the best performance that can be guaranteed off equilibrium.

Keywords: sequential auctions • online advertising • online learning • stochastic optimization • stochastic approximation • incomplete information • regret analysis • dynamic games

1. Introduction

Online ad spending has grown dramatically in recent years, reaching more than $70 billion in the United States in 2016 (eMarketer 2016b). A substantial part of this spending is generated through auction platforms where advertisers sequentially submit bids to place their ads at slots as these become available. Important examples include ad exchange platforms, such as Google’s DoubleClick and Yahoo’s RightMedia, and more centralized platforms, such as Facebook Exchange and Twitter Ads, as well as sponsored search platforms, such as Google’s AdWords and Microsoft’s Bing Ads. In such markets, advertisers may participate in many thousands of auctions per day, whereby competitive interactions are interconnected by budgets that limit the total expenditures of advertisers throughout their campaign. The complexity of this competition, together with the frequency of opportunities and the time scale on which decisions are made, are key drivers in the rise of automated bidding algorithms that have become common practice among advertisers (eMarketer 2016a).

One operational challenge that is fundamental for managing budgeted campaigns in online ad markets is to balance present and future bidding opportunities effectively. For example, advertisers face the analytical and computational challenge of timing the depletion of their budget with the planned termination of their campaign because running out of budget before the campaign’s planned termination (or alternatively, reaching the end of the campaign with unutilized funds) may result in significant opportunity loss. The challenge of accounting for future opportunities throughout the bidding process has received increasing attention in the online advertising industry: platforms now recommend advertisers to “pace” the rate at which they spend their budget and provide budget-pacing services that are based on a variety of heuristics. 7

Budget pacing is particularly challenging owing to the many uncertainties that characterize the competitive landscape in online ad markets. First, although advertisers can evaluate, just before each auction, the value of
the current opportunity (in terms of, e.g., likelihood of purchase), advertisers do not hold such information about future opportunities that may be realized throughout the campaign. Second, advertisers typically know very little about the extent of the competition they face in online ad markets. They typically do not know the number of advertisers that will bid in each auction, or important characteristics of their competitors, such as their budgets and their expectations regarding future opportunities. In addition, there is uncertainty about the varying levels of competitors’ strategic and technical sophistication: whereas some firms devote considerable resources to develop complex algorithms that are designed to identify and respond to strategies of competitors in real time to maximize their campaign utility, other, less resourceful firms may adopt simple bidding strategies that may be independent of idiosyncratic user information, and that may even seem arbitrary.

The main questions we address in this paper are these: (i) How should budget-constrained advertisers compete in repeated auctions under uncertainty? (ii) What type of performance can be guaranteed with and without typical assumptions on competitors’ behavior? (iii) Can a single class of strategies constitute an equilibrium (when adopted by all advertisers) while achieving the best performance that can be guaranteed off equilibrium? These questions are of practical importance and draw attention from online advertisers and ad platforms.

1.1. Main Contribution
The main contribution of the paper lies in introducing a new family of practical and intuitive bidding strategies that dynamically adapt to uncertainties and competition throughout an advertising campaign only on the basis of observed expenditures, and in analyzing the performance of these strategies off equilibrium and in equilibrium. In more detail, our contribution is along the following lines.

1.1.1. Formulation and Solution Concepts. We formulate the budget-constrained competition between advertisers as a sequential game of incomplete information, in which advertisers know neither their own valuation distribution nor the budgets and valuation distributions of their competitors. To evaluate the performance of a given strategy without any assumptions on competitors’ behavior, we quantify the portion it can guarantee of the performance of the best (feasible) dynamic sequence of bids one could have selected with the benefit of hindsight. The proposed performance metric, referred to as \(\gamma\)-competitiveness, extends that of no-regret strategies (commonly used in unconstrained settings) to a more stringent benchmark that allows for dynamic sequences of actions subject to global constraints. By showing that the maximal portion that can be guaranteed is, in general, less than one, we establish the impossibility of a “no-regret” notion relative to a dynamic benchmark.

1.1.2. Adaptive Pacing Strategies. We introduce a new class of practical bidding strategies that adapt the bidding behavior of advertisers throughout the campaign, solely on the basis of the sample path of expenditures they exhibit. We refer to this class as adaptive pacing strategies, because these strategies dynamically adjust the “pace” at which the advertiser spends its budget. This pace is controlled by a single parameter: a dual multiplier that determines the extent to which the advertiser bids below its true values. This multiplier is updated according to an intuitive primal–dual scheme that extends single-agent learning ideas for the purpose of balancing present and future opportunities in uncertain environments. In particular, these strategies aim at maximizing instantaneous payoffs while guaranteeing that the advertiser depletes its budget “close” to the end of the campaign.

1.1.3. Performance and Stability. We analyze the performance of adaptive pacing strategies under different assumptions on the competitive environment. On the one hand, when competitors’ bids are independent and identically distributed over auctions (we refer to this setting as stationary competition), the performance of this class of strategies converges, in a long-run average sense, to the best performance attainable with the benefit of hindsight. On the other hand, when competitors’ bids are arbitrary (we refer to this setting as arbitrary competition), we establish, through matching lower and upper bounds on \(\gamma\)-competitiveness, the asymptotic optimality of these strategies; that is, no other strategy can guarantee a larger portion of the best performance in hindsight. Together, these results establish performance guarantees and asymptotic optimality in both the “optimistic” and “pessimistic” scenarios. When all bidders follow adaptive pacing strategies, competing bids are nonstationary and endogenous. We show that in such a scenario the sequences of dual multipliers (and the resulting payoffs) converge, and characterize a regime (that is well grounded in practice) under which these strategies constitute an approximate Nash equilibrium: the benefit from unilaterally deviating to any fully informed dynamic strategy diminishes to zero as the number of auctions and the market size grow large, even when advertisers know their value distribution as well as the value distributions and budgets of their competitors up front, and even when they can acquire real-time information on the past value realizations, bids, and payoffs of their competitors.

Notably, without using any prior knowledge of the competitive setting (that is, whether it is stationary, arbitrary, or of simultaneous learning), these strategies asymptotically attain the best performance that is
achievable when advertisers have ex ante knowledge of the nature of the competition. Our results (summarized in Table 1) establish a connection between performance and market stability in our setting: advertisers can follow bidding strategies that constitute an approximate equilibrium and that, at the same time, asymptotically ensure the best performance that can be guaranteed off equilibrium.

To assess the value that may be captured in practice by adaptive pacing strategies, we show empirically, using data from a large online ad auction platform, that these strategies often achieve performance that is very close to the best performance that could have been achieved in hindsight.

1.2. Related Work

Our study lies in the intersection of marketing, economics, operations, and computer science. Although we focus on the broad application area of online advertising, our analysis includes methodological aspects of learning, games of incomplete information, and auctions theory.

1.2.1. Sequential Games of Incomplete Information.

There is a rich literature stream on the conditions under which observation-based dynamics may or may not converge to an equilibrium. For an extensive overview of related work, which excludes budget considerations and mostly focuses on discrete action sets and complete information, see the books of Fudenberg and Levine (1998) and Hart and Mas-Colell (2013). Although convergence to equilibrium has appealing features, it generally does not imply any guarantees on performance along the strategy’s decision path. One important notion that has been used to evaluate performance along the decision path in many sequential settings is Hannan consistency (Hannan 1957), also referred to as universal consistency and, equivalently, no-regret. A strategy is Hannan consistent if, when competitors’ actions are arbitrary, it guarantees a payoff at least as high as any static action that could have been selected in hindsight. An important connection between Hannan consistency and convergence to equilibrium is established by Hart and Mas-Colell (2000), who provide a no-regret strategy that, when adopted by all players, induces an empirical distribution of play that converges to a correlated equilibrium.

In the present paper we advance similar ideas by establishing a connection between performance and equilibrium in a setting with global constraints that limit the sequence of actions players may take. Because in such a setting repeating any single action may be infeasible as well as generate poor performance, we replace the static benchmark that appears in Hannan consistency with the best dynamic and feasible sequence of actions. We characterize the maximal portion of this dynamic benchmark that may be guaranteed, provide a class of strategies that achieve this portion, and demonstrate practical settings in which these strategies constitute an ε-Nash equilibrium among fully informed dynamic strategies. This equilibrium notion is particularly ambitious under incomplete information because it involves look-ahead considerations and is hard to maintain even in more information-abundant settings (see, e.g., Brafman and Tennenholtz 2004 and Ashlagi et al. 2012).

1.2.2. Learning and Equilibrium in Sequential Auctions.

Our work relates to other papers that study learning in various settings of repeated auctions. Iyer et al. (2014) adopt a mean-field approximation to study repeated auctions in which bidders learn about their own private value over time. Weed et al. (2016) consider a sequential auction setting, where bidders do not know the associated intrinsic value, and adopt multiarmed bandit policies to analyze the underlying exploration-exploitation trade-off. Bayesian learning approaches are analyzed, for example by Hon-Snir et al. (1998) in the context of sequential first-price auctions, as well as by Han et al. (2011) under monitoring and entry costs. These studies address settings significantly different from ours and, in particular, do not consider budget constraints. In our setting, items are sold via repeated second-price auctions, and bidders observe current private values when bidding (but are uncertain about future values). Although truthful bidding is weakly dominant in a static second-price auction, a budget-constrained bidder needs to learn and account for the (endogenous) option value of future auctions, which depends directly on its budget, together with unknown factors, such as its value distribution as well as the budgets, value distributions, and strategic behavior of its competitors.

1.2.3. Repeated Auctions with Budgets. Balseiro et al. (2015) introduce a fluid mean-field approximation to study the outcome of the strategic interaction between budget-constrained advertisers in a complete information setting. They show that stationary strategies that shade bids using a constant multiplier can constitute

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an approximate Nash equilibrium, but these strategies rely on the value distributions and the budget constraints of all the advertisers. The present paper focuses on the practical challenge of bidding in these repeated auctions and, in particular, when such broad information on the entire market is not available to the bidder, and when competitors do not necessarily follow an equilibrium strategy. Although we do not impose a fluid mean-field approximation in the present paper, under such an approximation our proposed strategies could be viewed as converging under incomplete information to an equilibrium in fluid-based strategies. In a related work, Conitzer et al. (2018) introduce a pacing equilibrium concept that also relies on complete information. They provide evidence through numerical experiments that the adaptive pacing strategies introduced here would converge to an equilibrium in their setting as well. Convergence under realistic (incomplete) information of the simple dynamics introduced here to the different notions of equilibria mentioned above provides some support to these equilibria as solution concepts. A few other budget management approaches have been recently considered in the literature; see, for example, Charles et al. (2013), Karande et al. (2013), and Balseiro et al. (2017).

Zhou et al. (2008) study budget-constrained bidding for sponsored search in an adversarial setting and provide an algorithm with a competitive ratio depending on upper and lower bounds on the value-to-weight ratio. Our results are not directly comparable to theirs. Considering a stationary setting, Badanidiyuru et al. (2013) study multiarmed bandit problems in which the decision maker may be resource-constrained and discuss applications to bidding in repeated auctions with budgets whereby arms correspond to possible bids. In our paper the information structure is different because advertisers observe their values before bidding. Our paper also relates to the literature on contextual multiarmed bandit problems with resource constraints, in which the context corresponds to the impression value (see, e.g., Badanidiyuru et al. 2014). The algorithm of Badanidiyuru et al. (2014) can be applied to our setting by discretizing the action space, with performance guarantees that are worse than the guarantees obtained in the present paper. Borgs et al. (2007) study heuristics for bidding in repeated auctions in which advertisers simultaneously update bids on the basis of past campaigns and establish convergence across campaigns. By contrast, we study strategies that converge throughout a single campaign. Finally, Jiang et al. (2014) analyze numerically an approximation scheme that is similar to ours, but from the perspective of a single agent that bids in the presence of an exogenous and stationary market. To the best of our knowledge, the present paper is the first to propose a class of practical budget-pacing strategies with performance guarantees in stationary and adversarial settings, and concrete notions of optimality under competition.

1.2.4. Stochastic and Convex Optimization. The bidding strategies we suggest are based on approximating a Lagrangian dual objective by constructing and following subgradient estimates in the dual space. In this respect our strategies relate to the class of mirror descent schemes introduced by Nemirovski and Yudin (1983); see also Beck and Teboulle (2003) and chapter 2 of Shalev-Shwartz (2012), as well as the potential-based gradient descent method in chapter 11 of Cesa-Bianchi and Lugosi (2006). More broadly, these strategies belong to the class of stochastic approximation methods that have been widely studied and applied in a variety of areas; for a comprehensive review see Benveniste et al. (1990), Lai (2003), and Kushner and Yin (2003), as well as Araman and Caldentey (2011) for more recent applications in revenue management. This strand of the literature focuses on single-agent learning methods for adjusting to uncertain environments that are exogenous and stationary. Exceptions that consider the performance of stochastic approximation schemes in nonstationary environments include the formulations in section 3.2 of Kushner and Yin (2003) and chapter 4 of Benveniste et al. (1990), as well as Besbes et al. (2015). Notably, these frameworks consider underlying environments that change exogenously, and decision makers that operate as monopolists. In other related works, Rosen (1965) demonstrates finding an equilibrium in concave games by applying a gradient-based method in a centralized manner with complete information on the payoff functions, and Nedic and Ozdaglar (2009) study the convergence of distributed gradient-based methods adopted by cooperative agents.

The present paper contributes to this literature by extending, in two key aspects, single-agent learning ideas in a setting of practical importance. First, our proposed strategies use expenditure observations to construct subgradient estimates period by period, each time for a different component of the dual objective we approximate. This facilitates efficient learning throughout the campaign, rather than between campaigns. Second, the proposed strategies are analyzed in settings that include both uncertainty and competition. In particular, when these strategies are adopted by all advertisers, the resulting environment is both endogenous and nonstationary.

Our performance analysis in the presence of arbitrary competition relates to the literature initiated by Zinkevich (2003) on online convex optimization in an adversarial framework, where underlying cost functions can be selected at any point in time by an adversary, depending on the actions of the decision maker. (This constitutes a more pessimistic environment than the traditional stochastic setting, where a cost function is
picked a priori and held fixed.) For an overview of related settings, see chapter 2 of Shalev-Shwartz (2012).

1.2.5. Adaptive Algorithms. The challenge of designing strategies that adapt to unknown characteristics of the environment (in the sense of achieving ex post performance that is essentially as good as the performance that can be achieved when advertisers have ex ante knowledge of the nature of the competition) dates back to studies in the statistics literature (see Tsybakov 2008 and references therein) and has seen recent interest in machine learning. Examples include Mirrokni et al. (2012), who seek to design an algorithm for an online budgeted allocation problem that achieves an optimal competitive ratio in both adversarial and stochastic settings; Seldin and Slivkins (2014), who present an algorithm that achieves (near) optimal performance in both stochastic and adversarial multiarmed bandit regimes without prior knowledge of the nature of the environment; and Sani et al. (2014), who consider an online convex optimization setting and derive algorithms that are rate optimal regardless of whether the target function is weakly or strongly convex.

2. Incomplete Information Model

We model a sequential game of incomplete information with budget constraints, where K risk-neutral heterogeneous advertisers repeatedly bid to place ads through second-price auctions. We note that many of our modeling assumptions can be generalized and are made only to simplify exposition and analysis; some generalizations are discussed in Section 2.1.

In each time period \( t = 1, \ldots, T \) there is an available ad slot that is auctioned. The auctioneer (i.e., the platform) first shares with the advertisers some information about a visiting user and then runs a second-price auction with a reserve price of zero to determine which ad to show to the user. The information provided by the auctioneer heterogeneously affects the values advertisers perceive for the impression on the basis of their targeting criteria. The values advertisers assign to the impression they bid for at time \( t \) are denoted by the random vector \((v_{k,t})_{k=1}^K\), and are assumed to be independently distributed across impressions and advertisers, with a support over \( \prod_{k=1}^K [0, \bar{v}_k] \subseteq \mathbb{R}_+^K \). We denote the marginal cumulative distribution function of \( v_{k,t} \) by \( F_{k,t} \) and assume that \( F_{k,t} \) is absolutely continuous with bounded density \( f_{k,t} \). Advertiser \( k \) has a budget \( B_{k,t} \), which limits the total payments that can be made by the advertiser throughout the campaign. We denote by \( \rho_k := B_{k,t}/T \) the target expenditure (per impression) rate of advertiser \( k \). We assume that \( 0 < \rho_k \leq \bar{\rho}_k \) for each advertiser \( k \in \{1, \ldots, K\} \); otherwise the problem would be trivial because the advertiser would never deplete its budget. Each advertiser \( k \) is therefore characterized by a type \( \theta_k := (F_{k,t}, \rho_k) \).

In each period \( t \) each advertiser privately observes a valuation \( v_{k,t} \) and then posts a bid \( b_{k,t} \). Given the bidding profile \((b_{k,t})_{k=1}^K \in \mathbb{R}_+^K\), we denote by \( d_{k,t} \) the highest competing bid faced by advertiser \( k \):

\[
d_{k,t} := \max_{i: i \neq k} \{ b_{i,t} \}.
\]

We assume that advertisers have a quasilinear utility function given by the difference between the sum of the valuations generated by the impressions won and the expenditures corresponding to the second-price rule. We denote by \( z_{k,t} \) the expenditure of advertiser \( k \) at time \( t \):

\[
z_{k,t} := 1\{ d_{k,t} \leq b_{k,t} \} d_{k,t},
\]

and by \( u_{k,t} := 1\{ d_{k,t} \leq b_{k,t} \} (v_{k,t} - d_{k,t}) \) the corresponding net utility,\(^2\) After the bidding takes place, the auctioneer allocates the ad to the highest bidder, and each advertiser \( k \) privately observes its own expenditure \( z_{k,t} \) and net utility \( u_{k,t} \).

2.1. Information Structure and Admissible Budget-Feasible Bidding Strategies

We assume that at the beginning of the horizon advertisers have no information on their valuation distributions, nor on their competitors’; that is, the distributions \( \{ F_k : k = 1, \ldots, K \} \) are unknown. Each advertiser knows its own target expenditure rate \( \rho_k \), as well as the length of the campaign \( T \), but has no information on the budgets or the target expenditure rates of its competitors. In particular, each advertiser does not know the number of competitors in the market, their types, or its own type.

We next formalize the class of (potentially randomized) admissible and budget-feasible bidding strategies. Let \( y \) be a random variable defined over a probability space \((\mathcal{Y}, \mathcal{Y}, \mathbb{P}_y)\). We denote by \( \mathcal{H}_{k,t} \) the history available at time \( t \) to advertiser \( k \), defined by

\[
\mathcal{H}_{k,t} := \sigma \left( (v_{k,t}, z_{k,t}, u_{k,t})_{t=1}^{t-1}, v_{k,t}, y \right),
\]

for any \( t \geq 2 \), with \( \mathcal{H}_{k,1} := \sigma (v_{k,t}, y) \). A bidding strategy for advertiser \( k \) is a sequence of mappings \( \beta = (\beta_1, \beta_2, \ldots) \), where the functions \( \beta_t : \mathbb{R}_+ \times \mathcal{Y} \rightarrow \mathcal{R}_+ \) and \( \beta_t : \mathbb{R}_+^{k(t-1)+1} \times \mathcal{Y} \rightarrow \mathcal{R}_+ \) for \( t = 2, \ldots, T \) map available histories to bids \( \{ b_{k,t} : t = 1, \ldots, T \} \). A strategy is admissible, or nonanticipating, if it depends only on information that is available to the bidder; that is, if in each period \( t \) the strategy \( \beta_t \) is measurable with respect to the filtration \( \mathcal{H}_{k,t} \). We say that \( \beta \) is budget feasible if it generates expenditures that are constrained by the available budget; that is, \( \sum_{t=1}^T 1\{ d_{k,t} \leq b_{k,t} \} d_{k,t} \leq \rho_k T \) for any realized vector of highest competitors’ bids \( \mathbf{d}_k := (d_{k,t})_{t=1}^T \in \mathbb{R}_+^T \). We denote by \( \mathcal{B}_k \) the class of admissible budget-feasible strategies for advertiser \( k \) with target expenditure rate
The product space of strategies for all advertisers.

2.2. Off-Equilibrium Performance

To evaluate the off-equilibrium performance of a candidate strategy $\beta \in \mathcal{B}_k$ without any assumptions on the competitors’ strategies, we define the sample-path performance given vectors of realized values $v_t = (v_{k,t})_{t=1}^T \in [0, \delta_k]^T$ and realized competing bids $d_k$ as follows:

$$\pi_k^\beta(v_t; d_k) := \mathbb{E}^\beta \left[ \sum_{t=1}^T 1(d_{k,t} \leq b_{k,t})(v_{k,t} - d_{k,t}) \right],$$

where the expectation is taken with respect to any randomness embedded in the strategy. 3

We quantify the minimal loss a strategy can guarantee relative to any sequence of bids one could have made with the benefit of hindsight. Given sequences of realized valuations $v_t$ and highest competing bids $d_k$, we denote by $\pi_k^H(v_t; d_k)$ the best performance advertiser $k$ could have achieved with the benefit of hindsight:

$$\pi_k^H(v_t; d_k) := \max_{x_t \in [0,1]} \sum_{t=1}^T x_t \cdot (v_{k,t} - d_{k,t})$$

s.t. $\sum_{t=1}^T x_t d_{k,t} \leq T \rho_k$,

where the binary variable $x_t$ indicates whether the auction at time $t$ is won by advertiser $k$. The solution of (1) is the best dynamic response (sequence of bids that could have been made) against a fixed environment characterized by $v_t$ and $d_k$. Notably, if $v_k$ and $d_k$ are such that the bidder is effectively unconstrained (that is, if $\sum_{t=1}^T 1(d_{k,t} \leq v_{k,t})d_{k,t} \leq \rho_k T$), then $\pi_k^H$ is achieved by truthful bidding (i.e., $b_{k,t} = v_{k,t}$); otherwise, $\pi_k^H$ is the solution to a knapsack problem (for an overview of knapsack problems see, e.g., Martello and Toth 1990). Although the identification of the best sequence of bids in hindsight does not account for the potential competitive reaction to that sequence, this benchmark is tractable and has appealing practical features. For example, the best performance in hindsight can be computed using only historical data and requires no behavioral assumptions on competitors (see, e.g., Talluri and van Ryzin 2004, chapter 11.3).

Nonanticipating strategies might not be able to achieve or approach $\pi_k^H$. For some $\gamma \in [1, \infty)$, a bidding strategy $\beta \in \mathcal{B}_k$ is said to be asymptotic $\gamma$-competitive (Borodin and El-Yaniv 1998) if

$$\limsup_{T \to \infty} \sup_{\beta_k \in \mathcal{B}_k} \frac{1}{T} \left( \pi_k^H(v_t; d_k) - \gamma \pi_k^\beta(v_t; d_k) \right) \leq 0.$$ 2

An asymptotic $\gamma$-competitive bidding strategy (asymptotically) guarantees a portion of at least $1/\gamma$ of the performance of any dynamic sequence of bids that could have been selected in hindsight. 4

2.3. Equilibrium Solution Concept

Given a strategy profile $\beta = (\beta_k)_{k=1}^K \in \mathcal{B}$, we denote by $\Pi_k^\beta$ the total expected payoff for advertiser $k$:

$$\Pi_k^\beta := \mathbb{E}^\beta \left[ \sum_{t=1}^T 1(d_{k,t}^\beta \leq b_{k,t}^\beta)(v_{k,t} - d_{k,t}^\beta) \right],$$

where the expectation is taken with respect to any randomness embedded in the strategies $\beta$ and the random values of all the advertisers, and where we denote the highest competing bid induced by competitors’ strategies $\beta_{-k} = (\beta_{-k})_{k \neq k}$ by $d_{k,t}^{\beta_{-k}} = \max_{i \neq k} b_{i,t}^\beta$. We say that a strategy profile $\beta \in \mathcal{B}$ constitutes an $\epsilon$-Nash equilibrium in dynamic strategies if each player’s incentive to unilaterally deviate to another strategy is at most $\epsilon$; that is, if for all $k$:

$$\frac{1}{T} \sup_{\beta_{-k} \in [\mathcal{B}^C_k]} \left( \Pi_k^{\beta_{-k}}(\beta_k^\beta) - \Pi_k^\beta(\beta_k^\beta) \right) \leq \epsilon,$$ 3

where $\mathcal{B}^C_k \supset \mathcal{B}_k$ denotes the class of strategies with complete information on market primitives. Although a more precise description of this class will be provided in Section 5, we note that this class also includes strategies with access to complete information on the types $(\theta_t)_{t=1}^K$ and on the bids made in past periods by all advertisers. Naturally, the solution concept in (3) takes into account the strategic response of the competitive environment to the actions of an advertiser.

2.4. Discussion of Model Assumptions

Although we focus on second-price auctions, our results hold for all deterministic dominant-strategy mechanisms that are incentive compatible and individually rational with nonnegative transfers. This broad class includes mechanisms such as second-price auctions with anonymous reserve prices, second-price auctions with personalized reserve prices, ranking mechanisms, and Myerson’s optimal mechanism. A question of theoretical and practical interest is how to extend the adaptive pacing approach that is suggested in this paper to nontruthful mechanisms such as first-price auctions.

Predicated on the common practice of running campaigns with daily budgets, we focus on synchronous campaigns that start and finish simultaneously. The synchronicity of the campaigns allows one to capture some key market features, while providing the tractability required to transparently highlight key problem characteristics. We note that the results in Section 3.3 that are established under arbitrary competition are agnostic to this assumption and hold even when campaigns start and finish at different times. An interesting...
research direction is to extend our convergence and equilibrium results to models with asynchronous campaigns, such as those in Iyer et al. (2014) and Balseiro et al. (2015).

Our model assumes that values are independent across time and advertisers and absolutely continuous. The assumption that values are independent across time is frequently used in the literature and is based on typical browsing behavior of internet users that leads to weak intertemporal correlations. Our model can be relaxed to accommodate correlation across advertisers’ values as long as certain technical conditions (such as smoothness of the dual objective function and strong monotonicity of the expenditure function) hold. Although we provide sufficient conditions that rely on values being independent, these conditions can be shown to hold when values are imperfectly correlated and when distributions have atoms at zero (capturing the possibility of advertisers not participating in every auction). We discuss the absolute continuity of values in Section 6.

In our model the number of impressions is known to all advertisers. In practice advertisers may infer the length of their campaign (in terms of total number of impressions) before its beginning, using user traffic statistics commonly provided by publishers and analytics companies. Nevertheless, our model can accommodate a random number of impressions through dummy arrivals that are valued at zero by the advertiser.

3. Off-Equilibrium Performance Guarantees
In this section we study the performance that can be guaranteed by bidding strategies with and without assumptions on the primitives (budgets, valuations) and strategies that govern competitors’ bids. We obtain a lower bound on the minimal loss one must incur relative to the best dynamic response in hindsight under arbitrary competition. We then introduce a class of adaptive pacing strategies that react to realized valuations and expenditures throughout the campaign and obtain an upper bound on the performance of these strategies under arbitrary and stationary competitive environments, thereby establishing asymptotic optimality in both settings.

3.1. Lower Bound on the Achievable Guaranteed Performance
Before investigating the type of performance an advertiser can aspire to, we first formalize the type of performance that cannot be achieved. The following result bounds the minimal loss one must incur relative to the best dynamic (and feasible) sequence of bids that could have been selected in hindsight, under arbitrary competition.

Theorem 1 (Lower Bound on the Achievable Guaranteed Performance). For any target expenditure rate \( \rho_k \in (0, \bar{\rho}_k] \) and number \( \gamma < \bar{\rho}_k / \rho_k \) there exists some constant \( C > 0 \) such that for any bidding strategy \( \beta \in \mathcal{B}_k \):

\[
\limsup_{T \to \infty} \sup_{\beta_k = \rho_k T} \frac{1}{T} \left( \pi^H_k(v_k; d_k) - \gamma \pi^\beta_k(v_k; d_k) \right) \geq C.
\]

Theorem 1 establishes that no admissible strategy can guarantee asymptotic \( \gamma \)-competitiveness for any \( \gamma < \bar{\rho}_k / \rho_k \) and therefore cannot guarantee a portion larger than \( \rho_k / \bar{\rho}_k \) of the total net utility achieved by a dynamic response that is taken with the benefit of hindsight. Because an advertiser should never pay more than its value, \( \bar{\rho}_k \) is the maximal possible expenditure and, thus, the ratio \( \rho_k / \bar{\rho}_k \) captures the relative “wealthiness” of the advertiser. This implies that bidding strategies may perform poorly relative to best performance in hindsight when budgets are small, but better performance might be guaranteed when budgets are bigger (when \( \rho_k \geq \bar{\rho}_k \) the advertiser can bid truthfully to guarantee optimal performance and 1-competitiveness). Thus, Theorem 1 establishes the impossibility of “no-regret” relative to dynamic sequences of bids, by showing that, in general, the performance of any learning strategy cannot be guaranteed to approach \( \pi^H_k \).

We next describe the main ideas in the proof of the theorem. We first observe, by adapting Yao’s principle (Yao 1977) to our setting, that to bound the worst-case loss of any (deterministic or not) strategy relative to the best response in hindsight, it suffices to analyze the expected loss of deterministic strategies relative to that benchmark, where the sequence of competing bids is drawn from a certain distribution (see Lemma A.1 in the online supplemental material). We next illustrate the idea behind the worst-case instance we construct. We assume that the advertiser knows in advance that the sequence of highest competing bids will be randomly selected from the set \( \{d^1, d^2\} \), where

\[
d^1 = \left( d^{\text{high}}, \ldots, d^{\text{high}}, \bar{\rho}_k, \ldots, \bar{\rho}_k \right),
\]

\[
d^2 = \left( d^{\text{high}}, \ldots, d^{\text{high}}, d^{\text{low}}, \ldots, d^{\text{low}} \right)
\]

for some \( \bar{\rho}_k \geq d^{\text{high}} > d^{\text{low}} > 0 \), and that values will be \( \bar{\rho}_k \) throughout the campaign. We establish that in this case one may restrict analysis to strategies that determine before the beginning of the campaign how many auctions to win at different stages of the campaign (see Lemma A.2). This presents the advertiser with the following trade-off: whereas early auctions introduce a return per unit of budget that is certain but low, later auctions introduce a return per unit of budget that may
be higher but may also decrease to zero. Because the number of auctions in the first stage, denoted by $\tau$, may be designed to grow with $T$, this trade-off may drive a loss that does not diminish to zero asymptotically. The proof of the theorem presents a more general construction that follows the ideas illustrated above and tunes the structural parameters to maximize the worst-case loss that must be incurred by any strategy relative to the best dynamic response in hindsight. Notably, because the above construction assumes that the advertiser’s values are known and fixed to $\bar{v}_k$, Theorem 1 implies that, in general, no strategy can achieve a competitive ratio lower than $\bar{v}_k/\rho_k$ (and, therefore, diminishing regret) even if advertisers know their own valuations.

### 3.2. Asymptotic Optimal Bidding Strategy

We introduce an adaptive pacing strategy that adjusts bids according to observations. In what follows we denote by $P_{[a,b]}(x) = \min\{\max\{x,a\},b\}$ the Euclidean projection operator on the interval $[a,b]$.

**Adaptive Pacing Strategy (A).** Input: a number $\epsilon_k > 0$.

1. Select an initial multiplier $\mu_{k,0}$ in $[0,\bar{\mu}_k]$, and set the remaining budget to $\bar{B}_{k,1} = B_k = \rho_k T$.

2. For each $t = 1, \ldots, T$:
   
   a. Observe the realization of the random valuation $v_{k,t}$, and post a bid:
   
   $$b_{k,t}^a = \min\{v_{k,t}/(1 + \mu_k), \bar{B}_{k,t}\}.$$ 

   b. Observe the expenditure $z_{k,t}$. Update the multiplier by
   
   $$\mu_{k,t+1} = P_{[0,\bar{\mu}_k]}(\mu_{k,t} - \epsilon_k (\rho_k - z_{k,t}))$$

   and the remaining budget by $\bar{B}_{k,t+1} = \bar{B}_{k,t} - z_{k,t}$.

The adaptive pacing strategy dynamically adjusts the pace at which the advertiser depletes its budget by updating a multiplier that determines the extent to which the advertiser bids below its true values (shades bids). The strategy consists of a sequential approximation scheme that takes place in the Lagrangian dual space and that is designed to approximate the best solution in hindsight defined in (1). The objective of the Lagrangian dual of (1) is $\sum_{t=1}^T x_{k,t}(v_{k,t} - (1 + \mu_d) d_{k,t}) + \mu P_V$, where $\mu$ is a nonnegative multiplier capturing the shadow price associated with the budget constraint. For a fixed $\mu$, the dual objective is maximized by winning all items with $v_{k,t} \geq (1 + \mu) d_{k,t}$, which is obtained by bidding $b_{k,t} = \min\{v_{k,t}/(1 + \mu), \bar{B}_{k,t}\}$, because advertiser $k$ wins whenever $b_{k,t} \geq d_{k,t}$. Therefore, one has

$$\pi_k^{H_1}(v_k; d_k) \leq \inf_{\mu \geq 0} \sum_{t=1}^T (v_{k,t} - (1 + \mu)d_{k,t})^+ + \mu P_V,$$

where we denote by $y^+ = \max\{y,0\}$ the positive part of a number $y \in \mathbb{R}$. The tightest upper bound can be obtained by solving the minimization problem on the right-hand side of (4). Because this problem cannot be solved without prior information on all the values and competing bids throughout the campaign, the proposed bidding strategy approximates (4) by estimating a multiplier $\mu_{k,t}$ and bidding $b_{k,t}^a = v_{k,t}/(1 + \mu_{k,t})$ in each round, as long as the remaining budget suffices.

The dual approximation scheme consists of estimating a direction of improvement and following that direction. More precisely, the sequence of multipliers follows a subgradient descent scheme, using the noisy point access each advertiser has to $\partial \psi_{k,t}(\mu_{k,t})$, namely, the left derivative of the $t$-period component of the dual objective in (4). In other words, in each period $t = 1, 2, \ldots$, one has

$$\partial \psi_{k,t}(\mu_{k,t}) = \rho_k - d_{k,t} 1\{v_{k,t} \geq (1 + \mu_{k,t}) d_{k,t}\} = \rho_k - d_{k,t} 1\{b_{k,t}^a \geq d_{k,t}\} = \rho_k - z_{k,t}.$$

In each period $t$ the advertiser compares its expenditure $z_{k,t}$ to the target expenditure rate $\rho_k$ that is “affordable” given the initial budget. Whenever the advertiser’s expenditure exceeds the target expenditure rate, $\mu_{k,t}$ is increased by $\epsilon_k(z_{k,t} - \rho_k)$, implying that the shadow price increases and that the associated budget constraint becomes more binding. On the other hand, if the advertiser’s expenditure is lower than the target expenditure rate (including periods in which the expenditure is zero) the multiplier $\mu_{k,t}$ is decreased by $\epsilon_k(\rho_k - z_{k,t})$, implying that the shadow price decreases and that the associated budget constraint becomes less binding. Then, given the valuation in that period, the advertiser shades its value using the current estimated multiplier.

The multiplier $\mu_{k,t}$ essentially captures a “belief” advertiser $k$ has at time $t$ regarding the shadow price associated with its budget constraint. Notably, the “correct” shadow price, reflecting the “correct” value of future opportunities, depends on unknown characteristics such as the value distribution of the advertiser, as well as the value distributions, budgets, and strategies of its competitors. Rather than aiming at estimating these unknown factors, the proposed strategy aims at learning the best response to these in terms of this shadow price.

**Assumption 1 (Appropriate Selection of Step Size).** The number $\epsilon_k$ satisfies $0 < \epsilon_k \leq 1/\bar{\mu}_k$, $\lim_{T \to \infty} \epsilon_k = 0$, and $\lim_{T \to \infty} T \epsilon_k = \infty$.

Assumption 1 adapts a standard step size condition that is common in the stochastic approximation literature (see, e.g., section 1.1 in Kushner and Yin 2003) to stationary step sizes and characterizes the range of rates at which step sizes should converge to zero (as a function of the horizon length) to allow for the convergence of the scheme, while guaranteeing sufficient
exploration. By doing so, this assumption prescribes a broad range for tuning the strategy’s step size in a manner that guarantees the asymptotic optimality results that follow. Even though Assumption 1 places some restrictions on the step size selection, it still defines a broad class of step sizes; for example, $\epsilon_k = c/T^\gamma$ satisfies Assumption 1 for any constants $0 < c \leq 1/\bar{\nu}_k$ and $0 < \gamma < 1$.

### 3.3. Performance Under Arbitrary Competition

The following result characterizes the performance that can be guaranteed by an adaptive pacing strategy under arbitrary competition.

**Theorem 2 (Asymptotic Optimality Under Arbitrary Competition).** Let $A$ be an adaptive pacing strategy with a step size $\epsilon_k$ satisfying Assumption 1 and $\hat{\mu}_k \geq \bar{\nu}_k/\rho_k - 1$. Then:

$$\limsup_{T \to \infty} \sup_{B_k = \rho_k T} \frac{1}{T} \mathbb{E}_{\nu_k(0), \bar{d}_k} \left[ \frac{\bar{\nu}_k}{\rho_k} \pi_k^A(\nu_k, \bar{d}_k) - \pi_k^H(\nu_k, \bar{d}_k) \right] = 0.$$ 

Moreover, when the step size $\epsilon_k$ is of order $T^{-1/2}$, the convergence is at rate $T^{-1/2}$.

Theorem 2 establishes that an adaptive pacing strategy is asymptotic ($\bar{\nu}_k/\rho_k$)-competitive (Borodin and El-Yaniv 1998), guaranteeing at least a fraction $\rho_k/\bar{\nu}_k$ of the total net utility achieved by any dynamic response that could have been selected with the benefit of hindsight. This establishes the asymptotic optimality of the strategy, because by Theorem 1 no other admissible bidding strategy can guarantee a larger fraction of the best performance in hindsight. (Recalling the worst-case instance used in Theorem 1, a larger fraction of the best performance in hindsight cannot be guaranteed even when advertisers know their valuations up front.)

Moreover, when the step size $\epsilon_k$ is of order $T^{-1/2}$, the convergence is at rate $T^{-1/2}$.

3.4. Performance Under Stationary Competition

The upper bound in Theorem 2 holds in every sample path (because the adaptive pacing strategy is deterministic) and is extremely robust. For example, it holds also when campaigns can start and finish at different times and under arbitrary sequences of valuations and competing bids, including sequences that are correlated across impressions or advertisers, as well as with dynamic sequences that are based on knowledge about the bidding strategy and its realized path. However, the established performance guarantee could be viewed as conservative, or “pessimistic,” in the sense that it holds under any sequences of valuations and competing bids, including worst-case instances such as the one described in the proof of Theorem 1. In particular, when the target expenditure rate $\rho_k$ is small relative to $\bar{\nu}_k$, the guaranteed performance is limited as well. In this subsection we demonstrate that for any fixed target expenditure rate, the asymptotic optimality of an adaptive pacing strategy can also be carried over (with much better performance) to more “optimistic” settings in which uncertainties are drawn independently from a stationary distribution. (See Iyer et al. 2014 and Balseiro et al. 2015 for discussions on the validity of assuming stationarity competition.)

We assume that the pairs $(\nu_k, \bar{d}_k)$ are independently drawn from some unknown stationary distribution. We denote by $\Psi_k(\mu) := \sum_{t=1}^T \left[ \mathbb{E}_{\nu_k, \bar{d}_k} \left( (\nu_k - (1 + \mu) \bar{d}_k)^+ \right) + \mu \bar{\nu}_k \right]$ the expectation of the dual objective function given in (4), with respect to values and competing bids.

**Theorem 3 (Asymptotic Optimality Under Stationary Competition).** Suppose that in each period $t$ the pair $(\nu_k, \bar{d}_k)$ is independently drawn from some stationary distribution such that the dual function $\Psi_k(\mu)$ is thrice differentiable with bounded derivatives and strongly convex with parameter $\lambda_k > 0$. Let $A$ be an adaptive pacing strategy with a step size $\epsilon_k$ satisfying Assumption 1, as well as $\epsilon_k < 1/(2\lambda_k)$ and $\hat{\mu}_k \geq \bar{\nu}_k/\rho_k - 1$. Then

$$\lim_{T \to \infty} \sup_{B_k = \rho_k T} \frac{1}{T} \mathbb{E}_{\nu_k, \bar{d}_k} \left[ \pi_k^H(\nu_k, \bar{d}_k) - \pi_k^A(\nu_k, \bar{d}_k) \right] = 0.$$ 

Moreover, when the step size $\epsilon_k$ is of order $T^{-1/2}$, the convergence is at rate $T^{-1/2}$.

Theorem 3 establishes that when valuations and competing bids are independent and identically drawn throughout time, an adaptive pacing strategy is long-run average optimal, in the sense that for any fixed $\rho_k > 0$, its average expected performance converges to the average expected performance of the best sequence of bids that could have been selected in hindsight. Together with Theorems 1 and 2, Theorem 3 establishes the asymptotic optimality of the adaptive pacing strategy in both the “pessimistic” and “optimistic”
scenarios when advertisers have no prior knowledge of whether the nature of the competition is “stationary” or “arbitrary,” and that selecting $\epsilon_k \sim T^{-1/2}$ guarantees a convergence rate of $T^{-1/2}$ in both frameworks. We leave the characterization of rate optimality in these frameworks as an open problem.

Theorem 3 is based on the assumption that the dual function is thrice differentiable and strongly convex. The first condition is technical and is required to perform Taylor series expansions, and the second condition is a standard one guaranteeing that the multipliers employed by an adaptive pacing strategy converge. Lemma C.1 in Online Supplemental Material Part C shows that these conditions hold when values and competing bids are independent and absolutely continuous with bounded densities and the valuation density is differentiable. The conditions in Theorem 3 can also be shown to hold when competing bids are (imperfectly) correlated across auctions.

To prove the theorem we first upper bound the expected performance in hindsight in terms of $\Psi(\mu^*)$, where $\mu^*$ minimizes the dual function. We then lower bound the performance of the adaptive pacing strategy in terms of $\Psi(\mu^*)$ by developing a second-order expansion around $\mu^*$ of the expected utility per auction when the strategy uses a multiplier $\mu_{k,t}$. This expansion involves the mean error $E[\mu_{k,t} - \mu^*]$ in the first-order term and the mean squared error $E[(\mu_{k,t} - \mu^*)^2]$ in the second-order term. Using the update rule we control the cumulative sum of these errors to establish that there exist some positive constants $C_1$, $C_2$, and $C_3$, independent of $T$, such that

$$E_{\nu_k, \mathbf{d}_k} \left[ \pi_k^A(\nu_k, \mathbf{d}_k) - \pi_k^H(\nu_k, \mathbf{d}_k) \right] \leq C_1 + \frac{C_2}{\epsilon_k} + C_3 T \epsilon_k.$$

This establishes asymptotic optimality whenever the sequence of step sizes satisfies Assumption 1 and establishes a convergence rate of $T^{-1/2}$ under a step size selection of order $T^{-1/2}$.

### 3.5. Empirical Proof of Concept Using Ad Auction Data

To analyze the value that may be captured in practice by an adaptive pacing strategy, we empirically evaluate the portion this strategy recovers of the best performance in hindsight, using data from a large online ad auction platform. The data set includes bids from 13,182 sequential online ad auctions, in which a total of 2,923 advertisers participated. The objective of the following analysis is to demonstrate the effectiveness of an adaptive pacing strategy in a realistic yet simple setting. In particular, an optimal tuning of the strategy’s parameters may depend on the available budget as well as competition characteristics. Obtaining the optimal characterization of these parameters is a challenging open problem that we leave as future work.

#### 3.5.1. Setup

For each advertiser $k$, we set the sequence of valuations $\nu_k$ to be the sequence of recorded bids submitted by that advertiser (where $\hat{\nu}_k$ is the highest recorded bid of advertiser $k$) and the sequence $\mathbf{d}_k$ to be the sequence of highest bids submitted by other advertisers. We randomly generated $\rho_k \in [0, \hat{\nu}_k]$ according to a uniform distribution, and for each target expenditure rate we approximated $\pi_k^H(\nu_k, \mathbf{d}_k)$ using a standard linear programming relaxation. Note that whenever an advertiser is effectively unconstrained, that is, $\sum_{t=1}^T \mathbf{1}(d_{k,t} \leq \nu_{k,t})d_{k,t} \leq \rho_k T$, the best performance in hindsight, $\pi_k^H$, can be implemented by truthful bidding. We then computed the performance of an adaptive pacing strategy $\pi_k^A$ and compared its performance with $\pi_k^H$. We focused on the 10 bidders with the largest cumulative expenditure (together, these advertisers were responsible for more than 98% of the overall expenditure), and for each of these advertisers we replicated the process 200 times. In all the instances we used an initial multiplier $\mu_1 = 0.5$ and a step size $\epsilon = T^{-1/2}$ (the choice of the initial multiplier had very limited impact on the performance).

#### 3.5.2. Results and Discussion

The plot that appears at the left-hand side of Figure 1 depicts the fractions of $\pi_k^H$ (as a function of $\rho_k/\hat{\nu}_k$) that were recovered by the adaptive pacing strategy. Notably, in all the instances the strategy significantly outperformed the payoff guarantees for an arbitrary competitive environment. Moreover, it approximately achieved $\pi_k^H$, except when the target expenditure rate was particularly small. The plot on the right-hand side of Figure 1 compares the fractions of $\pi_k^H$ that were recovered by the adaptive pacing strategy with those recovered by “truthful bidding,” a naïve strategy that bids the advertiser’s value until the budget is depleted (each dot corresponds to the performance of one advertiser under one target expenditure rate). We plotted these quantities as a function of $B_{k}/B^0_k$, where $B_k = \rho_k T$ is the available budget, and $B^0_k = \sum_{t=1}^T \mathbf{1}(d_{k,t} \leq \nu_{k,t})d_{k,t}$ is the minimal budget under which the advertiser is effectively unconstrained and truthful bidding is optimal (the plot focuses on instances with $B_{k}/B^0_k \leq 2$). One may expect that whenever $\rho_k < \hat{\nu}_k$ but $B_{k}/B^0_k \geq 1$, any nonanticipating strategy would incur a loss (relative to truthful bidding) that is associated with not having ex ante knowledge of being effectively unconstrained. However, in these cases the loss incurred by the adaptive pacing strategy is less than 1%. On the other hand, whenever $B_{k}/B^0_k < 1$, the advertiser is effectively constrained. On average, in these cases the adaptive pacing strategy outperforms truthful bidding by 120%. We attribute the relative deterioration in the empirical performance of adaptive pacing for small values of $\rho_k$ to the fact that, for fixed valuations and competing bids, a very small
budget suffices for winning only a few auctions (under such conditions one could expect the ability of learning policies to compete with the best sequence of bids in hindsight to be limited). We further note that one can likely improve the empirical performance of the strategy by customizing the selection of the tuning parameters for different advertisers. Overall, these results demonstrate the practical effectiveness of an adaptive pacing strategy even under moderate budgets and problem horizons, as well as realistic bidding behavior that may be subject to correlated preferences.

4. Convergence of Simultaneous Learning
We turn to study the dynamics that emerge when all advertisers follow adaptive pacing strategies simultaneously. We establish that the sequences of multipliers generated by the strategies converge to a tractable profile of multipliers and that the payoffs converge to the payoffs achieved under such a profile.

4.1. A Candidate Profile of Dual Multipliers
Under simultaneous adoption of adaptive pacing strategies, each advertiser follows the subgradient \( \partial_\psi k \cdot (\mu_k) = \rho_k - z_k \cdot \). Therefore, if the induced sequences of multipliers converge, one may expect the limiting profile to consist of multipliers under which each advertiser’s expected expenditure equals its target expenditure (whenever its budget is binding). Denote the expected expenditure per auction of advertiser \( k \) when advertisers shade bids according to a profile of multipliers \( \mu \in \mathbb{R}^k \) by \( G_k(\mu) := \mathbb{E}_r \{ (1 + \mu_k) d_k \leq v_k \} d_k \), where \( d_k = \max_{i : i \neq k} v_i / (1 + \mu_i) \) and the expectation is taken with respect to the values \( v_k \sim F_k \). Following the above intuition, we consider the vector \( \mu^* \), defined by the complementary conditions:

\[
\mu_k^* \geq 0 \quad \perp \quad G_k(\mu^*) \leq \rho_k, \quad \forall k = 1, \ldots, K,
\]

where \( \perp \) indicates a complementarity condition between the multiplier and the expenditure; that is, at least one condition must hold with equality. Intuitively, each advertiser shades bids if its expected expenditure is equal to its target rate. Conversely, if an advertiser’s target rate exceeds its expected expenditure, then it bids truthfully by setting the multiplier equal to zero.

We next provide sufficient conditions for the uniqueness of the vector \( \mu^* \). The vector function \( G : \mathbb{R}^K \rightarrow \mathbb{R}^K \) is said to be \( \lambda \)-strongly monotone over a set \( \mathcal{U} \subseteq \mathbb{R}^K \) if \( (\mu - \mu')^t (G(\mu) - G(\mu')) \geq \lambda \| \mu - \mu' \|^2 \) for all \( \mu, \mu' \in \mathcal{U} \). We refer to \( \lambda \) as the monotonicity constant.

**Assumption 2 (Stability).**

1. There exists a set \( \mathcal{U} = \prod_{k=1}^K [0, \tilde{\mu}_k] \) and a monotonicity constant \( \lambda > 0 \), independent of \( K \), such that the expected expenditure function \( G(\mu) \) is \( \lambda \)-strongly monotone over \( \mathcal{U} \).

2. The target expenditure rate satisfies \( \rho_k \geq \bar{v}_k / \tilde{\mu}_k \) for every bidder \( k \).

The first part of Assumption 2 requires the expenditure function to be strongly monotone over a set of feasible multipliers. In the case of a single agent, the strong monotonicity of the expenditure function is equivalent to the strong convexity of the dual objective function (see Theorem 3). Therefore, the monotonicity condition is a natural extension of the strong convexity condition to a simultaneous learning setting. The monotonicity condition is similar to other common conditions that guarantee uniqueness and stability in
concave (unconstrained) games; see, for example, Rosen (1965), as well as the discussion on convergence of stochastic approximation methods in multidimensional action spaces in section 1.10, part 2 of Benveniste et al. (1990). To provide some intuition on this condition, consider a scenario in which the expected expenditure per auction of all advertisers is above their target expenditure rate (that is, $G_k(\mu) > \rho_k$ for all $k$). Because the expenditure function $G_k$ is nonincreasing in $\mu_k$ for all $k$, each advertiser can get closer to its target expenditure rate by unilaterally increasing the multiplier by a small amount. This condition guarantees that if all advertisers simultaneously increase their multipliers by a small amount, then all advertisers get closer to their target expenditure rates. The second part of the assumption that budgets are not too small and guarantees that $\mu^* \in \mathbb{R}_+$. This part suggests a simple rule of thumb for setting a sufficiently large upper bound $\bar{\mu}_k$. In Online Supplemental Material Part C (see Proposition C.3), we show that Assumption 2 guarantees the existence of a unique vector of multipliers $\mu^*$. As we further discuss in Section 6, the uniqueness of $\mu^*$ has practical benefits that may motivate a market designer to place restrictions on the target expenditure rates of advertisers to guarantee that Assumption 2 holds.

4.2. Convergence of Dual Multipliers

To analyze the sample path of multipliers resulting from simultaneous adoption of adaptive pacing strategies, we consider some conditions on the step sizes selected by the different advertisers.

**Assumption 3** (Joint Selection of Step Sizes). Let $\bar{\varepsilon} = \max_{c_k \in \{1,...,K\}} \varepsilon_{c_k}$, $\bar{\xi} = \min_{c_k \in \{1,...,K\}} \xi_{c_k}$, $\bar{\delta} = \max_k (\bar{\delta}_k)$, and let $\lambda$ be the monotonicity constant that appears in Assumption 2. The profile of step sizes $(\varepsilon_{c_k})_{c_k=1}^{K}$ satisfies the following:

i. $\bar{\varepsilon} \leq 1/\bar{\delta}$ and $\bar{\varepsilon} < 1/(2\lambda)$,

ii. $\lim_{T \to \infty} \varepsilon_T^2/\bar{\xi} = 0$,

iii. $\lim_{T \to \infty} T \varepsilon_T^2/\bar{\varepsilon} = \infty$.

Assumption 3 details conditions on the joint selections of step sizes (extending the individual conditions that appear in Assumption 1), essentially requiring the step sizes selected by the different advertisers to be “reasonably close” to one another. Nevertheless, the class of step sizes defined by Assumption 3 is quite general and allows for flexibility in the individual step size selection. For example, step sizes $\varepsilon_{c_k} = c_k T^{-j}$ satisfy Assumption 3 for any constants $c_k > 0$ and $0 < j_k < 1$ such that $\max_k \gamma_k \leq 2 \min_k \gamma_k$ and $2 \max_k \gamma_k \leq 1 + \min_k \gamma_k$. Moreover, when step sizes are equal across advertisers and satisfy $\varepsilon < 1/(2\lambda)$, Assumption 3 coincides with Assumption 1.

The following result states that, when Assumption 3 holds, the multipliers selected by advertisers under simultaneous adoption of adaptive pacing strategies converge to the vector $\mu^*$. In what follows we use the standard norm notation $\|x\|_p := \left(\sum_{k=1}^{K} |x_k|^p\right)^{1/p}$ for a vector $x \in \mathbb{R}^K$.

**Theorem 4** (Convergence of Dual Multipliers). Suppose that Assumption 2 holds, and let $\mu^*$ be the profile multipliers defined by (5). Assume that all advertisers follow adaptive pacing strategies with step sizes that together satisfy Assumption 3. Then,

$$\lim_{T \to \infty} \frac{1}{T} \sum_{t=1}^{T} E \left[\|\mu_t - \mu^*\|_2^2 \right] = 0.$$

Theorem 4 adapts standard stochastic approximation techniques (see, e.g., Nemirovski et al. 2009) to accommodate the expenditure feedback form and the possibility of different step sizes across agents. In the proof of the theorem we establish that under Assumption 2 there exist positive constants $C_1$ and $C_2$, independent of $T$, such that

$$E \left[\|\mu_t - \mu^*\|_2^2 \right| \{\bar{B}_{k,t+1} \geq \bar{\delta}_k \ \forall k\} \right] \leq C_1 \bar{\varepsilon} \left(1 - 2\frac{\lambda}{\varepsilon}\right)^{-1} + C_2 \bar{\varepsilon}^2 \frac{\bar{\varepsilon}}{\varepsilon},$$

for any $t \in \{1,...,T\}$, where the dependence of $C_1$ and $C_2$ on the number of advertisers is of a linear order. This implies that insofar as the respective primal solutions are feasible given the remaining budgets, the vector of multipliers $\mu$ converges in $L_2$ to the vector $\mu^*$. We then argue that the strategies do not deplete the budget too early and establish the result using the conditions in Assumption 3. We note that the fastest rate of convergence within this class of strategies is of order $T^{-1/2}$ and is achieved under the symmetric step-size selection of the form $\varepsilon \sim T^{-1/2}$.

Owing to primal feasibility constraints, the convergence of multipliers does not guarantee convergence in performance. However, the former has stand-alone practical importance in allowing the market designer to predict the bidding behavior of advertisers. This implication is discussed in Section 6.

4.3. Convergence in Performance

We next study the long-run average performance achieved when all advertisers follow adaptive pacing strategies. Recall that Theorem 4 implies that the vector of multipliers $\mu$ selected by advertisers under these strategies converges to the vector $\mu^*$. However, because in every period the remaining budget may potentially constrain bids, the convergence of multipliers to $\mu^*$ does not guarantee performance that is close to the
performance achieved when using the fixed vector $\mu^*$ throughout the campaign. Nevertheless, establishing convergence in performance is a key step toward equilibrium analysis, and the performance that is obtained using the fixed vector $\mu^*$ is a natural candidate for such a convergence. Given a vector of multipliers $\mu$, we denote

$$
\Psi_k(\mu) := T \left( \mathbb{E}_\nu \left[ (v_k - (1 + \mu_k)d_k)^* \right] + \mu_k\rho_k \right),
$$

where $d_k$ is defined as in Section 4.1. Recalling (4), $\Psi_k(\mu^*)$ is the dual performance of advertiser $k$ when in each period $t$ each advertiser $i$ bids $b_{it} = v_{it}/(1 + \mu_i^*)$.

Balseiro et al. (2015) show that $\Psi_k(\mu^*)$ coincides with the expected performance achieved when all advertisers shade bids according to $\mu^*$ and are allowed to bid even after budgets are depleted.

**Theorem 5 (Convergence in Performance).** Suppose that Assumption 2 holds, and let $\mu^*$ be the profile of multipliers defined by (5). Let $A = (A_k)_{k=1}^K$ be a profile of adaptive pacing strategies, with step sizes that together satisfy Assumption 3. Then, for each $k \in \{1, \ldots, K\}$,

$$
\limsup_{T \to \infty} \frac{1}{T} \left( \Psi_k(\mu^*) - \Pi_k^A \right) \leq 0.
$$

In the proof we establish that when all advertisers follow adaptive pacing strategies, deviations from target expenditure rates are controlled in a manner such that advertisers very often deplete their budgets “close” to the end of the campaign. Therefore the potential loss along the sample path due to budget depletion is relatively small compared with the cumulative payoff. We establish that there exist some positive constants $C_1$ through $C_4$, independent of $T$, such that

$$
\Psi_k(\mu^*) - \Pi_k^A \leq C_1 + C_2 \frac{\epsilon^{1/2}}{\sqrt{T}} + C_3 \frac{\epsilon T}{\sqrt{T}} + C_4 \frac{1}{\epsilon}.
$$

This allows for the analysis of various step size selections and establishes convergence in performance whenever the sequence of step sizes satisfies Assumption 3. In particular, (7) implies that a symmetric step-size selection of order $\epsilon \sim T^{-1/2}$ guarantees a long-run average convergence rate of order $T^{-1/4}$. We conjecture that such parametric selection further guarantees a convergence rate of $T^{-1/2}$ and demonstrate this numerically in Online Supplemental Material Part D.

**Illustrative Numerical Analysis of Convergence Under Simultaneous Learning.** In Online Supplemental Material Part D, we demonstrate the convergence established in Theorems 4 and 5 through a sequence of numerical experiments that simulate the sample path of multipliers and performance of bidders that simultaneously follow adaptive pacing strategies. Our results demonstrate the asymptotic convergence of dual multipliers and performance in a variety of instances and under different selections of step sizes that satisfy Assumption 3. A step size selection of $\epsilon \sim T^{-1/2}$ led to superior convergence rates of $T^{-1/2}$, for both multipliers and performance.

5. **Approximate Nash Equilibrium in Dynamic Strategies**

In this section we study conditions under which adaptive pacing strategies constitute an $\epsilon$-Nash equilibrium. First, we note that although a profile of adaptive pacing strategies converges in multipliers and in performance, such a profile does not necessarily constitute an approximate Nash equilibrium. We then detail one extension of our formulation (motivated in the context of ad auctions) whereby in each period each advertiser participates in one of multiple auctions that take place simultaneously. We show that in such a regime adaptive pacing strategies constitute an $\epsilon$-Nash equilibrium in dynamic strategies.

An advertiser can potentially gain by deviating from a profile of adaptive pacing strategies if this deviation: (i) reduces competition in later auctions by causing competitors to deplete their budgets earlier; or (ii) reduces competition before the budgets of the competitors are depleted by inducing competitors to learn “wrong” multipliers. Under Assumption 3, the self-correcting nature of adaptive pacing strategies prevents competitors from depleting budgets too early, and thus the first avenue is not effective. However, when the same set of advertisers frequently interact, an advertiser can cause competitors to excessively shade bids by driving them to learn higher multipliers and, therefore, can reduce competition and increase profits throughout the campaign.

In practice, however, the number of advertisers bidding in an online ad market is typically large and, owing to the high frequency of auctions as well as sophisticated targeting technologies, advertisers often participate only in a fraction of all auctions that take place (Celis et al. 2014). As a result, each advertiser often competes with different bidders in every auction (Balseiro et al. 2014). This limits the impact a single advertiser can have on the expenditure trajectory of other advertisers and, therefore, the benefit from unilaterally deviating to strategies such as the ones described above can be expected to be small. To demonstrate this intuition, we next detail one extension of our basic framework, whereby in each period multiple auctions take place simultaneously and each advertiser is “matched” with one of these auctions according to its targeting criteria and the attributes associated with the auctioned impressions. We establish that in markets with such
characteristics adaptive pacing strategies constitute an $\varepsilon$-Nash equilibrium.

5.1. Parallel Auctions Model

In each time period $t = 1, \ldots, T$ we now assume that there are $M$ ad slots sold simultaneously in parallel auctions. This can correspond to multiple publishers simultaneously auctioning inventory or a publisher with multiple ad slots running one auction per slot. We assume that each advertiser participates in one of the $M$ auctions independently at random. We denote by $m_{k,t} \in \{1, \ldots, M\}$ the auction advertiser $k$ participates in at time $t$. In each period $t$ the random variable $m_{k,t}$ is independently drawn from an exogenous distribution $\alpha_k = \{\alpha_{k,m}\}_{m=1}^M$ where $\alpha_{k,m}$ denotes the probability that advertiser $k$ participates in auction $m$. The type of advertiser $k$ is therefore given by $\theta_k := (\alpha_k, \rho_k, \alpha_k)$. The definition of $d_{k,t}$, the highest competing bid faced by advertiser $k$, is adjusted to be

$$d_{k,t} := \max \left\{ \max_{i \neq k} \left\{ \sum_{i=m}^m b_{i,t} \right\}, 0 \right\}.$$ 

Notably, all the results we established thus far hold when advertisers compete in $M$ parallel auctions as discussed above. We next establish that under this extension adaptive pacing strategies constitute an $\varepsilon$-Nash equilibrium in dynamic strategies, in the sense that no advertiser can benefit from unilaterally deviating to any other strategy (including ones with access to complete information) when the number of time periods and bidders is large. We note that by applying a similar analysis, these strategies could also be shown to constitute an $\varepsilon$-Nash equilibrium in dynamic strategies under other frameworks that have been considered in the literature, such as those in Iyer et al. (2014) and Balseiro et al. (2014), in which advertisers’ campaigns do not necessarily start and end at the same time. Additionally, the results of this section generalize to more sophisticated matching models, such as ones that allow the advertisers to participate in multiple slots per period (when the number of slots each advertiser participates in per period is small relative to $M$) and the distribution of values to be auction dependent.

We denote by $\mathcal{K}^A \supseteq \mathcal{K}$ the space of nonanticipating and budget-feasible strategies with complete information. In particular, strategies in $\mathcal{K}^A$ may have access to all types $\theta_k$, as well as past value realizations, bids, and expenditures of all competitors. To analyze unilaterals deviations from a profile of adaptive pacing strategies we consider conditions on the likelihood that different advertisers compete in the same auction in a given time period. Let $\delta_{k,i} := P\{m_{k,t} = m_{i,t}\} = \sum_{m=1}^M \alpha_{k,m} \alpha_{i,m}$ denote the probability that advertisers $k$ and $i$ compete in the same auction in a given time period, and let $a_k = (a_k)_{i \neq k} \in [0,1]^{K-1}$. We refer to these as the matching probabilities of advertiser $k$.

**Assumption 4** (Limited Interaction). The matching probabilities $(a_k)_{k=1}^K$ and profile of step sizes $(\varepsilon_k)_{k=1}^K$ satisfy

i. \( \lim_{M,K \to \infty} \max_{k=1, \ldots, K} \|a_k\|_2 < \infty, \)

ii. \( \lim_{M,K,T \to \infty} \exp(-\varepsilon) \max_{k=1, \ldots, K} \|a_k\|_2^2 = 0 \).

The first condition on the matching probabilities in Assumption 4 is that each advertiser interacts with a limited number of different competitors per time period, even when the total number of advertisers and auctions per period is large. The second condition ties the matching probabilities with the profile of step sizes selected by advertisers, essentially implying that advertisers can interact with more competitors per time period when the “learning rates” of the strategies are similar to one another. The class of matching probabilities and step sizes defined by Assumption 4 includes many practical settings. For example, when each advertiser participates in each auction with the same probability, one has $\|a_k\|_2 \approx K^{1/2}/M$ because $\alpha_{i,m} = 1/M$ for each advertiser $i$ and auction $m$. When the expected number of bidders per auction $k := K/M$ is fixed (implying that the number of parallel auctions is proportional to the number of players), we have $\|a_k\|_2 \approx \kappa/K^{1/2}$, and the first condition is satisfied. The second condition states that in such a case the difference between the learning rates of different advertisers should be of order $o(K^2)$.

**Theorem 6** ($\varepsilon$-Nash Equilibrium in Dynamic Strategies). Suppose that Assumptions 2 and 4 hold. Let $A$ be a profile of adaptive pacing strategies with step sizes that together satisfy Assumption 3. Then,

$$\limsup_{T,K,M \to \infty} \sup_{\delta \in \mathcal{K}^A} \frac{1}{T} \left( \Pi_k^A - \Pi_k^A \right) \leq 0.$$

Theorem 6 establishes that adaptive pacing strategies (with step sizes and matching probabilities that together satisfy Assumptions 3 and 4) constitute an $\varepsilon$-Nash equilibrium within the class of dynamic strategies with access to complete information on market primitives, and perfect information on all the events that took place in prior periods. In particular, the benefit from unilaterally deviating from a profile of adaptive pacing strategies diminishes to zero in large markets, even when advertisers know their value distribution up front (as well as the value distributions and budgets of their competitors), and even when they can acquire real-time information on the past value realizations, bids, and payoffs of their competitors.
The key idea of the proof lies in bounding the benefit in terms of \(\Psi_k(\mu^*)\) from unilaterally deviating to an arbitrary strategy. Under Assumption 4 the impact of advertisers on one another is limited because each advertiser interacts with a limited number of different competitors in each time period. Thus, competitors learn the stable multipliers \(\mu^*\) regardless of the actions of a deviating advertiser, and an advertiser cannot reduce competition by inducing competitors to learn a “wrong” multiplier. We show that the benefit from deviating to any other strategy is small when multipliers are “close” to \(\mu^*\). We bound the performance of an arbitrary budget-feasible strategy using a Lagrangian relaxation in which we add the budget constraint to the objective with \(\mu^*_k\) as the Lagrange multiplier. This yields a bound on the performance of any dynamic strategy, including strategies with access to complete and perfect information. Thus, we obtain that there exist constants \(C_1\) through \(C_5\), independent of \(T, K,\) and \(M\), such that for each profile of advertisers satisfying Assumption 2 and any strategy \(\beta \in \mathcal{B}^{(1)}_k\):

\[
\Pi_k^{\beta,A^1} - \Psi_k(\mu^*) \leq C_1 + C_2\|a_k\|_2 K^{1/2} \varepsilon^{1/2} \varepsilon^{3/2} + C_3\|a_k\|_2 K^{1/2} \frac{\varepsilon T}{\varepsilon^2} + C_4 \frac{1}{\varepsilon} + C_5\|a_k\|_2 \frac{\varepsilon T}{\varepsilon}
\]

for each advertiser \(k\). This allows one to evaluate unilateral deviations under different step sizes and to establish the result under the conditions in Assumptions 3 and 4.

Theorem 6 shows that a profile of adaptive pacing strategies constitutes an approximate equilibrium when there is a large number of players who do not interact too frequently. In some markets, however, a small group of advertisers may interact with each other in many auctions, thereby violating the assumptions of the theorem. Using a fluid model it is possible to numerically analyze the benefit from unilaterally deviating from a profile of adaptive pacing strategies in markets with a few advertisers. In numerical experiments we observed that the suboptimality of adaptive pacing strategies diminishes fast as the number of players increases. In particular, the benefit from deviating is typically small even in markets with few advertisers who interact frequently.

6. Concluding Remarks
6.1. Summary and Implications
In this paper we introduced a family of adaptive pacing strategies, in which advertisers adjust the pace at which they spend their budget according to their expenditures. We established that such strategies are asymptotically optimal when the competitive environment is stationary, but also when competitors’ bids are arbitrary. We further demonstrated that in large markets, these strategies constitute an approximate Nash equilibrium in dynamic strategies. Notably, adaptive pacing strategies maintain these notions of optimality without requiring any prior knowledge of whether the nature of competition is stationary, arbitrary, or simultaneously learning. This implies that given characteristics that are common in online ad markets, an advertiser can essentially follow an approximate equilibrium bidding strategy while ensuring the best performance that can be guaranteed off equilibrium.

Although we focus is on the advertisers’ perspective, our results have implications on market design as well. Given access to the advertisers’ types, our convergence results provide practical tools for predicting the bidding behavior of advertisers (through the vector \(\mu^*\)) and their payoff (through the function \(\Psi\)). Because these predictions rely on the uniqueness of \(\mu^*\), this motivates the market designer to place restrictions on target expenditure rates that satisfy Assumption 2. In addition, our equilibrium analysis indicates that platforms (such as Facebook, Google, and Twitter) that provide budget pacing services based on historical information on all advertisers may also offer, under conditions that are common on online ad markets, “private” budget-pacing services that do not use information on competing advertisers, with practically no loss of optimality.

6.2. Future Directions and Open Problems
We next list several avenues for future research and discuss limitations and potential extensions. First, our results rely on distribution of values being absolutely continuous and do not easily extend to the case of discrete values. In this case, multiple advertisers might submit the same bid, and how ties are broken is important. Conitzer et al. (2018) argue that, on top of shading bids, at equilibrium advertisers use mixed strategies in which they randomize over bids. We leave the analysis of discrete values as a future research direction.

Our results establish the asymptotic optimality of adaptive pacing strategies in both the “pessimistic” and “optimistic” scenarios when advertisers have no prior knowledge of whether the competitive nature is stationary or arbitrary, and that selecting \(c_k \sim T^{-1/2}\) guarantees a convergence rate of \(T^{-1/2}\) in both frameworks. We conjecture that in either of these frameworks, this rate cannot be improved by any admissible and budget-feasible strategy (random or not), even when advertisers know up front the environment in which they are competing (i.e., stationary or arbitrary). Kleinberg (2005) provides some evidence that this rate is tight in the stationary setting, showing that no algorithm can achieve a rate better than \(B^{-1/2}\) in a variation of the secretary problem in which the decision maker is allowed to choose the best \(B\) secretaries. Although this result assumes that values are drawn without
replacement from an unknown set of numbers (i.e., a random permutation model), we believe that similar results apply to a setting in which values are drawn independently from an unknown distribution.

The update rule of the adaptive pacing strategy takes a simple additive form, but one may consider other update rules. One approach is to use nonlinear update rules such as exponentiated gradient descent, in which multipliers are updated multiplicatively. One may also use rules that depend on the entire history of observations rather than only on the most recent period, for example, an update rule in which the gradient step at time $t$ depends on the difference between the target expenditure rate and the average expenditure up to time $t$, rather than only on the expenditure in period $t$. Another approach is to use methods that approximate the Hessian of the dual objective function based on all observed gradients, such as the online Newton method (Hazan et al. 2007) or the adaptive subgradient method (Duchi et al. 2011).

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Endnotes

1 A descriptive discussion of the budget pacing service Facebook offers advertisers appears at https://developers.facebook.com/docs/ marketing-api/pacing. See also recommendations for advertisers in online blogs powered by Twitter Ads (https://dev.twitter.com/ads/campaigns/budget-pacing), Google’s ad exchange platform DoubleClick (https://support.google.com/ds/answer/2682462), as well as exactExtract (http://exactdrive.com/news/how-budget-pacing-helps-your-online-advertising-campaign). All websites accessed October 2017.

2 Our results hold for random or lexicographic tie-breaking rules. To simplify exposition we assume that ties are broken in favor of the decision maker.

3 Although the excess budget remaining after period $T$ is not included in this payoff, it can be accounted for without impacting our results.

4 Although in the definition of $\pi_k^*(v_i; d_i)$, the vectors $v_i$ in $[0, \bar{v}_i] I^T$ and $d_i$ in $R^T_+$ are fixed in advance, we also allow the components of these vectors to be selected dynamically according to the realized path of $\beta$; that is, $v_{i,t}$ and $d_{i,t}$ could be measured with respect to the filtration $\sigma\{v_{j,t-1}, d_{j,t-1}, v_{i,t}, d_{i,t-1}\}$. This allows nonoblivious or adaptive adversaries and, in particular, valuations and competing bids that are affected by the player’s strategy. For further details see chapter 7 of Cesa-Bianchi and Lugosi (2006).

5 The heuristic of bidding $b_{i,t} = v_{i,t}/(1 + \mu^t)$ until the budget is depleted, with $\mu^t$ optimal for Figure 4, can be shown to be asymptotically optimal for the hindsight problem (1) as $T$ grows large (see, e.g., Talluri and van Ryzin 1998).

6 We restrict the formal definition of the adaptive pacing strategy to stationary step sizes only to simplify and shorten analysis, and the strategy can be adjusted to allow for time-varying step sizes through a more general form of Assumption 1. Nevertheless, we note that the asymptotic optimality notions we consider are defined for the general class of admissible budget-feasible strategies and achieved within the subclass of strategies with stationary step sizes.

7 In Online Supplemental Material Part C, we show that this part of the assumption is implied by the diagonal strict concavity condition defined in Rosen (1965), prove that it holds in symmetric settings, and demonstrate its validity in simple cases.

References


