A Characterization of the Random Arrival Rule for Bankruptcy Problems

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Abstract

It is known that no additive division rules exist for bankruptcy problems. In this paper, we study a restricted additivity property which we call “feasible set additivity” (FSA). This property requires division rules to be additive when the set of feasible allocation vectors for a sum of problems does not include allocations that were unfeasible when considering each problem separately.

In addition, we characterize the random arrival rule as the only division rule satisfying FSA and equal treatment of equals for two and three-agent cases. We also show that this characterization holds when the endowment is small enough in relation to the claims, while the question of whether it holds in general remains open.

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The problem of how to distribute the liquidation value of a firm to its creditors has been studied with an axiomatic approach since [\textsuperscript{1}]O’Neill [1982]. A bankruptcy problem (or claims problem) is a situation in which \( N \) agents have claims over an endowment and the sum of their claims exceeds the value of the endowment. An arbitrator decides how to allocate the endowment among the agents using a division rule, which is a function that assigns an allocation such that each agent receives a non-negative award not exceeding his/her claim and the sum of awards equals the endowment.

Suppose that a firm with presence in Mexico and the USA goes bankrupt and that its creditors \( A \) and \( B \) are also present in both countries. The Mexican branch is worth $7 and owes $5 to \( A \) and $3 to \( B \); the American branch is worth $10, and owes $2 to \( A \) and $12 to \( B \). Should these problems be solved separately in each country or should they be solved as a single problem where the firm is worth $17 (the sum of the value of the Mexican and the American branches), that owes $7 and $15 to \( A \) and \( B \) respectively?

It turns out that solving the dispute as a single claims problem or keeping it as two separate problems may lead to different allocations of the endowment. This was noted by

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Bergantiños and Méndez-Naya (2001), who show that no additive division rules exist. To illustrate why this is the case, consider the following examples:

- The Mexican branch is worth $4, owes $10 to A and 0 to B, whereas the American branch is worth $6, owes 0 to A, and $10 to B; if these problems are solved separately, A and B would receive $4 and $6 respectively.

- The Mexican branch is worth $5, owes $10 to A and 0 to B, while the American branch is worth $5, owes 0 to A, and $10 to B. If these problems are solved separately, A and B would receive $5 each.

Note however that, when aggregating the Mexican and American branches into a single entity, both situations lead us to the same problem in which the consolidated firm is worth $10, and owes $10 to A and $10 to B. However, if there was an additive division rule, it would yield different allocations for the same problem. Hence there is no additive rule.

In the last example the arbitrator was bound by the fact that since one agent had a claim of zero, the total endowment had to be awarded to the other agent; however by aggregating the two problems, many new possible awards would be made feasible for the arbitrator (namely, dividing $10 between A and B), most of which were not admissible when considering the two disputes separately. We propose a weakening of additivity which we call feasible set additivity (FSA). FSA states that the rule should be additive when the set of possible awards in the aggregate problem is equivalent to the set of possible awards when considering each “component” problem.

Our main result is the characterization of the random arrival rule, as the only division rule that satisfies FSA and equal treatment of equals for the two and three-agent cases. Moreover, we prove this characterization in some subdomains of the N-agent case, although the question of whether this holds in general remains open.

Moreover, it is well-known that the random arrival rule corresponds to the Shapley value of the corresponding coalitional game. If our characterization holds in general, this result would link the random arrival rule and the Shapley value in terms of the axioms involved in their characterizations.

Other characterizations of the random arrival rule include Albizuri et al. (2010) who show that it is the only extension of the contested garment rule studied by Annavi and Maschler (1985) that satisfies a weakening of the consistency property, and Hwang (2015), who proves analogue characterizations to those provided for the Shapley value by Myerson (1980) in terms of the balanced contributions axiom, and by Hart and Mas-Colell (1989) in terms of the marginal contributions of a potential function. Regarding FSA, Sánchez-Pérez (2018) introduced it in a recent paper under the name “weak additivity” and independently found some of the results presented in this paper.

Other division rules have been characterized on the basis of additivity-like properties in various papers, for example Bergantiños and Vidal-Puga (2004), Marchant (2008), Alcalde et al. (2014), Flores-Szwagrzak et al. (2017), Arin et al. (2017) and Sánchez-Pérez (2018). For a full review of the literature and a more comprehensive survey of division rules and their properties, the reader is referred to Thomson (2003) and (2015).

2This property states that agents with equal claims should receive equal awards.
3See O’Neill (1982).
4The consistency property requires a rule to be invariant for certain problems that consist of subsets of the original agents. Albizuri et al. (2010) study a weakening of this property which they call “bilateral U-consistency”. 
The rest of the paper is organized as follows: Section 1 presents the model and Section 2 present the axioms along with the main results. For brevity, we have omitted the proofs from the main body and have included them in the Appendix.

1. The model

A claims (or bankruptcy) problem consists of $N$ agents, each holding a claim over an amount $E$ called the endowment.

**Definition 1.** A claims problem with agent set $N$ is a pair $(c,E)$ such that $E \geq 0$, $c \in \mathbb{R}^N_+$ and $\sum_{i=1}^{N} c_i \geq E$. We denote by $C^N$ the set of all bankruptcy problems of $N$ agents.

A division rule assigns to each problem an awards vector such that each agent receives a non-negative amount that is no larger than his/her claim and the sum of awards exhausts the endowment.

**Definition 2.** A division rule is a function $\phi$ that associates to each problem $(c,E) \in C^N$, an awards vector $\phi(c,E) \in \mathbb{R}^N$ such that $0 \leq \phi(c,E) \leq c$ and $\sum_{i=1}^{N} \phi_i(c,E) = E$.

The requirements that each agent must be awarded an amount between zero and his/her claim, and that the sum of awards should equal the endowment can be thought of basic guidelines that any arbitrator should abide by when settling a claims problem. With these considerations in mind, given a problem the arbitrator has a set of possible award vectors:

**Definition 3.** Let $(c,E) \in C^N$. The feasible set of this problem is:

$$F(c,E) = \left\{ x \in \mathbb{R}^N : 0 \leq x \leq c \text{ and } \sum_{i=1}^{N} x_i = E \right\}$$

One way to solve claims problems is by lining up the agents to receive their payment according to some given order. Each agent receives his/her claim if there is enough money to cover it after the agents that preceded him/her on the line were compensated, or the remaining balance otherwise:

**Definition 4.** Let $\pi \in \Pi^N$ be a permutation of the set of agents. The serial dictator rule, $SD^\pi$, is defined as

$$SD^\pi_i(c,E) = \min \left\{ c_i, \max \left( E - \sum_{\{j: \pi(j) < \pi(i)\}} c_j, 0 \right) \right\}$$

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5 Abusing notation, we consider $N$ to be a natural number as well as the set of positive integers less or equal than $|N|$, depending on the context.

6 Vector inequalities: $x \leq y$ if and only if $x_i \leq y_i$ for all $i \in N$.

7 Some authors do not include the efficiency constraint in the definition of the feasible set, stating only that the sum of the awards may not exceed the value of the endowment. However, most authors include the efficiency constraint in the definition of a division rule.

8 We denote the set of permutations of $N$ by $\Pi^N$. 

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Most people would agree that it would be unfair to choose a particular permutation and solve a problem with the corresponding serial dictator rule. One way to eliminate this unfairness is to compute the serial dictator rule for each permutation and take the average over all permutations. O’Neill (1982) introduces this rule, which is known as the random arrival rule.

Definition 5. The random arrival rule (RAR) is the average of the serial dictator rule over all permutations of $N$:

$$RA(c, E) = \frac{1}{N!} \sum_{\pi \in \Pi^N} SD^\pi (c, E)$$

2. Axioms and Results

When facing two problems with the same set of claimants the arbitrator has the task of allocating the endowment of each problem. The arbitrator can solve the problems separately or aggregate them into a consolidated problem. Would the result be the same in both cases?

Definition 6. A division rule $\phi$ is additive if for each $(c, E)$ and $(c', E') \in C^N$ it holds that $\phi (c + c', E + E') = \phi (c, E) + \phi (c', E')$

It is known that no division rule is additive, and as Bergantiños and Méndez-Naya (2001) show. The reason is that the non-negativity and claims boundedness restrictions affect the feasible set of a problem, and when adding two different problems the feasible set of the consolidated problem may differ from the sum of feasible sets of the original problems. We propose requiring the rule to be additive only in situations when the feasible set of the aggregate problem is the sum of the feasible sets of the original problems.

Definition 7. A division rule $\phi$ satisfies feasible set additivity (FSA) if for all problems $(c, E), (c', E') \in C^N$ such that $F(c + c', E + E') = F(c, E) + F(c', E')$, it holds that $\phi (c + c', E + E') = \phi (c, E) + \phi (c', E')$

Example 1 in the following chart shows a case where the feasible set of the aggregate problem is the sum of the feasible sets of its components. Let $P_1 = (4, 6; 8)$ and $P_2 = (5, 3; 7)$, note that the extreme points of the feasible set of each problem (line segments in the two-agent case) are the serial dictator awards of the two permutations of $\{1, 2\}$ – namely $(4, 4)$ and $(2, 6)$ for $P_1$, and $(5, 2)$ and $(4, 3)$ for $P_2$. Moreover, we can see that when adding the serial dictator awards of $P_1$ and $P_2$ for each permutation, we get the serial dictator awards for $P_1 + P_2 = (6, 9)$ and $(9, 6)$, and that $F(P_1) + F(P_2) = F(P_1 + P_2)$. These observations are in fact general results.

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10The sum of the feasible sets is the Minkowski sum.
Lemma 1. Let \((c, E) \in \mathcal{C}^N\), then \(F(c, E)\) is a convex polytope and its set of extreme points is \(\{SD^\pi(c, E) : \pi \in \Pi^N\}\).

Our next result states that no feasible awards are lost when adding two problems, that is, the feasible set of the sum of two problems contains the sum of the feasible sets of the original problems. Moreover, for a given set of problems, the feasible set correspondence behaves additively if and only if the serial dictator rule for every permutation behaves additively.

Theorem 1. Let \((c^1, E^1), \ldots, (c^m, E^m) \in \mathcal{C}^N\) and let \((c, E) = \sum_{\ell=1}^m (c^\ell, E^\ell)\). Then:

1) \(F(c, E) \supseteq \sum_{\ell=1}^m F(c^\ell, E^\ell)\)

2) \(F(c, E) = \sum_{\ell=1}^m F(c^\ell, E^\ell)\) if and only if \(SD^\pi(c, E) = \sum_{\ell=1}^m SD^\pi(c^\ell, E^\ell)\) for all \(\pi \in \Pi^N\).

Example 2 in the previous chart shows a case where the feasible set of a sum of problems is not the sum of the individual feasible sets. Consider \(P_3 = (10, 0; 5)\) and \(P_4 = (0, 10; 5)\), since on each of these problems one of the agents has a claim of zero, the feasible sets consist of a single allocation, where \(F(P_3) = \{(5, 0)\}\) and \(F(P_4) = \{(0, 5)\}\). Thus, \(F(P_3) + F(P_4) = \{(5, 5)\}\). However, similarly to our example in the introduction, when we define a new problem as \(P_3 + P_4 = (10, 10; 10)\), we introduce additional feasible allocations and we have that \(F(P_3 + P_4) \neq F(P_3) + F(P_4)\).

Since no agent can receive more than the endowment, any claim greater than the endowment would be forgone, and one could ask the solution to be invariant if all claims are truncated at the endowment.

Definition 8. A division rule \(\phi\) satisfies invariance under claims truncation\(^{11}\) (ICT) if for any problem \((c, E)\), \(\phi(c, E) = \phi(\bar{c}, E)\), where \(\bar{c}\) is the vector of truncated claims, i.e. \(\bar{c}_i = \min\{c_i, E\}\) for all \(i \in N\).

\(^{11}\)This property was introduced by Curiel et al. (1987).
Given an agent, the amount leftover from the endowment after the rest of the agents were reimbursed is the minimal amount that he/she receives.

**Definition 9.** Let \((c, E) \in C^N\). The **minimal right** of agent \(i \in N\), \(MR_i(c, E)\) is the amount he/she would receive if all other agents receive their claim, or zero if this is not possible.

\[
MR_i(c, E) = \max \left( E - \sum_{j \neq i} c_j, 0 \right)
\]

We denote the minimal rights vector by \(MR(c, E)\).

Since each claimant receives at least his/her minimal right, one way to solve a problem is to first award agents their minimal rights, revise claims and endowment, and then solve the remaining problem.

**Definition 10.** A division rule \(\phi\) satisfies **minimal rights first**\(^{12}\) (MRF) if for every problem \((c, E) \in C^N\),

\[
\phi(c, E) = MR(c, E) + \phi \left( c - MR(c, E), E - \sum_{i=1}^{N} MR_i(c, E) \right)
\]

Another desirable property is that agents with equal claims should receive equal awards:

**Definition 11.** A division rule \(\phi\) satisfies **equal treatment of equals** (ETE) if for any problem \((c, E) \in C^N\) and for every pair of agents \(i, j \in N\) such that \(c_i = c_j\), \(\phi_i(c, E) = \phi_j(c, E)\)

Now we present two propositions relating FSA with the axioms that were introduced before\(^{13}\)

**Lemma 2.** If a division rule satisfies FSA, then it also satisfies MRF and ICT.

In particular, it can be shown that RAR satisfies FSA, hence it also satisfies MRF and ICT. Moreover, it also satisfies ETE.

**Lemma 3.** RAR satisfies FSA and ETE

The next two results are characterizations of RAR on two subdomains of the general \(N\)-agent problem space. First, we show that any rule satisfying FSA and ETE coincides with RAR for problems in which no claim exceeds the endowment \((c_j \leq E\) for all \(j\)), and such that the endowment is small enough in relation to the claims (specifically, when \(E \leq \frac{1}{N-1} \sum_{i=1}^{N} c_i\)). When these two conditions hold\(^{14}\) we can decompose a problem as follows:

\[
(c, E) = \sum_{j=1}^{N} \left( (E - c_j) \mathbf{1}_{N \setminus \{j\}}, E - c_j \right) + \left( \sum_{i=1}^{N} c_i - (N - 1)E \right) \left( \mathbf{1}_N, 1 \right)
\]

\(^{12}\)Curiel et al. (1987) originally introduced this property as the “minimal right property”.

\(^{13}\)Lemmas 2 and 3 also appear in Sánchez-Pérez (2018).

\(^{14}\)These conditions imply that \(\frac{1}{N-1} \sum_{i=1}^{N} c_i \leq E \leq \frac{1}{N-1} \sum_{i=1}^{N} c_i\). Note that as the number of agents increases, this condition becomes more demanding.
This decomposition satisfies FSA, and the problems on the RHS are such that: i) All agents have either a claim of zero, or the same claim as other agents with nonzero claims and ii) under any permutation, no agent arriving after the second position receives a positive award. These facts allow us to compute \( \phi(c, E) \) for any rule \( \phi \) that satisfies FSA and ETE, and when analyzing this computation we can see that \( \phi(c, E) = RA(c, E) \).

**Theorem 2.** Let \((c, E) \in C^N\) be a problem in which no claim exceeds the endowment \((c_j \leq E \text{ for all } j \in N)\) and \(\sum_{i=1}^{N} c_i \geq (N - 1)E\). Let \(\phi\) be a rule that satisfies feasible set additivity and equal treatment of equals, then

\[
\phi(c, E) = RA(c, E)
\]

Our next result states that any rule satisfying FSA and ETE coincides with RAR for problems in which no claim exceeds the endowment and no agent arriving after the second place receives a positive award under any permutation (i.e., that the sum of the claims for each pair of agents is no less than the endowment).

**Theorem 3.** Let \((c, E) \in C^N\) be a problem in which no claim exceeds the endowment \((c_j \leq E \text{ for all } j \in N)\) and \(c_i + c_j \geq E \text{ for all } i \neq j\). Let \(\phi\) be a rule that satisfies feasible set additivity and equal treatment of equals, then

\[
\phi(c, E) = RA(c, E)
\]

With this result we can show that if a rule satisfies FSA and ETE, it coincides with RAR for any two or three agent problem.

**Corollary.** Let \(\phi\) be a division rule in \(C^2\) or \(C^3\). Then \(\phi\) satisfies feasible set additivity and equal treatment of equals if and only if \(\phi\) is the random arrival rule.

We have proved that RAR is the only division rule that satisfies FSA and ETE for the two and three agent case, and for a subset of problems in the general \(N\)-agent case. So far we have not been able to prove that this characterization holds in the space of all problems and this remains an open question.

**Open question.** Is the random arrival rule the only division rule in \(C^N\) that satisfies feasible set additivity and equal treatment of equals?

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**References**


Appendix

In what follows, we include the proofs of the propositions that were omitted in the main body.

Lemma 1. Let \((c, E) \in C^N\), then \(F(c, E)\) is a convex polytope and its set of extreme points is \(\{SD^\pi(c, E) : \pi \in \Pi^N\}\).

Proof. Since \(F(c, E)\) is a subset of \(\mathbb{R}^N\) defined by linear inequalities and equalities, it is a convex polytope. Let \(x \in F(c, E)\). We know that \(x\) is an extreme point of \(F(c, E)\) if and only if \(N\) of the following \(2N + 2\) inequalities are binding and linearly independent:

\[
x_i \leq c_i \quad \text{for all } i \in N
\]

\[
-x_i \leq 0 \quad \text{for all } i \in N
\]

\[
\sum_{i=1}^{N} x_i \leq E
\]

\[-\sum_{i=1}^{N} x_i \leq -E
\]

Since \(x \in F(c, E)\), the last two inequalities hold with equality. Since they are not linearly independent, we only count them as one linearly independent inequality binding at \(x\). Therefore, \(x \in Ext(F(c, E))\) if and only if the set \(\{i : x_i = c_i \text{ or } x_i = 0\}\) contains at least \(N - 1\) elements.

Suppose that \(x = SD^\pi(c, E)\) for some permutation \(\pi \in \Pi^N\). Then, by definition \[\{i : x_i = c_i \text{ or } x_i = 0\}\) contains at least \(N - 1\) elements, hence \(x \in Ext(F(c, E))\).

Now suppose that \(x \in Ext(F(c, E))\), then \(A = \{i : x_i = c_i \text{ or } x_i = 0\}\) contains at least \(N - 1\) elements. Define \(A_1 = \{i \in A : x_i = c_i\}\) and \(A_2 = \{i \in A : x_i = 0 \text{ and } x_i \neq c_i\}\) and let \(\pi\) be any permutation such that \(\pi(i) \leq |A_1|\) for \(i \in A_1\) and \(\pi(i) > |A_1| + 1\) for \(i \in A_2\). Then, it follows that \(x = SD^\pi(c, E)\).

Theorem 1. Let \((c^1, E^1), \ldots , (c^m, E^m) \in C^N\) and let \((c, E) = \sum_{\ell=1}^{m} (c^\ell, E^\ell)\). Then:

1) \(F(c, E) \supseteq \sum_{\ell=1}^{m} F(c^\ell, E^\ell)\)

2) \(F(c, E) = \sum_{\ell=1}^{m} F(c^\ell, E^\ell)\) if and only if \(SD^\pi(c, E) = \sum_{\ell=1}^{m} SD^\pi(c^\ell, E^\ell)\) for all \(\pi \in \Pi^N\).

Proof. For 1):

Let \(x^\ell \in F(c^\ell, E^\ell)\) for all \(\ell = 1, \ldots , m\) and let \(x = \sum_{\ell=1}^{m} x^\ell\). To show that \(x \in F(c, E)\), note that

\[
\sum_{j=1}^{N} x_j = \sum_{j=1}^{N} \sum_{\ell=1}^{m} x_j^\ell = \sum_{\ell=1}^{m} \sum_{j=1}^{N} x_j^\ell = \sum_{\ell=1}^{m} E^\ell = E
\]

Since \(x^\ell \in F(c^\ell, E^\ell)\), we have that \(0 \leq x^\ell \leq c^\ell\) for all \(\ell\). Adding up these inequalities for \(\ell = 1, \ldots , m\) we conclude that \(0 \leq x \leq c\). Therefore \(x \in F(c, E)\).

Now for 2):
First suppose that \( SD\pi \(c, E\) = \sum_{\ell=1}^{m} SD\pi \(c^\ell, E^\ell\) \) for all permutations \(\pi\). By lemma 15 and the Krein-Milman theorem, we have that \( F\(c, E\) = \text{conv} \(\{SD\pi \(c, E\) : \pi \in \Pi^N\}\) \)

Let \(x \in F\(c, E\)\). Then there exist \(\gamma_\pi \geq 0\) for all \(\pi \in \Pi^N\) such that \(\sum_{\pi \in \Pi^N} \gamma_\pi = 1\) and

\[
x = \sum_{\pi \in \Pi^N} \gamma_\pi SD\pi \(c, E\) = \sum_{\pi \in \Pi^N} \gamma_\pi \left( \sum_{\ell=1}^{m} SD\pi \(c^\ell, E^\ell\) \right) = \sum_{\ell=1}^{m} \left( \sum_{\pi \in \Pi^N} \gamma_\pi SD\pi \(c^\ell, E^\ell\) \right)
\]

Let \(y^\ell = \sum_{\pi \in \Pi^N} \gamma_\pi SD\pi \(c^\ell, E^\ell\)\) for all \(\ell = 1, \ldots, m\).

Since \(F\(c^\ell, E^\ell\) = \text{conv} \(\{SD\pi \(c^\ell, E^\ell\) : \pi \in \Pi^N\}\)\), it follows that \(y^\ell \in F\(c^\ell, E^\ell\)\) for all \(\ell\). So we have that

\[
x = \sum_{\ell=1}^{m} y^\ell \text{ with } y^\ell \in F\(c^\ell, E^\ell\) \text{ for all } \ell
\]

Hence, \(F\(c, E\) \subseteq \sum_{\ell=1}^{m} F\(c^\ell, E^\ell\)\) and applying Theorem 1, we conclude that \(F\(c, E\) = \sum_{\ell=1}^{m} F\(c^\ell, E^\ell\)\).

Now suppose that \(F\(c, E\) = \sum_{\ell=1}^{m} F\(c^\ell, E^\ell\)\). Since all the feasible sets are compact and convex subsets of \(\mathbb{R}^N\), it follows that \(\text{Ext}(F\(c, E\)) \subseteq \sum_{\ell=1}^{m} \text{Ext}(F\(c^\ell, E^\ell\))\), and from lemma 1 we have that

\[
\{SD\pi \(c, E\) : \pi \in \Pi^N\} \subseteq \sum_{\ell=1}^{m} \left\{SD\pi \(c^\ell, E^\ell\) : \pi \in \Pi^N\right\}
\]

(1)

Fix \(\pi \in \Pi^N\), by (1), there exist permutations \(\pi^1, \ldots, \pi^m \in \Pi^N\) such that

\[
SD\pi \(c, E\) = \sum_{\ell=1}^{m} SD\pi^\ell \(c^\ell, E^\ell\)
\]

In order to prove the theorem, we need to show that if we choose \(\pi^\ell = \pi\) for all \(\ell\), the last equation holds. For ease of notation let \(x = SD\pi \(c, E\)\) and \(y^\ell = SD\pi^\ell \(c^\ell, E^\ell\)\) for all \(\ell = 1, \ldots, m\).

For a permutation \(\pi\) and a problem \((c, E)\), define \(K\(\pi, c, E\) \in N\) as follows:

\[
SD\pi^\ell \(c, E\) = \begin{cases} 
c_i & \text{if } \pi(i) < K\(\pi, c, E\) \\
E - \sum_{j < \pi(i)} c_j & \text{if } \pi(i) = K\(\pi, c, E\) \\
0 & \text{if } \pi(i) > K\(\pi, c, E\)
\end{cases}
\]

(2)

\(^{15}\)We denote the convex hull of \(A \subseteq \mathbb{R}^N\) by \(\text{conv}(A)\).

\(^{16}\)In general, if \(A, B \subseteq \mathbb{R}^N\) are compact and convex, then \(\text{Ext}(A + B) \subseteq \text{Ext}(A) + \text{Ext}(B)\) See Husain and Tweddle (1970) for a proof.
Let \( i \in N \). If \( \pi(i) < K(\pi, c, E) \), from equation 2 we have that

\[
c_i = x_i = \sum_{\ell = 1}^{m} y_i^\ell \leq \sum_{\ell = 1}^{m} c_i^\ell = c_i
\]

Therefore, \( y_i^\ell = c_i^\ell \) for all \( \ell \). Now consider the case where \( \pi(i) > K(\pi, c, E) \), so that

\[
0 = x_i = \sum_{\ell = 1}^{m} y_i^\ell \geq \sum_{\ell = 1}^{m} 0 = 0
\]

So \( y_i^\ell = 0 \) for all \( \ell \). Finally, if \( \pi(i) = K(\pi, c, E) \), since \( y_i^\ell \in F(c, E) \), it follows that

\[
y_i^\ell = E^\ell - \sum_{j \neq i} y_i^\ell = E^\ell - \sum_{\pi(j) < \pi(i)} \sum_{j \neq i} c_i^\ell
\]

From where we conclude that \( y^\ell = SD^\pi(c^\ell, E^\ell) \) and

\[
SD^\pi(c, E) = \sum_{\ell = 1}^{m} SD^\pi(c^\ell, E^\ell)
\]

\[\square\]

**Lemma 2.** If a division rule satisfies FSA, then it also satisfies MRF and ICT.

**Proof.** First, we show that if a rule satisfies FSA, then it satisfies MRF. Let \((c, E) \in C^N\). We claim that

\[
F(c, E) = F\left(c - MR(c, E), E - \sum_{i = 1}^{N} MR_i(c, E)\right) + F\left(MR(c, E), \sum_{i = 1}^{N} MR_i(c, E)\right)
\]

By Theorem 1, we only need to verify the “\( \subseteq \)” inclusion. Let \( x \in F(c, E) \). Note that since \( \sum_{i = 1}^{N} x_i = E \), then

\[
x_i = E - \sum_{j \neq i} x_j \geq E - \sum_{j \neq i} c_j \text{ for all } i \in N
\]

Since \( x \geq 0 \), it follows that \( x_i \geq MR_i(c, E) \) for all \( i \in N \). Therefore, both of the following conditions hold:

\[
0 \leq x - MR(c, E) \leq c - MR(c, E)
\]

\[
\sum_{i = 1}^{N} (x_i - MR_i(c, E)) = E - \sum_{i = 1}^{N} MR_i(c, E)
\]

Hence, \( x - MR(c, E) \in F\left(c - MR(c, E), E - \sum_{i = 1}^{N} MR_i(c, E)\right) \).

It is easy to see that \( F\left(MR(c, E), \sum_{i = 1}^{N} MR_i(c, E)\right) = \{MR(c, E)\} \), thus

\[
F(c, E) \subseteq F\left(c - MR(c, E), E - \sum_{i = 1}^{N} MR_i(c, E)\right) + F\left(MR(c, E), \sum_{i = 1}^{N} MR_i(c, E)\right)
\]
Now, let \( \phi \) be a rule that satisfies FSA, then

\[
\phi(c, E) = \phi \left( c - \sum_{i=1}^{N} MR_i(c, E) \right) + \phi \left( \sum_{i=1}^{N} MR_i(c, E) \right)
\]

Since \( \phi \left( \sum_{i=1}^{N} MR_i(c, E) \right) \in F \), it must be that

\[
\phi(c, E) = MR(c, E) + \phi \left( c - \sum_{i=1}^{N} MR_i(c, E) \right)
\]

Now we show that if a rule satisfies FSA, then it satisfies ICT.

We claim that \( F(c, E) = F(\bar{c}, E) \) where \( \bar{c}_j = \min(E, c_j) \). Since \( \bar{c} \leq c \), it is clear that \( F(\bar{c}, E) \subseteq F(c, E) \); to see that the converse inclusion holds, let \( x \in F(c, E) \).

Since \( 0 \leq x_i \leq c_i \) for all \( i \) and \( \sum_{i=1}^{N} x_i = E \), it follows that \( x_i \leq E \) as well. Therefore, \( 0 \leq x_i \leq \bar{c}_i \) for all \( i \), which implies that \( x \in F(\bar{c}, E) \).

Note that \( (c, E) = (\bar{c}, E) + (c - \bar{c}, 0) \) and \( F(c - \bar{c}, 0) = \{0\} \). Therefore,

\[
F(c, E) = F(\bar{c}, E) + F(c - \bar{c}, 0)
\]

By FSA, \( \phi(c, E) = \phi(\bar{c}, E) + \phi(c - \bar{c}, 0) = \phi(\bar{c}, E) + 0 \), so \( \phi \) satisfies ICT.

**Lemma 3.** RAR satisfies FSA and ETE

**Proof.** By Theorem 1, serial dictator rules satisfy FSA. Moreover, it is easy to see that if a set of division rules satisfy FSA, then any convex combination of these rules also satisfies FSA. Since the random arrival rule is a particular convex combination of serial dictator rules, it satisfies FSA.

The preservation of certain properties under convex operations is studied in Thomson and Yeh (2001). We thank an anonymous referee for suggesting this simple approach for the proof.

**Theorem 2.** Let \( (c, E) \in \mathcal{C}^N \) be a problem in which no claim exceeds the endowment \( (c_j \leq E \text{ for all } j \in N) \) and \( \sum_{i=1}^{N} c_i \geq (N - 1)E \). Let \( \phi \) be a rule that satisfies feasible set additivity and equal treatment of equals, then

\[
\phi(c, E) = RA(c, E)
\]

**Proof.** Since \( E - c_j \geq 0 \) for every \( j \in N \), \( (E - c_j, I_{N \setminus \{j\}}, E - c_j) \in \mathcal{C}^N \)

Also, since \( \sum_{i=1}^{N} c_i - (N - 1)E \geq 0 \),

\[
\left( \sum_{i=1}^{N} c_i - (N - 1)E \right) (\bar{1}, 1) \in \mathcal{C}^N
\]
Note that
\[(c, E) = \sum_{j=1}^{N} ((E - c_j) \vec{1}_{N\setminus\{j\}}, E - c_j) + \left( \sum_{i=1}^{N} c_i - (N-1)E \right) (\vec{1}, 1) \quad (3)\]

We claim that for every permutation \(\pi \in \Pi^N\),
\[SD_\pi (c, E) = \sum_{j=1}^{N} SD_\pi \left( (E - c_j) \vec{1}_{N\setminus\{j\}}, E - c_j \right) + SD_\pi \left( \left( \sum_{i=1}^{N} c_i - (N-1)E \right) (\vec{1}, 1) \right) \quad (4)\]

Let \(\pi \in \Pi^N\) and let \(i \in \mathbb{N}\) be such that \(\pi(i) = 1\). Note that the following conditions hold:
\[SD_\pi i (c, E) = c_i\]
\[\sum_{j=1}^{N} SD_\pi j \left( (E - c_j) \vec{1}_{N\setminus\{j\}}, E - c_j \right) = \sum_{j \neq i} (E - c_j) = (N-1)E - \sum_{j \neq i} c_j \quad (5)\]
\[SD_\pi i \left( \left( \sum_{i=1}^{N} c_i - (N-1)E \right) (\vec{1}, 1) \right) = \sum_{j=1}^{N} c_j - (N-1)E\]

Thus, equation 4 holds when \(\pi(i) = 1\). Note that since \(c_j \leq E\) for every \(j\), then \(\sum_{j=1}^{N} c_\pi_j \leq (N - 2)E\)

Since \(\sum_{i=1}^{N} c_i - (N-1)E \geq 0\), it must be that \(c_\pi_1 + c_\pi_2 \geq E\). Thus if \(\pi(i) = 2\), \(SD_\pi i (c, E) = E - c_\pi_1\). A computation similar to the one in equation 5 allows us to verify that condition 4 holds if \(\pi(i) = 2\) as well.

To see that condition 4 holds when \(\pi(i) > 2\), it suffices to see that no agent arriving after the third place will receive a positive award in any of the problems involved in equation 3. By theorem 1, it follows that:
\[F (c, E) = \sum_{j=1}^{N} F \left( (E - c_j) \vec{1}_{N\setminus\{j\}}, E - c_j \right) + F \left( \left( \sum_{i=1}^{N} c_i - (N-1)E \right) (\vec{1}, 1) \right) \]

Since \(\phi\) satisfies ETE, for every \(j \in \mathbb{N}\) we have that:
\[\phi \left( (E - c_j) \vec{1}_{N\setminus\{j\}}, E - c_j \right) = \frac{1}{N-1} (E - c_j) \vec{1}_{N\setminus\{j\}} = RA \left( (E - c_j) \vec{1}_{N\setminus\{j\}}, E - c_j \right) \]

And
\[\phi \left( \left( \sum_{i=1}^{N} c_i - (N-1)E \right) (\vec{1}, 1) \right) = \frac{1}{N} \left( \sum_{i=1}^{N} c_i - (N-1)E \right) \vec{1} \]
\[= RA \left( \left( \sum_{i=1}^{N} c_i - (N-1)E \right) (\vec{1}, 1) \right) \]

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Therefore, by FSA
\[
\phi(c, E) = \sum_{j=1}^{N} \phi\left((E - c_j) \mathbf{1}_{N \setminus \{j\}}, E - c_j\right) + \phi\left(\left(\sum_{i=1}^{N} c_i - (N - 1)E\right) \langle \mathbf{1}, 1 \rangle\right) \\
= \sum_{j=1}^{N} RA\left((E - c_j) \mathbf{1}_{N \setminus \{j\}}, E - c_j\right) + RA\left(\left(\sum_{i=1}^{N} c_i - (N - 1)E\right) \langle \mathbf{1}, 1 \rangle\right) = RA(c, E)
\]
Thus \(\phi(c, E) = RA(c, E)\)

To prove Theorem 3, we rely on Lemma 3 in Hwang (2015), which states that for any problem \((c, E)\) and \(i \in N\) we have that:
\[
RA_i(c, E) = \sum_{S \subseteq N \atop i \in S} d_S(N, E, c) \tag{6}
\]
Where \(d_S(N, E, c)\) are the Harsanyi dividends for coalition \(S\) in problem \((c, E)\), namely:
\[
d_S(N, E, c) = \frac{1}{|S|} \sum_{T \subseteq S} (-1)^{|S|-|T|} \left( E - \sum_{j \notin T} c_j \right)^+_+
\]

**Theorem 3.** Let \((c, E) \in C^N\) be a problem in which no claim exceeds the endowment \((c_j \leq E\) for all \(j \in N\)) and \(c_i + c_j \geq E\) for all \(i \neq j\). Let \(\phi\) be a rule that satisfies feasible set additivity and equal treatment of equals, then

\[\phi(c, E) = RA(c, E)\]

**Proof.** If \(\sum_{i=1}^{N} c_i \geq (N - 1)E\), then, by theorem 2 we know that \(\phi(c, E) = RA(c, E)\).

Suppose that \(\sum_{i=1}^{N} c_i < (N - 1)E\). From equation 3 it follows that:
\[
(c, E) + \left((N - 1)E - \sum_{i=1}^{N} c_i\right) \langle \mathbf{1}, 1 \rangle = \sum_{j=1}^{N} \left((E - c_j) \mathbf{1}_{N \setminus \{j\}}, E - c_j\right)
\]

Repeating the argument used to prove equation 4 we can show that for every permutation \(\pi\),
\[
SD^\pi(c, E) + SD^\pi\left(\left((N - 1)E - \sum_{i=1}^{N} c_i\right) \langle \mathbf{1}, 1 \rangle\right) = \\
SD^\pi\left(c + \left((N - 1)E - \sum_{i=1}^{N} c_i\right) \mathbf{1}, NE - \sum_{i=1}^{N} c_i\right) = \sum_{j=1}^{N} SD^\pi\left((E - c_j) \mathbf{1}_{N \setminus \{j\}}, E - c_j\right)
\]
By theorem 1, it follows that:

\[
F(c, E) + F\left(\left[(N - 1)E - \sum_{i=1}^{N} c_i\right]\right)(\vec{1}, 1) = \\
F\left(c + \left[(N - 1)E - \sum_{i=1}^{N} c_i\right]\vec{1}, NE - \sum_{i=1}^{N} c_i\right) = \sum_{j=1}^{N} F\left((E - c_j)\vec{1}_{N\setminus\{j\}}, E - c_j\right)
\]

Since \(\phi\) satisfies FSA, the following two equations hold:

\[
\phi\left(c + \left[(N - 1)E - \sum_{i=1}^{N} c_i\right]\vec{1}, NE - \sum_{i=1}^{N} c_i\right) = \phi(c, E) + \phi\left(\left[(N - 1)E - \sum_{i=1}^{N} c_i\right]\vec{1}, 1\right) \tag{7}
\]

\[
\phi\left(c + \left[(N - 1)E - \sum_{i=1}^{N} c_i\right]\vec{1}, NE - \sum_{i=1}^{N} c_i\right) = \sum_{j=1}^{N} \phi\left((E - c_j)\vec{1}_{N\setminus\{j\}}, E - c_j\right) \tag{8}
\]

By combining equations 7 and 8 and applying ETE, we get:

\[
\phi(c, E) = \sum_{j=1}^{N} \left(\frac{E - c_j}{N - 1}\right)\vec{1}_{N\setminus\{j\}} - \frac{1}{N} \left((N - 1)E - \sum_{j=1}^{N} c_i\right)\vec{1} \tag{9}
\]

We can verify that equations 6 and 9 are equivalent, hence \(\phi(c, E) = RA(c, E)\).

**Corollary.** Let \(\phi\) be a division rule in \(C^2\) or \(C^3\). Then \(\phi\) satisfies ETE and FSA if and only if \(\phi\) is the random arrival rule.

**Proof.** For \((c, E) \in C^2\), we have that \(c_1 + c_2 \geq E\) by definition. Now let \((c, E) \in C^3\). Since \(\phi\) satisfies MRF, we can assume without loss of generality that \(MR(c, E) = 0\), i.e., \(c_i + c_j \geq E\) for all \(i \neq j\).

In both cases, by theorem 3 we have that \(\phi(c, E) = RA(c, E)\).