Dynamic Pricing under Debt: Spiraling Distortions and Efficiency Losses

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First version: June 6, 2016
This version: March 6, 2017

Abstract

Firms often finance their inventory through debt and subsequently sell it to generate profits and service the debt. Pricing of products is consequently driven by both inventory and debt servicing considerations. We show that limited liability under debt induces sellers to charge higher prices and to discount products at a lower pace. We find that these distortions result in revenue losses that compound over time, leading to some form of performance spiral down. We quantify the extent to which these inefficiencies can be mitigated by practical debt contract terms that emerge as natural remedies from our analysis, and find debt amortization or financial covenants to be the most effective, followed by debt relief and early repayment options.

Keywords. dynamic pricing, debt, pricing distortions, channel efficiency, firm value, spiral-down.

1 Introduction

Recent years have witnessed a surge in the adoption of dynamic pricing practices across a wide range of industries, fueled in no small part by the increased availability of data and inexpensive computing, and by a growing body of focused research. The canonical models studied in the academic literature consider a firm endowed with inventory that is adjusting prices dynamically so as to maximize revenues from sales. Although such an approach is well aligned in spirit with the typical material flows of many firms, whereby inventory is built up and subsequently sold to customers, it remains agnostic to the associated financial flows, whereby inventory is predominantly financed through debt that is then serviced using the sales revenues. This could lead to potentially misleading conclusions, as it is well known that the presence of debt and its associated limited liability can significantly distort a decision maker’s incentives and actions, leading to efficiency losses (Jensen and Meckling, 1976; Myers, 1977). This raises the natural question of how debt would affect pricing decisions and their efficiency, as measured

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through expected sales revenues. Also, given the inherent dynamic nature of pricing decisions, how would these distortions and losses evolve dynamically over time?

For example, consider a real-estate firm that undertakes a new development project. As is common in this industry, the firm would finance the large upfront costs through a loan, which would be repaid through the sales of individual units. Construction loans for real-estate (tract) development in the U.S. usually cover up to 80% of project costs (CH, 2013, pages 10-12), and currently amount to a total of $1.1 trillion, with $201.7 billion raised during 2015 alone (Federal Mortgage Data, 2016). Since it is customary to pre-sell only a subset of the units before securing the loan (CH, 2013, pages 28-30), the firm would then have considerable flexibility in setting the subsequent prices, and this would be done under a heavy debt burden. Would this lead to lower prices (and a boost in sales), higher prices (and larger margins), or distortions of no particular pattern? And what would that imply for the total sales revenues and the efficiency of the entire channel?

Similarly, consider a liquidation house that undertakes a going-out-of-business (GOB) sale for a bankrupt retailer, whereby remaining inventory is liquidated through successive markdowns over a fixed period of time. The liquidator would commonly purchase the inventory upfront through a large, debt-financed transaction (Craig and Raman, 2015). For instance, one week prior to administering the liquidation of Borders Group in 2011, two liquidation houses, Hilco and Gordon Brothers, joined forces to purchase the book retailer’s entire store assets for approximately $270 million (Foley et al., 2013). Similar transactions occurred in several other liquidations, including those of CompUSA, Dots Stores, and Sports Authority. Would debt then induce more or less aggressive markdowns over time?

To address these questions, we anchor our analysis around the dynamic pricing formulation of Gallego and van Ryzin (1994). Although this classical model ignores several practical considerations that can shape pricing policies—such as competition or customers’ strategic behavior—it allows us to isolate the effect of debt on pricing policies and benchmark against a well-understood setting. More precisely, we consider a seller endowed with a given inventory of a single product who is dynamically adjusting prices in discrete time over a fixed planning horizon. The seller is faced with a debt payment at the end of the horizon, and collects only the residual revenues remaining after the debt is paid off. We compare the pricing policy and the revenues achieved by the seller with those of a seller without debt, who would follow a classical revenue-maximizing policy.

**Our findings.** To highlight the main insights, we first analyze the case in which the seller faces two periods and a linear demand function, and then generalize the results.

1. We formulate the seller’s decision problem as a dynamic program, where the value function depends on two state variables, remaining inventory and outstanding debt. Although the state-space extension is natural, the basic structural properties of the value function differ drastically from the classical dynamic pricing problem. More precisely, we find that the value function is
convex in the debt level and can become locally *convex in inventory*, i.e., the marginal value of an extra inventory unit can be increasing. This highlights the subtle but key impact that debt has on the nature of the problem and underscores the analytical challenges it introduces.

2. We show that a seller under debt sets higher prices than a revenue-maximizing seller. This occurs even when the debt amounts are small, and becomes more pronounced as debt levels increase. Although prices generally *tend* to grow with the debt, we find that this increase need not be monotonic, due to a subtle interplay between debt and inventory. In particular, when faced with a larger debt burden, a seller may prefer relying on more sales at lower prices to pay off the debt, whereas with a lower debt, he may rely on fewer sales at a higher price.

3. We show that time dynamics *maintain and amplify* these effects in a spiraling fashion. More precisely, the pricing policy under debt recommends less steep markdowns than the revenue-maximizing policy (to the extent that it marks up prices in expectation when inventory is ample, whereas the revenue-maximizing policy would maintain constant prices). In turn, this compounding distortion of the pricing policy gives rise to a degradation in efficiency, as relative revenue losses strictly increase in expectation over time.

4. We quantify the extent to which efficiency losses can be mitigated by practical debt contract terms that emerge as natural remedies from our analysis, namely early payment discounts, debt relief, debt amortization or financial covenants. We find an ordering, with early payment options being less effective than debt relief, which in turn is strictly less effective than debt amortization or financial covenants. None of the terms are able to fully restore efficiency.

We establish the robustness of our findings through several modeling extensions: by confirming some of the analytical results for the asset selling problem over an arbitrary number of periods and general demand functions (§5.1); by conducting numerical experiments for the general version of the problem (§5.2); by considering loans where borrowers are responsible for some of the losses when debt is not fully repaid (§A); and by studying a setting where debt is endogenously determined (§C).

Our results are consistent with anecdotal evidence in the popular press that documents underwhelming discounts or even price markups during GOB sales at Circuit City, Borders or Linens’n Things (see, e.g., Chang, 2009; Sakraida, 2011; White, 2016). They are also aligned with the empirical study by Genesove and Mayer (1997), who find that homeowners with larger debt loads set higher

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1 Financial covenants are common contingencies in debt contracts, requiring borrowers to maintain certain financial ratios, e.g., a maximum debt-to-equity ratio or a minimum cashflow-to-debt ratio (see Iancu et al., 2016 for a discussion).

2 Defending the practice, one of the liquidators’ senior executives stated: “We have to be economical on our discounts[... we have commitments to a lot of people—banks, creditors—who are expecting a certain amount of return.” (Chang, 2009).
asking prices for their houses, have them listed for a longer time on the market, and receive a higher price upon an eventual sale than owners with smaller debt loads (see our discussion in §1.1).

**Managerial implications.** Our finding of increased prices highlights the importance of controlling for slow markdowns in the presence of debt. Although this could in principle be enforced through price controls, these would be difficult to implement in practice. Instead, incentives to lower prices so as to counterbalance the distortions could be provided through various other means, for example, by allowing for repayments at lower prices—such as early payment discounts in trade credit—or by limiting the amount of available credit if sales fall short of projections (CH, 2013, page 20 and 28).

To limit potential revenue losses caused by increased leverage, lenders can control the size of the outstanding debt throughout the loan’s tenor. Under slow sales, all parties could benefit from debt restructuring through, e.g., a reduction in principal or interest (i.e., a “bondholder haircut”) or a tenor extension. For revolving real estate loans, leverage could also be controlled by enforcing borrowing limits tied to the sales rate or by directly limiting the investment size, e.g., by financing large developments in phases or by constraining the number of unsold units financed at any point of time (CH, 2013, page 29). To the latter point, however, our findings also issue a cautious warning against enforcing strict limits, since borrowers with larger amounts of inventory may actually prefer relying on more sales (at lower prices) to cover their debt, thus reducing distortions and improving efficiency.

To alleviate the compounding of distortions, lenders can frequently monitor sales performance, and require repayments that track the selling of units. This could be achieved through, e.g., suitable repayment schedules (debt amortization) or financial covenants requiring minimum levels of cashflow generated throughout the loan’s tenor (CH, 2013).

With regards to the efficacy of the aforementioned countermeasures, our analysis suggests that debt amortization or financial covenants tend to be a more effective way of improving efficiency than debt relief, which in turn could be more effective than early payment discounts. However, even these mechanisms may fail to fully alleviate distortions. This implies that, unlike flexibility in adjusting inventory levels (Iancu et al., 2016), flexibility in adjusting prices under debt can induce inefficiencies that are more difficult to mitigate through the design of common contractual terms and conditions.

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3. An immediate reason is the contract complexity in specifying contingent pricing policies. Prices are also often non-verifiable—such as when privately communicated to clients—making such controls hard to enforce in a court of law. Lastly, such controls would be tantamount to directly dictating operating decisions routinely taken during the course of business, which lenders usually refrain from doing (DeAngelo et al., 2002).

4. “A prudent development and construction loan policy includes requirements for principal curtailments to ensure periodic re-margining if sales [...] fall short of projections.” (CH, 2013, page 28)
1.1 Literature Review

Our paper is related to the extensive literature on pricing and revenue management, which is surveyed in Talluri and van Ryzin (2005) and Phillips (2005), among others. We build on a discrete-time counterpart of the classical dynamic pricing model of Gallego and van Ryzin (1994), by changing the decision maker’s objective to reflect the presence of debt. This relates our work to Levin et al. (2008), who consider a pricing problem in continuous-time with a risk-averse objective that mixes expected revenues with the probability of meeting a revenue target. They find that the optimal risk-averse policy involves discounts relative to the risk-neutral (i.e., revenue-maximizing) policy when revenues are immediately below the target. Besbes and Maglaras (2012) also consider revenue targets, but as constraints, with the classical objective of maximizing expected revenue; they derive the structure of the optimal policy for a deterministic problem, and argue that the prescription obtained also performs well even under a model with limited demand information. Our paper differs from these in both focus and model: our objective corresponds to the residual revenues in excess of the (debt payment) target, and we analyze both the distortions in the pricing policy, as well as the resulting efficiency losses. Furthermore, our insights are also qualitatively different, as the optimal policy becomes risk-seeking in our case, always pricing above the revenue-maximizing policy while the target is not met. Our work is also related to papers studying the dynamic evolution of (bid) prices and revenues. Pang et al. (2015) study the inter-temporal behavior of bid prices, finding an upward (downward) trend in time when the seller has multiple (a single) unit(s) in inventory. Cooper et al. (2006) document a downward spiral in revenues driven by incorrect assumptions about customer behavior. The present paper differs from these through its focus on debt.

Our work is also related to several papers in the operations management literature documenting operating distortions caused by debt. For instance, Xu and Birge (2004), Buzacott and Zhang (2004), Dada and Hu (2008), and Boyabatli and Toktay (2011) extend the newsvendor model to include financing considerations, and show how these can affect the firm’s optimal order quantity or choice of (flexible) capacity. Closer to our work, Chod (2015) shows how a firm that has already secured debt financing can engage in risk-shifting by ordering riskier products from suppliers, and studies the role of trade credit in alleviating the distortions. Such models are typically cast in a static setting, and are not focused on pricing decisions or on quantifying the dynamics of efficiency losses.

Several papers in the operations literature have also considered dynamic models with alternative objective functions motivated by financial considerations, e.g., Porteus (1972) (optimizing inventory policies while maintaining a cash safety level), Archibald et al. (2002) (maximizing the probability of survival of start-up firms), Possani et al. (2003) (maximizing survival probabilities of start-ups in manufacturing), Babich and Sobel (2004) (maximizing initial public offering cash-flows), Swinney et al. (2011) (optimizing capacity investments for start-ups and established firms), Gong et al. (2014) (exa-
ining inventory control under leverage), Li et al. (2013) (maximizing discounted dividends), and others. Closest to our work, Iancu et al. (2016) consider a two-period model of a firm endowed with inventory management capabilities, in the form of replenishments and partial liquidations (at fixed prices). They show that extra flexibility in managing inventory can lead to significant efficiency losses when the firm is financed through debt, but that such losses can be fully alleviated by common covenants present in debt agreements. In contrast to these papers, which deal almost exclusively with inventory management, we focus on pricing decisions, and on quantifying the time evolution of efficiency losses. We find that such losses always persist in our setting, even under the types of covenants considered in Iancu et al. (2016), and become more pronounced in time due to the problem’s dynamics. This suggests that pricing decisions induce losses of a qualitatively different nature than inventory decisions, requiring potentially more complex contractual terms to alleviate.

Our work is also related to a large body of finance and economics literature that deals with agency issues inherent when holding debt. Jensen and Meckling (1976) and Myers (1977) were among the first to challenge the classical Modigliani-Miller insight that a firm’s decisions are independent of its capital structure, by arguing how the equity holders of a leveraged firm could extract value from the debt holders by increasing the risk in the firm’s cashflows after the debt is in place. A large volume of subsequent literature in corporate finance has been devoted to examining how the design of the firm’s capital structure can recognize and control the resulting efficiency losses, known as the agency costs of debt. Within this literature, several recent papers quantified these costs using dynamic models of the firm, typically within a real options framework, see, e.g., Leland and Toft (1996), Leland (1998), Childs et al. (2005), Manso (2008) for a more in-depth review. The majority of these papers document costs of 0.5% – 1.5% of firm value. Closer to our work, Décamps and Faure-Grimaud (2002) study a model where the equity holders can liquidate the firm at a fixed set of “scraping times,” and numerically document larger agency costs. Décamps and Djembissi (2007) consider a firm with the ability to dynamically switch operations to a poor activity—with a higher volatility and lower expected returns—and numerically document costs of more than 7%. No papers in this stream model a firm that has dynamic pricing ability, which is the main emphasis of our work. Furthermore, to the best of our knowledge, the only paper in this literature that discusses some form of time evolution for the agency issues is Décamps and Faure-Grimaud (2002). This paper numerically documents that the distortions in the firm’s operating policy could either increase or decrease over time, and the agency costs of debt (measured at the initial time) could either increase or decrease with the introduction of additional decision points. Of key difference is that decision epochs in Décamps and Faure-Grimaud (2002) are separated by arbitrary time intervals, so that the firm’s profits/revenues across decision points are non-stationary. In contrast, we show both analytically and numerically that for a firm faced with stationary willingness-to-pay distributions, the pricing distortions always compound over time, giving
rise to actions and revenues that increasingly deviate from the system-optimal ones in expectation.

Also related are papers showing how debt can be used strategically as a pre-commitment tool to increase firm value. Brander and Lewis (1986) show how firms engaged in Cournot competition can use debt to pre-commit to larger production quantities, improving equilibrium outcomes. Note though that Faure-Grimaud (2000) argues how using an optimal renegotiation-proof contract can reverse the positive effect of debt documented by Brander and Lewis (1986). Chemla and Faure-Grimaud (2001) consider a monopolist selling to strategic consumers with private valuations for the good, and show how debt can reverse the negative effect of adverse selection, and allow the firm to charge larger prices. Some of our results are aligned with these findings: in Appendix C we also show how a more leveraged firm can charge prices that are closer to optimal, and improve its value. However, the main focus of our work is different—we discuss the dynamic evolution of pricing distortions and losses, and quantify the effectiveness of contractual mechanisms for alleviating the resulting inefficiencies.

Our pricing policy results are validated by the empirical work of Genesove and Mayer (1997), who find homeowners with larger debt loads (i.e., higher loan-to-value ratio) listing their homes at larger prices (as mentioned earlier). The effects are of significant magnitude, and are found to be more pronounced for investors than for individual owners. For the latter category, the authors rationalize the behavior using the model in Stein (1995), which relies on the liquidity constraints introduced by a required down-payment. For investors, Genesove and Mayer (1997) suggest an explanation relying on the limited liability effect. Our work formalizes this intuition providing rigorous theoretical backing.

Viewing the decision maker’s payoff as a bonus corresponding to the residual revenues in excess of a quota also relates our paper to a growing body of work focusing on Salesforce compensation. Basu et al. (1985) were the first to rationalize the existence of revenue quotas, by adopting the principal-agent framework of Holmstrom (1979) to argue that the optimal compensation package for a sales agent exerting unobservable effort may depend nonlinearly on the generated revenues. See also the related studies of Oyer (1998). Sales quotas have also been considered in the operations literature, with a focus on coordinating operating decisions with an agent’s compensation. For instance, Chen (2000) considers a dynamic model in which a manufacturer compensates a sales agent through an annual salary and a per-unit bonus when quantity sales exceed a target. The paper shows how the well-known “sales hockey stick” (SHS) effect can arise, whereby the agent’s optimal choice of delaying effort generates increasing sales towards the end of the horizon. We also refer the reader to Sohoni et al. (2010). While we also document distortions in the agent’s policy, in contrast with the SHS effect observed in this line of work, which could be viewed as a form of positive spiraling, we show that pricing distortions compound negatively over time, and we quantify how this generates a downward spiral in efficiency.
2 Problem Formulation

We present the base model we use to assess the effects of debt on pricing decisions and efficiency. We introduce the dynamics, followed by the seller’s problem and the metrics we use to quantify the effects.

Consider a decision maker (DM) in charge of selling a given number of units of a single product in discrete time over a horizon of $T$ periods. We denote by $Y \in \mathbb{N}$ the number of units available at the start of the selling horizon. In each period $t \in \{1, \ldots, T\}$, the DM sets a posted price $p_t$ selected from an interval of feasible prices $\mathcal{P} = [p, \overline{p}] \subseteq \mathbb{R}$. Subsequently, a single customer arrives with a willingness-to-pay (WTP) $W_t$. We assume $W_1, \ldots, W_T$ to be i.i.d., and let $\lambda : \mathcal{P} \to [0, 1]$ denote the corresponding demand function that gives the probability that the WTP exceeds a posted price, $\lambda(p_t) = \mathbb{P}(W_t \geq p_t)$.

The DM selects the posted prices according to a pricing policy $\mathbf{p} = (p_1, \ldots, p_T)$. In particular, the set of admissible pricing policies for the DM, denoted by $\mathcal{Q}$, consists of all non-anticipating policies for which the price $p_t$ is $\mathcal{F}_t$-measurable, where $\mathcal{F}_t = \sigma(W_1, \ldots, W_{t-1})$ denotes the information set available to the DM at the beginning of period $t = 1, \ldots, T$.

Let $Y_t$ denote the number of remaining units at the start of period $t$, which evolves according to:

$$Y_1 = Y, \quad Y_{t+1} = (Y_t - 1\{W_t \geq p_t\})^+, \quad t = 1, \ldots, T - 1.$$  

Consistent with typical dynamic pricing models, we assume that unmet demand is lost without any penalty, and unsold units at the end of the horizon are discarded without any salvage value. Consequently, the total revenues generated throughout the horizon are given by

$$\mathcal{R}(\mathbf{p}) := \sum_{t=1}^{T} p_t (Y_t - Y_{t+1}).$$

In this setting, a natural measure of efficiency is the total expected revenue $\mathbb{E}[\mathcal{R}(\mathbf{p})]$, which is the usual focus in dynamic pricing and revenue management models [Talluri and van Ryzin, 2005].

To introduce debt, we assume that the DM is faced with a debt repayment $B$ that is due at the end of the horizon. The generated revenues $\mathcal{R}(\mathbf{p})$ are used to first pay off the debt, and the DM only collects the residual revenues. The DM is shielded by limited liability, so that if revenues are insufficient to pay off the debt, he collects zero. That is, the DM’s payoff at the end of the horizon equals

$$(\mathcal{R}(\mathbf{p}) - B)^+.$$  

We shall say that the DM pays off or covers the debt in the event that $\mathcal{R}(\mathbf{p}) \geq B$.

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\textsuperscript{5}To facilitate exposition and avoid degeneracies, we assume that $\lambda(p) > 0 \forall p \in \text{int}(\mathcal{P})$. To capture demand functions with unbounded support (e.g., exponential), we allow $\overline{p} = \infty$, in which case we have $\mathcal{P} = [p, \infty)$.}
Decision maker’s policy. The DM selects his pricing policy, denoted by $p^\dagger$, so as to maximize his expected payoff, i.e.,

$$p^\dagger \in \arg \max_{p \in \mathcal{Q}} E \left[ (\mathcal{R}(p) - B)^+ \right].$$

We denote the expected revenue generated under such a policy by

$$J^\dagger := E[\mathcal{R}(p^\dagger)].$$

Note that the self-interested DM, by optimizing over an objective that is different from the expected revenue, follows a policy that might incur efficiency losses compared to what would be possible.

Optimal policy. An optimal policy maximizes expected revenue. Let $p^\star$ denote such a policy, i.e.,

$$p^\star \in \arg \max_{p \in \mathcal{Q}} E \left[ \mathcal{R}(p) \right],$$

which we shall also refer to as a revenue-maximizing policy. In accordance with our previous notation, let $J^\star$ be the expected revenue under $p^\star$ or the optimal revenue, i.e.,

$$J^\star := E \left[ \mathcal{R}(p^\star) \right].$$

By definition, we have that $J^\star \geq J^\dagger$. Note that for $B = 0$, the DM’s policy is also a revenue-maximizing policy, and consequently the inequality holds with equality. For $B > 0$, the policies generally differ, and a loss in expected revenues is likely when the DM’s policy is followed.

Policy comparison and efficiency loss. We aim to understand the pricing distortions and resulting efficiency loss induced by debt. We characterize the former by comparing the DM’s prices with the revenue-maximizing ones. We quantify the latter as the loss in expected revenue due the DM’s policy, relative to the optimal revenue. Formally, the efficiency loss in our setting is given by

$$L := \frac{J^\star - J^\dagger}{J^\star}.$$

Note that $L$ is the standard way of defining and measuring normalized efficiency losses in the academic literature, and has been used extensively in economics, finance, and operations management (see, e.g., Perakis and Roels, 2007). However, $L$ remains an eminently static measure, aggregating losses across time and across all future (uncertain) states of the world. In order to quantify the dynamic progression of revenue losses over time as the customers’ uncertain WTP is being realized, we introduce efficiency loss metrics that capture time and path dependence. In particular, let $J^\dagger_t (J^\star_t)$ be the conditional
expected total revenue under the DM’s (optimal) policy at time $t$, given that $\mathcal{F}_t$ has realized, i.e.,

$$J^\dagger_t := \mathbb{E}[\mathcal{R}(p^\dagger) | \mathcal{F}_t] \quad \text{and} \quad J^* := \mathbb{E}[\mathcal{R}(p^*) | \mathcal{F}_t], \quad t = 1, \ldots, T.$$  

Similarly, let $\mathcal{L}_t$ be the resulting conditional efficiency loss, dependent on the information at time $t$,

$$\mathcal{L}_t := \frac{J^* - J^\dagger}{J^*}, \quad t = 1, \ldots, T. \quad (1)$$

Note that the conditional expected revenues $J^\dagger$, $J^*$ and efficiency losses $\mathcal{L}_t$ are all $\mathcal{F}_t$-measurable random variables. We emphasize here that $\mathcal{L}_t$ depends on both policies $p^*$ and $p^\dagger$. It compares the total expected revenues generated by either policy conditional on a given history of willingness-to-pay values up to $t - 1$. Note that the inventory and debt levels under $p^*$ and $p^\dagger$ can be different for some history realizations. In particular, $\mathcal{L}_t$ corresponds to an updated measurement of the efficiency loss at time $t$, conducted under additional information. To facilitate the comparison, we also define the associated expected efficiency loss at time $t$,

$$L_t := \mathbb{E}[\mathcal{L}_t], \quad t = 1, \ldots, T.$$  

This generalized definition of efficiency loss is the exact counterpart of $L$ extended to a future time, when measurements are conducted under additional information. Note that, since no additional information is available at $t = 1$ ($\mathcal{F}_1 = \emptyset$), $J^* = J^*$, $J^\dagger = J^\dagger$, and $L_1 = L$, so that our definitions are consistent.

Model discussion. One way to interpret our model is that it corresponds to a setting where a debt contract is already in place between an inventory-selling firm and some debt holders. In particular, the debt holders are entitled to a debt repayment $B$; their expected payoff, which equals $\mathbb{E}[\min(B, \mathcal{R}(p))]$, is commonly referred to as debt value in the corporate finance literature (see page 75 of [Tirole, 2006]). The expected payoff for the firm’s equity holders, which equals $\mathbb{E}[\mathcal{R}(p) - B]^+$, is commonly referred to as equity value. Thus, the DM in our model acts in the interest of the firm’s equity holders, which is the fiduciary duty of corporate managers (see page 56 of [Tirole, 2006]). The sum of the expected payoffs to the debt and equity holders, which equals $\mathbb{E}[\mathcal{R}(p)]$, is referred to as firm value, and constitutes the de facto way of measuring the efficiency of contracts in economics and finance (in the operations literature, it corresponds to the “supply chain profit/payoff”). The efficiency losses $L$ are commonly referred to as agency costs of debt in the corporate finance literature.

Our model adopts the established paradigm in corporate finance that the equity holders/the DM have zero liability, so that they suffer no losses or penalty when revenues fail to cover the debt (see,}[6]

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[6]Our model implicitly assumes that the manager in charge of the firm’s operating policy (e.g., pricing) inherits the same fiduciary duty—an assumption that is consistent with typical models in corporate finance.
e.g., page 75 or page 115 of Tirole, 2006). Zero-liability loans are commonly referred to as non-recourse in practice (Wells Fargo, 2016, page 2). In Appendix A we consider a setting of nonzero liability: if unable to pay off the debt, the DM/equity holders incur losses (due to, e.g., recourse by the lender or bankruptcy and reputational costs when the firm goes into default) that are proportional to the shortfall, so that the DM’s payoff becomes $E[(R(p) - B)^+ - k E[(B - R(p))^+]]$, for some $k \in [0,1]$.

That generated revenues are used to first repay the debt is consistent with the literature (Myers, 1977), as well as with practice, where lockbox arrangements or escrow/impound accounts are routinely set up in conjunction with use-of-proceeds covenants to allow collecting debt payments from operational cashflow (see pp. 2 of Wells Fargo, 2016 and §7:6.1 of Hilson, 2013).

In the present paper, we assume for simplicity that there is no discounting, or equivalently that the risk-free interest rate is zero (see, e.g., page 115 of Tirole, 2006). Furthermore, we model discrete decision epochs to simplify the analysis. We emphasize however, that these choices do not affect qualitatively the insights obtained.

**Notation.** We use script font notation to denote random variables (e.g., $\mathcal{W}_t, \mathcal{B}_t,...$), and use comparisons (such as “increasing/decreasing,” “greater/smaller,” etc.) in their non-strict sense. Furthermore, we say that $f : [a, b] \to \mathbb{R}$ is piece-wise convex (respectively, piece-wise increasing) if there exists a finite set of points $a_0 = a < a_1 < a_2 < \cdots < a_m = b$ such that the restriction of $f$ on $(a_i, a_{i+1}]$ is convex (respectively, increasing), for any $i \in \{0, \ldots, m - 1\}$.

### 3 Dynamic Program and Properties of the Value Function

In our analysis, we restrict attention to Markov policies, a choice that is without loss of generality given our problem’s structure. A sufficient state representation at the start of any period $t$ is given by the remaining inventory units $\mathcal{Y}_t$ and the outstanding debt, which is equal to the initial debt $B$ less the revenues generated throughout periods $1, \ldots, t - 1$. Formally, the outstanding debt at the start of period $t$, denoted by $\mathcal{B}_t$, evolves as

$$\mathcal{B}_1 = B, \quad \mathcal{B}_{t+1} = \mathcal{B}_t - p_t \mathbf{1}\{\mathcal{W}_t \geq p_t, \mathcal{Y}_t > 0\}, \quad t = 1, \ldots, T - 1.$$  

Positive values of $\mathcal{B}_t$ correspond to the additional revenues that the DM needs to generate in periods $t, \ldots, T$ to pay off the debt. Negative values of $\mathcal{B}_t$ indicate that the DM has already generated enough revenues to pay off the debt, in which case $-\mathcal{B}_t$ corresponds to a payoff that the DM has already secured. In other words, when the DM faces a positive (negative) outstanding debt at some period, this means that some amount (no amount) of the additional revenues he will generate in the remaining periods must be withheld to service the debt.

We use $V_t(b, y)$ to denote the DM’s optimal expected payoff at the start of period $t$ when his
outstanding debt is $b$ and he has $y$ remaining inventory units; we also refer to $V_t$ as the value function. It can be readily seen that the value function satisfies the following recursion:

$$V_t(b, y) = \max_{p \in \mathcal{P}} \{\lambda(p)V_{t+1}(b-p, y-1) + (1-\lambda(p))V_{t+1}(b, y)\}, \quad y \geq 1, \ t = 1, \ldots, T \quad (2)$$

$$V_t(b, 0) = (-b)^+, \ t = 1, \ldots, T+1, \quad V_{T+1}(b, y) = (-b)^+. \quad (3)$$

Note that the debt payment causes the terminal payoff function $V_{T+1}$ to be convex in the outstanding debt $b$, a feature that differentiates our model from classical models in the operations management literature dealing with maximization of concave profits (or minimization of convex costs). When $B = 0$, one obtains the classical dynamic pricing recursion for a seller with no debt who maximizes expected revenues [Gallego and van Ryzin, 1994]. Thus, the case $B = 0$ serves as our benchmark, from which distortions will be measured.

In this setting, the DM’s price in period $t$, denoted by $p_t^+(b, y)$, satisfies

$$p_t^+(b, y) \in \arg \max_{p \in \mathcal{P}} \{\lambda(p)[V_{t+1}(b-p, y-1) - V_{t+1}(b, y)]\}, \quad y \geq 1.$$ 

As per our discussion above, for negative outstanding debt at the start of period $t$, the DM collects all revenue generated in the remaining periods $t, \ldots, T$. Consequently, the DM’s price would coincide in such cases with the revenue-maximizing price, denoted by $p_t^*(y)$. That is, $p_t^+(b, y) = p_t^*(y)$ for $b \leq 0$. If for some $t, b \geq \min\{y, T-t+1\}p$, where $\overline{p} = \sup_{p \in \mathcal{P}} p$, the DM’s payoff is equal to zero almost surely, in which case we take $p_t^+(b) = \overline{p}$ without loss of generality.

Next, we provide a set of structural properties for the value function.

**Lemma 3.1** (Properties of the DM’s value function). *We have that*

i.) $V_t(b, y)$ *is convex, decreasing in the outstanding debt $b$ and decreasing in $t$.*

ii.) $b + V_t(b, y)$ *is increasing in $b$.*

iii.) at time $t$, the probability of paying off the debt is given by $-\frac{\partial V_t}{\partial b}(b, y)$, for $b > 0$.

Part i.) confirms that the DM’s expected payoff decreases as the debt burden increases, or as less time is available to pay it off. It also shows that the recursion preserves the convexity of the terminal value function $V_{T+1}$ in $b$, i.e., $V_t(\cdot, y)$ is convex for all $t$ and $y$. The intuition behind this result is that an increasing debt, because it becomes unlikely to be paid off, marginally reduces the DM’s expected payoff by a decelerating amount. The convexity of $V_t$ also makes the characterization of the DM’s prices analytically challenging. However, we are able to derive some insightful structural properties that facilitate our subsequent analysis.
Part ii.) shows that the DM’s payoff decreases ‘slowly’ with the debt $b$. In particular, the marginal decrease is always less than 1. When interpreting $b$ as revenue that the DM needs to ‘return,’ this result suggests that the DM is expected to return only a fraction of an additional unit of debt, a testament to the DM’s limited liability.

Part iii.) shows that the probability of covering the debt is precisely given by the marginal decrease rate of the DM’s expected payoff at time $t$ with respect to the outstanding debt—a result that follows from an application of the Envelope Theorem.

We observe that there are no obvious structural properties of the value function with regard to the dependence on the inventory level. Indeed, the presence of debt fundamentally changes the nature of the dynamic pricing problem studied in Gallego and van Ryzin (1994). While it is well known that units have decreasing marginal returns in the absence of debt, the value function $V_t(b, y)$ will generally not be concave in $y$ for $b > 0$. To see why, note that an additional unit could determine whether the DM achieves a positive payoff or not, and hence $V_t(b, y)$ might have increasing marginal returns in $y$ in some region of $(b, y)$, as the following example illustrates.

**Example 3.1** (Increasing marginal returns in inventory). We illustrate that the value function might have decreasing (increasing) marginal returns in inventory for small (high) values of $b$. Suppose that $T = 2$, $Y = 2$ and $\lambda(p) = \alpha - \beta p$, for some $\alpha \in (0, 1]$, $\beta > 0$. For $b \leq 0$, the DM’s policy is aligned with the revenue-maximizing policy and we have that $V_{T-1}(b, 2) - V_{T-1}(b, 1) < V_{T-1}(b, 1) - V_{T-1}(b, 0)$.\footnote{This result follows from Gallego and van Ryzin (1994).}

By continuity, the inequality continues to hold for small enough positive values of $b$. However, for $b \in (\alpha \beta, 2 \alpha \beta)$, we have that $V_{T-1}(b, 2) - V_{T-1}(b, 1) > V_{T-1}(b, 1) - V_{T-1}(b, 0)$.\footnote{Note that for $b \in (\alpha \beta, 2 \alpha \beta)$ we have that $V_{T-1}(b, 1) \leq (\alpha \beta - b)^+ = 0$ and hence $V_{T-1}(b, 1) - V_{T-1}(b, 0) = 0$. In conjunction with (2)-(3), this also implies that $V_{T-1}(b, 2) = \max_{\epsilon \leq p \leq \frac{\alpha \beta}{\beta}} \lambda(p)V_T(b - p, 1) \geq \lambda(b + \epsilon)V_T(\epsilon, 1) > 0$, for small enough $\epsilon > 0$.}

The above discussion highlights the complexity of the interplay between the debt faced by the DM and the inventory level at his disposal. In §4, we analyze a problem with two periods, fully characterizing the distortions and the associated dynamics. We then deal with the multi-period case in §5 by generalizing the theoretical results for the asset selling problem, and establishing the robustness of the insights for instances with an arbitrary number of units.

### 4 The Two Period Case

Consider the case in which the DM has two periods to go, i.e., $T = 2$. Despite being the minimal instance in which dynamics can be analyzed, this setting already provides a rich enough model to understand the key distortions introduced by debt, their time-evolution, and their impact on the
efficiency loss. We make the following assumption about the WTP, which is intended primarily for facilitating the analysis, and is relaxed in some of the subsequent sections.

**Assumption 4.1.** For some \( \alpha \in (0, 1] \) and \( \beta > 0 \), the WTP is uniformly distributed on \( \mathcal{P} = [0, \alpha/\beta] \) such that \( \lambda(p) = \alpha - \beta p \).

Our first result characterizes the DM’s pricing policy.

**Proposition 4.1 (Pricing under debt).** Let \( T = 2 \) and suppose that Assumption 4.1 holds. For all \( b > 0 \),

i.) the DM’s price is strictly higher than the revenue-maximizing price, i.e., \( p^\dagger_t(b, y) > p^*_t(y) \) for all \( y \geq 1 \) and \( t \).

ii.) \( p^\dagger_T(b, y) \) is increasing in \( b \), for all \( y \geq 1 \).

iii.) \( p^\dagger_{T-1}(b, 1) \) is increasing in \( b \), and \( p^\dagger_{T-1}(b, y) \) is piecewise increasing in \( b \) for all \( y \geq 2 \).

Our result derives the distortions that debt induces on the pricing policy: the DM prices strictly higher than the revenue-maximizing price, even for arbitrarily small debt levels, and tends to increase prices further when facing an increasing debt. To understand the intuition behind the DM’s preference for prices higher than \( p^* \), despite them naturally reducing expected revenues, note that this increase also leads to an increase in the revenue variability. More precisely, pricing higher decreases the probability of a sale, but increases revenue if a sale does indeed occur; consequently, the probability mass in the revenue distribution gets more dispersed, as both the mass at the zero revenue point and the mass in the right tail of the distribution increase. Since the DM only achieves a positive payoff when the total revenue covers the debt, without being penalized when it falls short, he consequently attends to the right tail of the revenue distribution, and prefers a higher variability in outcomes; in turn, this makes higher prices preferable, despite the overall expected revenue loss.

Our insights are well aligned with the agency theory of debt developed in the corporate finance literature [Jensen and Meckling, 1976; Myers, 1977], which suggests that the manager of a leveraged firm would have the incentive to take excessively risky actions instead of firm-optimal ones, due to the limited liability inherent in the payoff structure. In our setting, we show that such excess risk taking is manifested through higher prices, which increase the revenue distribution’s dispersion.

It is worth comparing our results with those in Levin et al. (2008), who consider dynamic pricing with a risk-averse objective that mixes expected revenues with the probability of meeting a revenue target. Similar to Levin et al. (2008), we find that prices are not distorted once the target is met—if the debt is covered by the revenues generated to date, so that \( b \leq 0 \), the DM follows the revenue-maximizing policy, \( p^\dagger_t(b, y) = p^*_t(y) \). However, whereas Levin et al. (2008) find that the optimal risk-averse policy
may involve either markups (when revenues are far below the target) or discounts (immediately below the target) relative to the revenue-maximizing policy, we find consistent risk-seeking behavior and prices always above the revenue-maximizing ones, even when arbitrarily close to the “target.”

Our analysis also elicits an interesting interaction between inventory and outstanding debt, which may cause the DM’s price to not be monotonically increasing with the outstanding debt in some cases. In particular, $p^*_T(b, y)$ is only piece-wise increasing in $b$ for $y \geq 2$. To understand this, note that with two periods to go and sufficient inventory, the DM could either use a strategy aimed at covering the debt through a single unit sale, or an alternative strategy based on two unit sales. This interplay between inventory units and debt introduces the possibility of the DM’s price being discontinuous with the debt amount. Specifically, in the proof of Proposition 4.1 we show that there exists a $\hat{b} \in (0, \frac{\alpha}{\beta})$ such that the DM’s pricing policy in period $T - 1$ for $y \geq 2$ always has three regimes,

$$p^*_T(b, y) = \begin{cases} q_\ell(b) & b \in [0, \hat{b}], \\ q_m(b) & b \in (\hat{b}, \frac{\alpha}{\beta}], \\ q_h(b) & b \in (\frac{\alpha}{\beta}, 2\frac{\alpha}{\beta}], \end{cases}$$

where $q_\ell(\cdot)$, $q_m(\cdot)$, $q_h(\cdot)$ are increasing, $q_\ell(\hat{b}) > q_m(\hat{b}) > p^*_T(2)$, and $q_m(\frac{\alpha}{\beta}) = q_h(\frac{\alpha}{\beta})$. Specifically, the DM’s price has one discontinuity point at $b = \hat{b}$ and

$$p^*_T(b, y) \begin{cases} > b & b \in (0, \hat{b}], \\ < b & b \in (\hat{b}, 2\frac{\alpha}{\beta}]. \end{cases}$$

Thus, the DM plans to cover a low debt ($b \leq \hat{b}$) by selling a single unit priced sufficiently high, but plans to cover a high debt ($b > \hat{b}$) by selling two units, each priced lower than the debt. Note that this discontinuity cannot occur at time $T$, when the DM must rely on a single unit to pay off any outstanding debt. The driver of this phenomenon is the discrete nature of inventory and demand in conjunction with the “combinatorial” possibilities associated with selling the units at hand to maximize payoff under debt. In particular, revisiting the DM’s problem in (2), the function $\lambda(p)[V_{t+1}(b-p, y-1) - V_{t+1}(b, y)]$ can be multimodal for more than one unit at hand and multiple periods to go. Each mode can be associated with the type of strategies described above.

We illustrate the DM’s price at $T - 1$ as a function of the outstanding debt in Figure 1 for the demand function $\lambda(p) = 1 - 0.2 \times p$ and $y = 2$. Note that the revenue-maximizing price is $p^*_T(2) = 2.5$, and the associated optimal revenue is $J^* = 2.5$. We first observe that even a debt of half the optimal revenue ($b = 1.25$) leads to a 20% increase in price, a highly non-trivial distortion. The 45-degree line is also shown to highlight when the DM prices below or above the debt; this allows to discern the two
Figure 1: **DM’s pricing policy structure.** The demand function is linear given by \( \lambda(p) = 1 - 0.2 \times p \). The horizon is \( T = 2 \) and the DM has \( y = 2 \) units in inventory.

strategies discussed above. In particular, for any debt level below \( \hat{b} \approx 3.3 \), the DM relies on a strategy that covers the debt through the sale of a single unit. Above \( \hat{b} \), the regime switch takes place and the DM significantly decreases his price and now relies on two units to cover the debt.

Given that the DM’s policy significantly deviates from the revenue-maximizing one, efficiency losses emerge, which we quantify in our next result.

**Lemma 4.1 (Efficiency loss).** Let \( T = 2 \), \( p^* := \frac{\alpha}{2\beta} \), and suppose that Assumption 4.1 holds. Then, under any linear demand model, the efficiency loss \( L \) is piecewise increasing and piecewise convex in \( b \), and is bounded from below as follows:

\[
L \geq \begin{cases} 
0.093 \times \left( \frac{b}{p^*} \right)^2 & b \in [0, \hat{b}], \\
0.115 + 0.199 \times \frac{b-\hat{b}}{p^*} & b \in (\hat{b}, \frac{\alpha}{2\beta}], \\
0.407 + 0.222 \times \frac{b-\frac{\alpha}{2\beta}}{p^*} & b \in \left( \frac{\alpha}{2\beta}, \frac{2\alpha}{\beta} \right].
\end{cases}
\]

Furthermore, \( L \) is piecewise decreasing and piecewise convex in \( \alpha \), and piecewise increasing and piecewise convex in \( \beta \).

To understand the result, note first that \( p^* = p_{T-1}(2) = p_T(1) \) exactly corresponds to the revenue-maximizing price charged under ample inventory, i.e., in a setting where stock-outs never occur. Thus,
can be interpreted as a normalized measure of leverage. The lemma shows that efficiency losses always exist under debt, and that they grow with leverage, in a piecewise and convex fashion. In particular, under low debt \((b \leq \hat{b})\), the losses exhibit a convex growth with a quadratic rate. At intermediate debt \((b \in (\hat{b}, \frac{\alpha}{\beta})]\), the losses always exceed 11.5%, and again grow in a convex fashion, at a rate of roughly 20% per unit of leverage. At high debt (i.e., \(b > \frac{\alpha}{\beta}\)), losses always exceed 40.7% and grow convexly, at a rate of at least 22.2% per unit of leverage. These large values suggest that losses induced by pricing distortions can be quite substantial, and can grow quickly with leverage. The magnitude is also consistent with that documented in several corporate finance papers that consider borrowers endowed with the ability to alter both the expected value and the volatility of their cashflow, as is the case in our model (see Decamps and Djembissi (2007) and our discussion in §1.1).

The latter part of the lemma suggests that losses decrease and efficiency increases under a larger willingness to pay, i.e., larger \(\alpha\). This is intuitive, since larger WTP implies that the price increases inherent in the DM’s policy have less impact on the probability of selling, and thus on the expected revenue losses; furthermore, such revenue losses matter less relative to the optimal revenues, which are also higher under larger \(\alpha\). By a similar mechanism, efficiency losses are exacerbated by a greater price sensitivity, i.e., a larger \(\beta\), which facilitates the DM’s ability to engage in riskier behavior, and makes (even small) price changes have more drastic consequences on the generated revenue.

Figure 2 plots the efficiency loss \(L\) for our earlier example as a function of the debt, for different inventory levels. Interestingly, \(L\) is piecewise (but not globally) increasing and convex, exhibiting a downward jump when the debt exceeds \(\hat{b}\). As before, the root cause for this discontinuity is that the
DM relies on two sales for covering a debt $b > \hat{b}$. Consequently, he charges a lower price in period $T - 1$, which is closer to the revenue-maximizing one, and thus reduces efficiency losses (see Figure[1]).

4.1 Dynamics and Downward Efficiency Spiral

We now turn our attention to the dynamic evolution of the pricing distortions and efficiency losses we documented. Our first result provides a characterization for the price evolution under the DM’s policy.

Proposition 4.2 (Price evolution under debt). Let $T = 2$ and suppose that Assumption 4.1 holds. Then, under the DM’s policy,

i.) under ample inventory, the price increases in expectation over time by an amount that is increasing in the debt, i.e., $\mathbb{E}[p_t^+(B_T, Y_T)] - p_{T-1}^+(b, y)$ is positive and increasing in $b$ for $y \geq 2$.

ii.) conditional on no sale and under limited inventory, the price decreases over time by an amount that is decreasing in the debt, i.e., $p_{T-1}^+(b, y) - p_T^+(b, y)$ is positive and decreasing in $b$ for $y = 1$.

iii.) conditional on no sale and under ample inventory, the price increases over time by an amount that is increasing in the debt, i.e., $p_T^+(b, y) - p_{T-1}^+(b, y)$ is positive and increasing in $b$ for $y \geq 2$.

Part i.) states that, under ample inventory, the DM tends to mark up prices over time, in expectation. This result is in stark contrast with the classical insights in the dynamic pricing literature, where it is known that, under ample inventory, the revenue-maximizing policy maintains constant prices over time [Gallego and van Ryzin, 1994], i.e., $p_{T-1}^*(b, y) = p_T^*(B_T)$ almost surely for $y \geq 2$. Furthermore, our result shows that increased debt leads to higher expected markups, thus accentuating the distortion.

Parts ii.) and iii.) of the result focus on the case when no sale occurs. Under these circumstances, the classical insights in dynamic pricing suggest that a revenue-maximizing policy would always mark down prices. Part ii) shows that, under limited inventory, the DM’s policy would also mark down prices, albeit always at a slower rate than the revenue-maximizing policy, i.e., $p_{T-1}^+(b, y) - p_T^+(b, y) \leq p_{T-1}^*(b, y) - p_T^*(b, y)$. Part iii.), however, shows that, under ample inventory, the DM would actually mark up the price, highlighting another substantial departure from the revenue-maximizing policy. As before, increased debt accentuates the effects, leading to lower markdowns or higher markups.

Our results also lead to the following corollary for the case when no sale occurs.

Corollary 4.1. Let $T = 2$ and suppose that Assumption 4.1 holds. Then, conditional on no sale, the difference between the DM’s and the revenue-maximizing price increases over time in both absolute and relative terms, i.e.,

$$p_T^+(b, y) - p_T^*(y) \text{ and } \frac{p_T^+(b, y) - p_T^*(y)}{p_T^*(y)}$$

are both increasing in $t$. 18
By combining the results of Proposition 4.1, Proposition 4.2, and Corollary 4.1, we conclude that debt induces the DM to not only consistently price higher, but also to mark down prices at a slower pace or even to mark up compared to what would be (revenue-) optimal. These upward pricing distortions that are compounding over time raise the concern that the efficiency under the DM’s policy would also deteriorate over time, an issue to which we turn our attention next.

The evolution of generated revenues and the resulting efficiency losses over time is made subtle by the following two opposing forces. On the one hand, when no sales occur, the DM’s pricing policy increasingly deviates from the revenue-maximizing one, as per Corollary 4.1. This magnified deviation drives higher efficiency losses. On the other hand, when a sale occurs, the higher (than revenue-optimal) prices posted by the DM would generate more revenues in comparison, driving efficiency losses down. Note that all these future scenarios are accounted for by the efficiency loss $L$, which aggregates all possible sample paths. As time progresses, however, the conditional efficiency loss $L_t$ defined in (1) could drift either down or up, depending on the more or less favorable WTP realizations and associated sales. As such, the expected efficiency losses $L_t$, which aggregate these possible realizations of $L_t$ at time $t$, capture the evolution of performance over time. Recalling that $L = L_1$, our next result provides a characterization for this dynamic evolution.

**Proposition 4.3** (Downward efficiency spiral). Let $T = 2$ and suppose that Assumption 4.1 holds. Then,

$$L_{T-1} \leq L_T.$$  

The result establishes that, in expectation, the pricing distortions inherent in the DM’s policy under debt have a compounding effect, leading to a downward spiral in efficiency.

To illustrate this effect, we revisit our earlier example for $\lambda(p) = 1 - 0.2 \times p$, and consider the case in which $B = 1.5$. Figure 3(a) reports the expected revenue at time $T - 1$ under the DM’s policy ($J^* = 2.4$) and the revenue-maximizing policy ($J^\dagger = 2.5$). The resulting efficiency loss is $L_{T-1} = 4.12\%$. Figure 3(b) illustrates how all these quantities could evolve from $T - 1$ to $T$, depending on the realization of the WTP $\mathcal{W}_{T-1}$. In particular, the conditional expected revenue under the DM’s policy, $J^\dagger_T$, takes values 4.18 or 1.14, depending on whether the DM sells a unit ($\mathcal{W}_{T-1} \geq p^\dagger_{T-1}$, shaded region in second column) or not ($\mathcal{W}_{T-1} < p^\dagger_{T-1}$, non-shaded region in second column). Note that, since the WTP is uniformly distributed, the size of each region is also proportional to the probability of the associated event. Similarly, the conditional optimal revenue, $J^*_T$, takes values 3.75 or 1.25, depending on whether $\mathcal{W}_{T-1} \geq p^*_T$ (shaded region in third column) or not (non-shaded region in third column). This gives rise to three possible values for the conditional efficiency loss $L_{T-1}$, depending on the sales under the two different policies: $-11.5\%$ when a sale occurs under both policies, $70\%$ when a sale occurs only under the revenue-maximizing policy, and $9\%$ when no sale occurs under either policy. Weighing
these three events by their corresponding probabilities yields an expected efficiency loss \( L_T = 5.75\% \), which is larger than \( L_{T-1} = 4.12\% \).

Figure 3(b) also highlights that the revenue distribution under the DM’s policy gets considerably more dispersed compared to the optimal one: the “downside” worsens from 1.25 to 1.14, and the “upside” improves from 3.75 to 4.18.

To the best of our knowledge, these results constitute the first quantification of the dynamic evolution of efficiency losses induced by debt. We showed that debt introduces pricing distortions that compound over time, leading to a downward spiral in efficiency. This finding is particularly relevant because it suggests that any mechanism aimed towards reducing this inefficiency must necessarily entail some form of dynamic monitoring of the DM’s actions and performance throughout the planning horizon. In §6, we discuss some mechanisms and their effectiveness, after first confirming the robustness of our results, and extending them to a multi-period setting.

5 The Multi-Period Case

We now analyze problems with an arbitrary number of periods, and general demand functions. In §5.1, we extend our analytical results to a setting where a single item is sold, and in §5.2, we explore the case with an arbitrary number of units.
5.1 One Unit Case (Asset Selling)

Consider the special instance of our general model described in §2 where the DM is endowed with a single unit of inventory, i.e., \( Y = 1 \). We make no restrictions on the number of periods, and no longer assume that the demand function is linear. This special case corresponds to a variant of the well known asset selling problem where the seller posts prices instead of receiving offers, which we extend here by including debt. For an extensive comparison of the two classical models of posted prices and received offers in the absence of debt, we direct the interested reader to Arnold and Lippman (2001).

In this model, it is sufficient to keep track of whether a sale occurred or not. However, since we are interested in comparative statics with respect to the debt, we still retain it as part of the state.

For \( t \leq T \), let \( V_t(b) \) be the DM’s optimal expected payoff at \( t \) when the outstanding debt is \( b \) and no sale occurred up to \( t \), and let \( p^\dagger_t(b) \) be the price he posts. In case a sale occurs, the DM transitions to the terminal period \( T + 1 \), and generates a payoff of \( (p^\dagger_t(b) - b)^+ \). Otherwise, the DM transitions to the next period \( t + 1 \), without generating any payoff. The Bellman recursion in (2)-(3) can thus be rewritten as follows:

\[
V_t(b) = \max_{p \in \mathcal{P}} \{ \lambda(p) V_{T+1}(b - p) + (1 - \lambda(p)) V_{t+1}(b) \}, \quad t = 1, \ldots, T, \quad V_{T+1}(b) = (-b)^+.
\]  

(4)

Accordingly, the DM’s price at time \( t \) satisfies:

\[
p^\dagger_t(b) \in \arg \max_{p \in \mathcal{P}} \{ \lambda(p) V_{T+1}(b - p) + (1 - \lambda(p)) V_{t+1}(b) \}, \quad t = 1, \ldots, T.
\]  

(5)

As before, the DM’s price coincides with the revenue-maximizing price in the absence of debt, i.e., \( p^\dagger_t(0) = p^*_t \). For \( b \geq \overline{p} = \sup_{p \in \mathcal{P}} p \), the DM’s payoff is equal to zero almost surely and so is the maximand in (5), in which case we take \( p^\dagger_t(b) = \overline{p} \) for all \( t = 1, \ldots, T \) without loss of generality. For \( b < \overline{p} \), the DM can generate a positive payoff, provided that he posts a high enough price. In particular, for all \( p < b \), \( V_{T+1}(b - p) = 0 \) and the maximand in (5) is equal to \( (1 - \lambda(p)) V_{t+1}(b) \), an increasing function in \( p \). Thus, it is never optimal for the DM to post a price lower than the debt.

Lemma 5.1. The DM’s price is always larger than the debt, i.e., \( p^\dagger_t(b) \geq b \).

Using this property and substituting \( V_{T+1}(b - p) = p - b \) for \( p \geq b \), we can re-write (5) equivalently as

\[
p^\dagger_t(b) \in \arg \max_{p \in \mathcal{P}, p \geq b} \left\{ \lambda(p) \left[ p - (b + V_{t+1}(b)) \right] \right\}, \quad t = 1, \ldots, T, \quad \text{if } b < \overline{p}.
\]

(5)

For analytical tractability purposes, we make the following assumptions on the demand function.

Assumption 5.1. The demand function \( \lambda \) is differentiable on \( \mathcal{P} \) and log-concave.
This requirement is equivalent to the hazard rate of the WTP distribution being increasing, and is satisfied by many demand functions encountered in the literature, including linear, exponential, normal, generalized linear \( \lambda(p) = (a - bp)^n \), for \( a \in [0, 1] \), \( b \geq 0 \), \( n \geq 1 \), logit \( \lambda(p) = e^{-bp}/(1 + e^{-bp}) \) for \( b \geq 0 \), or Weibull-distributed WTP, etc.\(^9\)

Our next result further characterizes the DM’s pricing policy, and the way it is impacted by debt.

**Proposition 5.1** (Asset selling under debt). Under Assumption 5.1 and for all \( t = 1, \ldots, T \) and \( b > 0 \),

i.) the DM’s price is given by

\[
p_t^\dagger(b) = \begin{cases} 
\pi(b + V_{t+1}(b)) & b < \overline{p} \\
\overline{p} & b \geq \overline{p},
\end{cases}
\]

where \( \pi(x) = \arg\max_{p \in P} \lambda(p)(p - x) \); also, \( \pi(x) \geq x \) and \( 0 \leq \pi'(x) \leq 1 \).

ii.) the DM’s price is always strictly higher than the revenue-maximizing price, i.e., \( p_t^\dagger(b) > p_t^\star \).

iii.) the DM’s price is increasing in \( b \).

This lemma reinforces our earlier conclusions that pricing distortions under debt come in the form of higher prices. More precisely, parts ii.) and iii.) show that the DM always posts prices that are higher than the revenue-maximizing ones, and that the prices increase with the debt. As such, the results are direct counterparts of Proposition 4.1 for the case of multiple periods, and under a more general demand model.

Part i.) of the result provides a more detailed structure of the policy, which is possible when a single unit is being sold. In this case, note that the price is always continuously increasing in the debt level \( b \); this is unlike the case when multiple units are sold, when \( p_t^\dagger \) may exhibit a sharp drop as \( b \) increases and the DM switches from a strategy relying on a single sale to one relying on multiple sales to cover the debt. Furthermore, it can be seen that \( \frac{\partial p_t^\dagger}{\partial b} \leq 1 \), implying that extra units of debt would cause sub-unitary marginal price increases.

We now turn attention to the dynamics of the DM’s pricing policy.

**Proposition 5.2** (Price evolution under debt). Suppose that Assumption 5.1 holds and that \( \lambda \) is convex and \(-\lambda'\) is log-convex. Then, for all \( t = 2, \ldots, T \) and conditional on no sale to date,

i.) the DM marks prices down over time by an amount that is decreasing in the debt, i.e., \( p_{t-1}^\dagger(b) - p_t^\dagger(b) \) is positive and decreasing in \( b \).

---

\(^9\)This is exactly the condition discussed in Arnold and Lippman (2001) for an infinite horizon model, which guarantees unimodality of the profit function and natural comparative statics for the pricing decisions.
Figure 4: (a) Price evolution for the revenue-maximizing policy and the DM’s optimal policy under \( \lambda(p) = 1 - 0.2 \times p \) for \( T = 5 \), and for different levels of debt: ‘low’ \( b_l = p_T^* / 2 \), ‘medium’ \( b_m = p_T^* \) and ‘high’ \( b_h = 1.5 \times p_T^* \). (b) Evolution of price distortions over time.

ii.) the DM’s policy applies smaller markdowns than the revenue-maximizing policy, in both absolute and relative terms, i.e., \( p_{t-1}^* - p_t^* \) and \( (p_{t-1}^* - p_t^*) / p_{t-1}^* \leq (p_{t-1}^* - p_t^*) / p_{t-1}^* \).

iii.) the difference between the DM’s price and the revenue-maximizing price increases over time, in both absolute and relative terms, i.e., \( p_t^* - p_{t-1}^* \) and \( (p_t^* - p_{t-1}^*) / p_{t-1}^* \) are both increasing in \( t \).

It is well known that the revenue-maximizing policy for the asset selling problem would prescribe a sequence of prices that are decreasing over time (Arnold and Lippman, 2001). Proposition 5.2 shows that a debt-facing DM would also apply price markdowns, albeit smaller in magnitude, and decreasing with the debt level. This is in line with Proposition 4.2 obtained for the case of two periods. Furthermore, extending the intuition of Corollary 4.1, part iii.) of the result shows that the pricing distortions consistently increase over time, in an upward spiraling fashion. Note that several of the demand functions satisfying Assumption 5.1, such as linear, exponential, and a Weibull-distributed WTP with shape parameter \( k \leq 1 \) also satisfy the additional requirement of the lemma.

Figure 4 illustrates the findings of Proposition 5.1 and Proposition 5.2 by depicting the prices posted by the DM’s policy and the revenue-maximizing policy throughout time for different debt values, under a linear demand function. Specifically, Figure 4(a) shows that under higher debt values, the DM posts higher prices that decrease at a slower rate over time, and Figure 4(b) illustrates the compounding distortions. With a debt of \( b = p_T^* \) and five periods to go, the posted price is about 10% higher than the revenue-maximizing price. With two periods to go, the price is 25% higher.
The results in Proposition 5.1 and Proposition 5.2 are validated by empirical evidence documented in the literature on real estate markets. In particular, Genesove and Mayer (1997) show that homeowners with larger debt loads set higher asking prices for their houses, have them listed for a longer expected time on the market, and receive a higher price upon an eventual sale than owners with smaller debt loads. The documented effects are significant: compared to a loan-to-value ratio (LTV) of 80%, an LTV of 100% leads to a list price that is 4% higher, keeping the unit 15% longer on the market, and leading to a sale price that is also 4% higher.

We conclude our analysis of the asset selling problem by studying the efficiency losses and their dynamic evolution. This allows us to generalize our result of compounding efficiency losses in Proposition 4.3 to an arbitrary number of periods. To formally present our result, let \( J^\dagger_t \) be the expected revenues under the DM’s policy at the beginning of period \( t \), conditional on no sale in periods \( 1, \ldots, t-1 \), i.e.,

\[
J^\dagger_t := \mathbb{E}[\mathcal{R}(p^\dagger)|W_1 < p_1^\dagger(B), \ldots, W_{t-1} < p_{t-1}^\dagger(B)], \quad t = 1, \ldots, T.
\]

**Proposition 5.3 (Downward efficiency spiral).** Suppose that Assumption 5.1 holds and that \( J^\dagger_t / J^\dagger_{t+1} \) is increasing in \( b \) for any \( t = 1, \ldots, T-1 \). Then

\[
L_1 \leq L_2 \leq \ldots \leq L_T.
\]

Before commenting on the result, we first note that the monotonicity requirement on \( J^\dagger_t / J^\dagger_{t+1} \) is satisfied by any linear or exponential demand function (see Proposition F.1 in Appendix F). However, this requirement has a natural interpretation, and we expect it to hold more broadly. To understand this, note that \( J^\dagger_t / J^\dagger_{t+1} \) can be interpreted as the revenues that the DM is expected to generate when facing debt \( b \) and having \( T - t + 1 \) (\( T - t \)) periods to sell the asset. Clearly, both \( J^\dagger_t \) and \( J^\dagger_{t+1} \) are decreasing in \( b \), owing to the increasing pricing distortions that higher debt levels induce. However, when debt levels increase, it is also natural to expect \( J^\dagger_{t+1} \) to exhibit a higher relative decrease rate than \( J^\dagger_t \), due to the fact that pricing distortions compound across time (or, put differently, a DM with fewer periods to go would distort prices “more”). Thus, it is quite natural to expect \( J^\dagger_t / J^\dagger_{t+1} \) to be increasing in \( b \).

As we remarked in Section 4.1, the evolution of the efficiency loss \( L_t \) is made subtle by the higher prices charged by the DM: it decreases or increases depending on whether a sale occurs or not, respectively. Figure 5 illustrates all possible values of \( \mathcal{L}_t \) and the various sample paths that may occur for a specific problem instance. While the figure confirms that the expected efficiency loss \( L_t \) is increasing over time, it is also worth observing that the “variability” of \( \mathcal{L}_t \) also increases over time, with a wide range of possible outcomes in the last period \( t = 5 \). Note that \( \mathcal{L}_t \) can be negative (whenever the DM successfully sells the unit, which occurs at a higher price than the revenue-maximizing one) but can
Possible evolutions of efficiency loss $L_t$

Figure 5: Efficiency loss evolution under $\lambda(p) = 1 - 0.2 \times p$, for $T = 5$ and $b = p^*_T$. Each path in the tree depicts a possible evolution of $L_t$. In nodes with three edges, the upper edge denotes the event $\{p^*_t \leq W_t < p^*_t\}$ (when only the revenue-maximizing policy achieves a sale), the middle edge corresponds to the event $\{W_t < p^*_t\}$ (when no policy achieves a sale), and the lower node to the event $\{W_t \geq p^*_t\}$ (when both policies achieve a sale). Conditional on the revenue-maximizing policy having achieved a sale, nodes have two edges, for the events $\{W_t < p^*_t\}$ (upper) and $\{W_t \geq p^*_t\}$ (lower). The width of each edge is proportional to the probability of the corresponding event. The dots on the dashed line depict $L_t$. The probabilities of all possible outcomes for $L_5$ are displayed on the right.

also be as high as 80% (when the DM does not sell in the first four periods, but would have sold had he applied the revenue-maximizing policy).

### 5.2 Multi-unit Case

Recall from our discussion in Section 4 that a DM endowed with multiple inventory units can follow several strategies for covering the debt, relying on the sale of different numbers of units. This combinatorial feature introduces multi-modality in the DM’s objective function, which in turn generates discontinuous pricing policies, making a general multi-unit, multi-period setting not amenable for analysis. Consequently, we explore this setting numerically. We find that all our findings persist. For details, we refer the interested reader to Appendix B.
6 Managerial and Debt Contract Design Implications

Three main insights that our analysis yields are that (i) under debt, a DM would always tend to charge higher prices than the revenue-maximizing ones, (ii) that these price distortions would tend to increase with the debt load, and that (iii) these effects would compound over time, generating an upward spiral in prices, and a downward spiral in efficiency. These insights can be used to inform managers and lenders about specific contract terms that could alleviate the inefficiencies we studied.

Early repayment option. Offering an early repayment option could be a counterbalancing force for the DM’s tendency to charge higher prices. These options, which are routinely included in some forms of lending such as trade credit, typically stipulate a discount for any early repayments, thus encouraging borrowers to repay some portion of the principal so as to save on interest. In our model, this could induce the DM to charge lower prices in earlier periods, so as to achieve sales and take advantage of the earlier (reduced) repayment.

Debt relief. Since a larger debt tends to exacerbate distortions, reducing the debt burden in case of distress could avert further losses. Such reductions in principal and/or interest are informally referred to as “bondholder haircuts” and routinely occur in practice as part of a “debt workout,” i.e., a debt restructuring agreement between the creditor(s) and the borrower conducted outside the court system (for more details, see, e.g., Arent Fox LLP, 2008 and World Bank Report, 2011).

Debt amortization. In order to avert the compounding of pricing distortions and efficiency loss, another natural recommendation would be to require debt repayments throughout the entire horizon according to an amortization schedule. This is consistent with real-estate lending practice, where “construction loans that finance multiple units or phases must be structured to ensure that repayment appropriately follows unit sales” (CH, 2013, page 28). An equivalent mechanism to debt amortization could be the inclusion of financial covenants that require the borrower’s revenues to exceed certain thresholds (see Iancu et al., 2016 and Chapter 7 of Hilson, 2013). The threat of missing a repayment or tripping a covenant could induce the DM to charge lower prices so as to improve his chances of making sufficient early sales.

We formally model and analyze these three contracting mechanisms in §6.1, assessing their potency in alleviating efficiency losses when benchmarked against our plain contract with a single terminal debt repayment. Our findings can be summarized as follows:

Plain contract = Early repayment option ≤ Debt relief < Debt amortization < Optimal.

A practical implication of these results is that debt contracts would benefit most from including provisions for dynamically monitoring a borrower’s sales performance through, e.g., an amortization schedule or financial covenants. Providing debt relief could be another, albeit less effective, means for improving
efficiency. Early repayment options, however, appear to be ineffective in this context. Interestingly, although a judicious debt amortization always improves performance in our model, it cannot completely restore optimality. In this sense, the result provides a cautious note, as it suggests that in order to fully alleviate efficiency losses due to pricing distortions, one may require more complex contractual specifications, going beyond simple repayment schedules or financial covenants such as those considered in Iancu et al. (2016). For a more extensive discussion of these findings and the intuition driving them, we refer the interested reader to §6.1.

Concluding, we would like to remark that the contracts we considered above were all based on terms commonly encountered in lending agreements. Our work suggests that a natural fix for the price distortions could also be to enforce controls that prevent slow markdowns. However, such controls could be difficult to enforce in practice, due to the inherent contract complexity in specifying contingent pricing policies. Furthermore, prices are often non-verifiable—such as when privately communicated to clients—and are also considered as operating decisions routinely taken during the course of business, which lenders usually refrain from constraining (DeAngelo et al., 2002).

6.1 Comparing Debt Contract Designs

We formalize the discussion above and now enrich the model with debt contracts that include other terms in addition to the terminal debt repayment. To introduce some notation, we use $\kappa$ to denote all the terms of a specific contract. Let $J(\kappa)$ be the expected revenues generated under the DM’s pricing policy induced by contract $\kappa$, and $D(\kappa)$ be the corresponding expected debt repaid, i.e., the debt value. In the context of our model from §4, where we analyzed a plain contract with a single terminal debt repayment, we have $\kappa = \{B\}$, and $J(\{B\}) = J^\dagger$.

An optimal, i.e., efficiency-maximizing, contract $\kappa$ would seek to maximize $J(\kappa)$. To ensure a meaningful comparison and to be consistent with corporate finance theory, we consider contracts that lead to the same debt value, which we denote with $d$. The expected revenues generated under the optimal contract are then determined by solving the following optimization problem:

$$\text{maximize}_{\kappa} \ J(\kappa)$$

subject to $D(\kappa) = d$. 

Returning again to the class of plain contracts, note that the formulation above would essentially require the debt repayment $B$ to be set so that the debt value equals $d$. Consequently, there would

---

10Requiring a fixed debt value can be thought of as the equilibrium outcome when the lending market is perfectly competitive, in which case lenders determine contract terms so as to recover the time-value of money (see page 115 of Tirole, 2006). A fixed debt value is also necessary to meaningfully compare contracts, because otherwise, a contract with no terms $\kappa = \{\}$ will trivially emerge as optimal and recover optimal revenues.
usually be a single plain contract to consider.

We now analyze the efficiency of optimal contracts within the three classes introduced in this section. As benchmarks, we consider the optimal revenues $J^*$ and the revenues $J^\dagger$ under the plain contract. We base the analysis on the model from \[4\] with $T = 2, Y = 2, \lambda(p) = \alpha - \beta p$, where we take $\alpha = 1$ for simplicity (all our results hold for general $\alpha$).

To study contracts with an early repayment option, we assume that the DM can make a debt payment $x$ (of his choice) at time $T - 1$. In so doing, his terminal debt repayment would be reduced from $B$ to $(B - \frac{x}{\gamma})^+$, for some discount parameter $\gamma \in (0, 1]$. Thus, $\kappa = \{B, \gamma\}$ for this class of contracts. Let $J^E$ denote the revenues under the optimal contract with an early repayment option. It can be readily seen that for $\gamma = 1$ we recover the plain contract, so that $J^\dagger = J(\{B, 1\}) \leq J^E$.

To capture debt relief, we allow debt holders to dynamically choose whether and by how much to reduce the DM’s outstanding debt at the end of period $T - 1$, so as to maximize the expected debt payment made at the end of period $T$. Thus, $\kappa = \{B, r\}$ for this class, where $r$ is an indicator variable of whether debt relief is allowed. Let $J^R$ denote the revenues under the optimal contract with debt relief. Again, $J^\dagger = J(\{B, 0\}) \leq J^R$.

To explore contracts with debt amortization, we assume that instead of facing a single required debt repayment $B$ at the end of period $T$, the DM faces two required debt repayments, namely $\theta B$ at the end of period $T - 1$, and $(1 - \theta)B$ at the end of period $T$, for some parameter $\theta \in [0, 1]$.\[11\] We assume that, upon failing to make the payment $\theta B$ at $T - 1$, the DM loses decision control, and the subsequent price in period $T$ is set so as to maximize expected revenues. This is reasonable in the context of debt agreements, where missing a loan installment or breaching a covenant can trigger an event of default, in which case a firm’s equity holders and debt holders would share decision rights as the firm is undergoing restructuring under Chapter 7 bankruptcy protection (Tchore, 2006; Hilson, 2013). Thus, $\kappa = \{B, \theta\}$ in this case. Let $J^A$ denote the revenues under the optimal contract with debt amortization. Note that $J^\dagger = J(\{B, 0\}) \leq J^A$.

The following result compares the three contracts described above.

**Proposition 6.1 (Contract comparison).** Under the setup described above,

$$J^\dagger = J^E \leq J^R < J^A < J^*.$$  

Interestingly, according to Proposition 6.1, early repayment discounts reduce efficiency, so that the optimal contract in that class actually offers no discount (and becomes a plain contract). To gain some intuition for this result, note that although a discount generally induces the DM to lower the price in

\[11\]This setting could be used to model either loan installments or financial covenants, since both mechanisms are equivalent to a minimum revenue threshold that the borrower must generate by the intermediate time.
period $T-1$, it also results in lower payments to debt holders (due to the discount), which may also occur more often. In fact, the discount may even fail to induce lower prices in period $T-1$, by causing the DM to switch from a less risky strategy relying on two sales to a riskier strategy relying on one to cover the “discounted” debt. In both cases, debt holders would anticipate the lower payments, and increase the debt repayment $B$, creating a negative feedback loop that ends up hurting efficiency. This finding is well aligned with practice, where commercial real estate lenders often discourage early repayments by including prepayment penalties intended to recover some of the lost interest (Wells Fargo, 2016).

In contrast, no negative feedback loop arises when relying on debt relief. The prospect of debt relief lowers the effective debt burden for the DM in period $T-1$, causing him to reduce his price. In turn, this actually reduces the need for a debt relief in the first place, and does not decrease the debt value. However, debt relief is not always effective: when the debt burden is low ($B < p^*_T$), debt holders never find it optimal to trim the debt even when the DM fails to make a sale at $T-1$, rendering the entire mechanism superfluous. This is aligned with findings in corporate finance documenting that the presence of renegotiable debt could actually induce the firm to take up more debt and thus engage in more risk shifting, thereby increasing inefficiencies (see Flor, 2011; Gorton and Kahn, 2000).

Finally, debt amortization emerges as the most effective mechanism: a judicious selection of a repayment schedule always improves performance, and surpasses debt relief. This confirms that the compounding effects are truly first-order, and are best mitigated by dynamically monitoring the borrower’s sales performance. An alternative potential explanation is that such mechanisms are qualitatively different from the former two. Early payment discounts and debt relief act as “carrots,” which debt holders may have to compensate with an increased debt repayment $B$. In contrast, by requiring a payment at $T-1$ and threatening to transfer decision control at $T$, an amortization schedule acts as a “stick,” which eliminates the need for a compensating mechanism. However, even the optimal amortization schedule cannot completely restore efficiency: although the revenue-maximizing policy is always followed in period $T$, the DM continues to charge higher prices than the revenue-maximizing ones at $T-1$ (see the proof of Proposition 6.1). In this sense, the result provides a cautious note, as it suggests that in order to fully alleviate efficiency losses due to pricing distortions, one may require more complex contractual specifications, going beyond simple repayment schedules or financial covenants such as those considered in Iancu et al. (2016).

12 “To compensate for early loan repayments, CMBS lenders and most life insurance companies usually require early termination amounts or enforce “make whole” / “yield maintenance” provisions which attempt to make up the lost revenue [...].” (Wells Fargo, 2016)

13 This also strengthens our assumption concerning the assignment of decision rights in period $T$ upon missing the debt repayment in period $T-1$, since we show that efficiency losses persist even if revenue-maximizing decisions are always followed after a violation.


7 Limitations and Future Directions

We conclude by highlighting certain limitations of our study, and fruitful directions for future research.

To start, our model treated the debt repayment as exogenous. In Appendix C, we take a first step toward investigating whether and how endogenizing debt decisions could nuance our insights. Specifically, we consider an extension to our model where before the start of the selling horizon, a cash-constrained DM needs to decide how to finance the inventory purchase through debt and equity. By endogenizing the DM’s and the lender’s decisions, we confirm that the main qualitative results derived in §4 persist in equilibrium. Confirming our results in an analytical model that endogenizes additional debt contract terms (interest rate, maturity, amortization schedule, financial or borrowing-base covenants Iancu et al., 2016) could be an interesting direction for future research.

We also assumed a non-recourse loan, i.e., a borrower with zero liability. In Appendix A, we consider an alternative specification of our asset selling model in §5.1 where a borrower in default faces losses proportional to the shortfall, so that his payoff becomes $E[(\mathcal{R}(p) - B)^+] - k E[(B - \mathcal{R}(p))^+]$ for some $k \in [0, 1]$. We were able to confirm that our main insights are robust: the DM’s policy posts prices higher than the revenue-maximizing ones and applies slower markdowns, leading to efficiency losses. As expected, distortions decrease as $k$ increases, and vanish for $k = 1$. However, a new regime emerges in the DM’s pricing policy: when debt is sufficiently high, the DM abruptly switches to the revenue-maximizing policy, and continues to follow it for the remaining planning horizon. Qualitatively, the DM acts as if he were unable to repay the debt, and thus relies on a strategy that minimizes his losses, or equivalently maximizes revenues.\footnote{In fact, since revenue-maximizing prices are lower than the debt here, following this policy actually yields a “self-fulfilling prophecy,” and guarantees bankruptcy.} Examining the equilibrium lending contracts in such a model (with $k$ as a design parameter) would be an interesting direction for future research. Along the same lines, it would also be relevant to consider settings where the DM could become risk-averse ($k > 1$), such as in privately-held companies or when managers have alternative considerations due to, e.g., taxes, risk-aversion, or compensation schemes penalizing losses. Since one could expect prices to be either higher or lower than revenue-maximizing ones (Levin et al., 2008), quantifying the efficiency losses could be quite interesting (and challenging).

Our discussion in §6 of contract terms that could potentially alleviate inefficiencies could also be enriched. For instance, when the DM fails to achieve sales, we could consider a renegotiation of the loan maturity. Since a DM faced with a longer selling horizon would be less prone to distorting prices, this could alleviate the efficiency losses, in a similar manner to debt relief. An alternative useful lever for deterring the DM from charging higher prices could be a minimum quantity sales requirement. It is known in the marketing literature that salespersons faced with a compensation incentive tied to the total number of units sold would tend to provide customers with discounts, leading to prices...
that are lower than optimal (see, e.g., Over, 1998). A quantity sales requirement could thus provide a compensating effect to the revenue requirement induced by debt. This is well aligned with some of the common wisdom in real-estate lending, where lenders financing condominium developments often require a minimum number of units being (pre)sold before releasing their mortgage claim as part of the title conveyance process (CH, 2013, page 9). Understanding the relative effectiveness of these mechanisms would be an interesting direction for future research.

Finally, the DM may have access to the generated revenues throughout the selling horizon, and could use them for investment (e.g., ordering new items in retail, or repairing/upgrading units in real estate) or for consumption. To capture this, one could consider a more general utility model for the DM, and allow additional inter-temporal decisions involving the revenues. Although such decisions would introduce more opportunities for risk taking, and thus exacerbate the efficiency losses, they could also incentivize the borrower to reduce prices so as to generate additional revenues earlier, and thus potentially reduce pricing distortions. Understanding how these effects trade off against each other would be a very interesting extension for future research.

References


Online Appendix
Dynamic Pricing under Debt:
Spiraling Distortions and Efficiency Losses

A Model with Nonzero Limited Liability

Our base model assumed zero liability under debt, so that the DM’s objective was given by $E[(\mathcal{R}(p) - B)^+]$. In practice, failing to repay the debt often carries negative consequences for a borrower due to, e.g., recourse by the lender (Wells Fargo, 2016), or bankruptcy/reputation costs when the firm goes into default. To capture this limited (but nonzero) liability, we now assume that a DM faced with a debt payment $B$ maximizes the following objective function:

$$E[(\mathcal{R}(p) - B)^+] - k E[(B - \mathcal{R}(p))^+]$$

for some $k \in [0, 1]$.

The parameter $k$ controls the severity of the penalty when the DM fails to repay the debt. A choice of $k = 0$ recovers our familiar model, and $k = 1$ corresponds to maximizing the total revenues $\mathcal{R}(p)$.

In this context, the recursion characterizing the DM’s problem is given by:

$$\tilde{V}_t(b, y) = \max_{p \in \mathcal{P}} \left\{ \lambda(p)\tilde{V}_{t+1}(b - p, y - 1) + (1 - \lambda(p))\tilde{V}_{t+1}(b, y) \right\}, \quad y \geq 1, \ t = 1, ..., T \quad (A-1a)$$

$$\tilde{V}_t(b, 0) = \tilde{V}_{T+1}(b, y) = (-b)^+ - kb^+, \quad t = 1, ..., T + 1, \ y \geq 0. \quad (A-1b)$$

Let $\tilde{p}_t(b, y)$ denote the DM’s price in period $t$. Following similar arguments to our analysis in §3, it can be readily shown that the DM always charges the revenue-maximizing price once the debt is covered, i.e., $\tilde{p}_t(b, y) = p^*(y), \forall b \leq 0$. Furthermore, the following proposition provides certain structural properties for the DM’s value function.

**Lemma A.1.** We have that

i.) $\tilde{V}_t(b, y)$ is convex, decreasing in the outstanding debt $b$ and decreasing in $t$.

ii.) $b + \tilde{V}_t(b, y)$ is positive and increasing in $b$, and $b + \frac{\tilde{V}_t(b, y)}{k}$ is positive and decreasing in $b$.

iii.) $\tilde{V}_t(b, y)$ is decreasing in $k$.

**Proof of Lemma A.1.** Note that $\tilde{V}_{T+1}(b, y) = \max(-b, -kb)$ readily satisfies both properties. Assume by induction that these are also satisfied at time $t + 1$. We have that:

$$\tilde{V}_t(b, y) = \max_{p \in \mathcal{P}} \left\{ \lambda(p)\tilde{V}_{t+1}(b - p, y - 1) + (1 - \lambda(p))\tilde{V}_{t+1}(b, y) \right\}, \quad y \geq 1.$$
To show (i), note that since the maximand is convex and decreasing in \( b \) for any \( p \), \( \tilde{V}_t \) will also retain these properties. Also, \( \tilde{V}_t(b, 0) \geq \tilde{V}_{t+1}(b, 0) \) readily holds for \( y = 0 \); for any \( y \geq 1 \), we have \( \tilde{V}_{t+1}(b, y) \leq \tilde{V}_{t+1}(b - p, y - 1) \) holding for at least some \( p \in \mathcal{P} \), so that \( \tilde{V}_t(b, y) \geq \tilde{V}_{t+1}(b, y) \) follows. Assuming that \( \frac{\partial \tilde{V}_t}{\partial b} \) and \( \frac{\partial^2 \tilde{V}_t}{\partial b^2} \) are well defined, and that \( -1 \leq \frac{\partial^2 \tilde{V}_{t+1}}{\partial b^2} \leq -k \) holds by induction, we then readily obtain that \( -1 \leq \frac{\partial \tilde{V}_t}{\partial b} \leq -k \) by a direct application of the Envelope theorem. Part (iii) readily follows by induction.

As with our base model, the DM’s value function decreases with the outstanding debt \( b \), and additional units of debt reduce his payoff by a diminishing amount. Reflective of the non-zero (but limited) liability, the payoff always decreases at rates faster than \( k \), and can now become negative.

As expected, this new model retains the analytical complexity of our earlier model. To characterize the DM’s pricing policy and the impact of the nonzero liability, we thus restrict attention to the special case of asset selling discussed in \( \text{[6]} \). Furthermore, since \( Y = 1 \), we limit attention to the interesting case when \( b < \bar{p} \), so that the DM has a fractional probability of covering the debt. The following result characterizes the DM’s policy in this setting.

**Lemma A.2.** Under Assumption 5.1 and for all \( b > 0 \),

i.) there exist thresholds \( 0 \leq \bar{B}_T \leq \bar{B}_{T-1} \leq \cdots \leq \bar{B}_1 \) such that the DM’s pricing policy and value function are respectively given by:

\[
\bar{p}_t(b) = \begin{cases} 
\pi(b + \tilde{V}_{t+1}(b)), & \text{if } 0 \leq b \leq \bar{B}_t \\
\bar{p}_t, & \text{if } \bar{B}_t < b < \bar{p}
\end{cases}
\]

\[
\tilde{V}_t(b) = \begin{cases} 
\tilde{V}_{t+1}(b) + h(b + \tilde{V}_{t+1}(b)), & \text{if } 0 \leq b \leq \bar{B}_t \\
-kb + k\tilde{V}_t(0), & \text{if } \bar{B}_t < b < \bar{p},
\end{cases}
\]

where \( \pi(x) := \arg \max_{p \in \mathcal{P}} \lambda(p)(p - x) \) and \( h(x) := \max_{p \in \mathcal{P}} \lambda(p)(p - x) \).

ii.) at low debt values (i.e., \( b \leq \bar{B}_1 \)), the DM’s price \( \bar{p}_t(b) \) is higher than the revenue-maximizing price, and is increasing in the debt \( b \) and decreasing with the penalty \( k \).

iii.) at large debt values (i.e., \( b > \bar{B}_1 \)), the DM’s price \( \bar{p}_t(b) \) exactly corresponds to the revenue-maximizing price, and is thus unaffected by the debt \( b \) or the penalty \( k \).

**Proof of Lemma A.2.** The DM’s problem in period \( t \in \{1, \ldots, T\} \) is given by:

\[
\tilde{V}_t(b) = \tilde{V}_{t+1}(b) + \max_{p \in \mathcal{P}} \lambda(p)[\tilde{V}_{T+1}(b - p) - \tilde{V}_{t+1}(b)] \quad (\text{since } \tilde{V}_{T+1}(b) = \max(-b, -kb))
\]

\[
= \tilde{V}_{t+1}(b) + \max \left\{ \max_{p \geq b} f_t^d(p, b), \max_{p \leq b} f_t^m(p, b) \right\},
\]

where \( f_t^d(p, b) := \lambda(p)[p - b - \tilde{V}_{t+1}(b)] \quad f_t^m(p, b) := \lambda(p)[k(p - b) - \tilde{V}_{t+1}(b)] \).
We analyze each of the problems above separately. First, recall from Proposition 5.1 that for \( x \geq 0 \), we have \( 0 \leq \pi'(x) \leq 1 \). Therefore, since \( b + \tilde{V}_{t+1}(b) \geq 0 \) and \( b + \tilde{V}_{t+1}(b)/k \geq 0 \) by Lemma A.1, we have

\[
q^f_t(b) := \arg \max_{p \in \mathcal{P}} f^f_t(p, b) = \pi(b + \tilde{V}_{t+1}(b)), \quad q^m_t(b) := \arg \max_{p \in \mathcal{P}} f^m_t(p, b) = \pi\left(b + \frac{\tilde{V}_{t+1}(b)}{k}\right).
\]

By Lemma A.1 and since \( \pi'(x) \geq 0 \), we have that \( q^f_t(b) \) is increasing and \( q^m_t(\ell) \) is decreasing in \( b \). To determine the DM’s optimal price \( \tilde{p}_t(b) \), we distinguish two cases, depending on the sign of \( \tilde{V}_{t+1} \).

- If \( \tilde{V}_{t+1}(b) \geq 0 \), then \( b \leq q^f_t(b) \leq q^m_t(b) \), so that \( \tilde{p}_t(b) = q^f_t(b) \).
- If \( \tilde{V}_{t+1}(b) < 0 \), then \( q^m_t(b) < q^f_t(b) \). We claim that the optimal policy involves a threshold, such that \( q^f_t(b) \) is charged for \( b \) below the threshold, and \( q^m_t(b) \) is charged for \( b \) above the threshold. First, note that \( \tilde{p}_t(b) = q^f_t(b) \) for \( b \leq q^m_t(b) \), and \( \tilde{p}_t(b) = q^m_t(b) \) for \( b > q^f_t(b) \).

Let us define

\[
g_t(b) := f^m_t(q^m_t(b), b) - f^f_t(q^f_t(b), b) = k h(b + \tilde{V}_{t+1}(b)/k) - h(b + \tilde{V}_{t+1}(b)),
\]

where \( h(x) := \max_{p \in \mathcal{P}} \lambda(p)(p - x) \). To show that the policy is a threshold one, we first argue that \( g \) is monotonic increasing. This follows since \( h(x) \) is decreasing, and by Lemma A.1 \( b + \tilde{V}_{t+1}(b)/k \) is decreasing and \( b + \tilde{V}_{t+1}(b) \) is increasing in \( b \). Furthermore, \( g_t(0) < 0 \) since \( \tilde{V}_{t+1}(0) \geq 0 \), and \( g_t(b) > 0 \) if \( b > q^f_t(b) \) (which holds at large \( b \), since \( \pi' \leq 1 \)). Thus, there exists a threshold \( \tilde{B}_t \geq 0 \) given by:

\[
g_t(\tilde{B}_t) = 0
\]

price above such that the DM’s pricing policy is exactly given by

\[
\tilde{p}_t(b) = \begin{cases} 
q^f_t(b), & \text{if } b \leq \tilde{B}_t \\
q^m_t(b), & \text{if } b > \tilde{B}_t.
\end{cases}
\]

We also claim that \( \tilde{B}_t \geq \tilde{B}_{t+1}, \forall t \in \{1, \ldots, T - 1\} \). To see this, note that \( g_t(b) \leq g_{t+1}(b), \forall b \geq 0 \), since \( \tilde{V}_{t+1}(b) \geq \tilde{V}_{t+2}(b) \), by Lemma A.1. Thus, since \( g_t \) and \( g_{t+1} \) are increasing, \( \tilde{B}_t \geq \tilde{B}_{t+1} \).

To complete the proof, note that the expression for \( \tilde{p}_t(b) \) and \( \tilde{V}_t(b) \) for the case \( b \leq \tilde{B}_t \) follows from the arguments above. For \( b > \tilde{B}_t \), we prove by induction that for any \( t \in \{1, \ldots, T\} \),

\[
\tilde{V}_t(b) = -kb + k\tilde{V}_t(0), \forall b > \tilde{B}_t.
\]

First, note that this trivially holds at \( t = T + 1 \) with \( \tilde{B}_{T+1} := 0 \), since \( \tilde{V}_{T+1}(b) = -kb, \forall b \geq 0 \). Assume
by induction that the property also holds at time $t + 1$. We have:

$$
\forall b > \tilde{B}_t : q_t^{in}(b) = \pi(b + \tilde{V}_{t+1}(b)/k) = (\text{since } \tilde{B}_t \geq \tilde{B}_{t+1}) = \pi\left(b + \frac{-kb + k\tilde{V}_{t+1}(0)}{k}\right) = \pi(\tilde{V}_{t+1}(0)) = p_t^*.
$$

Replacing this in the expression for $\tilde{V}_t(b)$, we obtain

$$
\tilde{V}_t(b) = -kb + k\tilde{V}_{t+1}(0) + kh(\tilde{V}_{t+1}(0)) = -kb + k\tilde{V}_t(0),
$$

which completes the induction and the proof of parts (i) and (iii).

To prove (ii), it can be readily seen that $\tilde{p}_t(b) = q_t(b)$ is increasing in $b$ for $b \leq \tilde{B}_t$. Since $\tilde{V}_t(b)$ is decreasing with $k$ by Lemma A.1, so is

The result suggests that the presence of a non-zero liability carries certain nontrivial implications on the DM’s pricing policy, depending on the required debt repayment $b$. Two regimes emerge. When the debt is not too large, the DM charges prices that increase with the debt, and exceed the revenue-maximizing price. Qualitatively, this exactly corresponds to the main distortion documented in our base model, whereby the DM shifts risk by charging high(er) prices. However, prices now decrease with the magnitude of $k$, which confirms the intuition that transferring more liability to the DM successfully reduces his risk shifting incentives.

Interestingly, when debt is sufficiently high, the DM’s price exactly equals the revenue-maximizing price. This occurs when the debt exceeds a certain time-dependent threshold $\tilde{B}_t$, and the switch is sudden: the DM’s price exhibits a downward jump, from a value that exceeds $\tilde{B}_t$ to $p_t^* < \tilde{B}_t$. Once the switch occurs, the DM then continues to follow the revenue-maximizing policy for the remaining planning horizon (since $\tilde{B}_t \geq \tilde{B}_\tau, \forall \tau \geq t$). Qualitatively, in this regime the DM effectively acts as if he were unable to repay the debt, and therefore relies on a strategy that seeks to minimize his losses, or equivalently maximize revenues. (In fact, since $b > p_t^*$ holds here, following the revenue-maximizing policy actually yields a “self-fulfilling prophecy,” guaranteeing bankruptcy.) This regime is new, and is entirely caused by the non-zero liability, which acts as a severe threat for the DM. We note that the debt required to generate this regime is very high: $\tilde{B}_t > \tilde{V}_t(0)$, so that the debt repayment would exceed the expected revenues that could be generated over the remaining horizon.$^{15}$

Regarding the time-dynamics of the pricing policy, similar arguments to those in Proposition 5.2 can be used to confirm that when the DM still relies on the risk-shifting strategy, he would reduce prices over time, but the markdowns would always be lower than the revenue-maximizing ones, and would decrease with the debt. Thus, the pricing distortions and efficiency losses would again compound

$^{15}$Since loans might not be issued under such unfortunate circumstances, it would be interesting to study whether this regime survives in equilibrium, once the debt repayment is endogenized. We leave this interesting analysis for future research.
over time, leading to a spiraling behavior.

To summarize our findings, the presence of the non-zero (limited) liability successfully reduces some of risk-shifting incentives driving the DM’s decisions. However, unless the entire liability is transferred to the DM (i.e., \( k = 1 \)), the pricing distortions and the associated efficiency losses persist, albeit with a diminished magnitude.

B Numerical Experiments on Multi-unit Case

In our experiments, we considered several demand functions that all yielded consistent findings—including ones that do not satisfy the requirements in Assumption 5.1. Below, we present the case of a logit demand function \( \lambda(p) = \frac{e^{1-p}}{1+e^{1-p}} \).

**Pricing policy.** In Figure 6, we depict the DM’s price \( p^\dagger_{T-4} \) and the revenue-maximizing price \( p^\star_{T-4} \) as functions of the outstanding debt, for two starting inventory levels, \( y = 5 \) and \( y = 3 \). Consistent

![Figure 6](chart.png)

Figure 6: The DM’s pricing policy structure. The demand curve is \( \lambda(p) = \frac{e^{1-p}}{1+e^{1-p}} \), and the time horizon is \( T = 5 \).
with our findings in §4, we observe that the DM’s prices are piecewise increasing in the debt; but with many units in inventory and many periods, there are now multiple discontinuity points, as we alluded to in our earlier discussion.

In contrast with the two-period case depicted in Figure 1, it is no longer possible to exactly associate strategies that rely on selling $k$ units to cover the debt with prices that lie between $b/(k-1)$ and $b/k$. The reason is that, with more than two periods to go, strategies become increasingly complex, as they also need to account for multiple future contingent strategies that possibly rely on more or less units, depending on sales realizations. This precisely illustrates how the underlying combinatorial structure dramatically increases the complexity of the DM’s pricing policies in the general case, defying an analytical characterization.

In Figure 7, we depict the time-evolution of prices on sample paths where no sale occurs, for different levels of debt. Similar to the analytical results in §4 the DM’s policy always entails slower markdowns than the revenue-maximizing policy, and actually may prescribe markups. Furthermore, price distortions increase monotonically over time. However, due to the discontinuities in the DM’s price as a function of debt, price distortions are not monotonic in the debt level.

![Figure 7: (a) Price evolution for the revenue-maximizing policy and the DM’s policy under $\lambda(p) = e^{1-p}/(1 + e^{1-p})$, for $T = 5$ and $y = 3$, and for different levels of debt: ‘low’ $b_l = p^*_T/2$, ‘medium’ $b_m = p^*_T$, and ‘high’ $b_{h} = 1.5 \times p^*_T$. (b) Evolution of price distortions over time.](image)

Finally, Figure 8 depicts all possible sample paths corresponding to the evolution of the DM’s price over time, as well as the corresponding expected prices. We observe that when inventory is ample, as
Figure 8: Efficiency loss evolution under $\lambda(p) = e^{1-p}/(1+e^{1-p})$, for $T = 5$, $y = 5$ and $b = 1.75 \times p_T^*$. Each edge in the tree depicts a possible evolution of $p_t^*(B_t, \mathcal{Y}_t)$. The width of the edge is proportional to the probability of the edge. The dots on the dashed line depict $E[p_t^*(B_t, \mathcal{Y}_t)]$.

in the case depicted, the expected price increases over time. Furthermore, the range of possible price values also significantly expands, particularly on the paths on which few sales to date occur, where the DM becomes more and more aggressive with pricing decisions as time progresses.

Efficiency losses. In Figure 9 we explore the evolution of $L_t$ and the various paths that may be taken. In particular, consistent with all our analytical results thus far, we observe that the expected efficiency loss $L_t$ is increasing over time, and the “variability” of $L_t$ also increases over time, with a wide range of possible outcomes in the last period $t = 5$.

C Endogenizing Debt

In this section, we formulate a model where debt is endogenously determined. We base our analysis on the setup in §4, but assume that the DM is no longer endowed with inventory; instead he can purchase an inventory bundle of $Y = 2$ units at a cost of $c$, before the start of the selling season. The DM has limited available equity $\bar{e}$ that he can use to pay for the purchase. To exclude uninteresting cases, we assume that the optimal revenues that could be generated from this purchase would exceed the costs, i.e., $J^* \geq c$, and that the DM’s available equity is insufficient to cover the purchase, i.e., $\bar{e} < c$.

If the DM decides to proceed with the purchase by investing $e \leq \bar{e}$ of his own equity, he may be able to obtain a loan for the remaining amount $c - e$ from a lender. We make the standard assumption that
Each edge in the tree depicts a possible evolution of $L_t$. The width of the edge is proportional to the probability of the edge. The dots and dashed lines depicts $L_t$.

Figure 9: Efficiency loss evolution under $\lambda(p) = e^{1-p}/(1 + e^{1-p})$, for $T = 5$, $y = 5$, and $b = 1.75 \times p_T^*$. Each edge in the tree depicts a possible evolution of $L_t$. The width of the edge is proportional to the probability of the edge. The dots and dashed lines depicts $L_t$.

the debt market is perfectly competitive [Tirolo (2006, page 115)]. That is, when the borrowed amount is $c-e$, the lender would set the required repayment $B$ (of principal plus interest) at the loan’s maturity so as to break even. More precisely, using the terminology and notation of §2, the lender would anticipate that if the required repayment were set to $B$, the borrower would follow a pricing policy $p^*(B, Y)$, which would yield an expected repayment (i.e., a debt value) of $D(B, Y) := \mathbb{E}[\min\{B, R(p^*(B, Y))\}]$. The lender would then set $B$ so that

$$D(B, Y) = c - e.$$  \hfill (C-2)

Note that this equation may not have a solution $B$ for particular values of $e$, in which case the lender would be unwilling to extend a loan. Intuitively, this could occur if the expected revenues are low and
the borrowed amount \( c - e \) is high.

By not pursuing the inventory purchase, the DM would achieve a profit of zero. Alternatively, by injecting equity \( e \), borrowing \( c - e \), and facing a repayment of \( B \), his profit would be given by \( V_1(B, Y) - e \). His decision problem before the start of the selling season can then be formulated as:

\[
\max \left\{ 0, \max_{0 \leq e \leq \bar{e}} D(B, Y) = e - e \right\}
\]

(C-3)

Several outcomes are possible. If the inner optimization problem is infeasible (i.e., when (C-2) is infeasible for any \( e \)), we say that lenders refuse to lend. If the inner optimization is feasible, but has a negative optimal value, we say that the DM finds the purchase unprofitable. In both of these cases, the DM generates zero profit. Finally, if the optimal value in (C-3) is strictly positive, we say that the inventory purchase goes through.

C.1 One-Period Case

We first analyze the one-period case under a linear demand model, i.e., \( T = 1 \) and \( \lambda(p) = \alpha - \beta p \), for some \( \alpha \in (0, 1) \) and \( \beta > 0 \). Using our analysis from the proof of Proposition 4.1 (with \( b \equiv B \) to retain the familiar notation), it can readily be seen that the lender’s expected collected payment is equal to

\[
D(b) = \lambda(p_T^*(b, 1))b = \frac{1}{2}b\lambda(b).
\]

The break-even equation (C-2) yields 

\[
\beta b^2 - \alpha b + 2(c - e) = 0.
\]

Consequently, lenders refuse to lend unless

\[
e \geq c - \frac{\alpha^2}{8\beta},
\]

where the right-hand side can be interpreted as the minimum equity level that lenders expect the DM to inject. When the DM injects more than this minimum level, lenders set the repayment amount to

\[
b(e) = \frac{\alpha - \sqrt{\alpha^2 - 8(c - e)\beta}}{2\beta},
\]

and the DM’s profit can be expressed as

\[
V_1(b(e), 1) - e = \frac{\alpha \left( \alpha + \sqrt{\alpha^2 - 8(c - e)\beta} \right) - 4(c + e)\beta}{8\beta}.
\]

It can be readily checked that \( V_1(b(e), 1) - e \) is increasing in \( e \). Thus, the DM injects all available equity \( \bar{e} \). The inventory purchase is then profitable for the DM as long as \( V_1(b(\bar{e}), 1) - \bar{e} \geq 0 \).
Figure 10: Outcomes for $T = 1$, $c = 1$, $\alpha = 1$ and different values of $\beta$ and $\bar{e}$. Cyan means that the inventory purchase goes through; Yellow means that lenders refuse to lend; Red means that the DM finds the purchase unprofitable.

To summarize our findings for the case of $T = 1$: the DM always injects $\bar{e}$ and achieves a profit of $V_1(b(\bar{e}), 1) - \bar{e}$, unless his available equity $\bar{e}$ is low. In particular, if $\bar{e} < c - \frac{\alpha^2}{8\beta}$, the lenders refuse to lend and if $V_1(b(\bar{e}), 1) - \bar{e} < 0$, the DM finds the purchase unprofitable.

We illustrate the above cases graphically in Figure 10 for fixed values of $c = 1$ and $\alpha = 1$, and all possible values of $\bar{e}$ and $\beta$, namely $0 \leq \bar{e} < 1$ and $0 < \beta \leq \frac{1}{4}$

C.2 Two-Period Case

We now consider the two period case we analyzed in §4, i.e., $T = 2$ and $Y = 2$ and $\lambda(p) = \alpha - \beta p$, for some $\alpha \in (0, 1]$ and $\beta > 0$. In this case, the break-even equation (C-2) becomes a fourth-order equation.
polynomial equation in $b$, when the DM prices so as to cover the debt in one period. In case he prices $q_m$ so as to cover the debt in two periods, it also involves square root terms as in the definition of $q_m$, see (E-6). Consequently, we were unable to obtain closed-form expressions and tackle the DM’s problem (C-3) analytically. Instead, we performed a numerical study for fixed values of $c = 2$ and $\alpha = 1$, where we considered various possible values for $0 \leq \bar{e} < 2$ and $0 < \beta \leq \frac{1}{4}$, Figure (a) depicts the outcomes. We distinguish the following cases, depending on the value of $\beta$:

- For small values of $\beta$’s (namely $\leq 0.133$), lenders set a low enough repayment amount $b$, so that the DM prices to cover it in one period, for any value of $\bar{e}$.

- For intermediate values of $\beta$’s (namely $0.133 \leq \beta \leq 0.211$), if $\bar{e}$ is too low, lenders might refuse to lend (yellow region). If $\bar{e}$ is slightly higher, then lenders set a high enough repayment amount $b$, and the DM prices to cover it in two periods (magenta region). If $\bar{e}$ is even higher, then lenders set a lower repayment amount $b$, and the DM prices to cover it in one period (cyan region).

- For high values of $\beta$’s (namely $\geq 0.211$), if $\bar{e}$ is too low, lenders refuse to lend (yellow region). If $\bar{e}$ is slightly higher, then lenders set a high enough repayment amount $b$, and the DM finds the purchase unprofitable (red region). If $\bar{e}$ is even higher, then lenders set a lower repayment amount $b$, and the DM prices to cover it in one period (cyan region).

These results are in line with our findings in the one period case above, and bear a similar interpretation. Importantly, however, they demonstrate that both pricing strategies of covering the debt in one or two periods (discussed in §4) could arise. Put differently, the discontinuity in the DM’s pricing strategy we elicited in §4 (see Figure 11) could arise. This is further illustrated in Figure 12 where we plot the optimal price $p^*_{T-1}(2)$ (green) and the DM’s price $p^*_{T-1}(b, 2)$ (blue) for fixed $\bar{e} = 0.4$, as we vary $\beta$. We observe that for high enough values of $\beta$, the repayment amount increases to the extent that the DM switches his pricing strategy, resulting in a discontinuity point. The dashed line corresponds to the repayment amount $b$, and helps to highlight the strategy switch.

An important feature that arises in the two period case is that the DM’s profit $V_1(b, 2) - e$ is no longer monotonic in the equity injected $e$. In other words, the DM may find it profitable to only invest a fraction of his initial equity, which would lead to larger profits than choosing whether to invest the entire equity.

To appreciate this point, it is useful to compare Figure 11(a) with Figure 11(b). In Figure 11(a), the DM can choose what amount of equity to inject, i.e., $e \in [0, \bar{e}]$. In Figure 11(b), the DM is only allowed to choose whether to inject all his available equity, i.e., $e \in \{0, \bar{e}\}$. The critical difference between the two figures occurs at intermediate values of $\beta$, namely $0.15 \leq \beta \leq 0.211$. In this range, there are particular values of the initial equity $\bar{e}$ such that the inventory purchase goes through when
Figure 11: Outcomes for $T = 2$, $Y = 2$, $c = 2$, $\alpha = 1$ and different values of $\beta$ and available equity $\bar{e}$. In (a), the DM chooses how much equity to invest ($e \in [0, \bar{e}]$). In (b), the DM chooses whether to invest all equity or not ($e \in \{0, \bar{e}\}$). Cyan (Magenta) means that the inventory purchase goes through, and the DM prices so as to cover the debt in one (two) periods; Yellow means that lenders refuse to lend; Red means that the DM finds the purchase unprofitable.

Figure 12: Optimal price $p^*_{T-1}(2)$ (green) and DM’s price $p^\dagger_{T-1}(b,2)$ (blue) as a function of $\beta$, for $T = 2$, $c = 1$, $\alpha = 1$, $\bar{e} = 0.4$. The dashed line depicts the associated repayment amount $b$. 
the DM can inject a fraction of the equity—corresponding to the magenta region in Figure 11(a)—but the inventory purchase does not go through when the DM only chooses whether to inject the entire equity—corresponding to the red region in Figure 11(b). To further illustrate this, Figure 13 considers the case $\beta = 0.175$, and plots the optimal equity $e \in [0, \bar{e}]$ that the DM would invest as a function of the available equity $\bar{e}$. Note that three regions emerge:

- For $\bar{e} < \frac{1}{6}$, the DM injects all his equity, but lenders refuse to lend (solid yellow line).
- For $\bar{e} > \frac{1}{2}$, the DM injects all his available equity, $e = \bar{e}$, and the debt he raises induces him to price so as to cover the debt in one period (dashed cyan line).
- For $\frac{1}{6} \leq \bar{e} \leq \frac{1}{2}$, the DM chooses to inject only $e = \frac{1}{6}$, i.e., he does not use all available equity and $e < \bar{e}$. In this case, the amount of debt he raises induces him to price so as to cover the debt in two periods (dotted magenta line).

This highlights the same phenomenon as our earlier discussion: at intermediate values of equity—when $\frac{1}{6} \leq \bar{e} \leq \frac{1}{2}$—the DM would prefer not retain some of his equity, and only invest $e = \frac{1}{6}$. To understand this preference, note that investing a lower equity requires the DM to raise a high debt; in turn, this higher debt allows him to credibly pre-commit to a pricing strategy that clears the debt in two periods.
Such a strategy is less risky, by relying on lower prices, as we discussed in §4. Consequently, this allows the lenders to charge a lower interest, with the DM eventually benefiting.

This discussion illustrates the strategic role that debt could play in our setting. Thus, the appropriate selection of capital structure leads to a better alignment of incentives, so that the DM implements an ex-ante higher-value pricing policy. This strategic use of debt as a pre-commitment mechanism is akin to other papers in the finance literature, e.g., see Titman (1984). More broadly, a positive effect of debt on firm value has also been documented in several other papers in the corporate finance literature, see, e.g., Brander and Lewis (1986) and Chemla and Faure-Grimaud (2001).

D Proofs for Section 3

Proof of Lemma 3.1. i.) If \( y = 0 \), \( V_t(b, y) = (-b)^+ \) is clearly convex, decreasing in \( b \). If \( y \geq 1 \), \( V_{T+1}(b, y) \) is similarly convex, decreasing in \( b \). Assuming that so is \( V_{t+1}(b, y) \) for some \( t = 1, \ldots, T \), then the recursion (2) yields that

\[
V_t(b, y) = \max_{p \in \mathcal{P}} \left\{ \lambda(p)V_{t+1}(b - p, y - 1) + (1 - \lambda(p))V_{t+1}(b, y) \right\}.
\]

For any \( p \in \mathcal{P} \), the maximand above is convex, decreasing in \( b \) because it is a convex combination of two such functions. Thus, \( V_t(b, y) \) is also convex, decreasing in \( b \).

To show that \( V_t(b, y) \) is decreasing in \( t \), note that it is immediate for \( y = 0 \), since \( V_t(b, y) = (-b)^+ \). For \( y \geq 1 \), by the recursion (2) it suffices to show that \( V_t(b, y) \leq V_t(b - p, y - 1) \), for some \( p \in \mathcal{P} \), for all \( t = 1, \ldots, T + 1 \). We shall show it for \( p = \overline{p} \). At \( T + 1 \), we get \( V_{T+1}(b, y) = (-b)^+ \leq (\overline{p} - b)^+ = V_{T+1}(b - \overline{p}, y - 1) \). Suppose that it is true at \( t + 1 \). Then, for \( y \geq 2 \)

\[
V_t(b - \overline{p}, y - 1) = \max_{p \in \mathcal{P}} \left\{ \lambda(p)V_{t+1}(b - \overline{p} - p, y - 2) + (1 - \lambda(p))V_{t+1}(b - \overline{p}, y - 1) \right\}
\geq \max_{p \in \mathcal{P}} \left\{ \lambda(p)V_{t+1}(b - p, y - 1) + (1 - \lambda(p))V_{t+1}(b, y) \right\}
= V_t(b, y).
\]

A similar argument can be employed for \( y = 1 \).

ii.) For \( b \leq 0 \), it suffices to show that \( V_t(b, y) = -b + V_t(0, y) \), \( t = 1, \ldots, T + 1 \). At \( T + 1 \), \( V_{T+1}(b, y) = \)
\((-b)^+ = -b = -b + V_{T+1}(0, y)\). Suppose now that it is true at some \(t + 1\). Then, (2) yields

\[
V_t(b, y) = \max_{p \in P} \{\lambda(p) (p - b + V_{t+1}(0, y - 1)) + (1 - \lambda(p)) (-b + V_{t+1}(0, y))\}
\]

\[
= -b + \max_{p \in P} \{\lambda(p)V_{t+1}(-p, y - 1) + (1 - \lambda(p))V_{t+1}(0, y)\}
\]

\[
= -b + V_t(0, y).
\]

For \(b > 0\), it suffices to show \(\text{iii.}\) below.

\(\text{iii.}\) At \(T + 1\), or for \(y = 0\), \(V_t(b, y) = \frac{\partial}{\partial b}V_t(b, y) = 0\) and the probability of covering the debt is 0.

For \(y \geq 1\), at \(T\) the DM generates no revenue and fails to cover the debt with probability 1, unless he charges \(p_T^+(b, y) \geq b\). Consequently, if \(\overline{p} < b\), we have that \(p_T^+(b, y) \leq \overline{p} < b\) and \(V_T(b, y) = \frac{\partial}{\partial b}V_T(b, y) = 0\). Otherwise, if \(\overline{p} \geq b\) the DM charges \(p_T^+(b, y) \in [b, \overline{p}]\) and covers the debt only if he makes a sale, i.e., with probability \(\lambda(p_T^+(b, y))\). Also, by the Envelope Theorem

\[
-\frac{\partial}{\partial b}V_T(b, y) = -\frac{\partial}{\partial b}\{\lambda(p)(p - b)\}\bigg|_{p=p_T^+(b, y)} = \lambda(p_T^+(b, y)).
\]

Thus, \(V_T\) is differentiable with respect to \(b\) and \(-\frac{\partial}{\partial b}V_T(b, y)\) is the probability of covering the debt. Assuming that \(V_{t+1}\) has the same properties, we can apply the Envelope Theorem to the recursion for \(t\) to obtain

\[
-\frac{\partial}{\partial b}V_t(b, y) = -\frac{\partial}{\partial b}\{\lambda(p)V_{t+1}(b - p, y - 1) + (1 - \lambda(p))V_{t+1}(b, y)\}\bigg|_{p=p_T^+(b, y)}
\]

\[
= -\lambda(p_T^+(b, y)) \frac{\partial V_{t+1}}{\partial b}(b - p_T^+(b, y), y - 1) - (1 - \lambda(p_T^+(b, y))) \frac{\partial V_{t+1}}{\partial b}(b, y).
\]

By the law of total probability (applied depending on whether a sale occurred at \(t\)), it follows that

\[-\frac{\partial}{\partial b}V_t(b, y)\] is the probability of covering the debt at \(t\).

\(\square\)

\section*{E Proofs for Section 4}

\textbf{Proof of Proposition 4.1.} To facilitate exposition, we first attend to parts \(\text{ii.}\) and \(\text{iii.}\).

\(\text{ii.}\) Since there is only one period left \(p_T^+(b, y) = p_T^+(b, 1)\), which is shown to be increasing in \(b\) in the analysis of the one-unit case, Proposition 5.1

\(\text{iii.}\) The fact that \(p_{T-1}^+(b, 1)\) is increasing in \(b\) follows again from Proposition 5.1

Next, we analyze \(p_{T-1}^+(b, 2)\), which involves the solution of the optimization problem in (2), for \(t = T - 1\) and \(y = 2\). To this end, a characterization of \(V_T(b, y)\) is required.

Consider first the case of \(b \leq \frac{\lambda}{p}\). Clearly, \(V_T(b, 0) = 0\). For \(y \geq 1\), \(V_T(b, y) = V_T(b, 1)\). Thus,
by the analysis of the one-unit case, for \( b \geq 0 \) \((F-22)\) yields that \( p^T_T(b, 1) = \frac{1}{2} \left( \frac{\alpha}{\beta} + b \right) \). Substituting into \((F-25)\), we get that \( V_T(b, 1) = \frac{\lambda^2(b)}{4\eta} \). For \( b < 0 \), as we argued in the proof of Lemma 3.1 ii.), \( V_T(b, 1) = -b + V_T(0, 1) \). Summarizing then,

\[
V_T(b, y) = \begin{cases} 
\frac{\lambda^2(b)}{4\eta} & b \geq 0, \ y \geq 1 \\
-b + V_T(0, 1) & b < 0, \ y \geq 1 \\
0 & y = 0.
\end{cases}
\]

Substituting for \( V_T(b, y) \) using the above equation, the maximand of \((2)\) for \( t = T - 1 \) and \( y = 2 \) becomes

\[
V_T(b, 1) + \begin{cases} 
f_\ell(p) := \lambda(p) (p - b + V_T(0, 1) - V_T(b, 1)) & p \in (b, \frac{\alpha}{\beta}] \\
f_m(p) := \lambda(p) (V_T(b - p, 1) - V_T(b, 1)) & p \in [0, b].
\end{cases}
\]

We next analyze the problems \( \max_{p \in (b, \frac{\alpha}{\beta}]} f_\ell(p) \) and \( \max_{p \in [0, b]} f_m(p) \) separately. We show that they have unique optimal solutions, denoted by \( p_\ell(b) \) and \( p_m(b) \), and optimal values denoted by \( F_\ell(b) \) and \( F_m(b) \) respectively. If we let \( \Delta F(b) := F_\ell(b) - F_m(b) \) we have

\[
p^T_{T-1}(b, 2) = \begin{cases} 
p_\ell(b) & \text{if } \Delta F(b) \geq 0 \\
p_m(b) & \text{otherwise.}
\end{cases}
\]

• For \( \max_{p \in (b, \frac{\alpha}{\beta}]} f_\ell(p) \), note that \( f_\ell \) is concave, quadratic attaining its maximum at

\[
q_\ell(b) := \frac{1}{2} \left( b + V_T(b, 1) - V_T(0, 1) + \frac{\alpha}{\beta} \right).
\]

The value \( q_\ell(b) \), which is quadratic in \( b \), is bigger than \( b \) if and only if \( b \leq b_\ell := \frac{\alpha}{\beta} - \sqrt{\frac{\alpha^2 + 4}{\beta}} \). Thus,

\[
p_\ell(b) = \begin{cases} 
q_\ell(b) & b \in [0, b_\ell] \\
b & b \in [b_\ell, \frac{\alpha}{\beta}].
\end{cases}
\]

• For \( \max_{p \in [0, b]} f_m(p) \), note that \( f_m \) is cubic. By solving the quadratic equation \((f_m)'(p) = 0\) we obtain the stationary points \( \frac{2\beta b - \alpha + \sqrt{4\beta^2 b^2 - 10\alpha \beta b + 7\alpha^2}}{3\beta} \). It can be readily checked that for \( b \in [0, \frac{\alpha}{\beta}] \) the point

\[
q_m(b) := \frac{2\beta b - \alpha + \sqrt{4\beta^2 b^2 - 10\alpha \beta b + 7\alpha^2}}{3\beta}
\]

is non-negative and a local maximizer, whereas the other point is non-positive and a local minimizer. Thus, \( f_m \) is increasing in \([0, q_m(b)]\) and decreasing in \([q_m(b), \infty)\). Since we are interested
in \( p \in [0, b] \), note that \( q_m(b) < b \Leftrightarrow 3\beta^2b^2 - 12\alpha\beta b + 6\alpha^2 < 0 \Leftrightarrow b > b_m := (2 - \sqrt{2})\frac{a}{\beta} \). Combining the last two observations,

\[
p_m(b) = \begin{cases} b & b \in [0, b_m] \\ q_m(b) & b \in (b_m, \frac{a}{\beta}] \end{cases}.
\]

We use these results to simplify (E-4). In particular, we consider different values for \( q \) which is a local maximizer. It can be readily checked that

- For \( 0 \leq b \leq b_m \) we have \( \Delta F(b) = F_\ell(b) - F_m(b) = f_\ell(q_\ell(b)) - f_m(b) > f_\ell(b) - f_m(b) = 0 \).
- For \( b_m < b_\ell \leq b \leq \frac{a}{\beta} \), we similarly get \( \Delta F(b) < 0 \).
- For \( b_m < b < b_\ell \) we have that \( \Delta F(b) = f_\ell(q_\ell(b)) - f_m(q_m(b)) \). Using tedious algebra, one can show that \( (\Delta F)' \) is increasing in \( b \) and negative at \( b_\ell \). Thus, \( \Delta F \) is decreasing in \( b \). Since \( \Delta F(b_m) > 0 \) and \( \Delta F(b_\ell) < 0 \), there exists a unique \( \hat{b} \in (b_m, b_\ell) \) such that \( \Delta F(b) \geq (\leq)0 \) for \( b \leq (\geq)\hat{b} \).

By combining the above we obtain that

\[
p_{T-1}^\dagger(b, 2) = \begin{cases} q_\ell(b) & b \in [0, \hat{b}] \\ q_m(b) & b \in (\hat{b}, \frac{a}{\beta}] \end{cases}.
\]

For \( b \in (\frac{a}{\beta}, 2\frac{a}{\beta}] \), we have that \( V_T(b, y) = 0 \) and thus

\[
p_{T-1}^\dagger(b, 2) \in \arg\max_{p \in [0, \frac{a}{\beta}]} \{\lambda(p)V_T(b - p, 1)\}.
\]

The objective function above, denoted by \( f_h \), evaluates to 0 for \( b - p > \frac{a}{\beta} \). Thus, we consider only prices \( p \geq b - \frac{a}{\beta} \). Then, \( f_h \) is cubic. Its stationary points are \( -\frac{\lambda(b)}{\beta} \), which is a local minimizer, and

\[
q_h(b) := \frac{b}{3} + \frac{\alpha}{3\beta},
\]

which is a local maximizer. It can be readily checked that \( q_h(b) \in [b - \frac{a}{\beta}, \frac{a}{\beta}] \) and as a result \( p_{T-1}^\dagger(b, 2) = q_h(b) \) for \( b \in (\frac{a}{\beta}, 2\frac{a}{\beta}] \).

Having characterized the three pricing regimes for \( p_{T-1}^\dagger(b, 2) \), it suffices to show that \( q_\ell, q_m, \) and \( q_h \) are all increasing. In particular, \( q_\ell \) is increasing by Lemma 3.1 vi). To show that \( q_m \) is increasing note that

\[
\frac{dq_m}{db}(b) = \frac{2}{3} + \frac{4\beta b - 5\alpha}{3\sqrt{4\beta^2b^2 - 10\alpha\beta b + 7\alpha^2}},
\]

\[
\frac{d^2q_m}{db^2}(b) = \frac{\alpha^2\beta}{(4\beta^2b^2 - 10\alpha\beta b + 7\alpha^2)^{\frac{3}{2}}} > 0.
\]

\(^{17}\)It can be readily checked that \( b_m < b_\ell \) for \( \alpha \leq 1 \).
Thus, $\frac{dn_h}{db}(b) \geq \frac{dn_h}{db}(0) = \frac{2}{b} - \frac{5}{3\sqrt{b}} > 0$. $q_h$ is linear and clearly increasing.

i.) By parts ii.) and iii.), we get that for $b > 0$, $p_{T-1}^\dagger(b, y) > p_T^\dagger(0, y) = \hat{p}_T^*(y)$ and $p_{T-1}^\dagger(b, 1) > p_{T-1}^\dagger(0, 1) = \hat{p}_T^*(1)$. For $p_{T-1}^\dagger(b, 2)$, we consider the three cases, depending on $\beta$. For $0 < b \leq \tilde{b}$, $p_{T-1}^\dagger(b, 2) = q_\ell(b) > q_\ell(0) = p_{T-1}^\dagger(0, 2) = \hat{p}_T^*(2)$. For $\tilde{b} < b \leq \frac{\beta}{\alpha}$, $p_{T-1}^\dagger(b, 2) = q_m(b) \geq q_m(0) = \frac{\sqrt{7} - 1}{3} > \frac{\alpha}{\beta} = p_{T-1}^\dagger(2)$.

Finally, for $\frac{\alpha}{\beta} < b \leq 2\frac{\alpha}{\beta}$, note that $p_{T-1}^\dagger(b, 2) = q_h(b) \geq q_h(\frac{\alpha}{\beta}) = q_m(\frac{\alpha}{\beta}) > p_{T-1}^\dagger(2)$ since $q_h$ is increasing, and the proof is complete.

We also note that the DM’s price $p_{T-1}^\dagger(b, 2)$ is higher than the debt $b$ for $0 \leq b \leq \tilde{b}$, and lower than the debt for $\tilde{b} < b \leq 2\tilde{b}$. To see this, note that for $b \leq \tilde{b}$ we have that $p_{T-1}^\dagger(b, 2) = q_\ell(b)$, which by its definition is greater than $b$. For $b \in (\tilde{b}, \frac{\alpha}{\beta})$ we have that $p_{T-1}^\dagger(b, 2) = q_m(b)$, which by its definition is less than $b$. Finally, our claim follows for $b > \frac{\alpha}{\beta}$ since prices are less than $\frac{\alpha}{\beta}$.

**Proof of Lemma 4.1.** We consider three cases, depending on $b$.

**Case 1.** When $b \in [0, \tilde{b}]$, the DM’s pricing policy in period $T − 1$ is given by $p_{T-1}^\dagger(b) = q_\ell(b)$, where $q_\ell$ is given by (E25) (refer to the proof of Proposition 4.1). Expressing $L_1 = 1 - \frac{J_{T-1}^\dagger}{J_{T-1}}$, it can be readily checked that:

$$
\begin{align*}
\frac{\partial^2 L_1}{\partial b^2} &= \frac{\alpha^2[(4 + 3\beta b)^2 - 12(1 + \beta b)\alpha + 2\alpha^2]}{8\alpha^2} \geq 0 \\
\frac{\partial^2 L_1}{\partial \alpha^2} &= \frac{\beta^2 b^2[\beta b(48 + 9\beta b - 8\alpha) + 24(4 - \alpha)]}{16\alpha^4} \geq 0 \\
\frac{\partial^2 L_1}{\partial \beta^2} &= \frac{b^2[(4 + 3\beta b)^2 - 12(1 + \beta b)\alpha + 2\alpha^2]}{8\alpha^2} \geq 0 \\
\frac{\partial L_1}{\partial b} &= \frac{\beta^2 b[3\beta^2 b^2 + 6\beta b(2 - \alpha) + 2(4 - \alpha)(2 - \alpha)]}{8\alpha^2} \geq 0 \\
\frac{\partial L_1}{\partial \alpha} &= \frac{-\beta^2 b[\beta b + 4(4 - \alpha) + 4(8 - 3\alpha)]}{16\alpha^3} \leq 0 \\
\frac{\partial L_1}{\partial \beta} &= \frac{\beta^2 b[3\beta^2 b^2 + 6\beta b(2 - \alpha) + 2(4 - \alpha)(2 - \alpha)]}{8\alpha^2} \geq 0.
\end{align*}
$$

The inequalities readily follow in each case, by recognizing that $\alpha, \beta, b \geq 0$ and $\alpha \leq 1$. This shows that $L_1$ is component-wise convex in $b, \alpha,$ and $\beta$, and has the desired monotonicity.

To prove the lower bound on $L_1$, we let $y := \frac{b}{\beta^2}$, and define

$$
g_\ell(y, \alpha) := \frac{L_1}{y^2} = \frac{128 - 32(3 - y)\alpha + (4 - y)(4 - 3y)\alpha^2}{512}.
$$
Note that $g_\ell$ is convex and quadratic in $y$, reaching its minimum at $\frac{8(2-\alpha)}{3\alpha} < 0$. Therefore,

$$g_\ell(y, \alpha) \geq g_\ell(0, \alpha) = \frac{(4-\alpha)(2-\alpha)}{32}.$$  

The latter function is convex and quadratic in $\alpha$, reaching its minimum at $\alpha = 3$. Thus, it is decreasing on $[0, 1]$, so we can conclude that $L_1/y^2 = g_\ell(y, \alpha) \geq g_\ell(0, 1) = \frac{3}{32} \approx 0.093$.

**Case 2.** When $b \in (\hat{b}, \alpha \beta]$, by the proof of Proposition 4.1 we have that $p^T_{T-1}(b) = q_m(b)$, which is given by (E-6). The efficiency loss can be written as

$$L_1(b, \alpha, \beta) = \frac{1}{54\alpha^2}
\left[ 123\beta^2b^2 + 16\beta^3b^3 - 240\alpha\beta b - 28\beta^3b^2\alpha + 106\beta\alpha^2 - 2\beta^2\alpha^2 b + 18\beta\alpha^3 +
(41\beta^2 b + 8\beta^3 b^2 - 40\beta\alpha - 4\beta^2\alpha b - 9\beta\alpha^2)\sqrt{4\beta^2 b^2 - 10\beta b\alpha + 7\alpha^2} \right].$$

As such, it can be readily checked that testing the positivity or negativity of a first-order or second-order partial derivative of $L_1$ with respect to $b, \alpha, \text{ or } \beta$ is equivalent to showing that

$$f_0(\alpha, \beta, b) \geq 0, \forall (\alpha, \beta, b) \in \mathcal{X} := \{(\alpha, \beta, b) \in \mathbb{R}^3 : f_i(\alpha, \beta, b) \geq 0, i = 1, \ldots, m \},$$

where $\{f_i\}_{i=0}^m$ are polynomial functions in the variables $b, \alpha, \beta$. This problem falls in the general class of polynomial optimization problems, which require testing the positivity of a polynomial objective on a feasible set given by a finite number of polynomial equalities and inequalities. Exact computational methods are available to produce certificates in such problems, using sum-of-squares (SOS) methods (see Parrilo, 2003 and references therein for details). We use these for every derivative above, and confirm that $\frac{\partial^2 L_1}{\partial b^2} \geq 0$, $\frac{\partial^2 L_1}{\partial \alpha^2} \geq 0$, $\frac{\partial^2 L_1}{\partial \beta^2} \geq 0$, $\frac{\partial L_1}{\partial b} \geq 0$, $\frac{\partial L_1}{\partial \alpha} \leq 0$, $\frac{\partial L_1}{\partial \beta} \geq 0$ always hold. Details are omitted for space considerations, but are available upon request.

To prove the bounds on $L_1$, we can again write $L_1 = g_m(y, \alpha)$, where $y := b/\beta_p$ and

$$g_m(y, \alpha) := \frac{1}{216}
\left[ 636 - 480y + 123y^2 + 184\alpha - 108y\alpha - 12\alpha^2 + 8y^3\alpha +
(-240 + 96y - 64\alpha + 8y\alpha + 8y^2\alpha)\sqrt{y^2 - 5y + 7} \right].$$

Before proceeding with the argument, it is useful to derive a set of bounds on the value of $y$. To this end, note that $b \geq \hat{b}$, and by the proof of Proposition 4.1 $\hat{b} \geq b_m := (2 - \sqrt{2})/2$. Thus, we have $y := \frac{2\beta}{\alpha} \geq 4 - 2\sqrt{2}$. Furthermore, since $b \leq \frac{\alpha}{2\beta}$, we also have $y \leq 2$. Using SOS techniques, it can then
be readily checked that for any \((y, \alpha) \in [4 - 2\sqrt{2}, 2] \times [0, 1]\),

\[
\begin{align*}
\frac{\partial g_m}{\partial \alpha} & \leq 0 \\
\frac{\partial^2 g_m}{\partial y^2} & \geq 0.
\end{align*}
\]  
(E-7a)

By letting \(y := 4 - 2\sqrt{2}\), we can therefore conclude that

\[
g_m(y, \alpha) \geq g_m(y, 1) \geq g_m(y, 1) + \frac{\partial g_m(y, 1)}{\partial y} \bigg|_{y=y} \cdot (y - y), \quad \forall y \in [\sqrt{2}, 2],
\]

which yields the desired bound when substituting the values.

**Case 3.** When \(b \in \left(\frac{1}{\beta}, \frac{2\alpha}{\beta}\right]\), by the proof of Proposition 4.1 we have that \(p_T(b) = q_h(b) = \frac{\beta b + \alpha}{3\beta}\).

The efficiency loss can be written as

\[
L_1(b, \alpha, \beta) = \frac{2\beta^3b^3 - 6\alpha\beta b + (15 - 8\alpha)\alpha^2 + 6(1 + \alpha)\beta^2b^2}{27\alpha^2}.
\]

As such, we have:

\[
\begin{align*}
\frac{\partial L_1}{\partial b} &= \frac{2\beta[(\beta b + 2\alpha - \beta b) - \alpha]}{9\alpha^2} \geq 0 \\
\frac{\partial L_1}{\partial \alpha} &= \frac{-8\alpha^3 + 6(2 + \alpha)\beta^2b^2 - 4\beta^3b^3 - 6\alpha\beta b}{27\alpha^3} \leq 0 \\
\frac{\partial L_1}{\partial \beta} &= \frac{2b(\beta b + 2\alpha - \beta b) - \alpha}{9\alpha^2} \geq 0 \\
\frac{\partial^2 L_1}{\partial b^2} &= \frac{4\beta^2(1 + \alpha - \beta b)}{9\alpha^2} \geq 0 \\
\frac{\partial^2 L_1}{\partial \alpha^2} &= \frac{4\beta b(\beta b + 3 + \alpha - \beta b) - \alpha}{9\alpha^4} \geq 0 \\
\frac{\partial^2 L_1}{\partial \beta^2} &= \frac{4b^2(1 + \alpha - \beta b)}{9\alpha^2},
\end{align*}
\]

where each of the inequalities follows by using the fact that \(\alpha \leq \beta b \leq 2\alpha \leq 2\).

To prove the bound on \(L_1\), we can again write \(L_1 = g_h(y, \alpha)\), where \(y := \frac{b}{p}\) and

\[
g_h(y, \alpha) := \frac{1}{108} \left[6(y^2 - 2y + 10) - \alpha(y + 2)(4 - y)^2\right].
\]
Since $b \in (\frac{4}{3}, 2\frac{\alpha}{\beta}]$, we always have $y \in (2, 4]$, and it can then be readily checked that for any such $y$,

\[
\frac{\partial g_h}{\partial \alpha} = \frac{(4 - y)^2(2 + y)}{108} \leq 0. \tag{E-8a}
\]
\[
\frac{\partial^2 g_h}{\partial y^2} = \frac{2 - \alpha(y - 2)}{18} \geq 0. \tag{E-8b}
\]

Therefore, we always have:

\[
\begin{align*}
\frac{\partial g_h(y, \alpha)}{\partial \alpha} \geq & g_h(y, 1) \geq g_h(2, 1) + \frac{\partial g_h(y, 1)}{\partial y} \bigg|_{y=2} \cdot (y - 2), \forall y \in (2, 4]
\end{align*}
\]

and the proof is complete. \(\square\)

**Proof of Proposition 4.2**

\(i\). We use the expressions for $p_T^{\dagger}(b, 2), t = T - 1, T$ derived in the proof of Proposition 4.1. We deal with the three regimes separately.

For $0 \leq b \leq \hat{b}$, we have that

\[
\mathbb{E}\left[p_T^{\dagger}(B_T, \mathcal{B}_T)\right] - p_{T-1}^{\dagger}(b, 2) = \lambda(q_\ell(b))p_T^{\dagger}(0, 1) + (1 - \lambda(q_\ell(b)))p_T^{\dagger}(b, 2) - q_\ell(b) = \frac{\beta b^2(2(1 - \alpha) + \beta b)}{16},
\]

which is clearly positive and increasing in $b$.

For $\hat{b} < b \leq \frac{\alpha}{\beta}$, we have that

\[
\mathbb{E}\left[p_T^{\dagger}(B_T, \mathcal{B}_T)\right] - p_{T-1}^{\dagger}(b, 2) = \lambda(q_m(b))p_T^{\dagger}(b - q_m(b), 1) + (1 - \lambda(q_m(b)))p_T^{\dagger}(b, 2) - q_m(b),
\]

and for $\frac{\alpha}{\beta} < b \leq 2\frac{\alpha}{\beta}$, we have that

\[
\mathbb{E}\left[p_T^{\dagger}(B_T, \mathcal{B}_T)\right] - p_{T-1}^{\dagger}(b, 2) = \lambda(q_h(b))p_T^{\dagger}(b - q_h(b), 1) + (1 - \lambda(q_m(b)))p_T^{\dagger}(b, 2) - q_h(b).
\]

Substituting for $q_m, q_h,$ and $p_T^{\dagger}$, and using similar arguments as in the proof of Lemma 4.1, one can show that the above differences are positive and increasing in $b$. Details are omitted for space considerations, but are available upon request.

\(ii\). The result follows from analysis of the one-unit case, Proposition 5.2.

\(iii\). We use the expressions for $p_T^{\dagger}(b, 2), t = T - 1, T$ derived in the proof of Proposition 4.1. We show that $g(b) := p_{T-1}^{\dagger}(b, 2) - p_T^{\dagger}(b, 2)$ is decreasing. For $b \in [0, \hat{b}]$, $g'(b) = (q_\ell)'(b) - \frac{1}{2} = \frac{1}{2} \frac{\partial}{\partial b} V_T(b, 1) \leq 0$, where the inequality follows from Lemma 3.1 \(iii\). For $b \in (\hat{b}, \frac{\alpha}{\beta}]$, $g'(b) = (q_m)'(b) - \frac{1}{2} \leq (q_m)'(\frac{\alpha}{\beta}) - \frac{1}{2} = \frac{1}{3} - \frac{1}{2} < 0$, where the first inequality follows from $q_m'$ being increasing and positive, as argued in the
shown that following the revenue-maximizing policy would result in pricing at 
arg max \( p \in \mathbb{R} \) \( J^* \) in both periods. Thus \( p_{T-1}^t(b, y) - p_T^t(b, y) \leq p_T^t(y) - p_T^*(y) = 0. \)

\[ J^* = \frac{\alpha}{2b} \]

**Proof of Proposition 4.1.** For \( y = 1 \) the result follows from analysis of the one-unit case, Proposition 5.3. For \( y = 2 \), according to Proposition 4.1 the price \( p^\dagger_{T-1}(b, 2) \) takes different expressions for \( b \) in \([0, \hat{b}], (\hat{b}, \hat{a}]\) and \((\hat{a}, 2\hat{a}]\). We argue for these cases separately. Note that in all cases, it can be readily shown that following the revenue-maximizing policy would result in pricing at \( \arg \max_{p \in \mathbb{R}} \{ p \lambda(p) \} = \frac{\alpha}{2b} \) in both periods. Thus \( J^* = \frac{\alpha}{2b} \).

- For \( b \in [0, \hat{b}] \), according to Proposition 4.1, \( p^\dagger_{T-1}(b, 2) = q_\ell(b) > b \). Thus, in case of a sale at \( T - 1 \), the DM covers his debt and charges \( p^\dagger_T(0, 1) \) at \( T \). Otherwise, he charges \( p^\dagger_T(b, 1) \). Combining these observations we get

\[ J^* = \frac{\alpha}{2b} \]

We now derive an expression for the expectation \( E[J^*_{\mathcal{T}}] \). Using the law of total expectation, in a similar fashion as in the proof of Proposition 5.3 we get

\[ E[J^*_{\mathcal{T}}] = \frac{\alpha}{2b} \frac{q_\ell(b) + \lambda(p^\dagger_{T-1}(2)) p^\dagger_T(0, 1)}{p^\dagger_{T-1}(2) + \lambda(p^\dagger_T(1))} \]

By substituting for all the prices in the expressions above and setting \( x := \beta b \in [0, \beta \hat{b}] \), after some tedious algebra we get

\[ J^* - E[J^*_{\mathcal{T}}] < 0 \]

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To see that the second multiplier above is negative, note that it is increasing for \( x \geq 0 \). Since \( \beta \hat{b} < \alpha \), we can upper bound the multiplier by evaluating it for \( x = \alpha \), which yields \( -\alpha(8+\alpha) < 0 \).

- For \( b \in \left( \hat{b}, \frac{\alpha}{2} \right) \), according to Proposition 4.1, \( p^\dagger_{T-1}(b, 2) = q_m(b) < b \). Thus, in case of a sale at \( T - 1 \), the DM still fails to cover his debt and charges \( p^\dagger_T(b - q_m(b), 1) \) at \( T \). Otherwise, he charges \( p^\dagger_T(b, 1) \). Combining these observations we get

\[
J^\dagger = \lambda(q_m(b)) \left( q_m(b) + \lambda \left( p^\dagger_T(b - q_m(b), 1) \right) \right) + (1 - \lambda(q_m(b))) \lambda \left( p^\dagger_T(b, 1) \right) p^\dagger_T(b, 1).
\]

Using the law of total expectation as above we get

\[
\mathbb{E} \left[ \frac{J^\dagger}{J^*} \right] = \lambda(q_m(b)) \frac{q_m(b) + \lambda \left( p^\dagger_T(b - q_m(b), 1) \right) p^\dagger_T(b - q_m(b), 1)}{p^\dagger_{T-1}(2) + \lambda \left( p^\dagger_T(1) \right) p^\dagger_T(1)}
\]

\[
+ \left( \lambda \left( p^\dagger_{T-1}(2) \right) - \lambda(q_m(b)) \right) \frac{\lambda \left( p^\dagger_T(b, 1) \right) p^\dagger_T(b, 1)}{p^\dagger_{T-1}(2) + \lambda \left( p^\dagger_T(1) \right) p^\dagger_T(1)}
\]

\[
+ \lambda \left( p^\dagger_{T-1}(2) \right) \frac{\lambda \left( p^\dagger_T(b, 1) \right) p^\dagger_T(b, 1)}{\lambda \left( p^\dagger_T(1) \right) p^\dagger_T(1)}.
\]

By substituting for all the prices in the expressions above and setting \( x := \beta b \in (\beta \hat{b}, \alpha] \), after some tedious algebra we get

\[
J^\dagger - \mathbb{E} \left[ \frac{J^\dagger}{J^*} \right] \cdot J^* = -\frac{2 - \alpha}{108 \beta (2 + \alpha)} \left( g_1(x) \sqrt{4x^2 - 10\alpha x + 7\alpha^2 + g_2(x)} \right),
\]

where \( g_1(x) := 8x^2 + (4\alpha + 48)x - 4\alpha(4\alpha + 15) \) and \( g_2(x) := 16x^3 + 4(\alpha + 12)x^2 - 2\alpha(27\alpha + 120)x + \alpha^2(46\alpha + 159) \). To see that \( g_1(x) \) is negative, note that it is increasing for \( x \geq 0 \) and evaluates to \(-4\alpha(\alpha + 3) < 0 \) for \( x = \alpha \). Thus, it suffices to show that

\[
g_2^2(x) - g_1^2(x)(4x^2 - 10\alpha x + 7\alpha^2) = 27(\alpha - x)w(x) \leq 0,
\]

or equivalently that \( w(x) \leq 0 \), where \( w(x) = 32x^4 + (8\alpha(1 + 2\alpha) + 165)x^3 - \alpha(4\alpha + 315)x^2 + \alpha^2(3 - 44\alpha(\alpha + 4))x + \alpha^3(4\alpha(11 + 3\alpha) + 3) \). The derivative of \( w \) is cubic in \( x \) and can be readily maximized over \([0, \alpha]\) to obtain that \( w'(x) \leq 0 \). Since \( w \) is then decreasing, we can upper bound it as follows

\[
w(x) \leq w(\beta \hat{b}) \leq w(b_m \beta) = w((2 - \sqrt{2})\alpha) < 0.
\]

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Note that \( w((2 - \sqrt{2})\alpha) \) depends only on \( \alpha \) and be readily maximized over \((0, 1]\) to obtain the last inequality above, which concludes the proof for this case.

- For \( b \in \left( \frac{\alpha}{\beta}, \frac{2\alpha}{\beta} \right] \), according to Proposition 4.1, \( p^*_T(b, 2) = q_h(b) \leq \frac{2}{\beta} < b \). Thus, in case of a sale at \( T - 1 \), the DM still fails to cover his debt and charges \( p^*_T(b - q_h(b), 1) \) at \( T \). Otherwise, it becomes infeasible for him to cover the debt. Combining these observations we get

\[
J^* = \lambda(q_h(b)) \left( q_h(b) + \lambda \left( p^*_T(b - q_h(b), 1) \right) p^*_T(b - q_h(b), 1) \right).
\]

Using the law of total expectation as above we get

\[
\mathbb{E} \left[ \frac{J^*}{J^*} \right] = \lambda(q_h(b)) \frac{q_h(b) + \lambda \left( p^*_T(b - q_h(b), 1) \right) p^*_T(b - q_h(b), 1)}{p^*_{T-1}(2) + \lambda \left( p^*_T(1) \right) p^*_T(1)}.
\]

By substituting for all the prices in the expression above we get

\[
J^* - \mathbb{E} \left[ \frac{J^*}{J^*} \right] J^* = \frac{(2 - \alpha)(\beta b + \alpha)(\beta b - 2\alpha)(\beta b - 2\alpha - 3)}{27\beta(2 + \alpha)} \geq 0,
\]

since \( \beta b - 2\alpha - 3 < \beta b - 2\alpha \leq 0 \) for \( b \leq \frac{2\alpha}{\beta} \).

\[\square\]

F Proofs for Section 5

Proof of Proposition 5.1 To show i), we follow the steps below.

Step 1: We first show that for all \( 0 \leq x < \underline{p} \), \( \lambda(p)(p - x) \) admits a unique maximizer in \( p \) over \( \mathcal{P} \), equal to \( \pi(x) \), such that \( \pi(x) \geq x \).

To this end, note that no \( p < x \) can be a maximizer of \( \lambda(p)(p - x) \) in \( p \) over \( \mathcal{P} \). This is clear if \( x \leq \underline{p} \). For \( x > \underline{p} \) and for all \( p \) such that \( \underline{p} < p < x < \overline{p} \), we get that \( \lambda(p)(p - x) < 0 \leq \lambda(\overline{p})(\overline{p} - x) \). Our claim that \( \pi(x) \geq x \) follows.

To show that \( \lambda(p)(p - x) \) admits a unique maximizer over \( [x \wedge \underline{p}, \overline{p}] \), it suffices to show that it is unimodal. If it has no stationary points in \( (x \wedge \underline{p}, \overline{p}) \), this is clearly the case. Otherwise, let \( \hat{p} \in (x \wedge \underline{p}, \overline{p}) \) be a stationary point, i.e., \( \hat{p} \) solves the FOC

\[
\frac{d}{dp} \lambda(p)(p - x) \bigg|_{\hat{p}} = \lambda'(\hat{p})(\hat{p} - x) + \lambda(\hat{p}) = 0.
\]

(F.9)
The second derivative at \( \hat{p} \) evaluates to
\[
\frac{d^2}{dp^2} \{ \lambda(p)(p - x) \} \bigg|_{\hat{p}} = \lambda''(\hat{p})(\hat{p} - x) + 2\lambda'(\hat{p}) = -\frac{1}{\lambda'(\hat{p})} \left( \lambda''(\hat{p})\lambda(\hat{p}) - 2(\lambda'(\hat{p}))^2 \right) < 0, \tag{F-10}
\]
where the second equality follows by substituting for \( \hat{p} - x \) from the FOC (note that if \( \lambda'(\hat{p}) = 0 \), we get \( \lambda(\hat{p}) = 0 \) from the FOC, a contradiction since \( \hat{p} \in (x \cap \mathbf{p}, \overline{\mathbf{p}}) \)). The negativity of the second multiplier above follows from log-concavity of \( \lambda \), which yields \( 0 \geq \lambda''(\hat{p})\lambda(\hat{p}) - (\lambda'(\hat{p}))^2 > \lambda''(\hat{p})\lambda(\hat{p}) - 2(\lambda'(\hat{p}))^2 \).

As such, all stationary points are local maxima. Thus, there exists a unique local maximum, which has to be the unique global maximizer.

Step 2: We show that \( 0 \leq \pi'(x) \leq 1 \). To this end, note that from (F-9) for \( p \to x \) the derivative of \( \lambda(p)(p - x) \) becomes positive. Thus, if it has no stationary points in \( (x \cap \mathbf{p}, \overline{\mathbf{p}}) \), it is increasing and \( \pi(x) = \overline{\mathbf{p}} \), which trivially satisfies our claim. Otherwise, \( \pi(x) \) is equal to the unique solution of the FOC. We calculate its derivatives using the Implicit Function Theorem. In particular, by taking the derivative of the FOC with respect to \( x \) we get
\[
\pi'(x) = \frac{\lambda'(\pi(x))}{\lambda''(\pi(x))(\pi(x) - x) + 2\lambda'(\pi(x))} = \frac{-\lambda'(\pi(x))^2}{\lambda''(\pi(x))\lambda(\pi(x)) - 2(\lambda'(\pi(x)))^2} \geq 0, \tag{F-11}
\]
where the second equality and the inequality follow from (F-10). Showing \( \pi'(x) \leq 1 \) is equivalent to \( \lambda''(\pi(x))\lambda(\pi(x)) - 2(\lambda'(\pi(x)))^2 \leq 0 \), which follows from log-concavity of \( \lambda \).

Step 3: We now complete the proof of \( i.) \) by deriving the DM’s price. For \( b \geq \overline{\mathbf{p}} \), as we argued in Section 5.1 we can take \( p_t^\dagger(b) = \overline{\mathbf{p}} \) without loss.

For \( b < \overline{\mathbf{p}} \), we have that \( p_t^\dagger(b) \in \arg \max_{p \in \mathbf{P}, p \geq b} \{ \lambda(p) [p - (b + V_{t+1}(b))] \} \) for all \( t = 1, \ldots, T \). Note first that posting a price \( \underline{p} < b + \epsilon < \overline{\mathbf{p}} \) (for \( \epsilon > 0 \) appropriately chosen) ensures a non-zero probability of covering the debt since \( \lambda(b + \epsilon) > 0 \). Thus, \( -V_t^\dagger(b) < 1 \) for all \( t = 1, \ldots, T \) by Lemma 3.1(iii). Consequently, \( b + V_{t+1}(b) \) is strictly increasing and \( b + V_{t+1}(b) < \overline{\mathbf{p}} + V_{t+1}(\overline{\mathbf{p}}) = \overline{\mathbf{p}} \). By our result in Step 1, we get that \( p_t^\dagger(b) = \pi(b + V_{t+1}(b)) \).

Finally, we also remark that for \( b < \overline{\mathbf{p}} \), we have that \( p_t^\dagger(b) > b \). To see this, note that if \( b < \underline{p} \), clearly \( p_t^\dagger(b) > b \). Otherwise, for \( \underline{p} \leq b < \overline{\mathbf{p}} \) note that for \( p = b \), the maximand above is less than equal to zero, whereas for \( p = b + V_{t+1}(b) + \epsilon' < \overline{\mathbf{p}} \) (for \( \epsilon' > 0 \) small enough) it is positive. Hence, again \( p_t^\dagger(b) > b \) and the proof is complete.

To show \( ii.) \) and \( iii.) \), we argue as follows. For \( b \geq \overline{\mathbf{p}}, i.) \) suggests that \( p_t^\dagger(b) = \overline{\mathbf{p}} \), which is trivially increasing in \( b \). For \( b < \overline{\mathbf{p}} \), we have that \( p_t^\dagger(b) = \pi(b + V_{t+1}(b)) \), which is increasing in \( b \) since \( b + V_{t+1}(b) \) is increasing in \( b \) (by Lemma 3.1(ii)) and so is \( \pi \) (by \( i.) \)). The proof is complete by recalling that \( p_t^\dagger(0) = p_t^* \). \( \square \)
Proof of Proposition 5.2. We first show that \( \pi(x) \) is concave. To that end, using \((F-11)\) and differentiating, we obtain:

\[
\pi''(x) = \frac{\left( [\lambda''(\pi(x))]^2 - \lambda'(\pi(x))\lambda'''(\pi(x)) \right) (\pi(x) - x) \pi'(x) + (1 - \pi'(x)) \lambda'(\pi(x)) \lambda''(\pi(x))}{(\lambda''(\pi(x))(\pi(x) - x) + 2\lambda'(\pi(x)))^2} \]

\[
\leq 0, \text{ by } -\lambda' \text{ log-convex}
\]

\[
\geq 0, \text{ by Proposition 5.1} \geq 0, \text{ by Thm. 5.1} \leq 0, \text{ by } \lambda \text{ convex}
\]

To show \(i\)., first note that for \(b \geq \overline{p} \), Proposition 5.1 suggests that \(p^*_t(b) = p^*_{t+1}(b) = \overline{p} \), and thus their difference is trivially decreasing in \(b \). For \(b < \overline{p} \), we express the prices \(p^*_t(b), p^*_{t+1}(b)\) using the function \(\pi\) and take the derivative of their difference. We obtain

\[
\frac{d}{db} \left( p^*_t(b) - p^*_{t+1}(b) \right) = (1 + V^*_{t+1}(b)) \pi'(b + V_{t+1}(b)) - (1 + V'_{t+2}(b)) \pi'(b + V_{t+2}(b)).
\]

Note that \(V_t(b)\) is decreasing in \(t\) (Lemma 3.1), and \(\pi'\) is positive and decreasing. Thus,

\[
\pi'(b + V_{t+2}(b)) \geq \pi'(b + V_{t+1}(b)) \geq 0.
\]

Also, since \(-V''_t(b)\) is less that one and decreasing in \(t\), we get

\[
1 + V^*_{t+2}(b) \geq 1 + V^*_{t+1}(b) \geq 0.
\]

Based on these two inequalities and the expression for the derivative above, we conclude that \(p^*_t(b) - p^*_{t+1}(b)\) is decreasing in \(b \).

To show \(iii\), simply recall that \(p^*_t(0) = p^*_t\) and that \(p^*_t(b) \geq p^*_t\). For \(iii\), we have that

\[
\frac{p^*_t(b)}{p^*_t} = \frac{(p^*_t(b) - p^*_{t+1}(b)) + p^*_{t+1}(b)}{(p^*_t - p^*_{t+1}) + p^*_{t+1}} \leq \frac{p^*_t - p^*_{t+1}}{p^*_t - p^*_{t+1}} \leq \frac{p^*_{t+1}(b)}{p^*_{t+1}},
\]

where \((*)\) follows from part \(i\), and \((***)\) is true since \(0 \leq p^*_t \leq p^*_{t+1}(b)\), by Proposition 5.1.

Proof of Proposition 5.3. Similarly to \(J^*_1\), we define \(J^*_2\) as the expected revenues under the revenue-maximizing policy at the beginning of period \(t\), conditional on no sale in periods \(1, \ldots, t - 1\). For clarity, we explicitly highlight the dependence of \(J^*_t\) on the debt, i.e., we write \(J^*_t(b)\). We first show that \(L_1 \leq L_2\), or equivalently that

\[
\frac{J^*_1(b)}{J^*_1} \geq \mathbb{E} \left[ \frac{J^*_2}{J^*_2} \right] = \mathbb{E} \left[ \frac{\mathbb{E}[\mathcal{D}(p^*) | \sigma(\mathcal{W}_1)]}{\mathbb{E}[\mathcal{D}(p^*) | \sigma(\mathcal{W}_1)]} \right]. \tag{F-12}
\]
We condition on the value of $\mathcal{W}_1$ in order to express the right-hand side above using the law of total expectation. In particular, since $p^*_1(b) \geq p_1^+$ by Proposition 5.1, we consider the events $\mathcal{E}_1 = \{\mathcal{W}_1 < p_1^+\}$, $\mathcal{E}_2 = \{p_1^+ \leq \mathcal{W}_1 < p_1^+(b)\}$ and $\mathcal{E}_3 = \{p_1^+(b) \leq \mathcal{W}_1\}$. Under $\mathcal{E}_1$, no sale occurs under either the DM’s or the revenue-maximizing policies. Under $\mathcal{E}_2$, a sale occurs only under the revenue-maximizing policy, whereas under $\mathcal{E}_3$ a sale occurs under both policies. For the revenue-maximizing (DM’s) policy, expected revenue equal $J_2^* (J_2^+(b))$ when no sale occurs, and $p_1^+(b)$ when a sale occurs. Combining all these facts, we get that

$$
\mathbb{E} \left[ \frac{\mathcal{R}(p^+)}{\mathcal{R}(p^*)} | \sigma(\mathcal{W}_1) \right] = \sum_{i=1}^{3} \mathbb{E} \left[ \frac{\mathcal{R}(p^+)}{\mathcal{R}(p^*)} | \sigma(\mathcal{W}_1) \right] | \mathcal{E}_i \] \mathbb{P}(\mathcal{E}_i)
$$

$$
= \frac{J_2^+(b)}{J_2^*} (1 - \lambda(p_1^*)) + \frac{J_2^*(b)}{p_1^+} (\lambda(p_1^*) - \lambda(p_1^+(b))) + \frac{p_1^+(b)}{p_1^*} \lambda(p_1^+(b))
$$

$$
= (1 - \lambda(p_1^*)) \left( \frac{1}{J_2^*} - \frac{1}{p_1^+} \right) J_2^*(b) + \frac{1}{p_1^*} J_2^+(b),
$$

where the last equality follows from $J_2^*(b) = \lambda(p_1^+(b))p_1^+(b) + (1 - \lambda(p_1^+(b)))J_2^+(b)$. Thus,

$$
\frac{J_2^+(b)}{J_2^*} - \mathbb{E} \left[ \frac{J_2^+}{J_2^*} \right] = J_2^*(b) \left[ \left( \frac{1}{J_2^*} - \frac{1}{p_1^+} \right) J_2^+(b) - (1 - \lambda(p_1^*)) \left( \frac{1}{J_2^*} - \frac{1}{p_1^+} \right) \right].
$$

Thus, the inequality in (F-12) is equivalent with $\left( \frac{1}{J_2^*} - \frac{1}{p_1^+} \right) \frac{J_2^+(b)}{J_2^*(b)} - (1 - \lambda(p_1^*)) \left( \frac{1}{J_2^*} - \frac{1}{p_1^+} \right) \geq 0$. For $b = 0$ the inequality holds with equality, since the two policies become the same. For $b > 0$, it suffices to show that the left-hand side is increasing in $b$. This is indeed the case since $\frac{J_2^+(b)}{J_2^*(b)}$ is increasing in $b$, and $J_2^* < p_1^+$, $p_1^*$ being the largest price charged under the revenue-maximizing policy.

Using identical arguments and the fact that $\frac{J_2^+(b)}{J_2^*(b)}$ is increasing in $b$, one can show that for all $t = 2, \ldots, T - 1$

$$
\frac{J_{t+1}^+(b)}{J_t^*} \geq \mathbb{E} \left[ \frac{J_{t+1}^+}{J_t^*} \right] | \mathcal{W}_1 = \ldots = \mathcal{W}_{t-1} = 0,
$$

where the conditioning event in the right-hand side ensures that no sale has occurred under either policy up until and including period $t - 1$.

We now show that for any $t = 2, \ldots, T - 1$ we have that $L_t \leq L_{t+1}$, or equivalently that

$$
\mathbb{E} \left[ \frac{J_t^+}{J_t^*} \right] \geq \mathbb{E} \left[ \frac{J_{t+1}^+}{J_{t+1}^*} \right].
$$

(F-14)
We condition again on the events \( \{ \mathcal{E}_i \}_{i=1,2,3} \) to express the left-hand side as follows

\[
\mathbb{E} \left[ \frac{\mathcal{J}_t^+}{\mathcal{J}_t^*} \right] = \frac{p_t^+(b)}{p_t^*} \lambda(p_t^+(b)) + \frac{J_{t+1}^+(b)}{p_t^*} (\lambda(p_t^*) - \lambda(p_t^+(b))) + \mathbb{E} \left[ \frac{\mathcal{J}_t^+}{\mathcal{J}_t^*} \middle| \mathcal{W}_1 < p_1^* \right] (1 - \lambda(p_1^*)),
\]

where we used the fact that \( \mathbb{E} \left[ \frac{\mathcal{J}_t^+}{\mathcal{J}_t^*} \middle| \mathcal{W}_1 < p_1^* \right] \) is equal to \( J_2^+(b) \) by definition. We can now use the same approach in order to express the expectation in the last term above, by conditioning on the value of \( \mathcal{W}_2 \). In particular, we get

\[
\mathbb{E} \left[ \frac{\mathcal{J}_t^+}{\mathcal{J}_t^*} \right] = \frac{p_t^+(b)}{p_t^*} \lambda(p_t^+(b)) + \frac{J_{t+1}^+(b)}{p_t^*} (\lambda(p_t^*) - \lambda(p_t^+(b))) + (1 - \lambda(p_t^*)) \left( \frac{p_{t+1}^+(b)}{p_{t+1}^*} \lambda(p_{t+1}^+(b)) + \frac{J_{t+2}^+(b)}{p_{t+1}^*} (\lambda(p_{t+1}^*) - \lambda(p_{t+1}^+(b))) + \mathbb{E} \left[ \frac{\mathcal{J}_{t+1}^+}{\mathcal{J}_{t+1}^*} \middle| \mathcal{W}_1 < p_1^*, \mathcal{W}_2 < p_2^* \right] (1 - \lambda(p_2^*)) \right).
\]

By applying the same approach recursively we obtain

\[
\mathbb{E} \left[ \frac{\mathcal{J}_t^+}{\mathcal{J}_t^*} \right] = \sum_{\tau=1}^{t-1} \phi_{\tau-1} \left[ \frac{p_{t+\tau}^+(b)}{p_{t+\tau}^*} \lambda(p_{t+\tau}^+(b)) + \frac{J_{t+\tau+1}^+(b)}{p_{t+\tau}^*} (\lambda(p_{t+\tau}^*) - \lambda(p_{t+\tau}^+(b))) \right] + \phi_{t-1} \frac{J_{t}^+(b)}{J_t^*},
\]

where we used that \( \mathbb{E} \left[ \frac{\mathcal{J}_t^+}{\mathcal{J}_t^*} \middle| \mathcal{W}_1 < p_1^*, \ldots, \mathcal{W}_{t-1} < p_{t-1}^* \right] = \frac{J_t^+(b)}{J_t^*} \), and \( \phi_0 := 1, \phi_\tau := \prod_{i=1}^{\tau} (1 - \lambda(p_i^*)) \) for \( \tau \geq 1 \). If we use the same expression for the right-hand side of the inequality (F-14) we want to show, we can express the difference of the two sides as

\[
\mathbb{E} \left[ \frac{\mathcal{J}_{t+1}^+}{\mathcal{J}_{t+1}^*} - \frac{\mathcal{J}_t^+}{\mathcal{J}_t^*} \right] = \phi_{t-1} \left( \frac{p_{t+1}^+(b)}{p_{t+1}^*} \lambda(p_{t+1}^+(b)) + \frac{J_{t+2}^+(b)}{p_{t+1}^*} (\lambda(p_{t+1}^*) - \lambda(p_{t+1}^+(b))) \right) + \phi_t \frac{J_{t+1}^+(b)}{J_{t+1}^*} - \phi_{t-1} \frac{J_t^+(b)}{J_t^*}
\]

\[
= \phi_{t-1} \left( \frac{p_{t+1}^+(b)}{p_{t+1}^*} \lambda(p_{t+1}^+(b)) + \frac{J_{t+2}^+(b)}{p_{t+1}^*} (\lambda(p_{t+1}^*) - \lambda(p_{t+1}^+(b))) \right) + (1 - \lambda(p_t^*)) \frac{J_{t+1}^+(b)}{J_{t+1}^*} - \frac{J_t^+(b)}{J_t^*}
\]

\[
= \phi_{t-1} \left( \mathbb{E} \left[ \frac{\mathcal{J}_{t+1}^+}{\mathcal{J}_{t+1}^*} \middle| \mathcal{W}_1 = \ldots = \mathcal{W}_{t-1} = 0 \right] - \frac{J_t^+(b)}{J_t^*} \right) \leq 0,
\]

where the inequality follows from (F-13) and the proof is complete.

\( \square \)

**Proposition F.1.** Suppose that the demand function is either linear or exponential. Then, \( J_t^+ / J_{t+1}^+ \) is increasing in \( b \) for any \( t = 1, \ldots, T - 1 \).

**Proof of Proposition F.1.** Throughout this proof, \( x' \) will denote the derivative \( \frac{d}{db} x \). To ease nota-
tion, we suppress the superscript † and the dependence on $b$, e.g., $p_t = p_t^\dagger(b)$, etc. Also, let

$$\lambda_t := \lambda(p_t^\dagger(b)), \quad t = 1, \ldots, T.$$  

We also define $\omega_t$ to be the probability of failing to cover the debt under the DM's policy at the beginning of period $t$, given that no sale has occurred until then. That is,

$$\omega_{t+1} := 1 \quad \text{and} \quad \omega_t := (1 - \lambda_t)\omega_{t+1}, \quad t = 1, \ldots, T. \quad (F-15)$$

Using our notation, Lemma 3.1iii.) can be expressed as

$$1 - \omega_t = -V_t', \quad t = 1, \ldots, T + 1. \quad (F-16)$$

Also, the expected revenue can be expressed as

$$J_t = \lambda_t p_t + (1 - \lambda_t)J_{t+1} = V_t + (1 - \omega_t)b, \quad t = 1, \ldots, T + 1. \quad (F-17)$$

By differentiating (F-17) and using (F-16) we get

$$J_t' = -b\omega_t', \quad t = 1, \ldots, T + 1. \quad (F-18)$$

We treat the two cases separately. 

**Case (1):** $\lambda(p) = e^{-\alpha p}$, $\alpha > 0$, $p \in [0, \infty)$ \(^{18}\) Using our notation, Proposition 5.1 yields that $p_t = \pi(b + V_{t+1})$, for all $t = 1, \ldots, T$. Using the fact that $\arg \max_{p \geq 0} e^{-\alpha p}(p - x) = \frac{1}{\alpha} + x$ and (F-17), we get that

$$p_t = \frac{1}{\alpha} + b + V_{t+1} = \frac{1}{\alpha} + b\omega_{t+1} + J_{t+1}, \quad t = 1, \ldots, T. \quad (F-19)$$

In conjunction with Lemma 3.1iii.), this also shows that $p_t$ is decreasing in $t$. By differentiating (F-19) and using (F-16), we get that

$$p_t' = \omega_{t+1}, \quad t = 1, \ldots, T, \quad (F-20)$$

which also yields

$$\lambda_t' = (e^{-\alpha p_t})' = -\alpha p_t'\lambda_t = -\alpha \omega_{t+1}\lambda_t, \quad t = 1, \ldots, T. \quad (F-21)$$

\(^{18}\)The proof can be generalized to the case where $\lambda(p) = \alpha_0 e^{-\alpha p}$, $\alpha_0 \in (0, 1)$, in a straightforward manner.
These allow us to express
\[
\left( \frac{J_{t-1}}{J_t} \right)' = \left( \frac{\lambda_{t-1}p_{t-1} + (1 - \lambda_{t-1})J_t}{J_t} \right)'
\]
\[
= \left( \frac{\lambda_{t-1}p_{t-1} - \lambda_{t-1}J_t}{J_t} \right)'
\]
\[
= \frac{\lambda_{t-1}}{J_t^2} \left( (p_{t-1} - J_t) \frac{\lambda_t}{\lambda_{t-1}} J_t + p_{t-1} J_t - p_{t-1} J_t' \right)
\]
\[
= \frac{\lambda_{t-1}}{J_t^2} \left( (p_{t-1} - J_t) \alpha J_t + p_{t-1} J_t - p_{t-1} J_t' \right) \quad \text{[by (F-21)]}
\]
\[
= \frac{\lambda_{t-1}}{J_t^2} \left( -\alpha (p_{t-1} - J_t) + p_{t-1} J_t - p_{t-1} J_t' \right) \quad \text{[by (F-19)]}
\]
\[
= \frac{\lambda_{t-1}}{J_t^2} \left( -\alpha b \omega t J_t + b \omega' \omega_{t-1} \right) \quad \text{[by (F-20) and (F-18)]}
\]
\[
= \frac{\lambda_{t-1} \omega_t b}{J_t^2} \left( -\alpha \omega_t J_t + \frac{\omega_t'}{\omega_t} \right)
\]
for all \( t = 2, \ldots, T \). Thus, it suffices to show that the inequality \( \frac{\omega_t'}{\omega_t} \geq \frac{\alpha \omega_t J_t}{p_t} \) holds for all \( t = 2, \ldots, T+1 \).

We will use induction. It is trivially true for \( T+1 \), since \( \omega_{T+1} = 1 \) and \( J_{T+1} = 0 \). We hypothesize now that it is true for some \( t+1 \). Then,
\[
\frac{\omega_t'}{\omega_t} = -\frac{\lambda_t}{1 - \lambda_t} + \frac{\omega_{t+1}}{\omega_{t+1}} \quad \text{[by differentiating (F-15)]}
\]
\[
\geq -\frac{\lambda_t}{1 - \lambda_t} + \frac{\alpha \omega_{t+1} J_{t+1}}{p_t} \quad \text{[by the induction hypothesis]}
\]
\[
\geq \alpha \omega_t J_t \left( \frac{\lambda_t}{1 - \lambda_t} + \frac{J_{t+1}}{p_t} \right) \quad \text{[by (F-21)]}
\]
\[
= \alpha \omega_t J_t \frac{\lambda_t p_t + (1 - \lambda_t) J_{t+1}}{(1 - \lambda_t) p_t} \quad \text{[by (F-15)]}
\]
\[
= \frac{\alpha \omega_t J_t}{(1 - \lambda_t)^2 p_t} \quad \text{[by (F-17)]}
\]
\[
\geq \frac{\alpha \omega_t J_t}{p_t} \quad \text{[since } 1 - \lambda_t < 1 \text{ and } p_t \text{ is decreasing in } t \]

and the proof for this case is complete.

**Case (2):** \( \lambda(p) = \alpha - \beta p, \alpha \in (0, 1], \beta > 0, p \in [0, \frac{\alpha}{\beta}] \). Let \( b < \frac{\alpha}{\beta} \). As before,
\[
p_t = \pi(b + V_{t+1}) = \frac{1}{2} \left( \frac{\alpha}{\beta} + b + V_{t+1} \right) = \frac{1}{2} \left( \frac{\alpha}{\beta} + b \omega_{t+1} + J_{t+1} \right), \quad t = 1, \ldots, T, \quad (F-22)
which also shows that \( p_t \) is decreasing in \( t \) and that

\[
p'_t = \frac{1}{2} \omega_{t+1}, \quad t = 1, \ldots, T \tag{F-23}
\]

\[
\lambda'_t = -\beta p'_t = -\frac{\beta}{2} \omega_{t+1}, \quad t = 1, \ldots, T \tag{F-24}
\]

By substituting for \( p_t \) in the main recursion

\[
V_t = \lambda_t (p_t - b) + (1 - \lambda_t)V_{t+1},
\]

we obtain

\[
V_t = V_{t+1} + \frac{\lambda_t}{2} \left( \frac{\lambda(b)}{\beta} - V_{t+1} \right), \quad t = 1, \ldots, T, \tag{F-25}
\]

which in conjunction with Lemma 3.1 also shows that

\[
\frac{\lambda(b)}{\beta} - V_{t+1} \geq 0, \quad t = 1, \ldots, T, \tag{F-26}
\]

Also, for all \( t = 2, \ldots, T \)

\[
(1 - \lambda_t)^2 \lambda_t \leq (1 - \lambda_t) \lambda_t \quad \text{[since } 1 - \lambda_t < 1]\]

\[
= \lambda_{t-1} - (\lambda_{t-1} - \lambda_t) - \lambda_t^2
\]

\[
= \lambda_{t-1} - \frac{\beta}{2} (V_{t+1} - V_t) - \lambda_t^2 \quad \text{[since } \lambda_t = \alpha - \beta p_t \text{ and (F-22)]}
\]

\[
= \lambda_{t-1} - \frac{\beta \lambda_t}{4} \left( V_{t+1} - \frac{\lambda(b)}{\beta} \right) - \lambda_t^2 \quad \text{[by (F-25)]}
\]

\[
= \lambda_{t-1} - \frac{\beta \lambda_t}{4} \left( V_{t+1} - \frac{\lambda(b)}{\beta} + \frac{4 \beta \lambda_t}{\beta} \right)
\]

\[
= \lambda_{t-1} - \frac{\beta \lambda_t}{4} \left( V_{t+1} - 2V_{t+1} + \frac{\lambda(b)}{\beta} \right) \quad \text{[since } \lambda_t = \alpha - \beta p_t \text{ and (F-22)]}
\]

\[
= \lambda_{t-1} - \frac{\beta \lambda_t}{4} \left( \frac{\lambda(b)}{\beta} - V_{t+1} \right)
\]

\[
\leq \lambda_{t-1}. \quad \text{[by (F-26)]}
\]
These allow us to express as above

\[
\left( \frac{J_{t-1}}{J_t} \right)' = \frac{1}{J_t^2} \left( (p_{t-1} - J_t) \lambda'_{t-1} J_t + \lambda_{t-1} p'_{t-1} J_t - \lambda_{t-1} p_{t-1} J_t' \right)
\]

\[
= \frac{1}{J_t^2} \left( -\frac{\beta}{2} (p_{t-1} - J_t) \omega_t J_t + \frac{1}{2} \lambda_{t-1} \omega_t J_t + \lambda_{t-1} p_{t-1} b \omega_t' \right)
\]

\[
= \frac{\lambda_{t-1} p_{t-1} b \omega_t}{J_t^2} \left( \left[ 1 - \frac{\beta (p_{t-1} - J_t)}{\lambda_{t-1}} \right] \frac{J_t}{2 p_{t-1} b} + \frac{\omega_t'}{\omega_t} \right)
\]

\[
= \frac{\lambda_{t-1} p_{t-1} b \omega_t}{J_t^2} \left( \left[ 1 - \frac{\beta (1 + b \omega_t)}{\lambda_{t-1}} \right] \frac{J_t}{2 p_{t-1} b} + \frac{\omega_t'}{\omega_t} \right)
\]

\[
= \frac{\lambda_{t-1} p_{t-1} b \omega_t}{J_t^2} \left( -\frac{\beta \omega_t J_t}{2 \lambda_{t-1} p_{t-1}} + \frac{\omega_t'}{\omega_t} \right),
\]

for all \( t = 2, \ldots, T \). Thus, it suffices to show that the inequality \( \omega_t' \geq \frac{\beta \omega_t^2 J_t}{2 \lambda_{t-1} p_{t-1}} \) holds for all \( t = 2, \ldots, T + 1 \). We will use induction. It is trivially true for \( T + 1 \), since \( \omega_{T+1} = 1 \) and \( J_{T+1} = 0 \). We hypothesize now that it is true for some \( t + 1 \). Then,

\[
\omega_t' = (V_t')'
\]

\[
= (-\lambda_t + (1 - \lambda_t)V_{t+1}')
\]

\[
= -\lambda_t \omega_{t+1} + (1 - \lambda_t) \omega_{t+1}'
\]

\[
\geq -\lambda_t \omega_{t+1} + (1 - \lambda_t) \frac{\beta \omega_{t+1}^2 J_{t+1}}{2 \lambda_t p_t}
\]

\[
= \frac{\beta}{2} \omega_{t+1}^2 + (1 - \lambda_t) \frac{\beta \omega_{t+1}^2 J_{t+1}}{2 \lambda_t p_t}
\]

\[
= \frac{\beta \omega_{t+1}^2 J_t}{2 \lambda_t p_t}
\]

\[
= \frac{\beta \omega_{t+1}^2 J_t}{2 (1 - \lambda_t)^2 \lambda_t p_t}
\]

\[
= \frac{\beta \omega_{t+1}^2 J_t}{2 \lambda_{t-1} p_{t-1}}
\]

[by the induction hypothesis]

\[
= \frac{\beta \omega_t^2 J_t}{2 \lambda_{t-1} p_{t-1}}
\]

[since \((1 - \lambda_t)^2 \lambda_t \leq \lambda_{t-1}\) and \(p_t\) is decreasing in \(t\)]

and the proof is complete.

\[\square\]

**G Proofs for Section 6**

**Proof of Proposition 6.1** Claim 1: \( J^i = J^E \). We first deal with contracts with early repayment option. The DM’s problem at \( T \) is the same with the one we analyzed in Section 4. Thus, the DM charges \( p_t^i (b, 1) \), and the value function \( V_T (b, y) \) is as in Proposition 4.1. At \( T - 1 \), the DM can follow
a strategy of covering the debt either in one, or two periods. We analyze these separately.

• When the DM plans on covering the debt in one period, his problem can be expressed as

\[
\max_{p \in (\gamma b, \beta)} f^E_t(p, \gamma),
\]

where \( f^E_t(p, \gamma) := \lambda(p) (p - \gamma b + V_T(0,1) - V_T(b,1)) \) in this case. The price he charges then is \( q^E_t(b, \gamma) := \frac{1}{2} \left( \gamma b + V_T(b,1) - V_T(0,1) + \frac{\alpha}{\beta} \right) \).

Having characterized the DM’s pricing policy, we can now express the debt value

\[
D(\{B, \gamma\}) = \lambda(q^E_t(B, \gamma)) \cdot \gamma B + (1 - \lambda(q^E_t(B, \gamma))) \cdot \lambda(p^\dagger_T(B,1)) \cdot B,
\]

where the first term corresponds to the expected (discounted) debt payment the debtholders receive if the DM makes a sale at \( T - 1 \), and the second term in case he makes a sale only at \( T \).

Similarly, expected revenues are

\[
J(\{B, \gamma\}) = \lambda(q^E_t(B, \gamma))(q^E_t(B, \gamma) + V_T(0,1)) + (1 - \lambda(q^E_t(B, \gamma))) \cdot \lambda(p^\dagger_T(B,1)) \cdot p^\dagger_T(B,1),
\]

where the terms have similar interpretation as above (see also the proof of Proposition 4.3 for similar derivations).

By equation (6b), note that \( B \) is essentially a function of \( \gamma \). In case this equation has multiple positive roots, we assume that the smallest one is always preferred; this is because higher debt repayment \( B \) will lead to higher efficiency losses for the regime where the DM charges a price to cover the debt in one period (see, for example, Figure 2 and the discussion in that Section). It can also be readily checked that

\[
D(\{0, \gamma\}) = D\left(\left\{2 \frac{\alpha}{\beta}, \gamma\right\}\right) = 0,
\]

which implies that

\[
\frac{\partial D(\{B, \gamma\})}{\partial B} \geq 0
\]

is a necessary condition for the smallest positive root of (6b).

We now argue that expected revenues \( J(\{B, \gamma\}) \) are non-decreasing in \( \gamma \), implying that \( \gamma = 1 \) would be an optimal early payment discount. In particular note that

\[
\frac{dJ(\{B, \gamma\})}{d\gamma} = \frac{\partial J(\{B, \gamma\})}{\partial \gamma} + \frac{\partial J(\{B, \gamma\})}{\partial B} \frac{dB}{d\gamma},
\]

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where the derivative of $B$ with respect to $\gamma$ can be obtained using the Implicit Function Theorem for \((6b)\)

\[
\frac{\partial D(\{B, \gamma\})}{\partial \gamma} + \frac{\partial D(\{B, \gamma\})}{\partial B} dB \frac{d\gamma}{d\gamma} = 0.
\]

One can then readily verify, using sum-of-square techniques, that the set

\[
\left\{ (\beta, B, \gamma, d) : \frac{dJ(\{B, \gamma\})}{d\gamma} \leq 0, D(\{B, \gamma\}) = d, \frac{\partial D(\{B, \gamma\})}{\partial B} \geq 0,\beta \in (0, 1], B \geq 0, \gamma \in (0, 1], d > 0 \right\}
\]

is empty.

- When the DM plans on covering the debt in two periods, we follow a similar approach. In particular, his problem can be expressed as

\[
\max_{p \in [0, \gamma_B]} f^E_m(p, \gamma),
\]

where $f^E_m(p, \gamma) := \lambda(p) \left( V_T(B - \frac{p}{\gamma}, 1) - V_T(b, 1) \right)$ in this case. The price he charges then is

\[
q^E_m(b, \gamma) := \frac{2\beta b \gamma + \alpha(1 - 2\gamma) + \sqrt{\alpha^2 + 2\alpha(\alpha - \beta b)\gamma + 4(\alpha - \beta b)^2\gamma^2}}{3\beta}.
\]

Having characterized the DM’s pricing policy, we can now express the debt value

\[
D(\{B, \gamma\}) = \lambda(q^E_m(B, \gamma)) \cdot \left[ q^E_m(B, \gamma) + \lambda \left( p^*_T \left( b - \frac{q^E_m(B, \gamma)}{\gamma}, 1 \right) \right) \cdot \left( b - \frac{q^E_m(B, \gamma)}{\gamma} \right) \right] + \left( 1 - \lambda(q^E_m(B, \gamma)) \right) \cdot \lambda(p^*_T(B, 1)) \cdot B.
\]

Similarly, expected revenues are

\[
J(\{B, \gamma\}) = \lambda(q^E_m(B, \gamma)) \cdot \left[ q^E_m(B, \gamma) + \lambda \left( p^*_T \left( b - \frac{q^E_m(B, \gamma)}{\gamma}, 1 \right) \right) \cdot p^*_T \left( b - \frac{q^E_m(B, \gamma)}{\gamma}, 1 \right) \right] + \left( 1 - \lambda(q^E_m(B, \gamma)) \right) \cdot \lambda(p^*_T(B, 1)) \cdot p^*_T(B, 1).
\]

We now argue that expected revenues $J(\{B, \gamma\})$ are non-decreasing in $\gamma$, implying that $\gamma = 1$ would be an optimal early payment discount in this case as well. In particular, one can then readily verify, using sum-of-square techniques, that the set \((G-27)\) is empty.

Since $\gamma = 1$ is without loss an optimal early payment discount in all cases, we have $J^* = J^E$.

**Claim 2:** $J^* \leq J^R$. We now deal with contracts with debt relief. If $r = 0$, we recover the plain
contract. If \( r = 1 \) debtholders have the option of adjusting the debt repayment at the beginning of period \( T \) so as to maximize debt value. That is, if the outstanding debt is \( b_T > 0 \), debtholders adjust debt repayment by solving

\[
\max_{0 \leq b \leq b_T} \lambda(\pi(b)) \cdot b.
\]

Thus, debt is adjusted to \( \min\{p^*_T, b_T\} \). Consequently, no debt relief will take place if the outstanding debt is low enough, \( b_T \leq p^*_T \).

As we shall prove later, the DM’s price at \( T - 1 \) under an optimal contract with debt relief with \( r = 1 \) is always such that it leads to no debt relief when a sale takes place. That is, the DM’s price at \( T - 1 \) is always higher than \( B - p^*_T \).

Using the above observation as a fact, we now analyze the cases where the DM follows a strategy of covering the debt in one or two periods at \( T - 1 \) separately. Note also that if \( r = 0 \), or if \( r = 1 \) and \( B \leq p^*_T \), debtholders would never adjust the debt. In other words, the contract \( \{B, 0\} \) is essentially a plain contract with debt repayment \( B \), and so \( \{B, 1\} \) when \( B \leq p^*_T \). We henceforth deal with contracts for which \( r = 1 \) and \( B > p^*_T \):

- **When the DM plans on covering the debt in one period, his problem can be expressed as**

\[
\max_{p \in (b, \frac{b}{2})} f^R_l(p),
\]

where \( f^R_l(p) := \lambda(p) (p - b + V_T(0, 1) - V_T(p^*_T, 1)) \) in this case. The price he charges then is \( q^R_l(b) := \frac{1}{2} \left( b + V_T(p^*_T, 1) - V_T(0, 1) + \frac{b}{p} \right) \).

Having characterized the DM’s pricing policy, we can now express the debt value

\[
D(\{B, 1\}) = \lambda(q^R_l(B)) \cdot B + (1 - \lambda(q^R_l(B))) \cdot \lambda(p^*_T(p^*_T, 1)) \cdot p^*_T,
\]  

(G-28)

where the first term corresponds to the debt payment the debtholders receive if the DM makes a sale at \( T - 1 \), and the second term in case he makes a sale only at \( T \)—after his debt is adjusted.

Similarly, expected revenues are

\[
J(\{B, 1\}) = \lambda(q^R_l(B))(q^R_l(B) + V_T(0, 1)) + (1 - \lambda(q^R_l(B))) \cdot \lambda(p^*_T(p^*_T, 1)) \cdot p^*_T(p^*_T, 1).
\]  

(G-29)

- **When the DM plans on covering the debt in two periods, we follow a similar approach. In particular, his problem can be expressed as**

\[
\max_{p \in [0, b]} f^R_m(p),
\]

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To complete the proof of this claim, we now revisit the possibility of the DM charging a price that is lower than $B$ at the beginning of $T$. Under such circumstances, whether the DM makes a sale at $T$ or not, his outstanding debt at the beginning of $T$ would remain higher than $p_T^*$, and thus trimmed to that level. Because the DM becomes indifferent about the price he charges at $T - 1$, we assume that he charges a revenue-maximizing price. We distinguish two cases:

\[
q_m^E(b) := \frac{4\beta b - 2\alpha + \sqrt{19\alpha^2 - 16\alpha\beta b + 4\beta^2 b}}{6\beta}.
\]

Having characterized the DM’s pricing policy, we can now express the debt value

\[
D(\{B, 1\}) = \lambda(q_m^R(B)) \cdot \left[q_m^R(B) + \lambda \left(p_T^+(b - q_m^R(B), 1)\right) (b - q_m^R(B))\right]
+ (1 - \lambda(q_m^R(B))) \cdot \lambda(p_T^+(p_T^*, 1)) \cdot p_T^*.
\]  

(G-30)

Similarly, expected revenues are

\[
J(\{B, 1\}) = \lambda(q_m^R(B)) \cdot \left[q_m^R(B) + \lambda \left(p_T^+(b - q_m^R(B), 1)\right) p_T^+(b - q_m^R(B), 1)\right]
+ (1 - \lambda(q_m^R(B))) \cdot \lambda(p_T^+(p_T^*, 1)) \cdot p_T^*.
\]  

(G-31)

We now show that $J^t \leq J^R$ need not always hold with equality, nor with strict inequality. To this end, consider the following instances:

- For $d = 2$, $\alpha = 1$ and $\beta = 0.13$, by solving (6b) we obtain that $\{4.11\}$ is an optimal plain contract, under which the DM charges a price $q_\ell = 5.15$, yielding revenues $J^t = J(\{4.11\}) = 3.26$. Under a contract with debt relief ($r = 1$ and $B > p_T^*$), by solving (6b) we obtain $B = 4.2$, $q_\ell^R = 5.22$ and $J(\{4.2, 1\}) = 3.28$ when the DM follows a single-period debt-covering strategy. Consequently, $J^t < J(\{4.2, 1\}) \leq J^R$.

- For $d = 2$, $\alpha = 1$ and $\beta = 0.1$, by solving (6b) we obtain that $\{3.32\}$ is an optimal plain contract, under which the DM charges a price $q_\ell = 5.97$, yielding revenues $J^t = J(\{3.32\}) = 4.74$. Under a contract with debt relief ($r = 1$ and $B > p_T^*$), by solving (6b) we obtain $B = 10.2$, $q_\ell^R = 9.17$ and $J(\{10.2, 1\}) = 2.7$ when the DM follows a single-period debt-covering strategy; $B = 13.3$, $q_\ell^R = 9.18$ and $J(\{13.3, 1\}) = 2.64$ when the DM follows a two-period strategy. Thus, the optimal contract with debt relief has $r = 0$, in particular, $\{3.32, 0\}$, and $J^t = J^R$ for this instance.

To complete the proof of this claim, we now revisit the possibility of the DM charging a price that is lower than $B - p_T^*$ at $T - 1$, under an optimal contract with debt relief, $r = 0$ and $B > p_T^*$. Under such circumstances, whether the DM makes a sale at $T - 1$ or not, his outstanding debt at the beginning of $T$ would remain higher than $p_T^*$, and thus trimmed to that level. Because the DM becomes indifferent about the price he charges at $T - 1$, we assume that he charges a revenue-maximizing price. We distinguish two cases:
When $B - p_T^* > p_T^*$, the revenue-maximizing price that is lower than $B - p_T^*$ is precisely $p_{T-1}^*(2) = p_T^*$. Thus, the debt value can be written in this case as

$$\tilde{D}({\{B, 1\}}) = \lambda(p_T^*) \cdot \left[ p_T^* + \lambda \left( p_T^*(p_T^*, 1) \right) p_T^* \right] + (1 - \lambda(p_T^*)) \cdot \lambda(p_T^*(p_T^*, 1)) \cdot p_T^*. $$

Similarly, expected revenues are

$$\tilde{J}({\{B, 1\}}) = V_T(0, 1) + \lambda(p_t^*(p_T^*, 1)) \cdot p_T^*(p_T^*, 1).$$

Suppose now that such a contract is optimal. It will therefore yield higher revenues than a debt relief contract $\{B', 1\}$ that induces a price $q_t^*(B') > p_T^*$. In other words,

$$\left\{ (\beta, B, B', d) : \tilde{J}({\{B, 1\}}) > J(\{B', 1\}), \quad \tilde{D}({\{B, 1\}}) = d, B > 2p_T^*, D(\{B', 1\}) = d, B' > p_T^*, \beta > 0, d > 0 \right\}$$

is non-empty, where the expressions for $J$ and $D$ are as in the single-period debt-covering strategy case above. However, one can readily use sum-of-squares techniques to show that the above set is, in fact, empty.

- When $B - p_T^* \leq p_T^*$, the revenue-maximizing price that is lower than $B - p_T^*$ is precisely $B - p_T^*$. Thus, the debt value can be written in this case as

$$\tilde{D}({\{B, 1\}}) = \lambda(B - p_T^*) \cdot \left[ B - p_T^* + \lambda \left( p_T^*(p_T^*, 1) \right) p_T^* \right] + (1 - \lambda(B - p_T^*)) \cdot \lambda(p_T^*(p_T^*, 1)) \cdot p_T^*. $$

Similarly, expected revenues are

$$\tilde{J}({\{B, 1\}}) = \lambda(B - p_T^*) \cdot (B - p_T^*) + \lambda(p_T^*(p_T^*, 1)) \cdot p_T^*(p_T^*, 1).$$

Suppose now that such a contract is optimal. It will therefore yield higher revenues than a debt relief contract $\{B', 1\}$ that induces a price $q_t^*(B') > p_T^*$. In other words,

$$\left\{ (\beta, B, B', d) : \tilde{J}({\{B, 1\}}) > J(\{B', 1\}), \quad \tilde{D}({\{B, 1\}}) = d, B \leq 2p_T^*, B > p_T^*, D(\{B', 1\}) = d, B' > p_T^*, \beta \in (0, 1], d > 0 \right\}$$

is non-empty, where the expressions for $J$ and $D$ are as in the single-period debt-covering strategy case above. However, one can readily use sum-of-squares techniques to show that the above set is, in fact, empty.
Claim 3: $J^R < J^A$. We now compare contracts with debt relief and contracts with debt amortization. We first claim that, in equilibrium, any debt amortization contract $\{B, \theta\}$ must satisfy $B \leq p^*_T$. To that end, consider two contracts $\{B, \theta\}$ and $\{B', \theta'\}$ such that $B \leq p^*_T < B'$. The DM’s pricing policy for a given debt amortization contract is fully characterized in Proposition [G.1]. When $B \leq p^*_T$, the DM charges a price $q^*_T(B) \geq B$ in period $T - 1$, and the price $p^*_T$ is always charged in period $T$. Therefore, the value of the debt and the expected revenues under the contract $\{B, \theta\}$ can be written as:

\[
D(\{B, \theta\}) = \lambda(q^1_T(B)) B + \left[1 - \lambda(q^1_T(B))\right] \lambda(p^*_T) B \tag{G-32a}
\]

\[
J(\{B, \theta\}) = \lambda(q^1_T(B)) q^1_T(B) + \lambda(p^*_T) p^*_T. \tag{G-32b}
\]

It is worth noting that the expressions above are independent of $\theta$. Under contract $\{B', \theta'\}$, two cases can arise, depending on whether $B' \leq \tilde{B}$ (see Proposition [G.1]):

- if $B' \leq \tilde{B}$, the DM charges a price $q^2_T(B') \geq B'$ at $T - 1$, and $p^*_T < B'$ is always charged at $T$. The value of the debt and the expected revenues are thus:

\[
D'_T(\{B', \theta'\}) := \lambda(q^2_T(B')) B' + \left[1 - \lambda(q^2_T(B'))\right] \lambda(p^*_T) p^*_T
\]

\[
J'_T(\{B', \theta'\}) := \lambda(q^2_T(B')) q^2_T(B') + \lambda(p^*_T) p^*_T.
\]

As with the contract $\{B, \theta\}$, these expressions do not depend on $\theta'$. Using Proposition [G.1] to express $D, J, D'_T$ and $J'_T$ as functions of $B, \theta, B', \theta', \alpha$ and $\beta$, we can then verify through sums-of-squares techniques that the set

\[
\{ (\alpha, \beta, B, B', d) : J'_T(\{B', \theta'\}) > J(\{B, \theta\}), D'_T(\{B', \theta'\}) = d, D(\{B, \theta\}) = d, \\
B' \geq p^*_T, B \leq p^*_T, \alpha \in [0, 1], \beta > 0, d > 0 \}
\]

is always empty, which proves that the contract $\{B, \theta\}$ dominates $\{B', \theta'\}$.

- if $\tilde{B} < B' \leq \frac{2a}{d}$, the DM charges a price $q^0_m(B') \leq B'$ at $T - 1$. Provided that this results in a sale, the DM then charges a price $p^*_T(B' - q^0_m(B'))$ at $T$; otherwise, the price $p^*_T$ is charged. The debt value and the expected revenues thus become:

\[
D'_m(\{B', \theta'\}) := \lambda(q^0_m(B', \theta')) \left[q^0_m(B', \theta') + \lambda(p^*_T(B' - q^0_m(B', \theta'))) \cdot (B' - q^0_m(B', \theta'))\right]
\]

\[+ \left[1 - \lambda(q^0_m(B', \theta'))\right] \lambda(p^*_T) p^*_T
\]

\[
J'_m(\{B', \theta'\}) := \lambda(q^0_m(B', \theta')) \left[q^0_m(B', \theta') + \lambda(p^*_T(B' - q^0_m(B', \theta'))) \cdot p^*_T(B' - q^0_m(B', \theta'))\right]
\]

\[+ \left[1 - \lambda(q^0_m(B', \theta'))\right] \lambda(p^*_T) p^*_T.
\]
Note that two sub-cases can be further distinguished depending on \( \theta' \), which determines \( q_m(B', \theta') \) per Proposition \( \text{G.1} \). For each case, using Proposition \( \text{G.1} \) to express \( D, J, D'_m, J'_m \) as functions of \( B, \theta, B', \theta', \alpha, \beta \) and \( \theta' \), we can verify again through sums-of-squares techniques that the set

\[
\begin{align*}
\left\{ \left( \alpha, \beta, B, B', \theta', d \right) : J'_m(\{B', \theta'\}) &> J(\{B, \theta\}), D'_m(\{B', \theta'\}) = d, D(\{B, \theta\}) = d, \\
B' &\geq p^*_r, \ B \leq p^*_r, \ \alpha \in [0, 1], \ \beta > 0, \ d > 0 \right\}
\end{align*}
\]

is also empty, which confirms that the contract \( \{B, \theta\} \) again dominates the contract \( \{B', \theta'\} \).

We now proceed to complete the proof that \( J^R < J^A \). Based on the argument above, we only need to consider contracts with debt amortization \( \{B, \theta\} \) with \( B \leq p^*_r \). Consider a contract allowing debt relief \( \{B', r\} \), with \( r \in \{0, 1\} \). Based on the argument in Claim 2, we distinguish two cases:

- For \( r = 1 \), let \( D'(\{B', 1\}) \) and \( J'(\{B', 1\}) \) denote the corresponding debt value and expected revenues, respectively. The expressions for \( D' \) and \( J' \) are available from Case 2, and are respectively given by either \( \text{(G-28)}-\text{(G-29)} \) or \( \text{(G-30)}-\text{(G-31)} \), depending on whether the DM plans on covering the debt with one or two sales. In each case, we can use sums-of-squares techniques to verify that the set

\[
\begin{align*}
\left\{ \left( \alpha, \beta, B, B', d \right) : J'(\{B', 1\}) &\geq J(\{B, \theta\}), \\
D'_m(\{B', 1\}) & = d, \ D(\{B, \theta\}) = d, \ B \leq p^*_r, \ \alpha \in [0, 1], \ \beta > 0, \ d > 0 \right\}
\end{align*}
\]

is always empty, where \( D(\{B, \theta\}) \) and \( J(\{B, \theta\}) \) are given by \( \text{(G-32a)}-\text{(G-32b)} \). This proves that a contract with debt amortization always strictly dominates one with debt relief here.

- For \( r = 0 \), the debt relief contract becomes a plain contract. Based on the analysis in the proof of Proposition \( \text{4.1} \) the debt value \( D'(\{B', 0\}) \) for this contract is given by:

\[
D'(\{B', 0\}) = \begin{cases} 
\lambda(q_\ell(B'))B' + \left[ 1 - \lambda(q_\ell(B')) \right] \lambda(p^*_T(B', 1))B', \quad &\text{if } B' \leq \hat{b} \\
\lambda(q_m(B')) \left[ q_m(B') + \lambda(p^*_T(B' - q_m(B'), 1)) \cdot (B' - q_m(B')) \right] \\
\quad + \left[ 1 - \lambda(q_m(B')) \right] \lambda(p^*_T(B', 1))B', \quad &\text{if } B' > \hat{b},
\end{cases}
\]

where \( q_\ell(B') \) and \( q_m(B') \) are given by \( \text{(E-5)} \) and \( \text{(E-6)} \), respectively. Similarly, the expected
Claim 4: Consider a contract with debt amortization Proposition G.1. Conversely, if $B > \hat{B}$ upon making a sale would be $p' > p^*$, then even if the DM actually charged $\beta$ in period $T$, the price charged in period $T$ will be $\tilde{p}' > p^*$, so that again $J(\{B, \theta\}) < J^*$. Thus, $J^* < J^*$. \qed

Proposition G.1. Consider a contract with debt amortization $\kappa = \{B, \theta\}$, and let $b_T$ denote the DM’s outstanding debt in period $T$. The DM’s prices are given by:

\[
\forall y, \tilde{p}_T(b_T, y) = \begin{cases} 
    p^\dagger(b_T, y), & \text{if } b_T \leq (1 - \theta)B \\
    p^*_T(1), & \text{otherwise}
\end{cases}
\]

\[
\tilde{p}_{T-1}(B, 2) = \begin{cases} 
    q^\dagger(B) := \frac{1}{\theta} \left[ B(2 - \alpha) + \frac{2\alpha}{\beta} \right], & \text{if } 0 < B \leq \frac{\alpha}{2\beta} \\
    q^\ddagger(B) := \frac{2B + \alpha(1 - \alpha)}{8\beta}, & \text{if } \frac{\alpha}{2\beta} < B \leq \hat{B} \\
    q^\theta_m(B, \theta), & \text{if } \hat{B} < B \leq \frac{2\alpha}{\beta},
\end{cases}
\]

where $q^\theta_m(B, \theta) := \begin{cases} 
    \frac{\alpha + B}{4\beta}, & \text{if } \theta \leq \frac{1}{3} + \frac{\alpha}{3\beta} \\
    \theta B, & \text{otherwise},
\end{cases}$
and \( \tilde{B} \in \left[ \frac{\alpha}{2\beta}, \frac{\alpha}{\beta} \right] \) depends on \( \alpha, \beta \) (and \( \theta \), if \( \theta > \frac{1}{3} + \frac{\alpha}{3\beta} \)). Furthermore,

\[
\tilde{p}_{T-1}(B, 2) > p_{T-1}^*(2) = \frac{\alpha}{2\beta} \\
\tilde{p}_{T-1}(B, 2) \geq B, \ \forall B \in (0, \tilde{B}) \\
\tilde{p}_{T-1}(B, 2) < B, \ \forall B \in (\tilde{B}, \frac{2\alpha}{\beta}].
\]

**Proof of Proposition G.7.** The price charged in period \( T \) is either \( p_T^*(1) \) (if \( b_T > (1-\theta)B \), i.e., the DM loses control) or \( p_T^*(b_T) \) (if the DM retains control). In the latter case, the value function at time \( T \) would thus be given by the (proof of) Proposition 4.1. We separate the analysis for period \( T-1 \) into different cases, depending on the value of \( B \).

**Case 1.** \( B \in [0, \frac{\alpha}{2\beta}] \). When no sale occurs at \( T-1 \), the DM loses control but still achieves a positive expected payoff of \( \lambda(p_T^*(1))(p_T^*(1) - b) = \frac{\alpha(1-2\beta)}{4\beta} \). Therefore, \( \tilde{p}_{T-1}(B, 2) \in \arg \max_p f(p) \), where

\[
f(p) := \begin{cases} 
    f_\ell(p) := \lambda(p)\left[p - B + \frac{\alpha}{2\beta}\lambda\left(\frac{\alpha}{2\beta}\right) - \frac{\alpha(2B-B)}{4\beta}\right], & p \in [B, \frac{\alpha}{\beta}] \\
    f_m(p) := \lambda(p)\left[\lambda^2(B-p) - \frac{\alpha(2B-B)}{4\beta}\right], & p \in (\theta B, B) \\
    f_h(p) := \lambda(p)\left[\frac{\alpha(2B-B)\lambda(B-p)}{4\beta} - \frac{\alpha(2B-B)}{4\beta}\right], & p \in [0, \theta B). 
\end{cases}
\]

We analyze each of the maximization problems separately, and compare their optimal values.

- For max_{\( p \in [B, \frac{\alpha}{\beta}] \)} \( f_\ell(p) \), note that \( f_\ell \) is a concave, quadratic function, achieving its maximum at

\[
q_\ell^1(B) := \frac{1}{4}\left[B(2 - \alpha) + \frac{2\alpha}{\beta}\right]. \tag{G-33}
\]

Furthermore, \( B \leq q_\ell^1(B) \leq \frac{\alpha}{\beta} \) always holds, so that \( q_\ell^1(B) \) is also the optimal constrained decision.

- For max_{\( p \in (\theta B, B) \)} \( f_m(p) \), note that \( f_m \) is cubic. By solving the equation \((f_m)'(p) = 0\), we obtain the critical points \(-\alpha + 2\beta + \sqrt{\frac{7\alpha^2 - 10\alpha \beta B + \beta^2 B^2}{3}}\). It can be checked that for \( B \in [0, \frac{\alpha}{2\beta}] \), the critical point given by a plus sign takes a value larger than \( B \) and corresponds to a local maximum, while the other critical point is negative, and corresponds to a local minimum. Therefore, the optimal decision is always \( p \to B \), resulting in an optimal value \( f_m(B) \).

- For max_{\( p \in [0, \theta B) \)} \( f_h(p) \), note that \( f_h \) is concave, quadratic, with a maximum achieved at \( \frac{\alpha}{2\beta} \). Since \( \frac{\alpha}{2\beta} \geq B \geq \theta B \), it is optimal to take a price \( p \to \theta B \), resulting in a value of \( f_h(\theta B) \).

We now compare the values above. It can be checked that \( f_m(B) = f_\ell(B) \), and \( f_m(\theta B) - f_h(\theta B) = \frac{1}{4}\beta B^2(1 - \theta)^2(\alpha - \theta \beta B) \geq 0 \) (when \( B \leq \frac{\alpha}{2\beta} \)). Therefore, the DM’s price in Case 1 is always \( q_\ell^1(B) \). To complete this case, it can be readily checked that \( q_\ell^1(B) \geq p_{T-1}^* = \frac{\alpha}{2\beta} \), and that \( q_\ell^1(B) \geq B \).
Case 2: \( B \in \left( \frac{\alpha}{3\beta}, \frac{\alpha}{2\beta} \right) \). When no sale occurs at \( T - 1 \), the DM achieves a payoff of zero. Therefore, \( \tilde{p}_{T-1}(B, 2) \in \arg \max_p f(p) \), where

\[
f(p) := \begin{cases} 
  f_\ell(p) := \lambda(p) \left[ p - B + \frac{\alpha}{2\beta} \lambda \left( \frac{\alpha}{2\beta} \right) \right], & p \in [B, \frac{\alpha}{2\beta}] \\
  f_m(p) := \frac{\lambda(p) \lambda^2(B - p)}{4\beta}, & p \in [\theta B, B) \\
  f_h(p) := \frac{\lambda(p) \alpha [\alpha - 2\beta(B - p)]}{4\beta}, & p \in [B - \frac{\alpha}{2\beta}, \theta B) \\
  0, & p \in [0, B - \frac{\alpha}{2\beta}).
\end{cases}
\]

We again analyze the maximization problems separately, and compare their optimal values to determine the DM’s pricing decision.

- The problem \( \max_{p \in [B, \frac{\alpha}{2\beta}]} f_\ell(p) \) parallels the one considered in Case 1. In particular, note that \( f_\ell \) is a concave quadratic that achieves its maximum at the value \( q_\ell^2(B) := \frac{4\beta B + \alpha (4 - \alpha)}{8\beta} \). While \( q_\ell^2(B) \leq \frac{\alpha}{2\beta} \) always holds, note that \( q_\ell^2(B) \geq B \) if and only if \( B \leq \frac{\alpha (4 - \alpha)}{4\beta} \). Thus, the optimal price and expected payoff in this case are respectively given by

\[
p_\ell(B) = \begin{cases} 
  q_\ell^2(B), & B \in \left( \frac{\alpha}{3\beta}, \frac{\alpha (4 - \alpha)}{4\beta} \right] \\
  B, & B \in \left( \frac{\alpha (4 - \alpha)}{4\beta}, \frac{\alpha}{2\beta} \right]
\end{cases} \quad F_\ell(B) = \begin{cases} 
  \frac{(\alpha (4 + \alpha - 4\beta B))^2}{64\beta}, & B \in \left( \frac{\alpha}{3\beta}, \frac{\alpha (4 - \alpha)}{4\beta} \right] \\
  \frac{\alpha^2(\alpha - 2\beta B)}{4\beta}, & B \in \left( \frac{\alpha (4 - \alpha)}{4\beta}, \frac{\alpha}{2\beta} \right].
\end{cases}
\]

- For \( \max_{p \in [\theta B, B]} f_m(p) \), note that \( f_m \) is cubic. By solving the equation \( (f_m)'(p) = 0 \), we obtain the critical points \( B - \frac{\alpha}{2\beta} \) and \( q_m^\theta(B) := \frac{\alpha + \beta B}{3\beta} \). When \( B \in \left( \frac{\alpha}{2\beta}, \frac{\alpha}{3\beta} \right) \), it can be checked that the former critical point is always negative and corresponds to a local minimum, while \( q_m^\theta(B) \) corresponds to a local maximum, and always satisfies \( q_m^\theta(B) < B \). Furthermore, \( q_m^\theta(B) \geq \theta B \) if and only if \( \theta \leq \frac{1}{3} + \frac{\alpha}{3\beta B} \). Therefore, \( f_m \) is increasing on \( [\theta B, q_m^\theta(B)] \) and decreasing for \( p \geq q_m^\theta(B) \), so that the optimal price and payoff are respectively given by

\[
p_m(B) = \begin{cases} 
  q_m^\theta(B), & \theta \leq \frac{1}{3} + \frac{\alpha}{3\beta B} \\
  \theta B, & \text{otherwise}
\end{cases} \quad F_m(B) = \begin{cases} 
  \frac{(2\alpha - \beta B)^2}{27\beta}, & \theta \leq \frac{1}{3} + \frac{\alpha}{3\beta B} \\
  \frac{(\alpha - \beta B)(1 - \theta)\beta B^2}{4\beta}, & \text{otherwise}.
\end{cases}
\]

- For \( \max_{p \in [B - \frac{\alpha}{2\beta}, \theta B]} f_h(p) \), note that \( f_h \) is concave, quadratic, with a maximum achieved at \( q_h^\theta(B) := \frac{\alpha + 2\beta B}{4\beta} \). Under \( B \in \left( \frac{\alpha}{2\beta}, \frac{\alpha}{3\beta} \right] \), it can be checked that \( q_h^\theta(B) \geq B - \frac{\alpha}{2\beta} \) always holds,
and \( q^\theta_h(B) < \theta B \) if and only if \( \theta > \frac{1}{2} + \frac{\alpha}{4\beta B} \). Thus, the optimal price and payoff are given by

\[
p_h(B) = \begin{cases} 
\theta B, & \theta \leq \frac{1}{2} + \frac{\alpha}{4\beta B} \\
q^\theta_h(B), & \text{otherwise}
\end{cases}
\]

\[
F_h(B) = \begin{cases} 
\frac{\alpha(\alpha-2(1-\theta)\beta B)(\alpha-\theta B)}{4\beta}, & \theta \leq \frac{1}{2} + \frac{\alpha}{4\beta B} \\
\frac{\alpha(3\alpha-2\beta B)^2}{32\beta}, & \text{otherwise}.
\end{cases}
\]

We now compare the optimal values in the problems above. Letting \( C \) denote the set of constraints \( \{\alpha \geq 0, \alpha \leq 1, \beta \geq 0, B \geq \frac{\alpha}{2\beta}, B \leq \frac{\alpha}{\beta}, \theta \geq 0, \theta \leq 1\} \), it can be checked that:

\[
\{ (\alpha, \beta, \theta, B) : F_h(B) > F_m(B), \ C \} = \emptyset.
\]

This requires testing three different conditions, depending on \( \theta \), and the corresponding sets can always be shown to be empty using sums-of-squares techniques. Therefore, \( F_m(B) \geq F_h(B) \) always holds. Similarly, it can also be tested using sums-of-squares techniques that:

\[
\emptyset = \{ (\alpha, \beta, \theta, B) : F'_f(B) > F'_m(B), \ B \leq \frac{\alpha}{4\beta}, \ C \}
\]

\[
\emptyset = \{ (\alpha, \beta, \theta) : F_f(\alpha) \leq F_m(\alpha) \}
\]

\[
\emptyset = \{ (\alpha, \beta, \theta) : F_f(\alpha) > F_m(\alpha), \ \alpha \geq 0, \alpha \leq 1, \beta \geq 0, \theta \geq 0, \theta \leq 1 \}
\]

These results imply that there exists a \( \tilde{B} \) such that \( F_f(B) > F_f(B) \) for \( B \in \left[ \frac{\alpha}{2\beta}, \tilde{B} \right] \) and \( F_f(B) \leq F_m(B), \forall B \in \left[ \tilde{B}, \frac{\alpha(4-\alpha)}{4\beta} \right] \). This exactly yields the pricing policy in the statement of the proposition. To complete the proof, note that \( \tilde{p}_{T-1}(B, 2) \geq \frac{\alpha}{2\beta} \) is immediate, and also \( q^2_f(B) \geq B, q^2_m(B) \leq B \).

**Case 3:** \( B \in (\frac{\alpha}{2\beta}, \frac{4\beta}{2\beta}) \). In this case, the DM must rely on two sales, and his payoff is always zero when failing to make the intermediate sale. Therefore, \( \tilde{p}_{T-1}(B, 2) \in \arg \max_p f(p) \), where

\[
f(p) := \begin{cases} 
0, & p > \frac{\alpha}{\beta} \\
\frac{\lambda(p)\lambda^2(B-p)}{4\beta}, & p \in [\theta B, \frac{\alpha}{\beta}] \\
\frac{\lambda(p)\lambda(\alpha-2\beta(B-p))}{4\beta}, & p \in [B - \frac{\alpha}{2\beta}, \theta B] \\
0, & p < B - \frac{\alpha}{2\beta}.
\end{cases}
\]

The two optimization problems for \( f_m \) and \( f_h \) are identical to those considered in \((G-34)\) for Case 2. Therefore, the price that maximizes \( f_m \) is \( p_m(B) \) and the corresponding payoff is \( F_m(B) \), as per \((G-36)\). Since \( p_m(B) \leq \frac{\alpha}{\beta} \), this remains optimal in Case 3, as well. For the problem of maximizing \( f_h \), a
candidate price is $p_h(B)$ given by (G-37). However, in addition to the conditions in Case 2, it can be readily checked that $p_h(B) \geq B - \frac{\alpha}{2\beta}$ holds if and only if $B \leq \frac{3\alpha}{2\beta}$. Letting $F_h(B) = f_h\left(\max\{p_h(B), B - \frac{\alpha}{2\beta}\}\right)$ and using sums-of-squares techniques, it can be verified that:

$$\left\{(\alpha, \beta, B) : F_h(B) > f_m(B), \quad \alpha \geq 0, \alpha \leq 1, \beta \geq 0, B \geq \frac{\alpha}{\beta}, B \leq \frac{2\alpha}{\beta}, \theta \geq 0, \theta \leq 1\right\} = \emptyset.$$  

This requires testing three cases, depending on $\theta$ and $B$; the feasible set in each case is empty. This proves that the DM’s price is $q_m(B)$. To complete the proof, note that $\frac{\alpha}{2\beta} \leq p_m(B) \leq \frac{\alpha}{\beta} \leq B$.  \hfill \square