The effect of interest rates on consumption in an income fluctuation problem

Ehud Lehrer† and Bar Light‡

July 16, 2018

ABSTRACT:

We examine the effect of a change in interest rates on an agent’s consumption and savings decisions when her income is fluctuating. In each period, a long-lived agent decides how much to save (i.e., invest in a risky bond) and how much to consume while her income and the rate of return on her savings are uncertain and depend on the state of the economy. We show that under the concavity of the consumption function, a condition that ensures that the substitution effect dominates the income effect, lower interest rates encourage the agent’s consumption across all states.

Keywords: Consumption; Savings; Interest rates; Income fluctuation problem; Dynamics.

JEL classification: C70; C78; D51; D58

*The authors acknowledge the support of the Israel Science Foundation, Grant #963/15.
†Tel Aviv University, Tel Aviv 69978, Israel and INSEAD, Bd. de Constance, 77305 Fontainebleau Cedex, France. e-mail: lehrer@post.tau.ac.il.
‡Graduate School of Business, Stanford University, Stanford, CA 94305, USA. e-mail: barl@stanford.edu
1 Introduction

The influence of the aggregate demand on the economy is undisputed. Given the dependence of the consumption function on current rates of return and future rates of return, it is worth examining how higher rates of return affect consumption in a standard consumption-savings problem. While many empirical studies have tried to estimate the effect of a change in interest rates on savings and consumption,\(^1\) we do not know of any analytical results in frameworks similar to ours that do not rely on closed form solutions\(^2\) or numerical methods. Our goal in this paper is to analytically examine the effect of higher interest rates on savings and consumption decisions in an income fluctuation problem. Even in a two-period model with no uncertainty, consumption may increase or decrease when interest rates increase. Both the income effect and the substitution effect influence the agent’s consumption choice. The income effect increases consumption if the agent has positive savings, since her savings are worth more with higher interest rates. The substitution effect decreases the agent’s consumption since higher interest rates raise the price of consumption. Thus, a necessary condition for a decrease in consumption in response to an increase in interest rates is that the substitution effect dominate the income effect. We offer a condition on the agent’s utility function that ensures such domination in Section 4.

In this paper, we examine the effect of higher interest rates on savings and consumption decisions in a standard consumption-savings model with income uncertainty, sometimes referred to as the savings problem (see Chamberlain and Wilson (2000) and Ljungqvist and Sargent (2004)) or the income fluctuation problem (see Schechtman and Escudero (1977)). The income fluctuation problem is fundamental in modern macroeconomics.\(^3\) In an income fluctuation problem, the agent receives a state-dependent income in each period. The states follow a stochastic process, so the agent’s income follows a stochastic process as well. The agent solves an infinite horizon consumption-savings problem. That is, the agent decides how much to save and how much to consume in each period. In contrast to complete markets models (e.g., Arrow-Debreu models) where the agent can insure herself against any realization, the income fluctuation problem is an incomplete markets model. The agent can transfer assets from one period to another only by investing in one asset, and either is not allowed to borrow or has some borrowing limit. Usually the interest rate in the income fluctuation problem is constant and is equal to or less than the discount factor (see Chamberlain and

---

\(^1\)For example, Campbell and Mankiw (1989), Attanasio and Weber (1993) and Di Maggio et al. (2017).

\(^2\)For example, Weil (1993) finds a closed form solution to the consumption function, showing that consumption increases with higher interest rates if the income effect dominates the substitution effect.

\(^3\)The income fluctuation problem is used to study many macroeconomic phenomena. For example, the permanent income hypothesis (Bewley, 1977), wealth distribution (Benhabib et al., 2015), and many more.
Wilson (2000)). In our model, the rates of return are uncertain and depend on the state of
the economy.

The conditions on the agent’s preferences that imply that savings increase with the inter-

est rates are closely related to the conditions that imply the uniqueness of an equilibrium
in Bewley-Huggett-Aiyagari general equilibrium models. The uniqueness of an equilibrium
in these models is not guaranteed (for examples of multiplicity, see Toda (2017) and Acik-
goz (2018)). In two contemporary papers, under similar, but stronger, conditions on the
agent’s preferences that ensure that the savings increase with the interest rate, Light (2017)
proves the uniqueness of an equilibrium in a discrete-time Aiyagari model\footnote{For a related result see Hu and Shmaya (2017).} and Achdou et al.
(2017) prove the uniqueness of an equilibrium in a continuous-time Aiyagari model. We be-
lieve that future research can apply the results presented in this paper to prove that the
aggregate demand for savings increases with the interest rate in general equilibrium models
with heterogeneous agents (e.g., Huggett (1993), Aiyagari (1994) and Krusell and Smith
(1998)). The property that the aggregate demand for savings increases with the interest
rate is important in many general equilibrium models with heterogeneous agents, because it
ensures the uniqueness of an equilibrium in these models.

The rest of the paper is organized as follows. Section 2 presents the income fluctuation
problem. Section 3 discusses a comparative statics theorem which shows how to determine
whether the endogenous variables (consumption and savings) are monotone with respect to a
change in some parameter. Our main theorem deals with the effect of higher rates of return
on consumption and savings decisions. In Section 4 we prove that if the consumption policy
function is concave in the agent’s cash-on-hand and the substitution effect dominates the
income effect, then higher rates of return increase the agent’s savings. In Section 5 we show
that an increase in the agent’s permanent income increases her consumption. In Section 6
we provide some final remarks.

2 The model: a dynamic consumption-savings prob-
lem

We consider a discrete time dynamic consumption-savings model. Let $S = \{s_1, \ldots, s_n\}$ be a
finite set of possible states of the economy. For notational convenience we assume that the
state of the economy determines the rate of return on the agent’s savings and the agent’s
labor income.\footnote{Since we study a single agent problem, there is no need to distinguish aggregate and idiosyncratic states
in our setting.}
At any time \( t = 1, 2, 3, \ldots \) a state \( s_t \) is realized and the agent gets an income that depends on this state. Let \( y : S \rightarrow \mathbb{R}_+ \) be the labor income function: \( y(s_t) \) indicates the agent’s income in state \( s_t \). We assume that the states are ordered in accordance with the corresponding incomes:

\[
y(s_1) < y(s_2) < \ldots < y(s_n).
\]

The evolution of the states follows a Markov chain with transition probabilities \( P = (P_{ij}) \). The probability of moving from state \( s_i \) to state \( s_j \) is denoted by \( P_{ij} \).

We now consider the agent’s consumption-savings decision problem. Suppose that the agent’s initial savings is \( a_0 \). In each period \( t = 1, 2, \ldots \) the agent receives an income \( y(s(t)) \) that depends on the state of the economy. The rate of return on her savings at time \( t \), \( R(s(t)) \), is determined by the state of the economy at time \( t \) and by the rate of return function \( R : S \rightarrow \mathbb{R}_+ \).

Denote the agent’s cash-on-hand at time \( t = 1 \) by \( x(1) = R(s(1))a_0 + y(s(1)) \). Suppose that at time \( t \) the agent’s cash-on-hand is \( x(t) \). Based on \( s(t) \) and \( x(t) \), the agent decides how much to consume at time \( t \), which we denote by \( c(t) \), and thereby how much to save for future consumption. Thus, her cash-on-hand at time \( t + 1 \) when \( s(t + 1) \) is the realized state is

\[
x(t + 1) = R(s(t + 1)) (x(t) - c(t)) + y(s(t + 1)).
\]

We assume that the agent cannot borrow\(^7\) and thus, \( x(t) \geq c(t) \) in every period. We denote by \( C(x) = [0, \min\{x, \overline{x}\}] \) the interval from which the agent may choose her consumption level where \( \overline{x} \) is an upper bound on the agent’s cash-on-hand that ensures compactness.\(^8\)

The agent’s utility from consumption in each period is given by a utility function \( u : [0, \infty) \rightarrow [0, \infty) \). We assume that \( u \) is strictly increasing, strictly concave, continuously differentiable and that \( u'(0) = \infty \). The agent’s utility derived from the sequence \( c(1), c(2), \ldots \) of consumption is the present discounted value \( \sum_{t=1}^{\infty} \beta^{t-1} u(c(t)) \), where \( \beta \in (0, 1) \) is the agent’s discount factor.

Let \( X = [0, \infty) \) and \( Z = X \times S \). For each initial cash-on-hand and state pair, a consumption strategy\(^9\) \( \pi \) and a Markov chain \( P \) induce a probability measure over the space

\(^6\)Note the distinction between \( s(t) \) which is the realized state at time \( t \), and \( s_i \) which is the \( i \)-th state as ordered in Eq. (1).

\(^7\)We will consider the case in which the agent can borrow in Section 4.

\(^8\)Since the utility function \( u \) is unbounded we assume that the cash-on-hand is bounded to avoid technical issues with the agent’s dynamic programming problem (see Li and Stachurski (2014) and Kuhn (2013) for conditions that ensure that the upper bound on the agent’s cash-on-hand never binds).

\(^9\)A consumption strategy is a function \( \pi \) that assigns to every finite history \( z^t = (z(1), \ldots, z(t)) \) in \( Z^t := \underbrace{Z \times \ldots \times Z}_{t \text{ times}} \) an action \( \pi(z^t) \) in \( C(x(k)) \), where \( z(t) = (x(t), s(t)) \).
of all infinite histories. We denote the expectation with respect to the probability measure by $E_{\pi}$. When the agent follows a consumption strategy $\pi$, and the initial cash-on-hand and state pair, is $(x, s)$ her expected present discounted value is

$$V_{\pi}(x, s) = E_{\pi}\left(\sum_{t=1}^{\infty} \beta^{t-1}u(\pi(z(1), \ldots, z(t)))\right).$$

Denote

$$V(x, s) = \sup_{\pi} V_{\pi}(x, s).$$

That is, $V(x, s)$ is the maximal expected utility that the agent can have when the initial cash-on-hand and state pair is $(x, s)$. We call $V : Z \to \mathbb{R}$ the value function and a strategy $\pi$ attaining it optimal.

We denote by $a$ the savings of the agent. For every $(x, s_i) \in Z$ and $a \in C(x)$ define the following function:

$$(2) \quad h(x, a, s_i, V) = u(x - a) + \beta \sum_{j=1}^{n} P_{ij}V(R(s_j) a + y(s_j), s_j).$$

Standard dynamic programming arguments (see for example Blackwell (1965) or Stokey and Lucas (1989)) show that the value function $V$ is the unique function that satisfies the Bellman equation,

$$(3) \quad V(x, s_i) = \max_{a \in C(x)} h(x, a, s_i, V).$$

Theorem 9.8 in Stokey and Lucas (1989) shows that the value function $V$ is continuous and concave. Thus $h$ itself is strictly concave in $a$, and as such has a unique maximum. Define $a(x, s)$ to be the savings level of the next period that attains the maximum of $h(x, a, s, V)$. That is,

$$(4) \quad a(x, s) = \arg\max_{a \in C(x)} h(x, a, s, V).$$

The argmax function defined in Eq. (4) is called the savings policy function. Note that the savings policy function induces an optimal consumption policy function $c(x, s)$ by the equation $c(x, s) = x - a(x, s)$.

### 3 A comparative statics theorem

Let $(I, \leq)$ be a partially ordered set: $I$ is a set and $\leq$ is a binary relation over $I$ that is reflexive, antisymmetric and transitive.\footnote{Formally, $\leq$ is a partial order if for every $e, e' \in I$ we have (i) $e \leq e$; (ii) $e \leq e'$ and $e' \leq e$ imply $e = e'$; and (iii) $e \leq e'$ and $e' \leq e''$ imply $e \leq e''$.} The set $I$ will serve as a set of parameters that
affect the consumption-savings problem faced by the agent. In the applications below the parameter $e$ will play several roles, such as the rate of return function $R$, and the labor income function $y$.

Throughout the discussion we assume that the parameter $e$ does not change the interval from which the agent chooses her level of consumption, i.e., for all $e \in I$, $C(x) = C(x,e)$. We slightly abuse the notations and allow an additional argument in the functions defined above. For instance, the value function of the parameterized consumption-savings problem $V \in B(Z \times I)$ is denoted by

$$V(x,s,e) = \max_{a \in C(x)} h(x,a,s,e,V).$$

Likewise, the savings policy function is denoted by $a(x,s,e)$, the consumption policy function by $c(x,s,e)$ and $h(x,a,s,e,V)$ is the $h$ function associated with the consumption-savings problem with parameter $e$, as defined above in Eq. (2).

We are interested in the question whether for every $e_1, e_2 \in I$ such that $e_2$ is greater than $e_1$ (i.e., $e_2 \geq e_1$) the savings related to $e_2$, $a(x,s,e_2)$ are greater than or equal to those related to $e_1$ (i.e., $a(x,s,e_1)$) for all $(x,s) \in Z$.

We introduce a condition that ensures that the answer to the last question is on the affirmative. For $f \in B(Z \times I)$ define

$$f'(x,s,e) := \frac{\partial f(x,s,e)}{\partial x}.$$

Denote by $u'$ the derivative of $u$ and by $h'(a,s,e,f)$ the derivative of $h$ with respect to $a$.

**Definition 1.** Consider a parameterized consumption-savings problem.\(^{11}\) We say that $h$ has the complementation-preserving (CP) property if for every differentiable function $f \in B(Z \times I)$, $f'(x,s,e)$ increasing in $e$ implies that $h'(a,s,e,f(x,s,e))$ is increasing in $e$ for all $(x,s) \in Z$ and $a \in C(x)$.

The CP property means that if $f$ has increasing differences\(^{12}\) in $(x,e)$, i.e., the cash-on-hand $x$ and the parameter $e$ are complementary; then $h$ has increasing differences in $(a,e)$, i.e., the savings $a$ and the parameter $e$ are also complementary. The next theorem\(^{13}\)

---

\(^{11}\)We omit the reference to the partially ordered set $(I, \preceq)$, the set of parameters.

\(^{12}\)Let $I$ be a partially ordered set and let $O \subseteq X \times I$. Let $f$ be a real valued function defined on $O$. $f$ is said to have increasing differences in $(x,e)$ if for all $e_2 \geq e_1$, $f(x,e_2) - f(x,e_1)$ is increasing in $x$. Thus, if $\partial f(x,e)/\partial x$ is increasing in $e$ then $f$ has increasing differences in $(x,e)$.

\(^{13}\)We note that a similar result to Theorem 1 and more general results can be found in Morand et al. (2015), Mirman et al. (2008), Light (2018) and Hopenhayn and Prescott (1992). For completeness, we provide the proof in the Appendix.
states that if \( h \) has the CP property, then the savings are increasing in the parameter \( e \), i.e., \( a(x,s,e) \) is increasing in \( e \) for all \((x,s) \in Z\). As we shall see in Sections 4 and 5 it is often easy to verify that \( h \) satisfies the CP property.

**Theorem 1.** Assume that \( h \) has the CP property. Then, for every \((x,s) \in Z\), \( a(x,s,e) \) is increasing and \( c(x,s,e) \) is decreasing in \( e \).

All the proofs are deferred to the Appendix.

Theorem 1 states that comparative statics results in consumption-savings dynamic models can be obtained by checking a simple property. Given a partially ordered set \( I \), when the fact that \( V'(x,s,e) \) is increasing in \( e \) for all \((x,s) \in Z\) implies that \( h'(x,a,s,e,V(x,s,e)) \) is increasing in \( e \) for all \((x,s) \in Z\) and \( a \in C(x) \), one can conclude that \( a(x,s,e) \) is increasing in \( e \) for all \((x,s) \in Z\). Similarly, when the fact that \( V'(x,s,e) \) is decreasing in \( e \) for all \((x,s) \in Z\) implies that \( h'(x,a,s,e,V(x,s,e)) \) is decreasing in \( e \) for all \((x,s) \in Z\) and \( a \in C(x) \), one can conclude that \( a(x,s,e) \) is decreasing in \( e \) for all \((x,s) \in Z\).

## 4 Comparing rate of return functions

In this section we present the comparative statics result regarding the effect of the rate of return function on consumption and savings. Let \( R_1, R_2 \) be two rate of return functions. We order the set of function \( R \) with the usual product order, i.e., we write \( R_2 \succeq R_1 \) if \( R_2(s) \succeq R_1(s) \) for each state \( s \in S \). Let \( c(x,s,R) \) be the optimal consumption policy function when the interest rate policy is \( R \). The following definition relates to a comment we made in the introduction regarding the fact that the substitution effect must dominate the income effect in order for an increase in interest rates to result in a decrease in consumption.

**Definition 2.** We say that the utility function exhibits substitution effect domination (SED) if \( c u'(c) \) is increasing on \( X \).

The SED is strongly related to the notion of relative risk aversion (RRA).\(^{14}\) Indeed, when \( u \) is twice continuously differentiable, \( u \) exhibiting SED is equivalent to the RRA being less than or equal to one. Our main theorem, Theorem 2, states that if the utility function exhibits SED and the consumption policy function is concave in the cash-on-hand, then \( R_2 \succeq R_1 \) implies \( c(x,s,R_1) \succeq c(x,s,R_2) \) for all \((x,s) \in X \times S\).

In order to provide a motivation for the condition that \( c u'(c) \) is increasing, we show that this condition is equivalent to an increasing differences condition on the utility function that implies that wealth and rates of return are complementary.

\(^{14}\)The measure of RRA is defined as \( R(c) = -\frac{cu''(c)}{u'(c)} \).
Suppose that an agent lives for one period and has an indirect utility from an amount of wealth \( a \) and a rate of return \( R \). This agent who lives for one period consumes all her wealth, and her consumption is given by \( Ra \) and her indirect utility function is given by \( u(Ra) \). When the rate of return rises it is not clear whether the marginal contribution of wealth to the agent’s indirect utility is positive. Let \( a_2 \geq a_1, R_2 \geq R_1 \). With a higher rate of return, the marginal contribution is higher in terms of consumption, i.e., \( R_2a_2 - R_2a_1 \geq R_1a_2 - R_1a_1 \). The question is what the marginal utility would then be.

Since the marginal utility from consumption is strictly decreasing, the difference in the LHS of the last inequality could, in principle, be smaller than the difference in the RHS in terms of utility: \( u(R_2a_2) - u(R_2a_1) \leq u(R_1a_2) - u(R_1a_1) \). The condition that \( u \) exhibit SED implies that the marginal contribution of wealth to the utility is increasing with the rate of return. In other words, \( u(R_2a_2) - u(R_2a_1) \geq u(R_1a_2) - u(R_1a_1) \), which is the property known as increasing differences in \((R,a)\).

To see the equivalence between increasing differences in \((R,a)\) and the SED property, fix \( a_1, a_2 \in [0, \infty) \) such that \( a_2 \geq a_1 \). Define the marginal contribution of wealth to the utility to be \( m(R) := u(Ra_2) - u(Ra_1) \). Note that \( m(R) \) is increasing in \( R \) if and only if \( a_2u'(Ra_2) \geq a_1u'(Ra_1) \), which in turn is true if and only if \( c_2u'(c_2) \geq c_1u'(c_1) \). Thus, the function \( u(Ra) \) has increasing differences in \((R,a)\) if and only if \( u \) exhibits SED.

As noted in the introduction, even in a two period consumption-savings model, it could be the case that savings decrease when the rate of return increases. This happens when the income effect dominates the substitution effect. The following example shows, however, that when \( u \) exhibits SED, the substitution effect dominates the income effect.

**Example 1.** Consider an agent who lives for two periods. The agent has \( x > 0 \) dollars at the start of the first period and she receives an income of \( y(s_j) \) with probability \( \theta_j \) in the second period. The agent decides how much to consume in each period. If the agent consumes \( 0 \leq c \leq x \) in the first period and her income in the second period is \( y(s_j) \) then the agent’s consumption in the second period is \( R(x - c) + y(s_j) \), where \( R \) is the rate of return on the agent’s savings \( x - c \). The agent chooses \( 0 \leq c \leq x \) to maximize her expected utility

\[
U(x, c, R) := u(c) + \sum_{j=1}^{n} \theta_j u(R(x - c) + y(s_j)).
\]

When the rate of return increases the income and substitution effects have their impact on the agent’s consumption choice. The substitution effect dominates the income effect if the agent’s consumption \( c(x, R) \) is increasing in \( R \). We show that if \( u \) exhibits SED, then \( c(x, R) \) is increasing in \( R \) for all \( x \in (0, \infty) \). Note that \( u \) exhibiting SED together with the fact that \( u' \) is decreasing imply that for all \( x \in (0, \infty) \) and each \( s \in S \) the function \( Ru(Rx + y(s)) \) is
increasing in \( R \). Let \( R_2 \geq R_1 \), we have

\[
U'(x, c, R_2) = u'(c) - \sum_{j=1}^{n} \theta_j R_2 u'(R_2(x - c) + y(s_j)) \\
\leq u'(c) - \sum_{j=1}^{n} \theta_j R_1 u'(R_1(x - c) + y(s_j)) = U'(x, c, R_1).
\]

Lemma 1 (see the Appendix) implies that \( c(x, R_1) \geq c(x, R_2) \) for all \( x \in (0, \infty) \). Therefore, if the utility function exhibits SED then the substitution effect dominates the income effect.

Given a rate of return function \( R \) we say that the consumption function is concave if for each state \( s \in S \), \( c(x, s, R) \) is concave in \( x \) on \( (0, \infty) \). Zeldes (1989) and Deaton (1991) have noted in numerical studies that when the agent’s labor income function is uncertain, the consumption function is concave in the cash-on-hand. Carroll and Kimball (1996) prove concavity of the consumption function analytically for the key class of hyperbolic absolute risk aversion (HARA) utility functions. Jensen (2017) offers a simpler proof of Carroll and Kimball’s concavity result and generalizes the result to a framework similar to ours, with borrowing constraints and a Markov earnings process. Concavity of the consumption function is also consistent with empirical evidence. Mian et al. (2013) provide detailed empirical evidence that the consumption function is concave in the cash-on-hand and show empirically that the average marginal propensity to consume (MPC) decreases with the cash-on-hand. They show that the average MPC for households living in ZIP codes with an average annual income of less than 35,000 dollars is three times as large as the MPC for households living in ZIP codes with an average annual income of more than 200,000 dollars. Jappelli and Pistaferri (2014) also find that households with low cash-on-hand exhibit a higher MPC than households with high cash-on-hand.

**Theorem 2.** Assume that \( u \) exhibits SED and that the consumption policy function is concave in \( x \). Then \( R_2 \geq R_1 \) implies that \( c(x, s, R_1) \geq c(x, s, R_2) \) for all \( x \in X \) and \( s \in S \).

Let \( \alpha \geq 0 \). A utility function is in the CRRA class if \( u(c) = c^{1-\alpha}/(1 - \alpha) \) for \( \alpha \neq 1 \) and \( u(c) = \ln(c) \) if \( \alpha = 1 \). It is known that the substitution effect dominates the income effect if the RRA is not greater than one,\(^{16}\) i.e., \( \alpha \leq 1 \). If \( u(c) \) is in the CRRA class then the consumption policy function is concave in the cash-on-hand.\(^{17}\) From the above, we can conclude the next corollary.

---

\(^{15}\)Recall that a utility function is in the class of HARA utility functions if its absolute risk aversion \( A(c) \) is hyperbolic. That is, \( A(c) := -\frac{u''(c)}{u'(c)} - \frac{1}{dc} \) for \( c > -\frac{b}{2} \)

\(^{16}\)Thus, \( u \) exhibits SED if \( \alpha \leq 1 \).

\(^{17}\)Recall that if a utility function is in the CRRA class it also belongs to the class of HARA utility functions.
Corollary 1. Let $\alpha \in (0, 1]$. Assume that $u(e) = e^{1-\alpha}/(1-\alpha)$ when $\alpha \neq 1$ and $u(e) = \ln(e)$ when $\alpha = 1$. Then, $R_2 \geq R_1$ implies $c(x, s, R_1) \geq c(x, s, R_2)$ for all $(x, s) \in X \times S$.

We end this section with the case in which the agent has negative cash-on-hand. Assume that $x_0 < 0$ is some borrowing limit. For a discussion on borrowing limits in Bewley models see Aiyagari (1994). Suppose that the agent with a cash-on-hand of $x$ chooses her savings level from the set $[x_0, x]$. In particular she may choose to save a negative amount. It is clear that when the agent has a negative cash-on-hand, an increase in the rates of return has a negative income effect. In this case the income effect and the substitution effect increase consumption when the interest rate is higher, making the next proposition intuitive.

Proposition 1. Let $x$ be such that $x_0 \leq x \leq 0$ (i.e., $x$ is non-positive). Then, $R_2 \geq R_1$ implies that $c(x, s, R_1) \geq c(x, s, R_2)$ for all $s \in S$.

5 Comparing labor income functions

Let $y_1, y_2$ be two labor income functions. We order labor income functions in a natural way: $y_2 \geq y_1$ if $y_2(s) \geq y_1(s)$ for each state $s \in S$. In this section we examine the impact on consumption of increasing the underlying labor income function. We first show that for two labor income functions $y_1, y_2$, if $y_2 \geq y_1$ then the agent consumes more under $y_2$ than under $y_1$ in each state of the economy and at every cash-on-hand level. That is, $c(x, s, y_2) \geq c(x, s, y_1)$ for every $(x, s) \in Z$. This result is related to the permanent income hypothesis (Friedman (1957)) which claims that an agent’s consumption is determined by the agent’s expected future income. We say that $P > 0$ if the probability to move from any state $s_i$ to any other one, $s_j$, is positive (i.e., $P_{ij} > 0$ for every $ij$). If $y_2 \geq y_1$, $y_1 \neq y_2$ and $P > 0$, then the agent’s present discounted value of expected future income under the labor income function $y_2$ is strictly higher than under the labor income function $y_1$. Thus, the permanent income hypothesis would predict that the agent’s consumption is higher in each state and at every cash-on-hand level. We show that it is indeed the case. This result is consistent with empirical evidence, as noted in Jappelli and Pistaferri (2010). We formulate these statements in the following theorem.

Theorem 3. Fix the interest rate policy $R$. Let $y_1 \neq y_2$ be two labor income functions with $y_2 \geq y_1$. Then,

(i) $c(x, s, y_2) \geq c(x, s, y_1)$ for all $(x, s) \in X \times S$.

(ii) If $P > 0$ then inequality (i) is strict for $(x, s) \in X \times S$ that satisfy $c(x, s, y_1) \in (0, x)$.

Consider two agents that are identical except for their income function: they have the same amount of cash-on-hand and the same utility function, but one has an income function
y_1$ and the other $y_2$. Assume that $y_2 \geq y_1$. We measure the utility inequality among the two agents by considering the relative values of their expected present discounted utility. Define, $r(x, s) := \frac{V(x, s, y_1)}{V(x, s, y_2)}$. The smaller $r(x, s)$ is, the larger the relative utility inequality. The next proposition shows that the utility inequality is getting smaller as the level of cash-on-hand is getting higher.

**Proposition 2.** Let $y_2 \geq y_1$. Then, the expected utility inequality $r(x, s)$ is decreasing in $x$ for each state $s \in S$.

6 Final remarks

In this paper we find conditions that guarantee that consumption increases as a result of lower rates of return in an income fluctuation problem. We believe that our results can be useful in studying other models. In particular, in showing that the aggregate demand for savings is increasing in the rate of return in general equilibrium models with heterogeneous agents.

7 Appendix

We first introduce the notations and preliminary results that are needed to prove Theorem 1. The *Envelope Theorem* (see Benveniste and Scheinkman (1979)) implies that the value function is differentiable. For each state $s \in S$ define

$$V'(x, s) := \frac{\partial V(x, s)}{\partial x}.$$ 

Denote by $u'$ the derivative of $u$. In addition, the Envelope Theorem states that for every $(x, s) \in Z$,

$$V'(x, s) \geq u'(c(x, s)),$$

with equality if $c(x, s) > 0$. The last equation is called the *envelope condition*.

Berge’s maximum theorem (see Aliprantis and Border (2006) theorem 17.31) implies that $c(x, s)$ is continuous in $x$. Thus, $V'$ is continuous as the composition of continuous functions. Let $h'(x, a, s_i, V)$ be the derivative of $h$ with respect to $a$. If the savings at $(x, s_i)$ is an interior point (i.e., $a(x, s_i) \in (0, x)$), then the savings policy function must satisfy the first order condition $h'(x, a(x, s_i), s_i, V) = 0$. That is,

$$-u'(x - a(x, s_i)) + \beta \sum_{j=1}^{n} P_{ij} R(s_j) V'(R(s_j) a(x, s_i) + y(s_j), s_j) = 0.$$ 

10
Theorem 1. Assume that $h$ has the CP property. Then, for every $(x, s) \in Z$, $a(x, s, e)$ is increasing and $c(x, s, e)$ is decreasing in $e$.

The proof uses the following lemma.

Lemma 1. Let $z : \mathbb{R} \rightarrow \mathbb{R}$ and $f : \mathbb{R} \rightarrow \mathbb{R}$ be strictly concave, continuously differentiable functions. Let $\phi > 0$. Denote $x_f = \text{argmax}_{x \in [0, \phi]} f(x)$ and $x_z = \text{argmax}_{x \in [0, \phi]} z(x)$. Then,

(i) Assume $x_z \in (0, \phi)$. Then $f'(x_z) \geq z'(x_z)$ if and only if $x_f \geq x_z$.

(ii) If for all $x \in [0, \phi]$ we have $f'(x) \geq z'(x)$, then $x_f \geq x_z$. Furthermore, if $x_z \in (0, \phi)$ and $f'(x) > z'(x)$ for all $x \in [0, \phi]$, then $x_f > x_z$.

Proof. Recall that if $f$ is continuously differentiable and concave then for all $x_1, x_2$ we have $f(x_1) \leq f(x_2) + f'(x_2) (x_1 - x_2)$. Furthermore, if $f : \mathbb{R} \rightarrow \mathbb{R}$ is strictly concave and continuous on a compact subset $[0, \phi] \subseteq \mathbb{R}$ then $f$ has a unique maximizer on $[0, \phi]$.

(i) First assume that $x_z \in (0, \phi)$ which implies from the optimality of $x_z$ that $z'(x_z) = 0$. Since $f$ is concave we have

$$f(x_f) \leq f(x_z) + f'(x_z)(x_f - x_z).$$

From the optimality of $x_f$ we have $f(x_f) > f(x_z)$, thus $f'(x_z)(x_f - x_z) \geq 0$ which implies that $f'(x_z) \geq 0 = z'(x_z)$ if and only if $x_f \geq x_z$. Furthermore, if $f'(x_z) > z'(x_z) = 0$ then $x_z \neq x_f$. For part (ii) consider two cases. The first is where $x_z = 0$. In this case $x_f \geq 0 = x_z$. The second case is $x_z = \phi$ which implies $z'(x) > 0$ for all $x \in (0, \phi)$. Thus, $f'(x) > 0$ for all $x \in (0, \phi)$, implying that $x_f = \phi$. 

Proof of Theorem 1. Assume that $f(x, s, e) \in B(Z \times I)$ is concave in the first argument and that the derivative of $f$ is increasing in $e$ (i.e., $e_2 \geq e_1$ implies $f'(x, s, e_2) \geq f'(x, s, e_1)$ for all $(x, s) \in Z$). The constant function $f \equiv 0$, for instance, satisfies these conditions.

Let $e_2 \geq e_1$, $(x, s) \in Z$ and $a \in C(x)$. A standard argument shows that $Tf$ is strictly concave and bounded. The envelope theorem implies that $Tf$ is differentiable. We now show that $(Tf)'$ is increasing in $e$. Define $a_f(x, s, e) = \text{argmax}_{a \in C(x)} h(x, a, s, e, f)$. Since $h$ has the CP property we have

$$h'(x, a, s, e_2, f(x, s, e_2)) \geq h'(x, a, s, e_1, f(x, s, e_1)).$$

Lemma 1 (ii) implies that $a_f(x, s, e_2) \geq a_f(x, s, e_1)$. Since for all $e \in I$ we have $C(x, e) = C(x)$, $a_f(x, s, e_2) \geq a_f(x, s, e_1)$ implies that $c_f(x, s, e_1) = x - a_f(x, s, e_1) \geq x - a_f(x, s, e_2) = c_f(x, s, e_2)$. The assumption $u'(0) = \infty$ implies that $c_f(x, s, e_2) > 0$. From the envelope condition and the concavity of $u$ we obtain

$$(Tf)'(x, s, e_2) = u'(c_f(x, s, e_2)) \geq u'(c_f(x, s, e_1)) = (Tf)'(x, s, e_1).$$
That is, \((Tf)'\) is increasing in \(e\).

Define the sequence \(f_n = T^n f\), \(n = 1, 2, \ldots\). Then \(f_n\) is strictly concave and \(f'_n\) is increasing in \(e\) for each \(n\). From the Banach Theorem\(^{18}\) \(f_n\) converges uniformly to \(V\). The envelope theorem implies that \(V\) is differentiable. Since \((Tf_n)' = u'(c_{f_n})\) for each \(n\) and\(^{19}\) \(c_{f_n} \to c\) we have \((Tf_n)'(x, s, e) = u'(c_{f_n}(x, s, e)) \to u'(c(x, s, e)) = V'(x, s, e)\). Thus, \(V'(x, s, e)\) is increasing in \(e\). From the same argument as in Eq. (5), \(c(x, s, e)\) is decreasing in \(e\) and \(d(x, s, e)\) is increasing in \(e\) for all \((x, s) \in Z\).

\[\square\]

**Theorem 2.** Assume that \(u\) exhibits SED and that the consumption policy function is concave in \(x\). Then \(R_2 \geq R_1\) implies that \(c(x, s, R_1) \geq c(x, s, R_2)\) for all \(x \in X\) and \(s \in S\).

We first need the following two Lemmas.

**Lemma 2.** Let \(f : [0, \infty) \to [0, \infty)\) be a concave function that satisfies \(f(0) = 0\). Then, the function \(\frac{k}{f(k)}\) is increasing on \((0, \infty)\).

**Proof.** Since \(f\) is concave then the function \(\frac{f(k) - f(k_1)}{k - k_1}\) is decreasing in \(k\) for a fixed \(k_1\). For \(k_1 = 0\) we obtain that \(\frac{f(k) - f(0)}{k - 0} = \frac{f(k)}{k}\) is decreasing in \(k\), implying that \(\frac{k}{f(k)}\) is increasing in \(k\).

**Lemma 3.** Fix the rate of return function \(R\) and the labor income function \(y\). Assume that \(u\) exhibits SED and that the consumption policy is concave. Then, for every \(s \in S\), \(a_1, a_2 \in X\) the function \(kV'(a_1k + a_2, s)\) is increasing in \(k\) on \(X\).

**Proof.** First we show that for every state \(s \in S\), \(xV'(x, s)\) is increasing in \(x\) on \(X\). Let \(s \in S\). The envelope condition implies

\[xV'(x, s) = xu'(c(x, s)) = \frac{x}{c(x, s)}c(x, s)u'(c(x, s)).\]

Let \(\alpha(x) := \frac{x}{c(x, s)}\) and \(\delta(x) := c(x, s)u'(c(x, s))\). Since \(c(x, s)\) is concave in \(x\) and \(c(0, s) = 0\), Lemma 2 implies that \(\alpha(x)\) is increasing in \(x\). Since \(c(x, s)\) is strictly increasing\(^{20}\) in \(x\), and \(cu'(c)\) is increasing on \(X\), \(\delta(x)\) is also increasing in \(x\) on \(X\). Thus, \(xV'(x, s)\) is increasing in \(x\) on \(X\) as the product of two positive increasing functions. Thus, \((a_1k + a_2)V'(a_1k + a_2, s) = a_1kV'(a_1k + a_2, s) + a_2V'(a_1k + a_2, s)\) is increasing in \(k\) on \(X\). Since \(V\) is concave in the first argument, \(a_2V'(a_1k + a_2, s)\) is decreasing in \(k\). This implies that \(kV'(a_1k + a_2)\) is increasing in \(k\) on \(X\), which proves the lemma.

\[\square\]

\(^{18}\)See Theorem 3.48 in Aliprantis and Border (2006).

\(^{19}\)See Theorem 3.8 in Stokey and Lucas (1989).

\(^{20}\)Recall that \(V'(x, s) = u'(c(x, s))\) is strictly decreasing in \(x\) for each \(s \in S\).
We conclude that $h$ has the CP property. Let $R_2 \geq R_1$ and assume that $V'(x, s, R_2) \geq V'(x, s, R_1)$ for all $(x, s) \in X \times S$. Fix $x \in X$, $s \in S$ and $a \in C(x)$. From Lemma 3 we have

$$R_2(s) V'(R_2(s) a + y(s), s, R_2) \geq R_1(s) V'(R_1(s) a + y(s), s, R_2).$$

This inequality and $V'(x, s_1, R_2) \geq V'(x, s_1, R_1)$ for all $(x, s_i) \in X \times S$ imply that

$$\beta \sum_{j=1}^{n} P_{ij} R_2(s_j) V'(R_2(s_j) a + y(s_j), s_j, R_2) \geq \beta \sum_{j=1}^{n} P_{ij} R_1(s_j) V'(R_1(s_j) a + y(s_j), s_j, R_2) \geq \beta \sum_{j=1}^{n} P_{ij} R_1(s_j) V'(R_1(s_j) a + y(s_j), s_j, R_1).$$

Adding $-u'(x - a)$ to both sides of the last inequality yields

$$h'(x, a, s_1, R_2, V(x, s, R_2)) \geq h'(x, a, s_1, R_1, V(x, s, R_1)).$$

We conclude that $h$ has the CP property. Theorem 1 proves the theorem.

\[\square\]

**Proposition 1.** Let $x$ be such that $x_0 \leq x \leq 0$. Then, $R_2 \geq R_1$ implies that $c(x, s, R_1) \geq c(x, s, R_2)$ for all $s \in S$.

**Proof.** Let $x \in [x_0, 0]$. Since $V'$ is decreasing and $a \leq x \leq 0$ we have

$$R_2(s) V'(R_2(s) a + y(s), s, R_2) \geq R_1(s) V'(R_1(s) a + y(s), s, R_2)$$

for each state $s \in S$. We can now continue as in the proof of Theorem 2 and prove the proposition. \[\square\]

**Theorem 3** Fix the interest rate policy $R$. Let $y_1 \neq y_2$ be two labor income functions with $y_2 \geq y_1$. Then,

(i) $c(x, s, y_2) \geq c(x, s, y_1)$ for all $(x, s) \in X \times S$.

(ii) If $P > 0$ (i.e., $P_{ij} > 0$ for every $i, j$) then inequality (i) is strict for $(x, s) \in X \times S$ that satisfy $c(x, s, y_1) \in (0, x)$.

**Proof.** Let $y_2 \geq y_1$. Assume $V'(x, s_i, y_2) \leq V'(x, s_i, y_1)$ for all $(x, s_i) \in X \times S$. Let $(x, s_i) \in X \times S$ and $a \in C(x)$. For each state $s \in S$ we have the following inequality

$$R(s) V'(R(s) a + y_1(s), s, y_1) \geq R(s) V'(R(s) a + y_1(s), s, y_2).$$

13
Thus, we have
\[
\sum_{j=1}^{n} P_{ij} R(s_j) V'(R(s_j) a + y_1(s_j), s_j, y_1) \geq \sum_{j=1}^{n} P_{ij} R(s_j) V'(R(s_j) a + y_1(s_j), s_j, y_2)
\]
\[
\geq \sum_{j=1}^{n} P_{ij} R(s_j) V'(R(s_j) a + y_2(s_j), s_j, y_2),
\]
where the second inequality follows from the strict concavity of \( V \) and the fact that \( y_2 \geq y_1 \).

Multiplying by \( \beta \) and adding \(-u'(x-a)\) to each side of the last inequality yields
\[
(6) \quad h'(x, a, s, y, V(x, s, y_1, V(x, s, y_2))) \geq h'(x, a, s, y, V(x, s, y_2)).
\]
This proves that \( h \) has the CP property. Theorem 1 implies that \( a(x, s, y_1) \geq a(x, s, y_2) \) and \( c(x, s, y_2) \geq c(x, s, y_1) \) for all \( (x, s) \in Z \). Note that inequality (6) is strict if \( y_2 \neq y_1 \), \( P > 0 \). Now apply Lemma 1 to prove that \( c(x, s, y_2) > c(x, s, y_1) \) for all \( (x, s) \) such that \( c(x, s, y_1) \in (0, x) \).

**Proposition 2.** Let \( y_2 \geq y_1 \). Then the expected utility inequality \( r(x, s) \) is decreasing in \( x \) for each state \( s \in S \).

**Proof.** First we show that \( r(x, s) \leq 1 \) for all \( (x, s) \in X \times S \). The proof is by induction. Let \( U_1 = U_2 = 0 \). Define \( U_i^t(x, s) = \max_{a \in C(x)} h(x, a, s, y, U_i^{t-1}) \) for all \( (x, s) \in Z, i = 1, 2 \) and \( t = 1, 2, \ldots \). Then for every \( t, U_i^t \) is strictly increasing, continuous, strictly concave and bounded. Assume that for some \( t \geq 1 \) we have \( U_2^t \geq U_1^t \). Let \( (x, s) \in X \times S \) and \( a \in C(x) \). We have
\[
U_2^{t+1}(x, s) \geq u(x - a) + \beta \sum_{j=1}^{n} P_{ij} U_2^t(R(s_j) a + y_2(s_j), s_j)
\]
\[
\geq u(x - a) + \beta \sum_{j=1}^{n} P_{ij} U_1^t(R(s_j) a + y_1(s_j), s_j),
\]
where the first inequality follows from the definition of \( U_2^{t+1} \). The second inequality follows from the induction hypothesis and the fact that \( U_1^t \) is increasing in the first argument. Taking the maximum in the RHS of the last inequality yields \( U_2^{t+1} \geq U_1^{t+1} \). We conclude that for every \( t \geq 1 \) we have \( U_2^t \geq U_1^t \).

The Banach fixed-point theorem and the continuity of \( V \) imply that \( V(x, s, y_2) \geq V(x, s, y_1) \). Thus, \( r(x, s) \leq 1 \). To prove that \( r(x, s) \) is increasing in \( x \), let \( s \in S \) and note that
\[
\frac{\partial r(x, s)}{\partial x} = \frac{V'(x, s, y_1) V(x, s, y_2) - V(x, s, y_1) V'(x, s, y_2)}{[V(x, s, y_2)]^2} \geq 0,
\]
where the inequality follows from \( V(x, s, y_2) \geq V(x, s, y_1) \geq 0 \) and from Theorem 3, which implies that \( V'(x, s, y_1) \geq V'(x, s, y_2) \geq 0 \).

\( \square \)
References


