Auctions with Liquidity Subsidies*

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Abstract

This paper proposes liquidity subsidies for improving allocative efficiency and price discovery in multi-unit auctions. In the proposed subsidy scheme, the market administrator divides some amount of subsidy revenue between agents proportional to their marginal contribution to the slope of auction aggregate demand at the equilibrium price. These subsidies cause agents to bid more aggressively, increasing the slopes of their submitted bid curves. This decreases bid shading, increases allocative efficiency, and lowers the variance of auction prices.

1 Introduction

This paper proposes liquidity subsidies for improving allocative efficiency and price discovery in multi-unit auctions. The market administrator divides some amount of revenue $R$ between agents, proportional to their marginal contribution to the slope of aggregate auction demand at the equilibrium price – that is, the slope of their submitted bid curves at the auction clearing price, divided by the slope of aggregate auction demand. I show that these liquidity subsidies cause agents to bid more aggressively, increasing the slopes of the demand curves that they submit. This increases the elasticity of residual supply faced by each bidder, which decreases bid shading and increases allocative efficiency, for small subsidies. Larger subsidy quantities cause agents to bid

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even more aggressively, demanding larger quantities than they would if they bid truthfully; while this harms allocative efficiency, it increases the slope of aggregate auction supply, causing auction prices to be more stable in equilibrium, which is independently desirable if auctions are used primarily for price discovery.

Formally, I show that agents’ best responses in a supply-function equilibrium bidding game are characterized by a simple Euler-Lagrange equation, in which agents optimally set the difference between marginal utility and marginal cost equal to a wedge which is a function of subsidies. When agents’ utility functions are quadratic, as most of the applied multi-unit auctions literature assumes, the Euler-Lagrange equations have simple linear solutions which allows me to analytically solve for auction equilibrium under subsidies.

Liquidity subsidies may be an attractive tool for market administrators in a variety of cases. First, in settings where bid shading is significant, small subsidies can alleviate shading, encouraging aggressive bidding and improving allocative efficiency. Quantitatively, I show that relatively small amounts of revenue can have large impacts on equilibrium outcomes. Second, in settings where auctions are primarily used not for reallocation but for price discovery, liquidity subsidies decrease the variance of auction prices by increasing the slope of auction supply. This makes the price discovery function of the auction more robust and less prone to manipulation, and thus may be socially valuable beyond the simple allocative effects of the auction.

1.1 Related literature

This paper is related to a number of strands of literature. It contributes to the large body of theory on uniform-price multi-unit auctions, starting with Wilson (1979); a number of papers discuss basic demand reduction incentives in these auctions, for example Ausubel et al. (2014). I propose a subsidy scheme which encourages aggressive bidding, and can potentially shift equilibrium behavior significantly; this is most closely related to McAdams (2002), who proposes a similar subsidy scheme. A number of other schemes for promoting aggressive bidding in multi-unit auctions are Kremer and Nyborg (2004), LiCalzi and Pavan (2005) and McAdams (2007).

More broadly, multi-unit auctions have found many applications in financial settings, and this is the subject of a large body of empirical and theoretical work. On the theory side a large literature on market microstructure, including for example Kyle (1985), Kyle (1989), Antill and Duffie (2017) and Du and Zhu (2017), has utilized double-auction models of
trading platforms. On the empirical side, a large body of work, surveyed in Kastl (2017), analyzes the performance of different market mechanisms for financial products, most commonly treasury auctions. Relative to existing theoretical and empirical papers, this paper takes an explicit market design approach, proposing a novel mechanism that can potentially be applied to many of these settings.

1.2 Outline

The remainder of the paper proceeds as follows. Section 2 introduces the model. Section 3 demonstrates the impact of subsidies on agents’ best-response bidding behavior. Section 4 analyzes equilibrium behavior under subsidies. Section 5 discusses various applications and extensions of these results.

2 Model

In this section, I exposit the main model used in this paper, which is an auction-plus-workup mechanism with quadratic inventory costs and Gaussian uncertainty. This is the basis for the equilibrium results of Section 4. However, the results in Section 3 will not exploit all of the structure of the model, hence some of these results apply to more general models.

2.1 Utilities

There is a single asset, with commonly known fundamental value $\pi$. The value $\pi$ is known to all market participants, but not to the market administrator. There are $n \geq 3$ identical risk-neutral dealers indexed by $i$.

Agents have independent private values, and all agents are risk-neutral. There are two classes of agents: “sophisticated traders” who are able to participate in either auctions or fixed-price protocols, and “unsophisticated traders” who I exogeneously constrain to participate only in the protocol in a reduced-form manner. In the initial analysis, I do not explicitly model uncertainty, instead fixing the value and cost functions of buyers and sellers and assuming these to be common knowledge; thus, the solution concept is ex-post equilibrium. Traders may be net buyers or net sellers depending on market conditions,
which I describe below. Traders begin with randomly distributed inventory $z_{i0} \sim N(0, \sigma_z^2)$ of the asset.

Agents’ utility consists of the fundamental value of the asset, $\pi$, plus a quadratic inventory cost term $\gamma z^2_i$. This is a simple version of a model studied in a number of papers (Vayanos, 1999; Rostek and Weretka, 2015; Du and Zhu, 2017; Duffie and Zhu, 2017). If an agent ends the trading game holding $z_i$ units of the asset, her utility is

$$\pi z_i - \gamma z^2_i - y_i$$

(1)

Where $y_i$ is any net payment made by agent $i$. As a result, the physical utility of an agent who begins with $z_{i0}$ units of the asset for buying $z$ additional units is

$$U(z) = \left( \pi (z_{i0} + z) - \gamma (z_{i0} + z)^2 \right) - \left( \pi z_{i0} - \gamma z_{i0}^2 \right)$$

$$U(z) = \pi z - 2\gamma z_{i0} z - \gamma z^2$$

(2)

2.2 Auctions

Sophisticated agents submit strictly decreasing demand schedules $z_i(p)$ into the auction stage, representing the net amount they are willing to purchase at price $p$. Alternatively, we can represent bids as strictly decreasing inverse demand curves $p_D(z)$. In addition, there is some completely inelastic shock $\epsilon \sim N(0, \sigma_\epsilon^2)$ to total supply in the auction stage. This may reflect, for example, bids from relatively small bidders, which come too close to the auction deadline for agents to respond to. The mechanism thus contains two kinds of uncertainty in quantities: endogeneous uncertainty, from the random starting positions $z_{i0}$, and exogeneous uncertainty from the inelastic shock $\epsilon$.

The market clearing price is the unique price that equates supply and demand:

$$p_{auc} = \left\{ p : \sum z_i(p) - \epsilon = 0 \right\}$$

Each agent then pays $pz_i(p)$, and purchases quantity $z_i(p)$.

From the perspective of agent $i$, in equilibrium, residual supply will be an affine function with constant slope, which is horizontally shifted due to uncertainty in the exogeneous supply shock $\epsilon$ and other agents’ inventory positions $z_{j0}$. That is, the agent
faces a residual supply curve

\[ p_{RS}(z, \eta) = \pi + dz - d\eta \]  (3)

Where the supply shock \( \eta \) is a random variable which combines uncertainty in \( \epsilon \) and \( z_{i0} \). I characterize \( \eta \) in detail in Section 4. For a given value of \( \eta \), agent \( i \) needs to pay \( p_{RS}(z, \eta) \) per unit to buy \( z \) units of the asset. This solution concept, in which agents best-respond to a residual inverse supply curve subject to horizontal supply shocks, is the supply function equilibrium concept originally proposed by Klemperer and Meyer (1989). I analyze agents’ best responses to these uncertain supply functions in the following Section.

3 Subsidies

Suppose some amount of revenue \( R \) is to be divided between participants. I propose that agent \( i \) is paid:

\[
\frac{z_i'(p)}{\sum_{j=1}^{n} z_j'(p)} R = \frac{\frac{1}{p_{Di}}}{\sum_{j=1}^{n} \frac{1}{p_{Dj}}} R
\]  (4)

Suppose that, in equilibrium, agent \( i \) bids against the strictly increasing residual supply curve:

\[ p_{RS}(z, \eta) = p_{RS}(z - \eta) \]

Where \( \eta \) is random. This is a generalization of the family of random affine residual supply curves in (3) of Section 2. In Appendix Subsubsection A.1, I show that agents’ submitted demand curves \( z(p) \) can alternatively be represented as functions \( z(\eta) \), specifying the amount demanded when the shock is \( \eta \). The constraint that demand functions are monotonically decreasing is equivalent to the constraint that \( 0 \leq z'(\eta) \leq 1 \), and also that

\[
\frac{1}{p_i} + \frac{1}{p'_i} + \frac{1}{p_{RS}} = z'(\eta)
\]

Hence, instead of choosing a demand curve, the agent can be thought of as choosing a function \( z(\eta) \) and receiving subsidy \( Rz'(\eta) \) if the supply shock is \( \eta \). In words, \( z'(\eta) \) is the pass-through of supply shocks to a given agent – the amount of additional quantity that
the agent demands when aggregate supply increases by one unit. Agents are rewarded for being willing to absorb more of residual supply.

The agent’s optimization problem can thus be written as:

$$\max_{z(\eta)} \int \left[ U(z(\eta)) - z(\eta) p(z(\eta); \eta) + R z'(\eta) \right] dF(\eta)$$  \hspace{1cm} (5)

This is the expectation of agents’ utility net of costs $U(z(\eta)) - z(\eta) p(z(\eta); \eta)$, plus the subsidy $R z'(\eta)$, over residual supply uncertainty $\eta$. This is an optimization problem involving an integral over the function $z(\eta)$ and its derivative $z'(\eta)$. It is thus a calculus of variations problem, and in Appendix Subsection A.2 I show that the optimal solution to this problem is characterized by a pointwise Euler-Lagrange FOC.

**Proposition 1.** The necessary and sufficient first-order condition for a strictly monotone solution to (5) is the Euler-Lagrange FOC:

$$v'(z(\eta)) - MC(z(\eta), \eta) = R \frac{\partial \log f}{\partial \eta}$$  \hspace{1cm} (6)

The term on the right in (6) is the percentage change of $f$ with respect to $\eta$. This equation is intuitive for the case of a unimodal distribution, such as a Gaussian, in which case the RHS is positive when $\eta$ is small and negative when $\eta$ is large. It says that marginal value should be higher than marginal cost, thus quantity lower than statically optimal, when $\eta$ is low and $f(\eta)$ is increasing in $\eta$, and marginal value should be lower than marginal cost, thus quantity higher than statically optimal, when $\eta$ is higher than its mode and $f(\eta)$ is decreasing in $\eta$. The net effect is to increase the slope of the demand function about the most likely equilibrium price points. Figure 1 illustrates the behavior of optimal demand curves as $R$ is increased, for a case in which the marginal value function is the cotangent function $u'(z) = \cot(z)$, residual supply is linear, and $\eta$ is Gaussian. Demand curves pivot around the most likely equilibrium price as $R$ is increasing.

My results are related to a number of proposals for combating collusive equilibria in uniform-price auctions. In particular, it is similar to McAdams (2002), differing only in that bidders in my mechanism are subsidized according to their marginal quantity, instead of according to rationed quantity as in McAdams (2002). The intuition behind these mechanisms is that, in uniform-price auctions, bidders are close to indifferent for different values of the slope of their submitted demand curves at the market clearing price; however,
this slope affects equilibrium residual demand and supply curve slopes, thus influencing
the optimal markups of other agents in equilibrium. Thus, small changes in rules that
influence incentives for agents’ demand and supply slopes, such as payments, rationing
rules Klemperer and Nyborg (2004), or adjustable supply by the market platform LiCalzi
and Pavan, 2005; McAdams, 2007 can significantly influence equilibrium outcomes.
Technical, the distinction between my results and the literature is that I am the first to
apply the “marginal subsidy” insights under Klemperer-Meyer uncertainty. I show that
this leads to very tractable best-response structure, and in Section 4 below I show that the
subsidy case can be solved explicitly in an example with quadratic utilities and Gaussian
uncertainty.

[Figure 1 about here.]

4 Equilibrium

4.1 Equilibrium characterization

I now consider the following setting. Utilities are as specified in 2. The market administra-
tor has a fixed amount of total revenue $R$ available to subsidize agents, and she commits
to implementing subsidies using the formula (4). I assume that workup quantity $\Delta$ is
infinitesimally small, and that bid-ask spreads $\tau$ are high enough that agents have no
incentives to purchase in the workup stage; I show how to set workup stage spreads
later in this section. I search for equilibria in which both demand functions and residual
supply functions are affine, and the residual supply shock $\eta$ is Gaussian (as it results from
combining Gaussian uncertainty in $z_{i0}$ and $\eta$). Hence, from the perspective of agent $i$,
residual supply is

$$
p_{RS} (z, \eta) = \pi + dz + d\eta
$$

From (5), if agent $i$ has inventory position $z_{i0}$, she chooses $z (\eta)$ to solve:

$$
\max_{z(\eta)} \int \left[ \left[ \pi z (\eta) - 2 \gamma z_{i0} z (\eta) - \gamma z (\eta)^2 \right] - z (\eta) [\pi + dz (\eta) + d\eta] + Rz' (\eta) \right] dF (\eta)
$$

Proposition 1 characterizes such equilibria given $R, \gamma, \sigma_z^2, \sigma_\epsilon^2, \pi$. 

7
Proposition 2. Let $\psi$ be the unique solution in the interval $[0, \gamma]$ to:

$$\frac{R}{\psi} = \left[ \left( \frac{n-2}{n-1} \right) \left( \frac{\gamma}{\gamma-\psi} \right) \right]^2 (n-1) \sigma_z^2 + \sigma^2$$

(7)

If $\psi < \frac{2\gamma}{n}$, then there is a linear equilibrium with all agents bidding:

$$p_D (z_i) = \pi + \frac{\gamma}{\gamma-\psi} (2\gamma - n\psi) z_i - \frac{n-1}{n-2} (2\gamma - n\psi) z_i$$

(8)

The auction price is distributed as:

$$(p_{auc} - \pi) \sim N \left( 0, \left[ \left( \frac{2\gamma-n\psi}{2\gamma-2\psi} \right) (2\gamma) \right]^2 \frac{\sigma_z^2}{n} + \left[ \left( \frac{n-1}{n-2} \right) \left( \frac{2\gamma-\psi n}{n} \right) \right]^2 \sigma^2 \right)$$

(9)

The equilibrium is characterized by $\psi = \frac{R}{\sigma^2}$, which I will call the subsidy ratio. This is implicitly determined by equation (7); while it has a relatively complicated analytical expression, it is a strictly increasing function of $R$. As I discuss in Section 3, the basic effect of subsidies is that they decrease the slopes of agents’ submitted inverse demand curves: the slope $\frac{dp}{dz_i} = \frac{n-1}{n-2} (2\gamma - n\psi)$ is a strictly decreasing function of the subsidy ratio $\psi$. In addition, the sensitivity of price to $z_i$, the slope $\frac{dp}{dz_{i0}} = \frac{\gamma}{\gamma-\psi} (2\gamma - n\psi)$, is also decreasing in $\psi$. The net result is to flatten agents’ demand curves and to decrease their variance around the mean price $\pi$, as shown in Figure 2.

In equilibrium without subsidies, agents shade their bids – the slope of price is $\frac{dp}{dz_i} = \frac{n-1}{n-2} (2\gamma)$, which is higher than the slope of marginal values $2\gamma$. Subsidies counteract this; for the choice $\psi = \frac{2\gamma}{n}$, the slope of the bid curve is equal to the slope of marginal values $2\gamma$. In Appendix C.1, I show that this leads to optimal reallocation of initial inventory between auction participants.

Interestingly, increasing $\psi$ further continues to decrease the slope of demand curve. Not all values of $\psi$ can sustain equilibria; if $\psi > \frac{2\gamma}{n}$, the slope $\frac{dp}{dz_i}$ becomes negative, hence bidders’ candidate equilibrium demand functions violate the monotonicity constraint. Intuitively, it is possible for subsidies to be large enough that bidders give up bidding to buy or sell quantity, instead bidding extremely elastic demand functions to maximize the subsidy revenues that they attain. As $\psi$ is increased past the value $\frac{2\gamma}{n}$, bidders actually bid curves further away from their true values, decreasing the efficiency of the auction in reallocating inventory between bidders. However, this makes aggregate auction
supply steeper; this increases the cost of price impact, and thus is beneficial for price discovery and combating manipulation. Correspondingly, expression (9), showing the variance of prices around its mean, is strictly decreasing in \( \psi \); increasing subsidy revenue monotonically increases price stability. This suggests that, if the goal of the market administrator is price discovery rather than maximizing the efficiency of reallocation, she may want to use subsidies higher than the choice \( \psi = \frac{\gamma}{n-1} \) which induces honest bidding.

Figure 3 shows the behavior of various equilibrium objects, as we vary \( n \) and the amount of subsidy revenue. Auction welfare is maximized at \( \psi = \frac{\gamma}{n-1} \), which translates to a revenue amount that is decreasing in \( n \). The standard deviation of prices is monotonically decreasing in prices, approaching 0 as \( \psi \to \frac{2\gamma}{n} \).

5 Discussion

This paper solves a model of auctions with liquidity subsidies. Quantitatively small subsidies based on agents’ equilibrium bid slopes cause agents to bid much more aggressively; this in turn decreases other agents’ incentives to shade bids, increasing allocative efficiency in equilibrium. This also increases the equilibrium slope of auction demand, which is beneficial in settings where the variance of auction prices is itself important.

These results could be useful in a number of settings. In settings where auctions are used to set price benchmarks, market administrators could charge fees to benchmark users, and then use the resultant revenue to subsidize auctions; For example, Duffie (2018b), Duffie (2018a), and Zhu (2018) have proposed using a double auction to determine the conversion rate between LIBOR and SOFR, and to allow agents to contract to convert LIBOR to SOFR at the auction rate. The auction administrator could charge fees for any parties using the auction price to convert contracts from LIBOR to SOFR, and then use these fees to subsidize auction participants, improving allocative efficiency, and perhaps more importantly, decreasing manipulability to improve price discovery.

A natural next step is to calibrate the model of this paper to data from auctions in practice, to calculate how large subsidies must be in order to substantially improve auction performance.
References


Duffie, Darrell. 2018a. “Compression Auctions, with an Application to LIBOR-SOFR Swap Conversion.”

Duffie, Darrell. 2018b. “Notes on LIBOR Conversion.”


Appendix

A  Proofs for Section 3

A.1  Pass-through

In this subsection, I show that

\[ -\frac{1}{p_i'} - \sum_{j=1}^{n} \frac{1}{p_j'} = z'(\eta) \]

Assume that in equilibrium, from the perspective of buyer i, residual supply shifts horizontally with shock \( \eta \), so that residual inverse supply is \( p_{RS} (z - \eta) \). Suppose a buyer chooses function \( z(\eta) \). Suppose the submitted inverse demand curve is \( p_D (z) \). Given \( z(\eta) \), we want to recover what \( p_D (z) \) is and the subsidy amount. To do this, note that \( z(\eta) \) is the amount demanded by an agent, meaning that residual supply and demand must cross at \( z(\eta) \). Hence,

\[ p_{RS} (z(\eta) - \eta) = p_D (z(\eta)) \]

Differentiating with respect to \( \eta \), we have

\[ p'_{RS} (z(\eta) - \eta) (z'(\eta) - 1) = p'_D (z(\eta)) z'(\eta) \]

Solving,

\[ z'(\eta) = \frac{p'_{RS}}{p'_{RS} - p'_D} \] (10)

Now, note that

\[ -\frac{1}{p_{Di}} = -\sum_{j=1}^{n} \frac{1}{p_{Dj'}} \]

From the definition of residual supply:

\[ z_{RS} (p) = \epsilon - \sum_{j \neq i} z_j (p, z_{j0}) \]
We have that
\[
\frac{1}{p'_{RS}} = - \sum_{j \neq i} \frac{1}{p'_{j}}
\]

Hence,
\[
\frac{-1}{p'_{Di}} - \sum_{j \neq i} \frac{1}{p'_{Di}} - \frac{1}{p'_{RS}} = \frac{-1}{p'_{Di}}
\]
\[
= \frac{-1}{p'_{Di} - p'_{RS}} = \frac{1}{p'_{Di} p'_{Dj} - p'_{RS}} = \frac{p'_{RS}}{p'_{RS} - p'_{Dj}} = z' (\eta)
\]

as desired. From this representation, we also find that, since \( p'_{Di} \leq 0 \) and \( p'_{RS} \geq 0 \), the monotonicity of \( p_{Di} \) is equivalent to \( z' (\eta) \) being bounded between 0 and 1.

Given any \( z (\eta) \) function, we can recover a demand function that induces \( z (\eta) \) by taking the inverse function \( \eta (z) \equiv z^{-1} (z) \), and then setting the demand function:

\[
p_{D} (z) = p_{RS} (z - \eta (z))
\]

By construction, the function \( p_{D} (z) \) crosses \( p_{RS} (z - \eta) \) at \( z (\eta) \) for all \( \eta \).

### A.2 Euler-Lagrange first-order conditions

The agent’s optimization problem can be written as:

\[
\max_{z(\eta)} \int \left[ U (z(\eta)) - z(\eta) p (z(\eta) ; \eta) + Rz' (\eta) \right] dF (\eta)
\]

(11)

Or, defining \( TC (z (\eta) , \eta) \equiv z (\eta) p (z (\eta) ; \eta) \), and \( dF (\eta) = f (\eta) d\eta \),

\[
\max_{z(\eta)} \int \left[ U (z(\eta)) - TC (z (\eta) , \eta) + Rz' (\eta) \right] f (\eta) d\eta
\]

The first-order condition for this problem is an Euler-Lagrange equation, of the form:

\[
\frac{\partial F}{\partial z} - \frac{d}{dx} \frac{\partial F}{\partial z'} = 0
\]
We have:

\[
\frac{\partial F}{\partial q} = [U'(z(\eta)) - MC(z(\eta), \eta)] f(\eta)
\]

\[
\frac{\partial F}{\partial q'} = Rf(\eta)
\]

\[
\frac{d}{dx} \frac{\partial F}{\partial q'} = R \frac{\partial f}{\partial \eta}
\]

Plugging in, we have:

\[
(U'(z(\eta)) - MC(z(\eta), \eta)) f(\eta) = R \frac{\partial f}{\partial \eta}
\]

or,

\[
U'(z(\eta)) - MC(z(\eta), \eta) = R \frac{\partial \log f}{\partial \eta}
\]

as desired. If \(U'(\cdot)\) is decreasing and \(MC(\cdot, \cdot)\) is increasing in \(z\) for all \(\eta\), the integrand in (11) is jointly concave in \(z\) and \(z'\), hence by Theorem 10.7.1 of Brunt (2004), the pointwise FOC is necessary and sufficient for function \(z(\eta)\) to be optimal. Hence, so long as \(0 < z'(\eta) < 1\), so that \(z(\eta)\) defines a monotone demand function, it is the optimal solution to (11).

### B Linear equilibrium characterization

#### B.1 Equilibrium slopes

As before, suppose that

\[
U'(z) = \pi - 2\gamma z_{i0} - 2\gamma z
\]

Now suppose that residual supply is some affine function

\[
p_{RS}(z, \eta) = \pi + dz - d\eta
\]

\[
TC(z, \eta) = z p(z, \eta) - z_F p(z, \eta)
\]

\[
MC(z, \eta) = p(z, \eta) + z p'(z, \eta) = \pi + 2dz - d\eta
\]
And, suppose that there is some subsidy revenue $R$. Suppose the agent chooses demand $z(\eta, z_{i0})$ when the residual supply shock is $\eta$ and her inventory shock is $\eta$. The Euler-Lagrange first order conditions in state that:

$$
[U'(z(\eta, z_{i0})) - MC(z(\eta, z_{i0}), \eta)] f(\eta) = R \frac{\partial f}{\partial \eta}
$$

Specializing to $\eta$ normal with mean 0, variance $\sigma^2_\eta$, and plugging in all other expressions, we have:

$$
[(\pi - 2\gamma z_{i0} - 2\gamma z(\eta, z_{i0})) - (\pi + 2dz(\eta, z_{i0}) - d\eta)] e^{-\frac{(\epsilon - \mu)^2}{2\sigma^2_\eta}} = R \left( -\frac{\eta}{\sigma^2_\eta} \right) e^{-\frac{(\epsilon - \mu)^2}{2\sigma^2_\eta}}
$$

Now define $\psi \equiv \frac{R}{\sigma^2_\eta}$. So, agents’ best responses are:

$$
(2\gamma z_{i0} - 2\gamma z(\eta, z_{i0})) - (2dz(\eta, z_{i0}) - d\eta) = -\psi \eta \quad (13)
$$

The variance $\sigma^2_\eta$ of $\eta$, and thus the quantity $\psi$, will be endogeneous in equilibrium. Hence, we will proceed in two steps. First, we will show that there exists a unique linear equilibrium for any $\psi$ sufficiently small. Then, we will characterize $\sigma^2_\eta$ in equilibrium, and show that for any $R$ there is a unique $\sigma^2_\eta$ which it can induce in equilibrium.

### B.2 Demand slopes

Fix $\psi$. Differentiating (13) with respect to $\eta$, we get:

$$
-2 (\gamma + d) \frac{dz}{d\eta} + d = -\psi
$$

$$
\frac{dz}{d\eta} = \frac{\psi + d}{2 (\gamma + d)}
$$

Now, defining demand as

$$
p_D(z) = a + cz_0 - bz
$$

We have, from (10):

$$
\frac{dz}{d\eta} = \frac{p'_R}{p'_R - p'_D} = \frac{d}{d + b}
$$
\[(d + b) \frac{dz}{d\eta} = d\]

\[b = \frac{d}{z'(\eta)} - d = d \left( \frac{1}{z'(\eta)} - 1 \right)\]

Hence,

\[b = d \left( \frac{2 (\gamma + d)}{\psi + d} - 1 \right) = d \left( \frac{2\gamma + d - \psi}{\psi + d} \right)\]

As before, residual supply is the horizontal sum of demand from \(n - 1\) agents. Thus,

\[
\frac{dp_{RS}}{dz} = -\frac{1}{n - 1} \frac{dp_D}{dz}
\]

\[\implies d = \frac{b}{n - 1}\]

Solving (14) and (15), we have:

\[d = \frac{2\gamma - \psi n}{n - 2}\]

\[p_{RS}(z, \eta) = \pi + \frac{2\gamma - \psi n}{n - 2} z - \frac{2\gamma - \psi n}{n - 2} \eta\]  

\[\text{B.3 Demand functions}\]

Given slopes, we also want to solve for the demand functions. Solving (13) for \(\eta\), we get:

\[\eta (d + \psi) = 2\gamma z_{i0} + (2\gamma + 2d) z (\eta, z_{i0})\]

Using \(\eta (z, z_{i0})\) to denote the inverse of \(z (\eta, z_{i0})\) fixing \(z_{i0}\), we can write:

\[\eta (z, z_{i0}) = \frac{(2\gamma + 2d) z + 2\gamma z_{i0}}{d + \psi}\]

Now, to get the optimal inverse demand function, we plug this expression for \(\eta\) into the residual supply function in (17), to get:

\[p_D = \pi + dz - d \frac{(2\gamma + 2d) z + 2\gamma z_{i0}}{d + \psi}\]
Substituting out d’s using (16), we have:

\[
p_D(z_i, z_{i0}) = \pi + \frac{\gamma}{\gamma - \psi} (2\gamma - n\psi) z_i - \frac{n-1}{n-2} (2\gamma - n\psi) z_i
\]

(18)

as desired. This is only a valid equilibrium if it is a strictly decreasing demand function; hence, in order for this to be a valid equilibrium, we need \( \psi < \frac{2\gamma}{n} \).

### B.4 Equilibrium residual inverse supply

To characterize \( \eta \), rearrange equilibrium demand functions to get:

\[
\frac{n-1}{n-2} (2\gamma - n\psi) z_i = (\pi - p(z_i, z_{i0})) + \frac{\gamma}{\gamma - \psi} (2\gamma - n\psi) z_{i0}
\]

Writing \( z_i(p, z_{i0}) \) to mean the inverse of \( p(z_i, z_{i0}) \), we have:

\[
z_i(p, z_{i0}) = \left( \frac{n-2}{n-1} \right) \left( \frac{1}{2\gamma - n\psi} \right) (\pi - p) + \left( \frac{n-2}{n-1} \right) \left( \frac{\gamma}{\gamma - \psi} \right) z_{i0}
\]

Residual supply facing any given agent, when the inelastic supply shock is \( \epsilon \) and other agents have inventory positions \( z_{j0} \), is:

\[
z_{RD}(p, z_{j0}, \epsilon) = -\sum_{j=1}^{n-1} z_j(p, z_{j0}) + \epsilon
\]

\[
z_{RD}(p, z_{j0}, \epsilon) = \left( \frac{n-2}{2\gamma - n\psi} \right) (\pi - p) + \left( \frac{n-2}{n-1} \right) \left( \frac{\gamma}{\gamma - \psi} \right) \sum_{i=1}^{n-1} z_{i0} + \epsilon
\]

Solving for price, we get residual inverse demand \( p_{RS}(z, z_{j0}, \epsilon) \):

\[
p_{RS}(z, z_{j0}, \epsilon) = \pi - \left( \frac{n-2}{2\gamma - n\psi} \right) z + \left( \frac{n-2}{2\gamma - n\psi} \right) \left[ \left( \frac{n-2}{n-1} \right) \left( \frac{\gamma}{\gamma - \psi} \right) \sum_{i=1}^{n-1} z_{i0} + \epsilon \right]
\]

Hence, residual inverse supply has constant slope \( \frac{n^2}{2\gamma - n\psi} \), and behaves as if it is subject
to a single inelastic quantity shock $\eta$, equal to the term in the rightmost square brackets:

$$
\eta = \left( \frac{n-2}{n-1} \right) \left( \frac{\gamma}{\gamma - \psi} \right) \sum_{i=1}^{n-1} z_{i0} + \epsilon
$$

(19)

Thus,

$$
\eta \sim N \left( 0, \left[ \left( \frac{n-2}{n-1} \right) \left( \frac{\gamma}{\gamma - \psi} \right) \right]^2 (n-1) \sigma_z^2 + \sigma^2 \right)
$$

(20)

B.5 Equilibrium given $R$

Now, we want to show that there is only one possible equilibrium for any given $R$. From the definition of $\psi$, we have:

$$
\sigma^2 = \frac{R}{\psi}
$$

(21)

Hence, fixing $R$, any equilibrium $\psi$ must simultaneously satisfy (21) and (20), or:

$$
\frac{R}{\psi} = \left[ \left( \frac{n-2}{n-1} \right) \left( \frac{\gamma}{\gamma - \psi} \right) \right]^2 (n-1) \sigma_z^2 + \sigma^2
$$

(22)

The LHS is strictly positive and the RHS is positive only for $\psi$ positive, so any $\psi$ satisfying (22) must be positive. Also, the LHS is unboundedly large for $\psi$ near 0 and is decreasing in $\psi$, whereas the RHS is increasing in $\psi$ on the interval $[0, \gamma]$ and becomes unboundedly large as $\psi \to \gamma$, and both the LHS and RHS are continuous. Hence, there is a unique solution $\psi$ in $[0, \gamma]$, which we will call $\psi(R)$. Solutions $\psi > \gamma$ cannot constitute an equilibrium, since then $\psi > \frac{2\gamma}{n}$. Thus, the only candidate equilibrium value of $\psi$ is $\psi(R)$. If $\psi(R) < \frac{2\gamma}{n}$, it induces strictly monotone demand functions, and thus supports a valid equilibrium.

B.6 Equilibrium price variation

Rearranging (18), we have:

$$
p_D - \pi = \frac{\gamma}{\gamma - \psi} (2\gamma - n\psi) z_{i0} - (n-1) \frac{2\gamma - \psi n}{n-2} z_i
$$

18
Hence, summing across $n$ agents,

$$n (p - \pi) = \frac{\gamma}{\gamma - \psi} (2\gamma - n\psi) \sum_{i=1}^{n} z_{i0} - (n - 1) \frac{2\gamma - \psi n}{n - 2} \sum_{i=1}^{n} z_i$$

Now, in equilibrium, market clearing requires net demands to be equal to the inelastic quantity shock $\epsilon$, so $\sum_i z_i = \epsilon$. Hence,

$$n (p - \pi) = \frac{\gamma}{\gamma - \psi} (2\gamma - n\psi) \sum_{i=1}^{n} z_{i0} - (n - 1) \frac{2\gamma - \psi n}{n - 2} \epsilon$$

$$p - \pi = \frac{\gamma}{\gamma - \psi} \frac{2\gamma - n\psi}{n} \sum_{i=1}^{n} z_{i0} - \frac{n - 1}{n - 2} \frac{2\gamma - \psi n}{n} \epsilon$$

Thus,

$$(p - \pi) \sim N \left( 0, \left[ \left( \frac{2\gamma - n\psi}{2\gamma - 2\psi} \right) (2\gamma) \right] \frac{\sigma_z^2}{n} + \left[ \left( \frac{n - 1}{n - 2} \left( \frac{2\gamma - \psi n}{n} \right) \right) \frac{\sigma_\epsilon^2}{\epsilon} \right] \right)$$

(23)

as desired.

C Linear equilibrium analysis

C.1 Auction allocative efficiency

From (6), we have:

$$(-2\gamma z_{i0} - 2\gamma z (\eta, z_{i0})) - (2dz (\eta, z_{i0}) - d\eta) = -\psi \eta$$

$$\Rightarrow z (\eta, z_{i0}) = \frac{\gamma}{\gamma + d} z_{i0} + \frac{d + \psi}{2\gamma + 2d} \eta$$

(24)

Plugging in $d = \frac{2\gamma - \psi n}{n - 2}$, we have:

$$z (\eta, z_{i0}) = \frac{n - 2}{n} \frac{\gamma}{\gamma - \psi} (-z_{i0}) + \frac{n}{n} \eta$$

(25)
Plugging in \(\eta\) from (19), we have:

\[
  z(\eta, z_{i0}) = \frac{n-2}{n} \frac{\gamma}{\gamma - \psi} (-z_{i0}) + \frac{(\frac{n-2}{n-1}) \left(\frac{\gamma}{\gamma - \psi}\right)}{n} \sum_{i=1}^{n-1} z_{i0} + \epsilon
\]  

(26)

At the allocatively efficient outcome, we require agents to keep the average of all initial inventory positions. Hence, we have:

\[
  z_{i0} + z(\eta, z_{i0}) = \frac{\epsilon + \sum_{i=1}^{n} z_{i0}}{n}
\]

Solving this, we get:

\[
  \psi = \frac{\gamma}{n-1}
\]

An alternative way to see this is that allocative efficiency requires agents to keep \(\frac{1}{n}\) of their endowments \(z_{i0}\), so to trade away \(\frac{n-1}{n}\) of their endowments. Hence, we need

\[
  \frac{dz}{dz_{i0}} = \frac{n - 1}{n}
\]

We have from (26) that

\[
  \frac{dz}{dz_{i0}} = \frac{n-2}{n} \frac{\gamma}{\gamma - \psi}
\]  

(27)

Hence when \(\psi = 0\), we have \(\frac{dz}{dz_{i0}} = \frac{n-2}{n}\), so agents trade too little of their endowments, which leads to inefficiency. Solving the below:

\[
  \frac{n-2}{n} \frac{\gamma}{\gamma - \psi} = \frac{n - 1}{n}
\]

We again get \(\psi = \frac{\gamma}{n-1}\). Moreover, (27) is strictly increasing in \(\psi\); as \(\psi\) increases past \(\frac{\gamma}{n-1}\), agents in fact trade too much, keeping inefficiently little of their initial endowments, which also leads to allocative inefficiency.
C.2 Auction welfare

Without loss of generality we can consider the expected utility of just one agent $i$. Ignoring transfers, the agent’s demand is:

$$z(\eta, z_{i0}) = \frac{n-2}{n} \gamma (z_{i0}) + \frac{\eta}{n}$$

Plugging in the definition of $\eta$ from (19), we have:

$$z(\eta, z_{i0}) = \frac{n-2}{n} \gamma (z_{i0}) + \frac{1}{n} \left( \left( \frac{n-2}{n-1} \right) \left( \gamma \psi \sum_{i=1}^{n-1} z_{i0} + \epsilon \right) \right)$$

Thus, we have:

$$z(\eta, z_{i0}) = \frac{n-2}{n} \gamma (z_{i0}) + \left( \frac{n-2}{n-1} \right) \left( \frac{\gamma}{\gamma - \psi} \right) \sum_{i=1}^{n-1} z_{i0} + \epsilon$$

Hence, the agent’s final allocation is her initial inventory $z_{i0}$ plus the amount she trades $z(\eta, z_{i0})$, or:

$$z_{i0} + z(\eta, z_{i0}) = \left( 1 - \frac{n-2}{n} \frac{\gamma}{\gamma - \psi} \right) z_{i0} + \left( \frac{n-2}{(n-1) n} \right) \left( \frac{\gamma}{\gamma - \psi} \right) \sum_{i=1}^{n-1} z_{i0} + \epsilon \quad (28)$$

Ignoring transfers, recall from (1) that utility is:

$$\pi z_i - \gamma z_i^2$$

Since $z_{i0}, \epsilon$ both have mean 0, we can ignore the linear $\pi z_i$ term. Since all terms in (28) are independent, this is just:

$$- \left[ \left( 1 - \frac{n-2}{n} \frac{\gamma}{\gamma - \psi} \right)^2 + \left( \frac{n-2}{(n-1) n} \right)^2 (n-1) \right] \sigma_z^2 + \frac{\sigma_\epsilon^2}{n} \quad (29)$$
# List of Figures

<table>
<thead>
<tr>
<th>Figure</th>
<th>Title</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Effect of subsidies on best responses</td>
<td>23</td>
</tr>
<tr>
<td>2</td>
<td>Equilibrium demand functions</td>
<td>24</td>
</tr>
<tr>
<td>3</td>
<td>Equilibrium Results</td>
<td>25</td>
</tr>
</tbody>
</table>
Figure 1: Effect of subsidies on best responses

Notes: More orange curves are higher values of R.
Figure 2: Equilibrium demand functions

Notes: Equilibrium demand functions for $\gamma = 1$, $n = 4$, $\pi = 0$, as subsidies increase. The left panel has $\psi = 0$, the center panel has $\psi = 0.2$, and the rightmost panel has $\psi = 0.4$. 
Figure 3: Equilibrium Results

Notes: Equilibrium normalized auction welfare and standard deviation of prices for $\sigma_e^2 = 1, \sigma_z^2 = 1, \gamma = 1$. 