1 The effect of cost shifters on equilibrium bid functions

This section adapts the proof techniques in Athey (2001) and Reny and Zamir (2004) to study the effect of competitors’ cost shifters on equilibrium bid functions. The technique consists in considering a model with a discrete set of bids and using the Kakutani fixed-point theorem to show existence of a set of strategies and cost shifters that satisfy the following two key properties:

- the vector of inverse bid functions \( \left[ \beta_j^{-1}(b) \right]_{j=1}^n \) equals a particular vector \( s \) for some bid \( b \in \mathbb{R} \)
- strategies are an equilibrium of the auction game with discrete bids.

As the grid of admissible bids becomes dense in the real line, the sequence of fixed points converges to a set of strategies and a vector of cost shifters. Extending the results in Reny and Zamir (2004), one can show that the limit strategies satisfy the first property and constitute an equilibrium for the continuous bid model.

The first subsection describes the discrete bid model and provides a series of preliminary results. The second subsection applies the fixed-point theorem to find an equilibrium for the game with discrete bids. The third and final subsection shows the existence of an equilibrium in the continuous bid auction. It also presents Proposition 1.10 which is the main result of this section and is invoked in the proof of Proposition 2 in the paper.

1.1 A discrete bid model

Let \( \mathcal{A}_i = \{ a_{i0} < a_{i1} < \ldots < a_{iM} \} \) be the set of available bids to bidder \( i \). Let \( [\underline{s}_i, \bar{s}_i] \subset [0, 1] \) be a subset of \( i \)’s signals. A monotone pure strategy \( \beta_i : [\underline{s}_i, \bar{s}_i] \to \mathcal{A}_i \) can be represented a step function (See Athey, 2001) that describes the points in \( [\underline{s}_i, \bar{s}_i] \) at which \( \beta_i \) jumps. The behavior of \( i \) at the jump points is inconsequential. Let \( \Sigma_i = \left\{ t \in [\underline{s}_i, \bar{s}_i]^M : \underline{s}_i \leq t_1 \leq t_2 \leq \ldots \leq t_M \leq \bar{s}_i \right\} ; t \in \Sigma_i \) represents \( \beta_i \) if \( t_m = \inf \{ \sigma : \beta_i(\sigma) \geq a_m \} \).
When competitors are restricted to select bids from a discrete set and employ monotone strategies \( \{ \beta_j \}_{j \neq i} \), their strategies can be represented by \( T_{-i} \in \Sigma_{-i} = \prod_{j \neq i} \Sigma_j \), where \( T_{-i} = \{ t_j \}_{j \neq i} \) and \( t_j \in \Sigma_j \). Similarly, \( T \in \Sigma = \prod_{i=1}^n \Sigma_i \) represents the strategies of all bidders. The event where \( i \) wins with bid \( b \) given competitors strategies represented by \( T_{-i} \) will be denoted by: \( \eta_i (b|T_{-i}) \). The expected utility of bidder \( i \) when competitors use strategies represented by \( T_{-i} \) will be denoted by \( U_i (b, s_i, z|T_{-i}) \) where \( z \) is a vector of cost shifters. This notation stresses that competitors are bidding from a discrete set of bids. It will be useful to consider the general case that allows bidder \( i \)'s utility to depend on \( z \) and \( t \) as the graph of \( b \) \( \pi \) of \( b \) as a function of \( T \). Consider \( \pi \) such that \( \pi \) is nondecreasing in (each component of) \( T \).

Define bidder \( i \)'s best response correspondence when restricted to choose from the set of bids \( A_i \) as:

\[
b_i^* (s_i, z, T_{-i}, A_i) = \arg \max_{b \in A_i} U_i (b, s_i, z|T_{-i}) .
\]

Define the subset of \( \Sigma_i \) that represents the monotone best response \( b_i^* (s_i, z, T_{-i}, A_i) \) as:

\[
T_i^{BR} (z, T_{-i}, A_i, [s_i, \pi_i]) = \{ t \in \Sigma_i : \forall s_i \in [s_i, \pi_i], t_m < s_i < t_{m+1} \implies a_m \in b_i^* (s_i, z, T_{-i}, A_i) \}.
\]

The following results stress key properties of the utility function \( U_i \) and correspondences \( b_i^* \) and \( T_i^{BR} \).

**Lemma 1.1.** If \( E (C_i|s, z) \) is nondecreasing in (each component of) \( z \) then \( b_i^* (s_i, z, T_{-i}, A) \) is nondecreasing in the strong set order in (each component of) \( z \).

**Proof:** Consider \( b' > b \) and \( z' > z \). Let \( \pi = \Pr (S_{-i} \geq \eta_i (b|T_{-i})|s_i) \) denote the probability of the event where \( i \) wins with bid \( b \) given competitors strategies represented by \( T_{-i} \). Define \( \pi' \) analogously for bid \( b' \)

\[
U_i (b', s_i, z'|T_{-i}) - U_i (b, s_i, z'|T_{-i}) = b' \pi' - b \pi + \int_{\{ \tau: \eta_i (b|T_{-i}) \leq \tau \leq \eta_i (b'|T_{-i}) \}} E (C_i|s_i, S_i = \tau, z') f_{S_{-i}|s_i} (\tau) d\tau \\
\geq b' \pi' - b \pi + \int_{\{ \tau: \eta_i (b|T_{-i}) \leq \tau \leq \eta_i (b'|T_{-i}) \}} E (C_i|s_i, S_i = \tau, z) f_{S_{-i}|s_i} (\tau) d\tau \\
= U_i (b', s_i, z|T_{-i}) - U_i (b, s_i, z|T_{-i})
\]

The function \( U_i (b, s_i, z|T_{-i}) \) exhibits increasing differences in \( (b, z) \); therefore \( b_i^* (s_i, z, T_{-i}, A_i) \) is nondecreasing in the strong set order in \( z \) (by Topkis Theorem).

**Lemma 1.2.** Because \( U_i (b, s_i, z|T_{-i}) \) is continuous in \( (s_i, z, T_{-i}) \), the graph of \( b_i^* (s_i, z, T_{-i}, A_i) \) as a function of \( (s_i, z, T_{-i}) \) is closed for any \( A_i \).

**Proof:** Consider a sequence \( (b^k, s^k, z^k, T^k_{-i}) \) that converges to \( (b, s_i, z, T_{-i}) \) such that \( b^k \in b_i^* (s^k, z^k, T^k, A_i) \). There is a \( K \), such that for all \( k > K \), \( b^k = b \) and \( U_i (b, s^k, z^k|T^k_{-i}) \geq 

\footnote{For economy of notation, I will refer to the graph \( \{(s_i, z, T_{-i}, b) \in [0, 1] \times Z_i \times \Sigma_{-i} \times \mathbb{R} : b \in b_i^* (s_i, z, T_{-i}, A_i) \} \) as the graph of \( b_i^* (s_i, z, T_{-i}, A_i) \) as a function of \( (s_i, z, T_{-i}) \).}
Lemma 1.3. If the graph of \( b_i^* (s_i, z, T_{-i}, \mathcal{A}_i) \) as a function of \((s_i, z, T_{-i})\) is closed for all \( s_i \in [\bar{s}_i, \bar{s}_i]\), then the graph \( T_i^{BR}(z, T_{-i}, \mathcal{A}_i, [s_i', \bar{s}_i']) \) as a function of \((z, T_{-i}, s_i', \bar{s}_i')\) is also closed for all \([s_i', \bar{s}_i']\) strictly included in \([\bar{s}_i, \bar{s}_i]\).

Proof: I follow the proof of Lemma 3 in Athey (2001) very closely. Consider a sequence \((s_i^k, \bar{s}_i^k, z^k, T_{-i}, t^k)\) that converges to \((s_i, \bar{s}_i, z, T_{-i}, t)\) such that \(t^k \in T_i^{BR}(z^k|T_{-i}, \mathcal{A}_i, [s_i^k, \bar{s}_i^k])\) for all \(k\). Consider signal \( s_i \in [\bar{s}_i, \bar{s}_i] \) such that \( t_m < s_i < t_{m+1} \) for some \( m \in \{0, \ldots, M\} \). Because \( t^k_m \) and \( t^k_{m+1} \) converge to \( t_m \) and \( t_{m+1} \), there is a \( K \) such that \( \forall k > K, t^k_m < s_i < t^k_{m+1} \), and thus \( a_m \in b_i^* (s_i, z^k, T_{-i}, \mathcal{A}_i) \). Because \( b_i^* \) has a closed graph, \( a_m \in b_i^* (s_i, z, T_{-i}, \mathcal{A}_i) \).

Lemma 1.4. If signals are affiliated; \( E(C_i|s, z) \) is bounded, nondecreasing in \( s_{-i} \) and strictly increasing in \( s_i \); and ties are precluded: \( U_i(b', s_i, z|T_{-i}) \geq 0, U_i(b', s_i, z|T_{-i}) \geq U_i(b, s_i, z, T_{-i}), (s_i' - s_i)(b' - b) > 0 \) imply \( U_i(b', s_i', z|T_{-i}) \geq U_i(b, s_i', z|T_{-i}) \).

Proof: The result holds by Proposition 2.3 in Reny and Zamir (2004). Assumption A.1.i is satisfied by boundedness and continuity conditions on \( E(C_i|s, z) \), A.1.ii by boundedness and risk neutrality, A.1.iii by monotonicity assumptions on the effect of \( s \) on \( E(C_i|s, z) \) and A.1.iv by risk neutrality. Affiliation and assumptions on the joint density functions ensure that Assumption A.2 also holds.

Lemma 1.5. Suppose that signals are affiliated and that \( E(C_i|s, z) \) is bounded, nondecreasing in \( s_{-i} \) and strictly increasing in \( s_i \). If \( \forall s_i \in [\bar{s}_i, \bar{s}_i], \exists a \in \mathcal{A}_i \) such that \( U_i(a, s_i, z|T_{-i}) \geq 0 \) then \( b_i^* (s_i, z, T_{-i}, \mathcal{A}_i) \) is nondecreasing in the strong set order with respect to \( s_i \in [\bar{s}_i, \bar{s}_i] \).

Proof: Let \( \bar{s}_i \leq s_i < s_i' \leq \bar{s}_i, b \in b_i^* (s_i, z, T_{-i}, \mathcal{A}_i) \) and \( b' \in b_i^* (s_i', z, T_{-i}, \mathcal{A}_i) \). Let \( a, a' \in \mathcal{A}_i \) such that \( U_i(a', s_i, z|T_{-i}) \geq 0 \) and \( U_i(a, s_i', z|T_{-i}) \geq 0 \). Suppose that \( b > b' \). Notice that \((s_i' - s_i)(b' - b) < 0\). Because \( b \in b_i^* (s_i, z, T_{-i}, \mathcal{A}_i) \)

\[ U_i(b, s_i, z|T_{-i}) \geq U_i(b', s_i, z|T_{-i}), \text{ and } U_i(b, s_i, z|T_{-i}) \geq U_i(a, s_i, z|T_{-i}) \geq 0. \]

Lemma 1.4 implies that \( U_i(b, s_i', z|T_{-i}) \geq U_i(b', s_i', z|T_{-i}) \) (\( b \) in this proof takes the place of \( b' \) in the lemma statement and vice versa). Thus \( b \in b_i^* (s_i', z, T_{-i}, \mathcal{A}_i) \). Similarly, because \( b' \in b_i^* (s_i', z, T_{-i}, \mathcal{A}_i) \)

\[ U_i(b', s_i', z, T_{-i}) \geq U_i(b, s_i', z, T_{-i}) \text{ and } U_i(b', s_i', z, T_{-i}) \geq U_i(a', s_i', z, T_{-i}) \geq 0 \]

Lemma 1.4 implies that \( U_i(b', s_i, z|T_{-i}) \geq U_i(b, s_i, z|T_{-i}) \) (\( s_i \) in this proof takes the place of \( s_i' \) in the lemma statement and vice versa). Thus \( b' \in b_i^* (s_i, z, T_{-i}, \mathcal{A}_i) \). It has been shown that for any \( s_i < s_i' \) in \([\bar{s}_i, \bar{s}_i]\), \( b \in b_i^* (s_i, z, T_{-i}, \mathcal{A}_i) \) and \( b' \in b_i^* (s_i', z, T_{-i}, \mathcal{A}_i) \) implies that \( \max \{b, b'\} \in b_i^* (s_i', z, T_{-i}, \mathcal{A}_i) \), \( \min \{b, b'\} \in b_i^* (s_i, z, T_{-i}, \mathcal{A}_i) \). \( b_i^* (s_i, z, T_{-i}, \mathcal{A}_i) \) is nondecreasing in \( s_i \) in the strong set order for \( s_i \in [\bar{s}_i, \bar{s}_i] \).
1.2 Fixed Point

Assume that $\mathcal{A} = \mathcal{A}_i = \{a_0 < a_1 < \ldots < a_M\}$ for all $i$ and that ties are broken using a priority rule. $a_M = \infty$ is equivalent to nonparticipation and $U_i(a_M, s_i, z, T_{-i}) = 0$ for all $(s_i, z, T_{-i})$. Fix $s \in [0, 1]^n$ and $\varepsilon \in [0, 1 - \max s]$. Bidders must bid $a_M$ when they receive a signal above $1 - \varepsilon$. (Athey (2001) and Reny and Zamir (2004) use the same device). $\mathcal{A}$ will be omitted from the notation in $b_i^+$ and $T_i^{BR}$. Similarly, when the set of signals $[s_i, \tilde{s}_i]$ is $[0, 1 - \varepsilon]$ it will be omitted from the notation in $T_i^{BR}$. Subsets $B \subset \mathcal{A}$ and $[s_i, \tilde{s}_i] \subset [0, 1]$ will not be omitted. Let $Z$ be the Cartesian product of intervals $[\underline{z}_i, \overline{z}_i]$.

Define the following correspondence:

$$b_i^+(s_i, z, T_{-i}) = b_i^+(s_i, z, T_{-i}) \quad \text{for all } z_i \in (\underline{z}_i, \overline{z}_i)$$

$$= b_i^+(s_i, z, T_{-i}) \cup \{b \in \mathcal{A} : b \leq \min b_i^+(s_i, z, T_{-i})\} \quad \text{for } z_i = \underline{z}_i$$

$$= b_i^+(s_i, z, T_{-i}) \cup \{b \in \mathcal{A} : b \geq \max b_i^+(s_i, z, T_{-i})\} \quad \text{for } z_i = \overline{z}_i$$

$b_i^+$ is an extension of the best response correspondence that includes all high bids when $z_i = \underline{z}_i$ and all low bids when $z_i = \overline{z}_i$. $b_i^+$ inherits the properties of $b_i^+$. If the graph of $b_i^+(s_i, z, T_{-i})$ as a function of $(z, T_{-i})$ is closed, the graph of $b_i^+(s_i, z, T_{-i})$ is also closed. If $b_i^+(s_i, z, T_{-i})$ is non decreasing in the strong set order in $s_i$, $b_i^+(s_i, z, T_{-i})$ is also nondecreasing.

The goal is to find a set of monotone strategies $T \in \Sigma$ and vector $z \in Z$, such that $T$ represents a set of strategies that constitute an equilibrium of the game and $t_{i, n} \leq s_i \leq t_{i, n+1}$ for all $i$. If $S = s$ is realized under conditions $z$, all bidders bid $a_{\tilde{m}}$.\footnote{The only exception is when $t_{i, n} = t_{i, n+1}$. In this case, bidder $i$ bids strictly below (above) $a_{\tilde{m}}$ for all signals below (above) $s_i$.} Let $B_m^- = \{a_0, a_1, \ldots, a_{\tilde{m}}\} \cup \{a_M\}$, and $B_m^+ = \{a_{\tilde{m}}, a_{\tilde{m}+1}, \ldots, a_M\}$.

$$\Gamma_i(z, T_{-i}) = \left\{(w_i, y_i) \in [\underline{z}_i, \overline{z}_i] \times \Sigma_i : \begin{array}{l}
\exists q : \{y_i, 1, \ldots, y_i, \tilde{m}, q\} \in T_i^{BR}(z, T_{-i}, B_m^-, [0, s_i]) \cup \\
\{y_i, \tilde{m}+1, \ldots, y_i, M\} \in T_i^{BR}(z, T_{-i}, B_m^+, [s_i, 1 - \varepsilon]) \cup \\
\min b_i^+(s_i, z_{-i}, w_i), T_{-i}) \leq a_{\tilde{m}} \leq \max b_i^+(s_i, [z_{-i}, w_i], T_{-i})
\end{array} \right\}. \tag{1}$$

$\Gamma_i$ is a correspondence that maps elements of $Z \times \Sigma_{-i}$ to subsets of $[\underline{z}_i, \overline{z}_i] \times \Sigma_i$. Let

$$\Gamma = \{\Gamma_1, \ldots, \Gamma_n\}. \tag{2}$$

$\Gamma$ is a correspondence that maps elements of $Z \times \Sigma$ onto subsets of the same set. Lemma 1.6 uses the Kakutani Fixed point theorem hold to show that $\Gamma$ has a fixed point. Lemmas 1.7 and 1.8 characterize the fixed point.

**Lemma 1.6.** The correspondence $\Gamma$: (i) is not empty, (ii) has a closed graph, (iii) is convex, and (iv) has a fixed point.
Proof: Part (i). By Assumption A.4 and Lemma 1.5, \( b_i^* (\sigma_i, z, T_{-i}, B_{\overline{m}}^-) \) is nondecreasing in the strong set order with respect to \( \sigma_i \in [0, s_i] \). It follows that \( T_i^{BR} (z, T_{-i}, B_{\overline{m}}^-[0, s_i]) \) is not empty. By the same argument \( T_i^{BR} (z, T_{-i}, B_{\overline{m}}^+, [s_i, 1 - \epsilon]) \) is not empty either. Let \( y_i = \{y_1, ..., y_M\} \in \Sigma_i \), where \( \{y_1, ..., y_m, q\} \in T_i^{BR} (z, T_{-i}, B_{\overline{m}}^+, [0, s_i]) \) and \( \{y_{m+1}, ..., y_M\} \in T_i^{BR} (z, T_{-i}, B_{\overline{m}}^+, [s_i, 1]) \). Now the focus is on finding an appropriate \( w_i \), \( b_i^* (s_i, z, T_{-i}) \) is nondecreasing in the strong set order with respect to \( z \); moreover, it is not empty and has a closed graph. If \( a_{\overline{m}} \leq \max b_i^* (s_i, z, T_{-i}) \), then \( (\tilde{z}_i, y_i) \in \Gamma_i (z, T_{-i}) \). If \( a_{\overline{m}} \geq \min b_i^* (s_i, z, T_{-i}) \), then \( (\tilde{z}_i, y_i) \in \Gamma_i (z, T_{-i}) \). If \( b_i^* (s_i, z, T_{-i}) \leq a_{\overline{m}} \leq \max b_i^* (s_i, w_i^1, T_{-i}) \), then \( (\tilde{z}_i, y_i) \in \Gamma_i (z, T_{-i}) \). If \( a_{\overline{m}} < \min b_i^* (s_i, w_i^1, T_{-i}) \), let \( w_i^2 = 0.5 (\tilde{z}_i + w_i^1) \) while if \( a_{\overline{m}} > \max b_i^* (s_i, w_i^1, T_{-i}) \), then \( w_i^2 = 0.5 (w_i^1 + \tilde{z}_i) \). Repeat this procedure for \( w_i^2 \). Either this procedure eventually reaches some \( k \) such that \( \min b_i^* (s_i, w_i^k, T_{-i}) \leq a_{\overline{m}} \leq \max b_i^* (s_i, w_i^k, T_{-i}) \) and \( (w_i^k, y_i) \in \Gamma_i (z, T_{-i}) \) or \( w_i^k \) converges to \( w_i \). For all \( k \) such that \( w_i < w_i^k \), \( a_{\overline{m}} < \min b_i^* (s_i, w_i^k, T_{-i}) \) whereas for all \( w_i^k < w_i \), \( \max b_i^* (s_i, w_i^k, T_{-i}) < a_{\overline{m}} \). Let \( \{w_i^k\}_q \) denote the subsequence such that \( w_i < w_i^k \) for all \( q \) and \( \{w_i^k\}_r \) denote the subsequence where \( w_i^k < w_i \) for all \( r \). By monotonicity in the strong set order \( \max b_i^* (s_i, w_i^k, T_{-i}) \) converges to \( b^+ \) and \( \min b_i^* (s_i, w_i^k, T_{-i}) \) converges to \( b^- \), where \( b^- < a_{\overline{m}} < b^+ \). Because \( b_i^* \) has a closed graph, then \( b^+ \in b_i^* (s_i, w_i, T_{-i}) \) and \( b^- \in b_i^* (s_i, w_i, T_{-i}) \). It follows that \( (w_i, y_i) \in \Gamma_i (z, T_{-i}) \). Let \( w = \{w_i\}_{i=1}^n \) and \( Y = \{y_i\}_{i=1}^n \) such that \( (w_i, y_i) \in \Gamma_i (z, T_{-i}) \), then \( (w, Y) \in \Gamma (z, T) \).

Part (ii). Consider a sequence \( (z^k, T^k, w^k, Y^k) \) that converges to \( (z, T, w, Y) \) such that \( (w^k, Y^k) \) \( \in \Gamma (z^k, T^k) \) for all \( k \). Consider bidder \( i \). For all \( k \), \( (w_i^k, y_i^k) \in \Gamma_i (z^k, T^k) \), and there is a \( q^k \) such that \( \{y_{i1}, ..., y_{im}, q^k\} \in T_i^{BR} (z^k, T^k, B_{\overline{m}}^+, [0, s_i]) \). Take a subsequence of \( q^k \) that converges to \( q \). \( U_i (b, s_i, z|[T_{-i}] \) is continuous in \( (z, T_{-i}) \). By Lemmas 1.2 and 1.3, \( \{y_{i1}, ..., y_{im}, q\} \in T_i^{BR} (z, T_{-i}, B_{\overline{m}}^+, [0, s_i]) \). By the same argument, \( \{y_{im+1}, ..., y_{iM}\} \in T_i^{BR} (z, T_{-i}, B_{\overline{m}}^+, [s_i, 1 - \epsilon]) \). For all \( k \), \( \min b_i^* (s_i, w_i^k, T_{-i}) \leq \max b_i^* (s_i, w_i^k, T_{-i}) \). A subsequence of \( \min b_i^* (s_i, w_i^k, T_{-i}) \) converges to \( b^- \) and a subsequence of \( \max b_i^* (s_i, w_i^k, T_{-i}) \) converges to \( b^+ \), where \( b^- \leq a_{\overline{m}} \leq b^+ \). Because \( b_i^* (s_i, w_i, T_{-i}) \) has a closed graph, \( b^- \in b_i^* (s_i, w_i, T_{-i}) \), which implies that \( \min b_i^* (s_i, w_i, T_{-i}) \leq a_{\overline{m}} \leq \max b_i^* (s_i, w_i, T_{-i}) \). It follows that \( (w_i, y_i) \in \Gamma_i (z, T_{-i}) \) for each \( i \), which implies that \( (w, Y) \in \Gamma (z, T) \).

Part (iii). Take \( (w_i, y_i), (w'_i, y'_i) \in \Gamma_i (z, T_{-i}) \). Let \( y_i'' = \lambda y_i + (1 - \lambda) y_i' \) and \( w_i'' = \lambda w_i + (1 - \lambda) w_i' \) for some \( \lambda \in [0, 1] \).

By Lemma 1.5, \( b_i^* (\sigma_i, z, T_{-i}, B_{\overline{m}}^-) \) and \( b_i^* (\sigma_i, z, T_{-i}, B_{\overline{m}}^+) \) are nondecreasing in the strong set order with respect to \( \sigma_i \in [0, s_i] \) and \( \sigma_i \in [s_i, 1] \), respectively. Lemma 2 in Athey (2001) ensures that both \( T_i^{BR} (z, T_{-i}, B_{\overline{m}}^-[0, s_i]) \) and \( T_i^{BR} (z, T_{-i}, B_{\overline{m}}^+, [s_i, 1 - \epsilon]) \) are convex. Let \( q'' = \lambda q + (1 - \lambda) q' \),

\(^3\)To keep notation simple in the proofs, \( z_{-i} \) will be omitted from \( b_i^* (s_i, [z_{-i}, z_i], T_{-i}) \).
it follows that 
\[
\{y''_1, ..., y''_{i_0}, q''\} \in T_i^{BR} (z, T_{-i}, B^-_{m}, [0, s_i]) \\
\{y''_{i_0+1}, ..., y''_M\} \in T_i^{BR} (z, T_{-i}, B^+_{m}, [s_i, 1 - \epsilon])
\]

Without loss, assume that \(w_i < w''_i < w'_i\). It is known that 
\[
\min b_i^+ (s_i, w_i, T_{-i}) \leq a_{\tilde{m}} \leq \max b_i^+ (s_i, w_i, T_{-i}) \\
\min b_i^+ (s_i, w_i', T_{-i}) \leq a_{\tilde{m}} \leq \max b_i^+ (s_i, w_i', T_{-i}) .
\]

By Lemma 1.1, \(b_i^+ (s_i, w_i, T_{-i}) \leq s b_i^+ (s_i, w''_i, T_{-i}) \leq s b_i^+ (s_i, w'_i, T_{-i})\). It follows that \(b_i^+ (s_i, w_i, T_{-i}) \leq s b_i^+ (s_i, w'_i, T_{-i})\), which implies:
\[
\min b_i^+ (s_i, w''_i, T_{-i}) \leq \min b_i^+ (s_i, w'_i, T_{-i}) \leq a_{\tilde{m}} \\
\quad a_{\tilde{m}} \leq \max b_i^+ (s_i, w_i, T_{-i}) \leq \max b_i^+ (s_i, w''_i, T_{-i})
\]

Therefore, \((w''_i, y''_i) \in \Gamma_i (z, T_{-i})\). Repeating the argument for all \(i\), it follows that \(\Gamma\) is convex.

Existence of a fixed point follows from the Kakutani Fixed Point Theorem. □

**Lemma 1.7.** \(\Gamma\) has a closed graph with respect to \(s\) and \(\epsilon\). The fixed point also has a closed graph.

**Proof:** Consider a sequence \((z^k, T^k, w^k, Y^k, s^k, \epsilon^k) \to (z, T, w, Y, s, \epsilon)\) where \((z^k, T^k) = \Gamma (w^k, Y^k|s^k, \epsilon^k)\) for all \(k\). Consider bidder \(i\). For all \(k\), \((z^k, t^k_i) \in \Gamma_i (w^k_i, Y^k|s^k_i, \epsilon^k)\) and there is a \(q^k\) such that \(\{y^k_1, ..., y^k_{i_0}, q^k\} \in T_i^{BR} (z^k|T^k_{-i}, B^-_{m}, [0, s^k_i])\). Take a subsequence of \(g^k\) that converges to \(g\). By Lemmas 1.2 and 1.3, \(\{y_{i_0+1}, ..., y_{i_M}\} \in T_{-i}^{BR} (z, T_{-i}, B^-_{m}, [0, s_i])\). By the same argument, \(\{y_{i_0+1}, ..., y_{i_M}\} \in T_{-i}^{BR} (z, T_{-i}, B^-_{m}, [s_i, 1 - \epsilon])\). For all \(k\), \(\min b_i^+ (s^k_i, w_i^k, T_i^k) \leq a_{\tilde{m}} \leq \max b_i^+ (s^k_i, w_i^k, T_i^k)\). A subsequence of \(\min b_i^+ (s^k_i, w_i^k, T_i^k)\) converges to \(b^-\) and a subsequence of \(\max b_i^+ (s^k_i, w_i^k, T_i^k)\) converges to \(b^+\), where \(b^- \leq a_{\tilde{m}} \leq b^+\). Because \(b_i^+ (s_i, w_i, T_{-i})\) has a closed graph, \(b^-, b^+ \in b_i^+ (s_i, w_i, T_{-i})\), which implies that \(\min b_i^+ (s_i, w_i, T_{-i}) \leq a_{\tilde{m}} \leq \max b_i^+ (s_i, w_i, T_{-i})\). It follows that \((w_i, y_i) \in \Gamma (z, T_{-i} | s, \epsilon)\) for each \(i\), which implies that \((w, Y) \in \Gamma (z, T|s, \epsilon)\). This proves the first part.

Consider a sequence \((z^n, T^n, s^n, \epsilon^n) \to (z, T, s, \epsilon)\) where \((z^n, T^n) \in \Gamma (z^n, T^n|s^n, \epsilon^n)\) for all \(n\). Because \(\Gamma\) has a closed graph in \((s, \epsilon)\), then \((z, T) \in \Gamma (z, T|s, \epsilon)\). This proves the second part. □

The following Lemma is one of the main results of this section. It describes the conditions under which a fixed point of \(\Gamma\) is an equilibrium of the discrete bid game. In particular, part (iv) states that if \(z_i\) is in the interior of the set \([z_i, \bar{z}_i]\), then each individual bidder is best-responding to the other bidder’s strategies. Therefore, the strategy profile \(T\) is an equilibrium.

**Lemma 1.8.** If \((z, T)\) is a fixed point of \(\Gamma\) then: (i) \(t_{i_0} \leq s_i \leq t_{i_{M+1}}\) for all \(i\), (ii) if \(z_i = z_{i_0}\) then \(a_{\tilde{m}} \leq \max b_i^+ (s_i, [z_i, \tilde{z}_i], T_{-i})\) and \(a_{m} \in b_i^+ (\sigma_i, z, T_{-i})\) for any \(\sigma_i > s_i\) such that \(t_{i_{\tilde{m}}} < \sigma_i < t_{i_{M+1}}\), (iii) if \(z_i = \tilde{z}_i\) then \(\min b_i^+ (s_i, [z_i, \tilde{z}_i], T_{-i}) \leq a_{\tilde{m}}\) and \(a_{m} \in b_i^+ (\sigma_i, z, T_{-i})\) for any \(\sigma_i < s_i\) such that
\( t_{im} < \sigma_i < t_{im+1} \), and (iv) if \( z_i \notin \{\tilde{z}_i, z_i\} \) then \( \min b_i^*(s_i, z, T_{-i}) \leq a_{\tilde{m}} \leq \max b_i^*(s_i, z, T_{-i}) \) and \( t_i \in T_i^{BR} (z, T_{-i}) \).

**Proof:** Part (i). \( t_{im} \leq s_i \leq t_{im+1} \) because \( \{t_{i1}, ..., t_{im}, q\} \in T_i^{BR} (z, T_{-i}, B_{m_i}^+ [0, s_i]) \) and \( \{t_{im+1}, ..., t_{im}\} \in T_i^{BR} (z, T_{-i}, B_{m_i}^-, [s_i, 1]) \).

Part (ii). If \( z_i = \tilde{z}_i \), then \( a_{\tilde{m}} \leq \max b_i^*(s_i, [z_{-i}, \tilde{z}_i], T_{-i}) \). Because \( b_i^* \) is monotone in the strong order in the signal, \( \sigma > s_i \) and \( a_m \in b_i^*(\sigma, z, T_{-i}, B_{m_i}^+) \) imply \( a_m \in b_i^*(\sigma, z, T_{-i}) \). Therefore, for any \( \sigma > s_i \) such that \( t_{im} < \sigma < t_{im+1} \), \( a_m \in b_i^*(\sigma, z, T_{-i}) \).

Part (iii). If \( z_i = \tilde{z}_i \), \( \min b_i^*(s_i, [z_{-i}, \tilde{z}_i], T_{-i}) \leq a_{\tilde{m}} \). Because \( b_i^* \) is monotone in the strong order in the signal, \( \sigma < s_i \) and \( a_m \in b_i^*(\sigma, z, T_{-i}, B_{m_i}^-) \) imply \( a_m \in b_i^*(\sigma, z, T_{-i}) \). Now suppose that \( q < \sigma < s_i \) so that \( a_m \in b_i^*(\sigma, z, T_{-i}, B_{m_i}^-) \). By monotonicity of \( b^* \), there is a \( m' \leq \tilde{m} \), such that \( a_{m'} \in b_i^*(\sigma, z, T_{-i}) \). It follows that \( U_i (a_{m'}, \sigma, z, T_{-i}) = 0 \). Because all bids have positive probability of winning, \( a_m - E (C_i | s_i, S_{-i} \geq \eta_l (a_{m'} | T_{-i}), z) = 0 \) which implies that \( U_i (a_{m'}, \sigma + \varepsilon, z, T_{-i}) < 0 \) and that \( a_{m'} \notin b_i^*(\sigma + \varepsilon, z, T_{-i}) \) for any \( \varepsilon > 0 \). It follows that \( q = s_i \). Therefore, for any \( \sigma < s_i \) such that \( t_{im} < \sigma < t_{im+1} \), \( a_m \in b_i^*(\sigma, z, T_{-i}) \).

Part (iv). If \( z_i \notin \{\tilde{z}_i, z_i\} \): \( \min b_i^*(s_i, z, T_{-i}) \leq a_{\tilde{m}} \leq \max b_i^*(s_i, z, T_{-i}) \). The results in (ii) and (iii) apply. Therefore, \( t_i \in T_i^{BR} (z, T_{-i}) \).

The following Lemma is the other important result of this section. It bounds the values that cost shifters \( z \in Z \) can take in equilibrium. Together with the previous Lemma, it will allow us to construct sets \( Z \) that are large enough so that only elements on their interior can be fixed points of the map \( \Gamma \) and the resulting strategies represented by \( T \) are an equilibrium of the discrete bid game. Let \( \Delta \) be maximum difference between two consecutive finite bid levels in \( A \).

**Lemma 1.9.** If \((z^k, T^k, \varepsilon^k) \to (z, T, 0)\) such that \((z^k, T^k) = \Gamma (z^k, T^k | \varepsilon^k)\) and \( \overline{c}_i (\tilde{z}_i, z_{-i}) < a_{\tilde{m}} < c_i (\tilde{z}_i, z_{-i}) \) for every bidder \( i \), then:

(i) \( c_i (z_i, z_{-i}) \leq a_{\tilde{m}} \) and \( z_i < \tilde{z}_i \) for all \( i \);

(ii) there is at most one bidder with \( a_m > \overline{c}_i (z_i, z_{-i}) + \Delta \),

(iii) for all other bidders \( z_j > \tilde{z}_i \)

(iv) finally, if there is bidder \( i \) with \( a_{\tilde{m}} > \overline{c}_i (z_i, z_{-i}) + \Delta \) who can win with probability one bidding \( a_0 \), and \( \tilde{z}_i \) is such that:

\[
\min_j c_{ij} (z_j, z_{-j}) - \Pr (S_{-i} \geq s_{-i} | s_i) \min_{j \neq i} c_{ij} (z_j, z_{-j}) - 2\Delta \leq \frac{1 - \Pr (S_{-i} \geq s_{-i} | s_i)}{1 - \Pr (S_{-i} \geq s_{-i} | s_i)} > \overline{c}_i (z_i, z_{-i}),
\]

then,

\[
\min_j c_{ij} (z_j, z_{-j}) - \Pr (S_{-i} \geq s_{-i} | s_i) \min_{j \neq i} c_{ij} (z_j, z_{-j}) - 2\Delta \leq \frac{1 - \Pr (S_{-i} \geq s_{-i} | s_i)}{1 - \Pr (S_{-i} \geq s_{-i} | s_i)} \leq \overline{c}_i (z_i, z_{-i})
\]

and \( z_i > \tilde{z}_i \).
Proof: Part (i). If \( z_i = \tilde{z}_i \), then \( c_i(z_i, z_{-i}) < \tilde{c}_i(\tilde{z}_i, z_{-i}) < a_{\tilde{m}} \) because \( \tilde{z}_i < z_i \). Now suppose \( z_i > \tilde{z}_i \). Parts (iii) and (iv) of Lemma 1.8 state that if \( z_i^k > \tilde{z}_i \), \( \min b_\ast^i(s_i, z_i^k, T_k^i) \leq a_{\tilde{m}} \). Therefore, bidder \( i \) makes non-negative profits by bidding \( a_{\tilde{m}} \) or some other lower amount. Because the probability of winning is positive, the markup has to be non-negative and, as a result, \( a_{\tilde{m}} \geq c_i(z_i^k, z_k^k) \). As \( k \to \infty \), continuity and monotonicity of \( c_i(\cdot, \cdot) \) imply that \( a_{\tilde{m}} \geq c_i(z_i, z_{-i}) \) and \( z_i < \tilde{z}_i \).

Because \( z_i < \tilde{z}_i \), parts (ii) and (iv) of Lemma 1.8 imply that for any \( \sigma > s_j \) such that \( t_{jm}^k < \sigma < t_{jm+1}^k \), \( a_m \in b_{\ast}^j(\sigma, z^k, T_{k-j}^k) \), i.e., \( T_k \) describes the bidders’ best responses.

Part (ii). Suppose that for some \( k \), \( a_{\tilde{m} - 1} > \tilde{c}_i(z_k^k, z_k^k) \). Let \( a_{m^k} \) be the highest finite bid submitted by \( i \) with some positive probability according to \( T_k^k \). Because \( i \)’s markups are positive for any signal and any bid above \( a_{\tilde{m}} \), \( a_{\tilde{m}} \leq a_{m^k} \), and \( i \) will bid at or below \( a_{m^k} \) with probability \( 1 - \varepsilon^k \), i.e., \( i \) will not voluntarily stay out. The same would be true for any \( j \neq i \) such that \( a_{m^k} > \tilde{c}_j(z_j^k, z_j^k) + \Delta \). In such case, when bidder \( i \) bids \( a_{m^k} \) the probability of winning is bounded above by \( \varepsilon^k \). Bidder \( i \) can bid instead \( a_{m-1} > \tilde{c}_i(z_i, z_{-i}) \), attain a positive mark up, and win with probability bounded below by \( \Pr(S_{-i} \geq s_{-i} \mid s_i) \). The difference in the markup between bidding \( a_{m^k} \) and \( a_{\tilde{m} - 1} \), is bounded by \( a_{M - 1} - a_{\tilde{m} - 1} \) which does not depend on \( k \). As \( \varepsilon^k \) approaches zero, bidder \( i \) will find this deviation profitable. Therefore, \( a_{\tilde{m} - 1} > \tilde{c}_i(z_i^k, z_i^k) \) implies that all other bidders should stay out voluntarily with positive probability and \( a_{\tilde{m}} \leq a_{m^k} \leq \tilde{c}_j(z_j^k, z_j^k) + \Delta \). By continuity of \( \tilde{c}_i(\cdot, \cdot) \), \( a_{\tilde{m}} > \tilde{c}_i(z_i, z_{-i}) + \Delta \) for at most one bidder.

Part (iii). For bidder \( j \) such that \( a_{\tilde{m} - 1} \leq \tilde{c}_j(z_j, z_{-j}) \), \( \tilde{c}_j(z_j, z_{-j}) < \tilde{c}_j(z_j, z_{-j}) \) which yields \( z_j > z_j^k \) by monotonicity of \( \tilde{c}_j(\cdot, \cdot) \).

Part (iv). Let \( b_\ast \) be the highest bid that ensures bidder \( i \) to win with probability one and let \( b \) be the bid that \( i \) bids in the interval \( [s_i, s_i + \varepsilon] \) for some positive \( \varepsilon \). By construction, \( b_\ast \geq a_0 \) and \( b \geq a_{\tilde{m}} \). By Parts (ii) and (iv) of Lemma 1.8 and the fact that \( z_i < \tilde{z}_i \) (part i), \( b \) is the optimal bid for \( i \) when the signal is \( s_i \).

\[
0 \leq U_i(b, s_i, z_i) - U_i(b_\ast, s_i, z_i) = \Pr(S_{-i} \geq \eta(b) \mid s_i) b - b_\ast + [1 - \Pr(S_{-i} \geq \eta(b) \mid s_i)] E(C_i \mid s_i, \eta(b) \not\leq S_{-i}, z_i) \\
\leq \Pr(S_{-i} \geq s_{-i} \mid s_i) b - b_\ast + [1 - \Pr(S_{-i} \geq s_{-i} \mid s_i)] E(C_i \mid s_i, \eta(b) \not\leq S_{-i}, z_i),
\]

where the last inequality follows because \( b \geq E(C_i \mid s_i, \eta(b) \not\leq S_{-i}, z_i) \) and \( \Pr(S_{-i} \geq s_{-i} \mid s_i) \geq \Pr(S_{-i} \geq \eta(b) \mid s_i) \).

By parts (i) and (iii), \( z_j < z_j^k < \tilde{z}_j \) for all \( j \neq i \) and they are best-responding. By the arguments in the first part of the Lemma, they are staying out voluntarily with some positive probability and \( b \leq b_\ast \leq \min \, j \neq i \tilde{c}_j(z_j, z_{-j}) + \Delta \). There is a bidder \( k \neq i \) that bids around \( b_\ast \) and makes non-negative profits. Therefore, \( \min_j c_j(z_j, z_{-j}) - \Delta \leq b_\ast \). Moreover, \( E(C_i \mid s_i, \eta(b) \not\leq S_{-i}, z_i) \leq \tilde{c}_i(z_i, z_{-i}) \).

Replacing in the expression above and rearranging:

\[
\frac{\min_j c_j(z_j, z_{-j}) - \Pr(S_{-i} \geq s_{-i} \mid s_i) \min_j \tilde{c}_j(z_j, z_{-j}) - 2\Delta}{[1 - \Pr(S_{-i} \geq s_{-i} \mid s_i)]} \leq \tilde{c}_i(z_i, z_{-i})
\]
Thus, by monotonicity of $\bar{c}_i (\cdot, \cdot)$, $z_i > \bar{z}_i$.

\[ \square \]

1.3 Continuous bid model

This section presents the main proposition of this document. The result generalizes Lemma 1.9 to the continuous bids case. It provides a way to determine whether a set of cost shifters $Z$ contains a vector $z \in Z$ such that there is an equilibrium with inverse bid functions $\beta^{-1}$ that go through a particular vector of signals $s$, i.e., there exist a $z \in Z$ and $b$ that $\beta^{-1}_j (b, z) = s_j$.

The result relies heavily on the discrete bid model. Consider the sequence of discrete bid games with bid sets given by $\mathcal{A}^q$ so that as $q \to \infty$, the set of bids becomes dense in $\mathbb{R}$. For each $q$, let $(z^q, T^q)$ be the limit of a sequence of fixed points $(z^{q, \varepsilon}, T^{q, \varepsilon}, \varepsilon) \to (z^q, T^q, 0)$ such that $(z^{q, \varepsilon}, T^{q, \varepsilon}) = \Gamma (z^{q, \varepsilon}, T^{q, \varepsilon}, \varepsilon)$ corresponding to the discrete set of bids $\mathcal{A}^q$. Let $\beta^q$ be the resulting strategies represented by $T^q$. The convex and bounded set $Z = \prod_{i=1,...,n} [\bar{z}_i, \bar{z}_i]$ is held fixed. Let $a^q_m \in \mathcal{A}^q$ be the reference bid corresponding to $q$, and let $a^q_m \to a$.

**Proposition 1.10.** There exist $(z, \beta)$ that are the limit of a convergent subsequence of $(z^q, \beta^q)$. Moreover, if $\bar{c}_i (\bar{z}_i, z_{-i}) < a < \bar{c}_i (\bar{z}_i, z_{-i})$ for every bidder $i$, then:

(i) $\underline{z}_i (z_i, z_{-i}) \leq a$ and $z_i < \bar{z}_i$ for all $i$;

(ii) there is at most one bidder with $a > \bar{c}_i (z_i, z_{-i})$ while for all other bidders $z_j > \bar{z}_j$;

(iii) if for every bidder $i$ with $a > \bar{c}_i (z_i, z_{-i})$:

\[
\frac{\min_j c_j (z_j, z_{-j}) - \Pr (S_{-i} \geq s_{-i} | s_i) \min_{j \neq i} c_j (z_j, z_{-j})}{1 - \Pr (S_{-i} \geq s_{-i} | s_i)} > \bar{c}_i (z_i, z_{-i}),
\]

then

\[
\frac{\min_j c_j (z_j, z_{-j}) - \Pr (S_{-i} \geq s_{-i} | s_i) \min_{j \neq i} c_j (z_j, z_{-j})}{1 - \Pr (S_{-i} \geq s_{-i} | s_i)} \leq \bar{c}_i (z_i, z_{-i}),
\]

$z_i > \bar{z}_i$, and the strategy profile $\beta$ is an equilibrium of the continuous bid auction game $\left\langle \left\{ u_j \right\}_{j=1}^n, F_S, z \right\rangle$ satisfying $\beta_j (\sigma) < a < \beta_j (\sigma')$ whenever $\sigma < s_j < \sigma'$ for all $j = 1, ..., n$.

**Proof:** By Helley’s Selection Theorem and by compactness of $Z$, there exist a subsequence of $\{(z^q, \beta^q)\}_q$ that converges to some $(z, \beta)$ as $q \to \infty$, where $z \in Z$ and $\beta$ is in the set composed of all $n$-tuples of monotone functions. To simplify notation, $\{(z^q, \beta^q)\}_q$ will denote such convergent subsequence in the remainder of the proof.

If $\bar{c}_i (\bar{z}_i, z_{-i}) < a < \bar{c}_i (\bar{z}_i, z_{-i})$, then $\bar{c}_i (\bar{z}_i, z_{-i}) < a_m < \bar{c}_i (\bar{z}_i, z_{-i})$ for all $q$ larger that some $Q$.

Part (i) in Lemma 1.9 implies, $c_i (z_i, z_{-i}) = a_m < a$, by continuity $c_i (z_i, z_{-i}) < a$ and by monotonicity $z_i < \bar{z}_i$. Part (ii) in Lemma 1.9 implies $a_m \leq \bar{c}_i (z_i, z_{-i}) + \Delta^q$ for at least $n - 1$ bidders. By continuity of $\bar{c}_i (\cdot, \cdot)$, $a > \bar{c}_i (z_i, z_{-i})$ for at most one bidder. For all other bidders $a \leq \bar{c}_j (z_j, z_{-j})$ and, by monotonicity of $\bar{c}_j (\cdot, z_{-j})$, $z_j > \bar{z}_j$. 

9
If there is bidder \( i \) with \( a > \bar{v}_i(z_i, z_{-i}) \) and the condition in the display holds, then for all \( q \) larger than some \( Q , a_{\delta_{i}} > \bar{v}_i(z_i, z_{-i}) \), and

\[
\min_j \xi_j (z^q_j, z^q_{-j}) - \Pr (S_{-i} \geq s_{-i} | s_i) \min_j \bar{v}_j (z^q_j, z^q_{-j}) \geq 2 \Delta \frac{q}{q + \Delta q} > \bar{v}_i (z_i, z_{-i}) .
\]

By part (iv) of Lemma 1.9, there is a bidder \( k \neq i \) such that:

\[
\min_j \xi_j (z^q_j, z^q_{-j}) - \Pr (S_{-i} \geq s_{-i} | s_i) \min_j \bar{v}_j (z^q_j, z^q_{-j}) \geq 2 \Delta \frac{q}{q + \Delta q} \leq \bar{v}_i (z^q_i, z^q_{-i})
\]

Therefore, \( z^q_i > \bar{z}_i \) and \( T^q \) represents equilibrium strategies of the discrete bid game. By continuity,

\[
\min_j \xi_j (z_j, z_{-j}) - \Pr (S_{-i} \geq s_{-i} | s_i) \min_j \bar{v}_j (z_j, z_{-j}) \leq \bar{v}_i (z_i, z_{-i})
\]

Thus, \( z_i > \bar{z}_i \).

I now show that \( \beta \) is an equilibrium of the continuous game when cost shifters are \( z \). The proof of the second part of Theorem 2.1 in Reny and Zamir (2004) would apply without modifications were it not for the different treatment of the restriction that bidders with signals above 1 \( \epsilon \) must bid \( \infty \). They assume that \( \epsilon \rightarrow 0 \) as \( q \rightarrow \infty \). I assume that for each \( q, (z^q, \beta^q) \) is the limit of \( (z^{q, \epsilon}, T^{q, \epsilon}) \) as \( \epsilon \rightarrow 0 \). This technical detail only changes the proof of (A.4) in that paper.

Take any \( j \), pick any \( \sigma_j \) such that \( \beta^q_j (\sigma_j + v) < \infty \) for some \( v > 0 \). For all \( q > Q, \beta^q_j (\sigma_j, x^q) < \infty \). Because \( \beta^q_j (\sigma_j, x^q) \) is a limit of strategies where all bids win with positive probability:

\[
0 \leq \beta^q_j (\sigma_j, z^q) - E \left( \xi_j | \sigma_j, S_{-j} \geq \eta_j \left( \beta^q_j (\sigma_j, z^q) | T_{-j}^q \right) \right) \leq \beta^q_j (\sigma_j, z^q) - E \left( \xi_j | \sigma_j, S_{-j} \geq \eta_j \left( \beta^q_j (\sigma_j, x^q) - \delta | T_{-j}^q \right) \right) \to \beta_j (\sigma_j, z) - E \left( \xi_j | \sigma_j, S_{-j} \geq \eta_j \left( \beta_j (\sigma_j, z) | \beta_{-j} \right) \right)
\]

The second inequality follows for any \( \delta > 0 \) for a sufficiently large \( q \) because the marginal cost is lower than the inframarginal. Take the limit as \( q \rightarrow \infty \) for each \( \delta \) such that \( \beta_j (\sigma_j) - \delta \) is not an atom in the distribution of \( M_j = \min_{k \neq j} \beta_k (S_j, z) \). Take the limit as \( \delta \) goes to zero to obtain last line. This is (A.4) in Reny and Zamir (2004). The rest of the proof there shows that \( \hat{\beta} \) is an equilibrium of the first-price auction game when cost shifters are \( z \).

It remains to show that in equilibrium the tie-curve passes through the vector of signals \( \hat{s} \). For each \( q, t^q_{j \hat{s}} \leq \hat{s}_j \leq t^q_{j \hat{s} + 1} \). Take any \( \sigma < \hat{s}_j, \beta^q_j (\sigma, z^q) \leq a \); take any \( \sigma' > \hat{s}_j, \beta^q_j (\sigma', z^q) \geq a \). It follows that \( \beta_j (\sigma, z) \leq a \) and \( \beta_j (\sigma', z) \geq a \). \( \square \)
2 Constructive Equilibrium Proof

This section shows that there is a large class of auction primitives that support an equilibrium in differentiable and strictly monotone strategies that generate the type of variation necessary for their identification. Consider any joint distribution of signals $F$ that satisfies assumptions A.1, A.2, and A.4. Consider a function $\gamma : [0, 1] \times X \to [0, 1]^n$ such that $\gamma (\cdot, x)$ is a monotone parametric curve on the space of signals $[0, 1]^n$. Let $\dot{\gamma} (t, x) = \frac{\partial}{\partial t} \gamma (t, x)$ and $J = \nabla_x \gamma (t, x)$, i.e., the Jacobian on $\gamma$ with respect to $x$ at all differentiability points. It will be convenient to introduce the normalization $t = 1 - \Pr (S \geq \gamma (t, x))$. This normalization implies $\nabla P J = 0$ and $\nabla P \dot{\gamma} = 1$ so that $\nabla P$, the gradient of the survival function of $S$, is an eigenvalue of $J$ with zero eigenvector.

The following matrix preliminaries will be used in the below. A $P$ ($P_0$) matrix is a complex square matrix with positive (nonnegative) principal minors. An $M$ ($M_0$) matrix is a complex square matrix of the form $A = t I - B$, $B \succeq 0$ with $t$ greater (greater or equal) to the spectral radius of $B$. $M$ ($M_0$) matrices are $P$ ($P_0$) matrices, but the converse is not true.

The following condition ensures monotonicity of each element of $\gamma$ with respect to $t$ and implies the non-vanishing competition condition.

**Condition 2.1.** The parametric curve $\gamma$ is strictly increasing and continuously differentiable with respect to its first argument. There exists a positive constant $\kappa$ such that for all $j, x, t$, $\gamma_j (t, x) \geq \kappa \gamma_j (t, x)$.

For a fixed $x_i$, let $\gamma_{x_i} : [0, 1] \times X_{-i} \to [0, 1]^n$ such that $\gamma_{x_i} (t, x_{-i}) = \gamma (t, [x_i, x_{-i}])$. The following condition requires that this map is a bijection so that it is always possible to find $(x_{-i}, t)$ such that $\gamma_{x_i} (t, x_{-i}) = s$ for any vector of signals $s$.

**Condition 2.2.** For every $i$, the map $\gamma_{x_i}$ is a bijection.

The next condition requires that $\gamma$ is differentiable with respect to $x$ when $t = 1$. Because $\gamma (1, x)$ is in the boundary of the set $[0, 1]^n$, the differentiability requirement will distinguish between the cases when $\gamma (1, x)$ is in a $(n - 1)$-th dimensional facet of the boundary and when it is not. When $s_j = 1$ and $s_{-j} < 1$, the $j$-th row of the Jacobian $J$ will be a vector of zeros. When $s_j = 1$ for more than one bidder, the map $\gamma$ will not be differentiable in $x$. The following condition takes these features of $\gamma$ into consideration.

**Condition 2.3.** If $t = 1$, and $\gamma_j (1, x) = 1$ for only one bidder, the parametric curve $\gamma$ is differentiable with respect to $x$ with Jacobian $J = \nabla_x \gamma (t, x)$ such that $-J_{[-j, -j]}$ is an $M$ matrix, $J_{[j, j]} = 0$ and $J_{[-j, j]} > 0$. If $t = 1$, and $\gamma_j (1, x) = 1$ for more than one bidder, the parametric curve $\gamma$ is semi-differentiable with respect to $x$.

The next Lemma will be used in the main result of this section. It shows conditions under which first-order conditions imply local and global optimality.

**Lemma 2.4.** Assume A.2, A.3, A.4 and A.5. If strictly monotone bid functions $\{\beta_i (\cdot)\}_{i=1}^n$ are such that (i) $b = \beta_i (s_i)$ satisfies the first-order condition for every signal $s_i \in [0, \phi_i]$ and (ii) bidder
i makes non-negative profits bidding \( \beta_i (\phi) \) when her signal is \( \phi \), (iii) \( \beta_i (\cdot) \) is continuous in \([0, \phi]\); then, for any signal \( s_i \leq \phi \): (a) bidder i makes non-negative profits bidding \( \beta_i (s_i) \), and (b) bidding \( \beta_i (s_i) \) is weakly preferable to any \( b' \geq \beta_i (0) \).

Proof. Consider the evolution of expected profits as \( s_i \) decreases from \( \phi_i \):

\[
V_i (s_i) = \int_{\{ \sigma : \sigma \geq s_{-i} (\beta (s_i)) \}} (\beta (s_i) - c_i (s_i, \sigma, x_i)) f (\sigma | s_i) d\sigma
\]

By the envelope theorem and differentiability assumptions in A.2 and A.3,

\[
\frac{d}{ds_i} V_i (s_i) = \left( (\beta (s_i) - E (C_i | S_{-i} \geq s_{-i}, s_i, x_i)) \frac{\partial \Pr (S_{-i} \geq s_{-i} | s_i)}{\partial s_i} \right)
\]

where \( s_{-i} = s_{-i} (\beta (s_i)) \). Affiliation (A.4) implies that \( \frac{\partial \Pr (S_{-i} \geq s_{-i} | s_i)}{\partial s_i} \geq 0 \) and, when full information costs are increasing in own costs (A.5), \( \frac{\partial}{\partial s_i} E (C_i | S_{-i} \geq s_{-i}, s_i, x_i) > 0 \). The right hand side is negative whenever \( (\beta (s_i) - E (C_i | S_{-i} \geq s_{-i}, s_i, x_i)) = 0 \). Therefore, if expected profits are non-negative at \( s_i = \phi_i \) (Hypothesis ii), they will also be non-negative at \( s_i < \phi_i \). This proves part (a).

Proposition 2.3 in Reny and Zamir (2004) shows that when signals are affiliated (A.4) and full information costs are non-decreasing in signals (A.5), the first price auction satisfies the individually rational tieless single crossing condition (IRT-SCC). The single crossing condition states that if bidder \( i \) prefers to bid \( b' \) to \( b \) when she receives signal \( s_i \) and she makes nonnegative profits when she does, she will also prefer to bid \( b' \) to \( b \) when:

- \( b' > b \) and she receives signal \( s_i' > s_i \), and
- \( b' < b \) and she receives signal \( s_i' < s_i \).

Suppose that bidder \( i \) has a profitable deviation to bid \( b' > b_{s_i} \) when she receives signal \( s_i \) and let \( b = \beta_i (s_i, x) \). By construction, she obtains non-negative expected profits if she bids \( b \). Any bid greater than \( b^* \) yields zero profits; therefore, \( b_{s_i} \leq b' \leq b^* \). Consider the case when \( b < b' \) \([b' < b]\). By continuity of the bidder’s objective function, implied by strict monotonicity of \( \beta_j (\cdot) \) for \( j \neq i \) and existence of a density of signals (A.2), there exist a \( \tilde{b} \) between \( b \) and \( b' \) such that bidder i's expected profits are strictly increasing \([strictly decreasing\] in the bid, i.e., there is a \( \delta > 0 \) such that bidder i prefers to bid \( \tilde{b} + \varepsilon \) to \( \tilde{b} \) when her signal is \( s_i \) for all \( 0 < \varepsilon < \delta \) \([-\delta < \varepsilon < 0]\). IRT-SCC implies that she will also prefer bid \( \tilde{b} + \varepsilon \) to \( \tilde{b} \) for all signals greater \([lesser\] than \( s_i \). This contradicts the existence of \( \tilde{s}_i \) being such that \( \tilde{b} = \beta (\tilde{s}_i) \) satisfies the first-order condition. However, by hypotheses (i) and (iii) such signal exists. This shows part (b). \( \square \)

The next proposition is the main result of this section. It shows that there under some regularity conditions on \( F \) and \( \gamma \), it is possible to construct a non-empty convex cone of functions \( b (\cdot, \cdot) \) such that observables \( H \) represented by \( (F, \gamma, b) \) are rationalizable by a set of full-information costs.
functions that are strictly increasing in signals if and only if \( b \) belongs to that cone. The resulting equilibrium strategies are continuous. Moreover, the regularity conditions on \( \gamma \) imply that for every bidder, cost shifter, and vector of signals, the observables satisfy non-vanishing competition (implied by Condition 2.1) and sufficient variation (implied by Condition 2.2). Inverse bid functions are strictly monotone by strict monotonicity of \( \gamma \) and continuity of bid functions.

**Proposition 2.5.** Suppose that \( F \) satisfies the testable implications of assumptions A.1 and A.4, and \( \gamma \) satisfies Conditions 2.1 – 2.3. There is a convex cone with non-empty interior of functions \( b : [0,1] \times X \to \mathbb{R} \) such that \( b \) belongs to the cone if and only if there are full-information costs \( \{c_i(s_i, s, x_i)\}_{i=1}^n \) that together with \( F \) are primitives for which strictly-monotone and continuous strategies \( \{\beta_i : \beta_i(\gamma_i(t,x), x) = b(t,x), \forall i, t, x\} \) are an equilibrium of the first-price auction; moreover, the vector \( \{c_i(s_i, s, x_i)\}_{i=1}^n \) satisfying such condition is unique.

**Proof.** The proof has three parts. The first part mimics Theorem 4. It shows that under conditions 2.1 and 2.2, there is a unique vector of full information costs \( \{c_i(s_i, s, x_i)\}_i \) that satisfy the necessary conditions for equilibrium. The second part Lemma 2.4 to show that if the implied full information costs are monotonic in signals and expected profits conditional on winning are non-negative, the necessary conditions are also sufficient. It also shows that the set of functions \( b(\cdot, \cdot) \) that imply monotonic signals with non-negative expected profits is a cone. The third and last part uses Condition 2.3 to show that the cone has a non-empty interior.

**Part 1.** Let \( B \) be the set of continuous functions \( [0,1] \times X \to \mathbb{R} \), and \( B \subset B \) the subset of functions that are strictly increasing and \( n \)-times differentiable in the first argument. Take any \( b \in B \). Suppose that bidder \( i \)'s competitors play strategies \( \beta_j(\gamma_j(t,x), x) = b(t,x) \). Bidder \( i \)'s problem can be written as:

\[
\max_t \int_{\{\sigma : \sigma \geq \gamma_i(t,x)\}} (b(t,x) - c_i(s_i, \sigma, x_i)) f(s_i) d\sigma. \tag{3}
\]

For \( \beta_i(\gamma_i(t,x), x) = b(t,x) \) to be a best-response, it is necessary that the following boundary and first order conditions hold:

- **Boundary Condition:** Full support of signals on \([0,1]^n\) implies that at \( t = 1 \), there will be at least one bidder \( j \) such that \( \gamma_j(1,x) = 1 \). Bidder \( i \neq j \) will only find it profitable to bid \( b(1,x) \) if it makes zero expected profits. This implies the following boundary condition: If \( s_i = \gamma_i(1,x) \) and \( \gamma_j(1,x) = 1 \), for some \( j \neq i \), then

\[
b(1,x) = E \left( c_i(s_i, S_{-i}, x_i) \mid S_{-i} \geq \gamma_{-i}(1,x), s_i, x_i \right)
\]

- **First Order Condition:** If \( \gamma_j(t,x) < 1 \) for all \( j \neq i \), then \( b(t,x) \) has to satisfy a first-order condition. Taking derivatives of expression (3) with respect to \( t \) and rearranging yields that
if \( s_i = \gamma_i(t, x) \), then:
\[
\begin{align*}
    b(t, x) \sum_{j \neq i} \gamma_j(t, x) \int_{\{\sigma : \sigma \geq \gamma_{-i}(t, x)\}} f(\sigma, \gamma_j(t, x) | s_i) \, d\sigma - \hat{b}(t, x) \int_{\{\sigma : \sigma \geq \gamma_i(t, x)\}} f(\sigma | s_i) \, d\sigma \\
    = \sum_{j \neq i} \gamma_j(t, x) \int_{\{\sigma : \sigma \geq \gamma_{-i}(t, x)\}} c_i(s_i, \gamma_j(t, x), \sigma, x_i) f(\sigma, \gamma_j(t, x) | s_i) \, d\sigma
\end{align*}
\]

In both equations above, the left hand side is a linear transformation \( \Lambda_{0i} \) of \( b \) and the right hand side is a linear transformation \( \Lambda_{1i} \) of \( c \). These linear transformations depend on the function \( \gamma \) and the joint distribution of signals \( F \). The transformation \( \Lambda_{0i} \) maps \( B \) to itself and \( \Lambda_{1i} \) maps the set of differentiable full information cost functions to \( B \). The boundary and first-order conditions can be written more succinctly as \( \Lambda_{0i} b = \Lambda_{1i} c_i \). Let \( \chi_i \) be the cost shifter part of the inverse map of \( \gamma_{x_i} \), i.e., \( x_i = \chi_i(x_i, s) \iff s = \gamma(1 - \Pr(S \geq s), x) \). Let \( \eta^i(t | x_i, s) \in [0, 1]^{n-1} \) be equal to \( y(t) \), the solution to the differential equation
\[
\begin{align*}
y(t_0) &= s_{-i} \\
y(t) &= \gamma_{-i} (1 - (1 - t) Pr(S_i \geq s_i), \chi_i(x_i, y(t))) k^i(t | x_i, s)
\end{align*}
\]
where \( t_0 = 1 - \Pr(S \geq s_i | S_i \geq s_i) \) and \( k^i(t | x_i, s) \) is a scalar that ensures \( t = 1 - \Pr(S \geq y(t) | S_i \geq s_i) \) for all \( t > t_0 \). Notice that
\[
E(C_i | s_i, S_{-i} \geq s_{-i}, x_i) \Pr(S_{-i} \geq s_{-i} | s_i) = \int_{\{\sigma, \sigma \geq \eta^i(t | x_i, s)\}} c_i(s_i, \sigma, x_i) f(\sigma | s_i) \, d\sigma.
\]
The right hand side is an invertible linear transformation \( \Lambda_{2i} \) of \( c_i \) that depends on \( \gamma \) and \( F \). \( \Lambda_{2i}^{-1} \) is the differential operator \( f^{-1} \frac{\partial^{n-1}}{\partial x_i \partial x_i} \). The next step is to find a linear transformation \( \Lambda_{3i} \) that maps \( B \) to the set of differentiable full information costs such that \( \Lambda_{2i} = \Lambda_{3i} \Lambda_{1i} \). Changing the variable of integration, the right hand side can be written as:
\[
\int_{t_0}^1 \left( \sum_{j \neq i} \eta^i_j(\tau | x_i, s) \int_{\{\sigma : \sigma \geq \eta^i_{-j}(\tau | x_i, s)\}} c_i(s_i, \eta^i_j(\tau | x_i, s), \sigma, x_i) f(\sigma, \eta^i_j(\tau | x_i, s) | s_i) \, d\sigma \right) \, d\tau
\]

The transformation \( \Lambda_{3i} \) of \( h \in B \) is
\[
\Lambda_{3i} h = \int_{1 - \Pr(S \geq s_i | S_i \geq s_i)}^1 h (1 - (1 - \tau) \Pr(S_i \geq s_i), \chi_i(x_i, \eta^i(\tau | x_i, s))) \, d\tau.
\]
We have that \( c_i = \Lambda_{2i}^{-1} \Lambda_{3i} \Lambda_{1i} c_i = \Lambda_{2i}^{-1} \Lambda_{3i} \Lambda_{0i} b \). Therefore, for each \( b \in B \), there is a unique collection of full information costs \( \{c_i(s_{-i}, s_i, x_i)\}_{i=1}^n \) that satisfy the necessary boundary and first-order conditions. The full information costs that satisfy the necessary conditions are given by \( c_i = \Lambda_i b \) where \( \Lambda_i = \Lambda_{2i}^{-1} \Lambda_{3i} \Lambda_{0i} \) is a linear transformation that is determined by the path \( \gamma \) and joint distribution \( F \).
Part 2. Let \( B' \subset B \) be the set of functions \( b \) such that for every \( i \): (i) \( c_i \) is strictly increasing in \( s_i \) and weakly increasing in \( s_{-i} \); and (ii) \( s_i \leq \gamma_i(1,x) \) implies that \( i \) makes non-negative expected profits, i.e., \((I - \Gamma \Lambda_i \Lambda_0)b \geq 0\), where \( \Gamma \) is a linear transformation defined by \((\Gamma c)(t,x) = c(\gamma(t,x),x_i)\). It is straightforward to verify that \( B' \) is a convex cone, i.e., if \( b,b' \in B' \) then \( \alpha b + \alpha' b \in B' \) for all positive scalars \( \alpha \) and \( \alpha' \).

Take any \( b \in B' \) and let \( c_i = \Lambda_i b \). Strictly monotone strategies \( \{\beta : \beta_i(\gamma_i(t,x),x) = b(t,x), \forall i, t,x\} \) satisfy the first-order optimality conditions and boundary conditions. Lemma 2.4 implies that for each \( x \), the strategies \( \beta_i(s_i,x) = b(\max \{t : \gamma_i(t,x) \leq s_i\},x) \) are an equilibrium for the game with primitives given by \( F \) and \( \{c_i(s_{-i},s_i,x)\}_i\).

Part 3. This part shows that \( B' \) has a non-empty interior. It will be shown that (i) the condition that \( E(C_i|s_i,S_{-i} \geq s_{-i},x_i) \) is strictly increasing in \( s_i \) imposes linear inequality restrictions on \( \nabla_x b(1,x) \); (ii) the condition of non-negative profits imposes linear inequality restrictions on \( \frac{\partial b(0,x)}{\partial t} \); and (iii) the condition on monotonicity with respect to all signals imposes linear inequality restrictions on \( \frac{\partial^n b(t,x)}{\partial x^n} \).

Let \( x \) be such that \( \gamma_j(1,x) = 1 \) and \( \gamma_{-j}(1,x) < 1 \). For \( s = \gamma(1,x) \), \( \Pr(S_{-i} \geq s_{-i}|s_i) = 0 \) for all \( i \neq j \) and the \( j \)th row of \( J = \nabla_x \gamma(1,x) \) is a vector of zeros because, by continuity of \( \gamma \), for all \( x' \) sufficiently close to \( x \), \( \gamma_j(1,x') = 1 \). Differentiating \( \gamma \) with respect to \( x \):

\[
ds_{-j} = J_{[-j,-j]} dx_{-j} + J_{[-j,j]} dx_j \]

For each \( i \neq j \), it is possible to find a \( dx^{(i)} \) such that \( dx^{(i)}_i = 0 \), \( ds_{-i} = 0 \) and \( ds_i = 1 \). To find it, set \( dx_i \) and \( ds_{-i} \) to zero in the last expression and solve for \( dx_{-i} \):

\[
\begin{bmatrix}
dx^{(i)}_{-ij} & \vdots & dx^{(i)}_j
\end{bmatrix} = \begin{bmatrix}
J_{[-ij,-ij]} & J_{[-ij,j]} \\
J_{[i,-ij]} & J_{ij}
\end{bmatrix}^{-1} \begin{bmatrix}
0 \\
1
\end{bmatrix}
\]

By Condition 2.3, \( J_{ij} > 0 \) and both \( J_{[i,-ij]} \) and \( J_{[-ij,j]} \) are nonnegative. Moreover, \( J_{[-ij,-ij]}^{-1} \) is also non-negative. Therefore, the vector \( dx^{(i)} \) such that \( dx_i = 0 \), \( ds_{-i} = 0 \) and \( ds_i = 1 \) has a non-negative and non-zero \( dx^{(i)}_{-i} \). The boundary condition implies \( b(1,x) = E(C_i|S_{-i} \geq \gamma_{-i}(1,x), \gamma_i(1,x),x_i) \).

The restriction that \( E(C_i|s_i,S_{-i} \geq s_{-i},x_i) \) is strictly increasing in \( s_i \) when \( \Pr(S_{-i} \geq s_{-i}|s_i) = 0 \) is satisfied if and only if \( \nabla_x b dx^{(i)} > 0 \). Let \( X \) be the matrix with column vector \( i \) given by \( dx^{(i)} \); the restriction can be written as \( \nabla_x bX > 0 \). Because \( X \) is a non-negative matrix with a strictly positive \( j \)th row, the set of functions \( b \) that satisfy this restriction has a non-empty interior. For example, it includes every \( b(1,\cdot) \) that is continuous and strictly increasing in cost shifters \( x \).

At \( t = 1 \) every bidder with zero probability of winning makes zero expected profit. One bidder \( i \) may have a non-zero probability of winning if \( s_i = 1 \) and \( s_{-i} < 1 \). This bidder’s expected profits
will be given by
\[ [b(1, x) - E(C_i|s_i, \gamma_{-i}(1, x)) \Pr (S_{-i} \geq \gamma_{-i}(1, x)|s_i)] \]

Using (4) this expression can be written as
\[ \int_{t_0} \left[ [b(1, \chi_i(t_0)) - b(1, \chi_i(\tau))] \sum_{j \neq i} \hat{\eta}_j^i (\tau) \int_{\sigma \geq \eta_j^i(\tau)} f (\sigma, \eta_j^i(\tau)|s_i) \, d\sigma + \hat{b}(1, \chi_i(\tau)) \int_{\sigma \geq \eta(\tau)} f (\sigma|s_i) \, d\sigma \right] \, d\tau \]

This notation omits \( x_i \) and \( s_0 \) because they are held constant. In particular, \( \eta_j^i(\tau) \) is short hand notation for \( \eta_j^i(\tau|x_i, s_0) \), \( s_0 = \gamma(1, x) \). \( \chi_i(\tau) \) is short hand for \( [x_i, \chi_i(x_i, \eta_j^i(\tau|x_i, s_0))] \). \( t_0 = -\Pr (S \geq s_0|S_{-i} \geq s_i) \). Taking derivatives with respect to \( t_0 \) yields:
\[ \left( \nabla_x b(1, x) \dot{\chi}_i(t_0) - \dot{b}(1, x) \right) \Pr (S_{-i} \geq \gamma_{-i}(1, x)|s_i). \]

When \( t_0 = 1 \), the boundary condition ensures that bidder \( i \) expected profits are zero. As \( t_0 \) decreases, profits increase if \( \dot{b}(1, x) \) is sufficiently large. Bidder \( i \)'s expected profits will be non-negative if
\[ \dot{b}(1, \chi_i(t_0)) \geq \nabla_x b(1, \chi_i(t_0)) \dot{\chi}_i(t_0) \]

whenever \( b(1, \chi_i(t_0)) = E(C_i|s_i, \eta_j^i(t_0)|s_i) \). This condition ensures that all bidders have non-negative expected profits at \( s = \gamma(1, x) \) for every \( x \).

Let \( \omega_i(s, x_i) = b(t, x) \) where \( t = 1 - \Pr (S \geq s) \) and \( x_{-i} = \chi_i(x_i, s) \). Because \( b(t, x) = \omega_i(\gamma(t, x), x_i) \), and \( \gamma_j(t, x) = \hat{\eta}_j^i \left( \frac{t}{\Pr (S_{-i} \geq s_i)} \right| x_i, s \right) \), the first-order conditions and equation (4) imply that \( E(C_i|s_i, S_{-i} \geq s_{-i}, x_i) \Pr (S_{-i} \geq s_{-i}|s_i) \) equals
\[ \int_{1 - \Pr (S \geq s|S_{-i} \geq s_i)} \left( \omega_i(\eta_j^i(\tau|x_i, s), x_i) \sum_{j \neq i} \hat{\eta}_j^i(\tau|x_i, s) \int_{\sigma \geq \eta_j^i(\tau|x_i, s)} f (\sigma, \eta_j^i(\tau|x_i, s)|s_i) \, d\sigma \right) \, d\tau 
- \left( \sum_{j} \hat{\eta}_j^i(\tau|x_i, s) \frac{\partial}{\partial s_j} \omega_i(\eta(\tau|x_i, s), x_i) \right) \int_{\sigma \geq \eta(\tau|x_i, s)} f (\sigma|s_i) \, d\sigma \right) \, d\tau 
\]

Applying the linear operator \( \frac{\partial^{n-1}}{\lambda_{i,k} \partial s_k} \) to both sides yields
\[ c_i(s, x_i) f(s_{-i}|s_i) = -\sum_j \left( \Pr (S_{-i} \geq s_{-i}|s_i) \left| \frac{\partial}{\partial s_j} \Pr (S \geq s|S_{-i} \geq s_i) \right|_{s \neq j} \right) \frac{\partial^{n-1}}{\lambda_{i,k} \partial s_k} \omega_i(s, x_i) + \tilde{K}_i \]

The right hand side is a weighted sum over terms of the form \( \frac{\partial^d}{\partial s_k^{d_i, \ldots, s_k^{d_i}}} \omega_i(s, x_i) \) where \( \sum_k \lambda_{k,d} = d \) (see Faà di Bruno’s formula). The term \( \tilde{K}_i \) collects all terms with \( d < n - 1 \) or \( d = n \) and \( \lambda_{k,d} > 1 \) for some \( k \).
where the equation yields:

\[
\frac{\partial^n \omega_i(s, x_i)}{\partial s_1 \ldots \partial s_n} \prod_j \gamma_j < \frac{K_k - c_i(s, x_i) \frac{\partial}{\partial s_k} f(s_{-i}|s_i)}{\Pr(S_{-i} \geq s_{-i}|s_i) \left| \frac{\partial \Pr(S \geq s)}{\partial s_k} \right|}.
\]

where \(K_k\) contains other terms with mixed partial derivatives of \(\omega_i(s, x_i)\) where \(d < n\) or \(d = n\) and \(\lambda_{k,d} > 1\) for some \(k\). The terms in \(K_k\) depend on the value of \(\frac{\partial^n \omega_i(s', x_i)}{\partial s_1 \ldots \partial s_n}\) for \(s' \geq s\) and \(s' \neq s\).

For every bidder \(i\), \(b(t, x) = \omega_i(\gamma(t, x), x_i)\). Applying the linear operator \(\frac{\partial^n}{\partial t^n}\) to both sides of the equation yields:

\[
\frac{\partial^n b(t, x)}{\partial t^n} = \frac{\partial^n \omega_i(s, x_i)}{\partial s_1 \ldots \partial s_n} \prod_j \gamma_j + K_i
\]

where \(K_i\) contains other terms with mixed partial derivatives of \(\omega_i(s, x_i)\) where \(d < n\) or \(d = n\) and \(\lambda_{k,d} > 1\) for some \(k\). Therefore, if

\[
\frac{\partial^n b(t, x)}{\partial t^n} < \min_i \left( \min_k \frac{K_k - c_i(s, x_i) \frac{\partial}{\partial s_k} f(s_{-i}|s_i)}{\Pr(S_{-i} \geq s_{-i}|s_i) \left| \frac{\partial \Pr(S \geq s)}{\partial s_k} \right|} + K_i \right)
\]

the full-information cost function that satisfies the boundary and first-order conditions is non-decreasing in all signals. The bound on \(\frac{\partial^n b(t, x)}{\partial t^n}\) can be determined at each point \((t, x)\) as a function of \(b(t', x')\) for \(\gamma(t', x') \geq \gamma(t, x)\). This concludes the proof that \(B'\) has a non-empty interior.

3 Consistency, Asymptotic Normality and Validity of the Bootstrap

This section shows that the the estimator used in this paper belongs to the class of estimators studied by Chen, Linton, and Van Keilegom (2003). Let \(H\) define the family of joint distributions with Gaussian Copula and write \(H = F^{n \times n} \times PD_n\), where \(F\) is the set of univariate densities and \(PD_n\) is the set of symmetric positive definite matrices with unit main diagonal. For bidder \(i\), the moment function for quantile \(\tau\) and instrument \(x_j\) is:

\[
m_{i,j,\tau}\left(\{b, x\}, \{\alpha_i, \nu_i\}, H\right) = \left[1 \left( b_i - \alpha_{i0}x_0 - \alpha_{i1}x_i - \alpha_{i2}k_i - \nu_i - \nu_{i\tau} \leq 0 \right) - \pi_i(\tau, x|H) \right] x_j
\]

where \(t = 1, ..., T_i\) indexes all the auctions where \(i\) participates, \(k_{it} = k_i(b_{it}, x_i|H)\) and \(\nu_{it} = \nu_i(b_{it}, x_i|H)\). Let

\[
M_i(\{\alpha_i, \nu_i\}, H) = E\left(\{m_{i,j,\tau}(\{B, X\}, \{\alpha_i, \nu_i\}, H)\}_{j,t}\right)
\]

\[
= E\left(\left[ F_{B_i|X} (\alpha_{i0}X_0 + \alpha_{i1}X_i + \alpha_{i2}k_i + \nu_i + \nu_{i\tau}) - \pi_i(\tau, x|H) \right] x_j\right)_{j,t}.
\]
At the true unknown \( \{ \alpha_i, \nu_i \} \in \Theta \) and \( H^* \in \mathbb{H} \), \( M_i (\{ \alpha^*, \nu^* \}, H^*) = 0 \). This paper estimates the model parameters \( (\alpha_i, \nu_i) \) for bidder \( i \) solving the following sample minimization problem:

\[
\min_{(\alpha_i, \nu_i)} \left\| M_{i,n} \left( \{ \alpha_i, \nu_i \}, \hat{H} \right) \right\| \tag{5}
\]

where \( \hat{H} \) is a first-stage estimate of \( H^* \) and \( M_{i,n} (\cdot) \) is a sample analog of \( M_i (\cdot) \):

\[
\frac{1}{T_i} \sum_{t=1}^{1} \left[ 1 \left( b_{it} - \alpha_{i0}x_{0it} - \alpha_{i1}x_{it} - \alpha_{i2k_{it}} - \nu_{it} \leq 0 \right) - \pi_i (\tau, x_t | \hat{H}) \right] x_t.
\]

The estimator (5) only depends on \( H \) through the functions \( \tau_i, \pi_i, \kappa_i \). Let \( \mathcal{H} \) denote the infinite dimensional parameter set with typical element equal to a triplet of functions \( h_i = (\tau_i, \pi_i, \kappa_i) \in \mathcal{H} \). As in Chen, Linton, and Van Keilegom (2003), \( M_i (\theta_i, h_i) \) with \( \theta_i = (\alpha_i, \nu_i) \) is continuous and differentiable in \( (\theta_i, h_i) \) even if \( m_{i,j,\tau} \) is not. The requirement that the pseudo-metric \( ||\cdot||_{\mathcal{H}} \) is a sup-norm with respect to \( \theta_i \) is trivially satisfied because, in the estimator defined by (5), \( h_i \) does not depend on \( \theta_i \). The paper only requires that \( ||\cdot||_{\mathcal{H}} \) is a pseudo-metric with respect to all other arguments. As a result it is possible to use the \( L_2 \) metric without imposing additional restrictions on the functional space \( \mathcal{H} \).

The estimator defined by (5) satisfy conditions (1.1) and (2.1) in Chen, Linton, and Van Keilegom (2003). The identification results ensure conditions (1.2) and condition (2.2) can be verified by differentiating \( M_i \) with respect to \( \theta_i \) around their true unknown value and verifying that the Jacobian has full rank and it is continuous with respect to \( \theta_i \). Let

\[
\Gamma_1 = \frac{\partial M_i (\{ \alpha^*, \nu^* \}, H^*)}{\partial \{ \alpha, \nu \}} = \{ E \left( f_{B_{ij}|X} (\alpha_{i0}^*X_0 + \alpha_{i1}^*X_i + \alpha_{i2k_{i}}^* + \nu_{i}^* + \nu_{i\tau}^*) X_j | X_0, X_i, k_i^* ,1) \} \}_{j,\tau}
\]

The fact that the distribution of bids is atomless ensures continuity and identification ensures full-rank.

The estimator of \( \pi_i \) is a deterministic function of \( \phi_i (X|H) \) that is just a non-parametric regression of participation on geographical coordinates. Stone (1980, 1982) showed that this object can be estimated at rate \( r = \frac{p-m}{2p+d} \), where \( p \) is the smoothness of the regression function, \( m \) is the order of the estimate and \( d \) is the dimension of the covariate space. In this case, \( m = 0, d = 2 \) and, under the assumption that the probability of participation conditional on geographical coordinates is twice differentiable, \( p = 2 \). For \( \tau \) bounded away from zero, \( \pi_i \) can be estimated at rate \( r = \frac{2-0}{4+2} = 1/3 \) which is faster than required by conditions (1.4) and (2.4). The estimator of \( \kappa_i \) is a deterministic function of the parametric estimates of the Gaussian copula and of all \( F_{B_{ij}|lat,lon} (b) \), which are just non-parametric regressions of an indicator of bidding above \( b \) on geographical coordinates which can also be estimated at a sufficiently fast rate. The term \( \epsilon_i \) is a function of the density of the minimum competitors’ bids conditional on geographical coordinates and own bid. In this case, \( m = 1 \) and \( d = 4 \). Under the assumption that the CDF of bids conditional on geographical coordinates and own bids has continuous fourth derivatives, \( p > 4 \), which yields an optimal convergence rate.
\( r > \frac{4 - 1}{8 + 4} = 1/4 \) which is exactly the rate required by conditions (1.4) and (2.4).

Conditions (1.3) and (2.3) can be verified by differentiating \( M_i \) with respect to \( h_i \) around its true unknown value. Define matrices \( \Gamma_{2,\pi} \), \( \Gamma_{2,k} \) and \( \Gamma_{2,t} \) so that

\[
\begin{align*}
\Gamma_{2,\pi} \times (\pi_i(\cdot | H) - \pi_i(\cdot | H^*)) &= E \left\{ \left( \phi_1(x | H^*) - \frac{\phi_1(X | H^*)}{\phi_1(X | H)} \right) X_j \right\}_{j,\tau} \\
\Gamma_{2,k} \times (k_i(\cdot | H) - k_i(\cdot | H^*)) &= E \left\{ \left( f_{B_i | X} (\alpha_{i0} x_0 + \alpha_{i1} x_i + \alpha_{i2} k_i + \nu_i) \right) X_j \left[ k_i(B_{it}, X_i | H) - k_i(B_{it}, X_i | H^*) \right] \right\}_{j,\tau} \\
\Gamma_{2,t} \times (t_i(\cdot | H) - t_i(\cdot | H^*)) &= E \left\{ \left( f_{B_i | X} (\alpha_{i0} x_0 + \alpha_{i1} x_i + \alpha_{i2} k_i + \nu_i) \right) X_j \left[ t_i(B_{it}, X_i | H) - t_i(B_{it}, X_i | H^*) \right] \right\}_{j,\tau}
\end{align*}
\]

Notice that the fact that the distribution of bids is atomless ensures conditions (2.3.1) and (2.3.2). The terms \( \phi_1 \), \( k_i \), and \( t_i \) are differentiable functions of standard series estimators. Ichimura and Newey (2017) show how to derive the influence functions for this type of estimators that satisfy conditions (2.5) and (2.5′). I follow their arguments in Example 2 that studies an closely related estimator.

The expectation of the second line is bounded by \( \sup_{\|H' - H\|_2 \leq \delta} \left| \pi_i(\tau, x | H) \right| + \sup_{\|H' - H\|_2 \leq \delta} \left| \pi_i(\tau, x | H) \right| \]

where \( (x, y) \) is short-hand for \( \min(x, y), \max(x, y) \). The first line is bounded by \( K_1(\tau, x) \delta \) where \( K_1(\tau, x) \) denotes the maximum directional derivative of \( \pi_i(\tau, x | H) \) with respect to \( H \) around the true \( H_0 \). The expectation of the second line is bounded by \( K_2(\tau) \delta \) where \( K_2(\tau) \) is proportional to the density of \( b \) and the expected directional derivative of \( k_i = k_i(B_{it}, x_i | H) \) and \( t_i = t_i(B_{it}, x_i | H) \) around the true parameter values \( \{\alpha, \nu\}_0 \) and \( H_0 \). Because the first stage estimator of \( H \) is restricted to have a Gaussian copula and the marginal distributions of bids are restricted to be continuous, \( E(K_1(\tau, X)) \) and \( K_2(\tau) \) are bounded for any \( \tau \in (0, 1) \). Therefore, condition (3.2) is satisfied with constants \( r = 2 \) and \( s_j = 1/2 \). Theorem 3 in Chen, Linton, and Van Keilegom (2003) shows that conditions (2.5) and (2.5′B) hold.

References


