Online Appendix for:
A Theory of Foreclosure and Wholesale Bundling
(Not for publication)

Enrique Ide and Juan-Pablo Montero

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Chapter 1

Baseline Model - Retail Market in Depth

1.1 Bertrand Retailers - The Retail Market Subgame

1.1.1 Retail Pricing Equilibrium - Characterization

We denote by $\bar{\pi}_R^*$ $R_i$’s profits before fixed fees (and $\pi_R^*$ for profits after fixed fees). When necessary, we apply the usual refinement in Bertrand environments that rules out retailers pricing below cost any product they expect to sell nothing of it with probability one. Finally, we say that an equilibrium in the retail market is essentially unique if all equilibria gives retailers the same profits, and each of the different groups of consumers obtain the same surplus.

The following four claims completely characterize the equilibrium in the retail market for any combination of wholesale prices ($w_{A1}, w_{A2}, w_{B1}, w_{B2}$).

Claim 1.1.1. Suppose $\min\{w_{A1}, w_{A2}\} < 1$, then if $w_{Ai} \leq w_{Aj}$, and $w_{Bi} \leq w_{Bj} \leq b$, there is an essentially unique pure-strategy equilibrium in the retail pricing subgame. It is characterized by $p^*_{Ai} = \min\{w_{Aj}, (1 + w_{Ai})/2\}$, $p^*_{Aj} = w_{Aj}$, $p^*_{Bi} = p^*_{Bj} = w_{Bj}$, and $p^*_{ABi} \geq p^*_{Aj} + p^*_{Bi}$, $p^*_{ABj} \geq p^*_{Aj} + p^*_{Bj}$. Hence, retailers’ equilibrium profits are $\bar{\pi}_R^* = 0$ and

$$\bar{\pi}_R^* = \left\{ \begin{array}{ll} (1 - w_{Ai})^2/4 + (w_{Bj} - w_{Bi}) & \text{if } (1 + w_{Ai})/2 \leq w_{Aj} \\ (w_{Aj} - w_{Ai})(1 - w_{Aj}) + (w_{Bj} - w_{Bi}) & \text{otherwise} \end{array} \right.$$

Claim 1.1.2. Suppose $\min\{w_{A1}, w_{A2}\} < 1$, then if $w_{Ai} > w_{Aj}$, $w_{Bi} \leq w_{Bj} \leq b$, and $w_{Ai} + w_{Bi} \leq w_{Aj} + w_{Bj}$, there is an essentially unique pure-strategy equilibrium in the retail pricing subgame. It is characterized by $p^*_{Ai} = w_{Ai}$, $p^*_{Aj} = \min\{w_{Ai}, (1 + w_{Aj})/2\}$, $p^*_{Bi} = p^*_{Bj} = w_{Bj}$, and $p^*_{ABi} = p^*_{ABj} = w_{Aj} + w_{Bj}$. Hence, retailers’ equilibrium profits are

$$\bar{\pi}_R^* = (w_{Bj} - w_{Bi}) - \mu(w_{Ai} - w_{Aj})(1 - w_{Aj})$$

$$\bar{\pi}_R^* = \left\{ \begin{array}{ll} (1 - \mu)(1 - w_{Aj})^2/4 & \text{if } (1 + w_{Aj})/2 \leq w_{Ai} \\ (1 - \mu)(w_{Ai} - w_{Aj})(1 - w_{Ai}) & \text{otherwise} \end{array} \right.$$
Claim 1.1.3. Suppose \( \min \{w_{A1}, w_{A2}\} < 1 \), then if \( w_{Ai} > w_{Aj} \), \( w_{Bi} \leq w_{Bj} \leq b \), and \( w_{Ai} + w_{Bi} \geq w_{Aj} + w_{Bj} \), there is an essentially unique pure-strategy equilibrium in the retail pricing subgame. It is characterized as follows. Define \( \Delta \equiv w_{Ai} - (1 + w_{Aj})/2 \), and \( \psi \equiv w_{Ai} - w_{Aj} - 1/\mu \).

(a) When \( w_{Aj} + 1/\mu \leq w_{Ai} \), then \( p_{Ai}^* = w_{Ai}, p_{Aj}^* = (1 + w_{Aj})/2 \). Furthermore,

\( (i) \) If \( w_{Bi} - \Delta + \psi \leq w_{Bi} \leq w_{Bj} \), then \( p_{Bi}^* = p_{Bj}^* = w_{Bj} \), \( p_{ABi}^* \geq p_{Ai}^* + p_{Bi}^* \), and \( p_{ABj}^* \geq p_{Aj}^* + p_{Bj}^* \). Hence, retailers’ equilibrium profits are \( \tilde{\pi}_{Ri} = (w_{Bj} - w_{Bi}) - \mu(w_{Bj} - w_{Bi})(1 - w_{Aj})/2 \), and \( \tilde{\pi}_{Rj} = (1 - w_{Aj})^2/4 \).

\( (ii) \) If \( w_{Bj} - \Delta - \psi/2 \leq w_{Bi} \leq w_{Bj} - \Delta + \psi \), then

\[
\begin{align*}
p_{Bi}^* &= w_{Bi} + (w_{Bj} - w_{Bi})/3 + 2[1 - \mu(1 - w_{Aj})]/3/\mu \\
p_{ABj}^* &= w_{Bi} + 2(w_{Bj} - w_{Bi})/3 + [1 + \mu(1 + 2w_{Aj})]/3/\mu
\end{align*}
\]

and \( p_{Bj}^* \in [w_{Bj}, +\infty) \), \( p_{ABi}^* \geq p_{Ai}^* + p_{Bi}^* \). Hence, retailers’ equilibrium profits are

\[
\begin{align*}
\tilde{\pi}_{Ri} &= [2 - \mu(1 - w_{Aj}) + \mu(w_{Bj} - w_{Bi})]^2/9\mu \\
\tilde{\pi}_{Rj} &= (1 - \mu)(1 - w_{Aj})^2/4 + [1 + \mu(1 - w_{Aj}) - \mu(w_{Bj} - w_{Bi})]^2/9\mu
\end{align*}
\]

(b) When \( (1 + w_{Aj})/2 < w_{Ai} \leq w_{Aj} + 1/\mu \), then \( p_{Ai}^* = w_{Ai}, p_{Aj}^* = (1 + w_{Aj})/2 \). Furthermore,

\( (i) \) If \( w_{Bj} - \Delta \leq w_{Bi} \leq w_{Bj} \), then \( p_{Bi}^* = p_{Bj}^* = w_{Bj} \), \( p_{ABi}^* \geq p_{Ai}^* + p_{Bi}^* \), and \( p_{ABj}^* \geq p_{Aj}^* + p_{Bj}^* \). Hence, retailers’ equilibrium profits are \( \tilde{\pi}_{Ri} = (w_{Bj} - w_{Bi}) - \mu(w_{Bj} - w_{Bi})(1 - w_{Aj})/2 \), and \( \tilde{\pi}_{Rj} = (1 - w_{Aj})^2/4 \).

\( (ii) \) If \( w_{Bj} - \Delta + \psi/2 \leq w_{Bi} \leq w_{Bj} - \Delta \), then \( p_{Bi}^* = p_{Bj}^* = w_{Bj} \), and \( p_{ABi}^* = p_{ABj}^* = w_{Ai} + w_{Bi} \). Hence, retailers’ equilibrium profits are

\[
\begin{align*}
\tilde{\pi}_{Ri} &= (w_{Bj} - w_{Bi}) - \mu(w_{Bj} - w_{Bi})(1 - w_{Ai} - w_{Bi} + w_{Bj}) \\
\tilde{\pi}_{Rj} &= (1 - \mu)(1 - w_{Aj})^2/4 + \mu(w_{Ai} + w_{Bi} - w_{Aj} - w_{Bj})(1 - w_{Ai} - w_{Bi} + w_{Bj})
\end{align*}
\]

\( (iii) \) If \( w_{Bi} \leq w_{Bj} - \Delta + \psi/2 \), then \( p_{Bi}^* = w_{Bi} + w_{Ai}/2 + (1 - \mu)/2\mu \), \( p_{Bj}^* = w_{Bj} \), and \( p_{ABi}^* = p_{ABj}^* = w_{Ai} + w_{Bi} \). Hence, retailers’ equilibrium profits are

\[
\begin{align*}
\tilde{\pi}_{Ri} &= (w_{Bj} - w_{Bi}) - \mu(w_{Bj} - w_{Bi})(1 - w_{Ai} - w_{Bi} + w_{Bj}) \\
\tilde{\pi}_{Rj} &= (1 - \mu)(1 - w_{Aj})^2/4 + \mu(w_{Ai} + w_{Bi} - w_{Aj} - w_{Bj})(1 - w_{Ai} - w_{Bi} + w_{Bj})
\end{align*}
\]

(c) When \( w_{Aj} < w_{Ai} \leq (1 + w_{Aj})/2 \), then \( p_{Ai}^* = p_{Aj}^* = w_{Ai} \). Furthermore,
(i) \( w_{Bj} - \Delta + \psi/2 \leq w_{Bi} \leq w_{Bj} \), then \( p_{Bi}^* = p_{Bj}^* = w_{Bj} \), and \( p_{ABi}^* = p_{ABj}^* = w_{Ai} + w_{Bi} \).

Hence, retailers’ equilibrium profits are

\[
\overline{\pi}_{Ri}^* = (w_{Bj} - w_{Bi}) - \mu(w_{Bj} - w_{Bi})(1 - w_{Ai} - w_{Bi} + w_{Bj})
\]

\[
\overline{\pi}_{Rj}^* = (1 - \mu)(w_{Ai} - w_{Aj})(1 - w_{Ai}) + \mu(w_{Ai} + w_{Bi} - w_{Aj} - w_{Bj})(1 - w_{Ai} - w_{Bi} + w_{Bj})
\]

(ii) \( w_{Bi} \leq w_{Bj} - \Delta + \psi/2 \), then \( p_{Bi}^* = w_{Bi} + w_{Ai}/2 + (1 - \mu)/2\mu \), \( p_{Bj}^* = w_{Bj} \), and \( p_{ABi}^* = p_{ABj}^* = w_{Ai} + w_{Bi} \). Hence, retailers’ equilibrium profits are

\[
\overline{\pi}_{Ri}^* = [1 - \mu(1 - w_{Ai})]^2/4\mu
\]

\[
\overline{\pi}_{Rj}^* = (1 - \mu)(w_{Ai} - w_{Aj})(1 - w_{Ai}) + \mu(w_{Ai} + w_{Bi} - w_{Aj} - w_{Bj})(1 - \mu(1 - w_{Aj}))/2
\]

Claim 1.1.4. Suppose \( \min \{w_{A1}, w_{A2}\} < 1 \), and \( \min \{w_{B1}, w_{B2}\} \leq b < \max \{w_{B1}, w_{B2}\} \), then there exists a (possibly mixed) Nash equilibrium of the retail pricing subgame.

1.1.2 Retail Pricing Equilibrium - Proofs of Claims 1.1.1-1.1.4

Preliminaries

First, in all the proofs we apply the standard refinement of ruling out retailers from pricing below cost any product they expect to make no sale with probability 1. Second, we say that an equilibrium is essentially unique if all equilibria give retailers the same profits, and each group of consumers obtains the same consumer surplus. Third, in terms of organization, we begin the proof for the case of \( \mu \in (0,1) \), and only afterwards, we consider the extreme cases of \( \mu = 0 \) and \( \mu = 1 \) and show that their corresponding equilibria can be obtained as the limits of the equilibrium analysis for \( \mu \in (0,1) \). Fourth, and given our strategy of proof, we merge the proof of Claims 1.1.1-1.1.3 into one. Finally, because we need not only to characterize the retail equilibrium for any combination of wholesale prices but also show that such characterization is (essentially) unique, the proof is quite long, although not particularly difficult, and divided in several steps that we explain next.

Proofs of Claims 1.1.1-1.1.3

Suppose \( \mu \in (0,1) \) and that wholesale prices are in the region defined by (i) \( \max \{w_{Bi}, w_{Bj}\} \leq b \) and (ii) \( \min \{w_{Ai}, w_{Aj}\} < 1 \). The case of \( \min \{w_{Bi}, w_{Bj}\} \leq b < \max \{w_{Bi}, w_{Bj}\} \) is considered in the next section, i.e., in the proof of Claim 1.1.4. From (i) we have that \( \min \{w_{Bi}, w_{Bj}\} \leq b \), which in turn implies that \( \min \{p_{Bi}^*, p_{Bj}^*\} \leq b \). Similarly, from (ii) and \( \mu < 1 \) we have that \( \min \{p_{ABi}^*, p_{ABj}^*\} < 1 \). Our strategy of proof is to first show that when \( w_{Ai} \neq w_{Aj} \) and \( w_{Bi} \neq w_{Bj} \) the retail-pricing subgame has a unique pure-strategy equilibrium. We then take the limit of such equilibrium (i.e., let \( w_{Ai} \rightarrow w_{Aj} \) and/or \( w_{Bi} \rightarrow w_{Bj} \)) and show that it coincides with the equilibrium of the game when \( w_{Ai} = w_{Aj} \) and/or \( w_{Bi} = w_{Bj} \), and that such equilibrium is also unique.
The characterization of the equilibrium when $w_{Ai} \neq w_{Aj}$ and $w_{Bi} \neq w_{Bj}$ follows two steps. In the **first step** of characterization we derive necessary and sufficient conditions for an arbitrary pair of retail prices to constitute a pure-strategy equilibrium (notice that the intersection of those conditions may be empty, implying that such pair of prices cannot constitute a pure-strategy equilibrium). The set of all prices that constitute a pure-strategy, with their respective conditions and consistent with (i) and (ii), is denoted by $E^\ast$. In the **second step**, we take a combination of wholesale prices within the region defined by (i) and (ii) and ask whether there exists an element of $E^\ast$ for such particular combination of wholesale prices. By construction, if such element exists, then the prices that define it constitute a pure-strategy equilibrium of the retail subgame given those wholesale prices. As we will see, for any combination of wholesale prices there will be at most a single element in $E^\ast$, implying that when it exists, the equilibrium is unique.

Let us begin with the **first step**, characterizing $E^\ast$. Suppose that $(p^\ast_{Ai}, p^\ast_{Bi}, p^\ast_{ABi})$ and $(p^\ast_{Aj}, p^\ast_{Bj}, p^\ast_{ABj})$, with $p^\ast_{ABi} \leq p^\ast_{Ai} + p^\ast_{Bi}$ and $p^\ast_{ABj} \leq p^\ast_{Aj} + p^\ast_{Bj}$, constitute a pure-strategy equilibrium. We then have four distinct cases to consider.

**Case I:** $p^\ast_{ABi} \leq p^\ast_{Ai} + \min\{b, p^\ast_{Bj}\}$ and $p^\ast_{ABj} \leq p^\ast_{Aj} + \min\{b, p^\ast_{Bj}\}$ Since $w_{Ai} \neq w_{Aj}$ and $w_{Bi} \neq w_{Bj}$, then $q_{Ai} \cdot q_{Aj} = 0$ and $q_{Bi} \cdot q_{Bj} = 0$, that is, it cannot be that both retailers are selling positive quantities of $A$ and of $B$. With this in mind, there are four possible equilibrium outcomes to consider depending on whether who sells good $A$ and $B$ on a standalone fashion:

1. When $Ri$ sells both goods and either
   (a) $Ri$ sells the bundle $AB$, or
   (b) $Rj$ sells $AB$, or
   (c) both retailers sell $AB$, or
   (d) neither retailer sells $AB$

2. When $Rj$ sells good $A$, $Ri$ sells good $B$, and either:
   (a) $Ri$ sells $AB$, or
   (b) $Rj$ sells $AB$, or
   (c) both retailers sell $AB$, or
   (d) neither retailer sells $AB$

3. When $Ri$ sells good $A$ and $Rj$ sells good $B$ (this mirrors possibility 2, so it is omitted)

4. When $Rj$ sells both goods (this mirrors possibility 1, so it is also omitted)
Consider first possibility 1 in all its subcases, from (a) through (d). If $R_i$ is the only one selling $A$, then $w_{Ai} \leq p_{Ai}^* \leq p_{Aj}^* - \epsilon_1 \{w_{Ai} \geq w_{Aj}\}$. If, on the other hand, $R_i$ is also the only one selling $B$, then $w_{Bi} \leq p_{Bi}^* \leq \min\{b, p_{Bj}^* - \epsilon_1 \{w_{Bi} \geq w_{Bj}\}\}$. We now want to argue that $p_{Ai}^* \leq w_{Aj}$ and $p_{Bi}^* \leq w_{Bj}$. Notice that $R_j$ is obtaining non-negative profits from selling bundles if we happen to be in either 1(b) or 1(c). Recall that since the combination of wholesale prices $(w_{Ai}, w_{Bi}, w_{Aj}, w_{Bj})$ is arbitrary for the time being, it can be that $w_{Ai} + w_{Bi} \geq w_{Aj} + w_{Bj}$, which would imply that $R_j$ would be selling bundles whenever $p_{ABi}^* = p_{ABj}^*$. But if $R_j$ is selling bundles, then $R_j$ could deviate to undercut one product at a time.

Take product $A$. If $p_{Ai}^* > w_{Aj}$, $R_j$ could deviate to $p_{Aj}^i = p_{Ai}^* - \epsilon, p_{Bj}^i \rightarrow \infty$, and $p_{ABj}^i = p_{ABj}^*$ (note that $p_{ABj}^i \leq p_{Aj}^i + p_{Bj}^i$) and strictly increase her profit by selling units of $A$ at a positive margin. Then, it must be true that in equilibrium $w_{Ai} \leq p_{Ai}^* \leq w_{Aj}$. But since $w_{Ai} \neq w_{Aj}$, we must have that $w_{Ai} < w_{Aj}$ and that $w_{Ai} < 1$, from the assumption $\min\{w_{Ai}, w_{Aj}\} < 1$, implying that $p_{Ai}^* \leq p_{Aj}^* - \epsilon_1 \{w_{Ai} \geq w_{Aj}\} = p_{Aj}^*$. Since the same deviation argument applies to product $B$, it must be true that in equilibrium $w_{Bi} \leq p_{Bi}^* \leq w_{Bj} \leq b$. But since $w_{Bi} \neq w_{Bj}$, we must have that $w_{Bi} < w_{Bj} \leq b$, implying that $p_{Bi}^* \leq p_{Bj}^* - \epsilon_1 \{w_{Bi} \geq w_{Bj}\} = p_{Bj}^*$.

We now argue that since $w_{Ai} < w_{Aj}$ and $w_{Bi} < w_{Bj}$, the only retailer selling bundles must be $R_i$ (possibility 1(a)). From $w_{Ai} < w_{Aj}$ and $w_{Bi} < w_{Bj}$, we know that $w_{Ai} + w_{Bi} < w_{Aj} + w_{Bj}$, so the only way for $R_j$ to be selling bundles in equilibrium is if $p_{ABj}^* < p_{ABi}^*$. But this cannot be the case. Suppose the contrary, that $w_{Ai} + w_{Bi} < w_{Aj} + w_{Bj} \leq p_{ABj}^* < p_{ABi}^*$. This implies that

$$\bar{\pi}_{R_i}^* = (1 - \mu)(p_{Ai}^* - w_{Ai})(1 - p_{Ai}^*) + (1 - \mu)(p_{Bi}^* - w_{Bi}) + \mu(p_{Bi}^* - w_{Bi}) \min\{1, p_{ABj}^* - p_{Bi}^*\}$$

where we know that $p_{Ai}^* \leq \min\{w_{Aj}, 1 - \epsilon\}$ and $p_{Bi}^* \leq w_{Bj}$. But consider the following deviation by $R_i$: $p_{Ai}^i = p_{Ai}^*, p_{Bi}^i = w_{Bj}$, and $p_{ABi}^i = p_{Ai}^i + p_{Bi}^i = p_{Ai}^i + w_{Bj} \leq w_{Aj} + w_{Bj} \leq p_{ABj}^*$. This deviation reports a payoff difference of

$$\bar{\pi}_{R_i}^i - \bar{\pi}_{R_i}^* = (1 - \mu)(w_{Bj} - p_{Bi}^*) + \mu(w_{Bj} - w_{Bi}) + \mu(w_{Bj} - w_{Bi}) \min\{1, p_{ABj}^* - p_{Bi}^*\}$$

which is strictly positive given that $p_{Ai}^* < 1$ and $p_{Bi}^* \leq w_{Bj}$. So, we arrive at a contradiction. Hence, in any equilibrium in which $R_i$ sells both $A$ and $B$, the following must be true: (i) $w_{Ai} < w_{Aj}$, (ii) $w_{Bi} < w_{Bj}$, (iii) $p_{Ai}^* \leq \min\{w_{Aj}, p_{Aj}^*\}$, (iv) $p_{Bi}^* \leq \min\{w_{Bj}, p_{Bj}^*\}$, (v) $p_{ABi}^* \leq \min\{w_{Aj} + w_{Bj}, p_{ABj}^*\}$, and (vi) $\bar{\pi}_{R_j}^* = 0$.

We now tackle the issue of existence and characterization of an equilibrium of the kind just described. The first thing to notice is that since $\bar{\pi}_{R_j}^* = 0$, $R_j$’s “local” best-response correspondences are “flat” in the region of (potential) equilibrium prices of $R_i$ (i.e., when $p_{Ai}^* \leq w_{Aj}, p_{Bi}^* \leq w_{Bj}$ and $p_{ABi}^* \leq w_{Aj} + w_{Bj}$): $p_{AJj}^{BR} \in [w_{Aj} + \infty), p_{BJj}^{BR} \in [w_{Bj} + \infty)$, and $p_{ABj}^{BR} \in [w_{Aj} + w_{Bj} + \infty)$. On the other hand, for any given set of $R_j$’s prices in that region,
$R_i$’s “local” best response $(p_{Ai}^{BR}, p_{Bi}^{BR}, p_{ABi}^{BR})$ is the solution to

\[
\max_{p_{Ai}, p_{Bi}, p_{ABi}} \left\{ (1 - \mu) (p_{Ai} - w_{Ai}) D_{Ai}^{(1-\mu)} (p_{Ai}, p_{Aj}^*) + (1 - \mu) (p_{Bi} - w_{Bi}) D_{Bi}^{(1-\mu)} (p_{Bi}, p_{Bj}^*) + \mu (p_{ABi} - w_{Ai} - w_{Bi}) D_{ABi}^{(\mu)} (p_{ABi}, p_{ABj}^*, p_{Bi}^*, p_{Bj}^*) \right\} + \mu (p_{ABi} - w_{Ai} - w_{Bi}) D_{ABi}^{(\mu)} (p_{ABi}, p_{ABj}^*, p_{Bi}^*, p_{Bj}^*)
\]

which yields

\[
\begin{align*}
p_{Ai}^{BR} &= \min \left\{ \frac{1 + w_{Ai}}{2}, p_{Aj}^* \right\}, \quad p_{Bi}^{BR} = \min \{ b, p_{Bj}^* \} , \quad \text{and} \\
p_{ABi}^{BR} &= \min \{ p_{Ai}^{BR} + p_{Bi}^{BR}, p_{ABj}^* \}
\end{align*}
\]

From these best responses, we find that there is a unique candidate for this type of equilibria to exist, which is:

\[
\begin{align*}
p_{Ai}^* &= \min \left\{ w_{Aj}, \frac{1 + w_{Ai}}{2} \right\}, \quad p_{Aj}^* = \begin{cases} [w_{Aj}, +\infty) & \text{if } (1 + w_{Ai})/2 \leq w_{Aj} \\ w_{Aj} & \text{otherwise} \end{cases} \\
p_{Bi}^* &= w_{Bj} = p_{Bj}^* \\
p_{ABi}^* &= p_{Ai}^* + p_{Bi}^*, \quad \text{and} \quad p_{ABj}^* \in [w_{Aj} + w_{Bj}, p_{Aj}^* + p_{Bj}^*]
\end{align*}
\]

Since $R_i$ is the only one selling, and $p_{ABi}^* = p_{Ai}^* + p_{Bi}^*$, retail bundling is irrelevant and we can equivalently write $p_{ABi}^* \geq p_{Ai}^* + p_{Bi}^*$ and $p_{ABj}^* \geq p_{Aj}^* + p_{Bj}^*$.

Finally, it is easy to see that the candidate above is indeed an equilibrium. $R_j$ does not have incentives to deviate, otherwise it would be selling below cost. $R_i$, on the other hand, does not have incentives either. She is maximizing “locally” by construction, so it only remains to check “global” deviations. A global deviation in this setting entails abandoning one or more groups of consumers in an effort to extract more surplus from other groups (e.g., dropping consumers purchasing only good $A$ by setting $p_{Ai} \to \infty$, in an effort to relax the $p_{ABi} \leq p_{Ai} + p_{Bi}$ constraint and raise the price of the bundle). But in this case, such deviations are clearly not profitable, since conditional on what $R_j$ is playing, $R_i$ is extracting the most she can from all group of consumers. Hence, an equilibrium in which $R_i$ sells both $A$ and $B$ exists only if $w_{Ai} < w_{Aj}$ and $w_{Bi} < w_{Bj} \leq b$. Moreover, among the class of such equilibria, the equilibrium is essentially unique: consumer surplus and retailers’ profits are the same across equilibria, with $\pi_{R_j}^* = 0$ and

\[
\pi_{R_i}^* = \begin{cases} (1 - w_{Ai})^2/4 + (w_{Bj} - w_{Bi}) & \text{if } (1 + w_{Ai})/2 \leq w_{Aj} \\ (w_{Aj} - w_{Ai})(1 - w_{Aj}) + (w_{Bj} - w_{Bi}) & \text{otherwise} \end{cases}
\]

This concludes the analysis of possibility 1, which, as we discuss at the end, will serve as proof Claim [1.1.1].

Consider now possibility 2(a), that is, when $R_j$ sells good $A$, and $R_i$ sells good $B$ and the bundle $AB$. If $R_j$ is the only one selling $A$, then $w_{Aj} \leq p_{Aj}^* \leq p_{Ai}^* - \epsilon 1 \{ w_{Aj} \geq w_{Ai} \}$, which implies that $p_{Aj}^* < 1$ and $w_{Aj} < 1$. If, on the other hand, $R_i$ is the only one selling $B$, then $w_{Bi} \leq p_{Bi}^* \leq \min \{ b, p_{Bj}^* - \epsilon 1 \{ w_{Bi} \geq w_{Bj} \} \}$, which implies that $p_{Bi}^* \leq b$. In addition, if
$R_i$ is also the only one selling bundles, then $1 - p^*_{ABi} + p^*_{Bi} > 0$, and $w_{Ai} + w_{Bi} \leq p^*_{ABi} \leq p^*_{ABj} - \epsilon 1 \{w_{Ai} + w_{Bi} \geq w_{Aj} + w_{ Bj}\}.$

Now, since $R_i$ is the only one selling bundles and $B$, it must be true that $p^*_{Bi} \leq w_{Bj}$; otherwise $R_j$ would deviate to $p'_{Aj} = p^*_{Ai}$, $p'_{Bj} = p^*_{Bi} - \epsilon > w_{Bj}$, and $p'_{ABj} \to \infty$, strictly increasing her payoff. But then $w_{Bi} \leq p^*_{Bi} \leq w_{Bj}$, which implies that $w_{Bj} < w_{Bj}$ and that $p^*_{Bi} \leq p^*_{Bj}$. On the other hand, since $1 - p^*_{ABi} + p^*_{Bi} > 0$, it must be true that $p^*_{ABi} \leq w_{Aj} + w_{Bj}$; otherwise $R_j$ would deviate to $p'_{Aj} = p^*_{Ai}$, $p'_{Bj} = p^*_{Bi} \to \infty$, and $p'_{ABj} = p^*_{ABi} - \epsilon$, strictly increasing her payoff from $\pi'_R_j = (1 - \mu)(p^*_{Aj} - w_{Aj})(1 - p^*_{Aj})$ to $\pi'_R_j = \pi'_R_j + \mu(p^*_{ABi} - w_{Aj} - w_{Bj})(1 - p^*_{ABi} + p^*_{Bj})$ as $\epsilon \to 0$. But if $p^*_{ABi} \leq w_{Aj} + w_{Bj}$, then $w_{Ai} + w_{Bi} \leq p^*_{ABi} \leq w_{Aj} + w_{Bj}$, which implies that $w_{Ai} + w_{Bi} \leq w_{Aj} + w_{Bj}$. Moreover, since $R_i$ is the only one selling bundles, it must be true that $w_{Ai} + w_{Bi} < w_{Aj} + w_{Bj}$ and $p^*_{ABi} \leq p^*_{ABj}$.

We now use the fact that both $R_i$ and $R_j$ must be maximizing “locally” in any equilibrium given a set of prices chosen by the rival, that is,

$$p^*_{Aj} \in \arg \max_{p^*_{Aj}} \{(1 - \mu)(p_{Aj} - w_{Aj})D_{Aj}^{(1-\mu)}(p_{Aj}, p^*_{Ai})\}$$

which yields $p^*_{Aj} = \min\{(1 + w_{Aj})/2, p^*_{Ai} - \epsilon 1 \{w_{Aj} > w_{Ai}\}\}$, $p^*_{Bj} \geq w_{Bj}$, and $p^*_{ABj} \geq w_{Aj} + w_{Bj}$, and

$$(p^*_{Bi}, p^*_{ABi}) \in \arg \max_{p^*_{Bi}, p^*_{ABi}} \{(1 - \mu)(p_{Bi} - w_{Bi})D_{Bi}^{(1-\mu)}(p_{Bi}, p^*_{Bj}) + \mu(p_{ABi} - w_{Ai} - w_{Bj})D_{ABi}^{(\mu)}(p_{ABi}, p^*_{ABj}, p_{Bi}, p^*_{Bj}) + \mu(p_{Bi} - w_{Bi})D_{Bi}^{(\mu)}(p_{ABi}, p^*_{ABj}, p_{Bi}, p^*_{Bj})\}$$

which yields $p^*_{Bi} = \min\{b, p^*_{Bj}\}$, $p^*_{ABi} = \min\{(1 + w_{Ai})/2 + p^*_{Bi}, p^*_{ABj}\}$, and $p^*_{Ai} \geq w_{Ai}$.

Notice that in equilibrium the restriction $p^*_{ABi} \leq p^*_{Ai} + p^*_{Bi}$ cannot be strictly binding (that is, a strictly positive Lagrange multiplier) since $R_i$ could always relax it by deviating to a higher $p_{Ai}$ as she is not selling units of $A$. We will now show that $p^*_{Ai} \leq w_{Ai}$ necessarily. Suppose not. Then $R_i$ could deviate to $p'_{Ai} = p^*_{Aj} - \epsilon > w_{Ai}, p'_{Bj} = from the fact that $w_{Aj} < 1$; (ii) $p^*_{ABi} \leq p^*_{Ai} + \min\{b, p^*_{Bi}\}$, which also holds since $p^*_{ABi} = w_{Aj} + w_{Bj} \leq p^*_{Ai} + w_{Bj}$
from the fact that \( w_{Aj} \leq p_{Aj}^* \); and (iii) \( p_{ABj}^* \leq p_{Aj}^* + \min\{b, p_{Bj}^*\} \), which also holds since 
\[ w_{Aj} + w_{Bj} \leq p_{Aj}^* + w_{Bj} \] 
from the fact that \( w_{Aj} \leq p_{Aj}^* \). Then, a candidate for an equilibrium in which \( Rj \) sells \( A \) and \( Ri \) sells units of \( B \) and of \( AB \) exists when \( w_{Aj} < w_{Ai} \), \( w_{Bi} < w_{Bj} \leq b \), and \( w_{Ai} + w_{Bi} < w_{Aj} + w_{Bj} \). Moreover, among this class the candidate is essentially unique and given by:

\[
p_{Aj}^* = \min\left\{ w_{Ai}, \frac{1 + w_{Aj}}{2} \right\}, \quad p_{Ai}^* = \begin{cases} \left[ w_{Ai}, +\infty \right) & \text{if } (1 + w_{Aj}) / 2 \leq w_{Ai} \\ w_{Ai} & \text{otherwise} \end{cases}
\]

\[
p_{Bi}^* = w_{Bj} = p_{Bj}^*, \quad \text{and } p_{ABi}^* = p_{ABj}^* = w_{Aj} + w_{Bj}
\]

leading to

\[
\bar{\pi}_{Ri}^* = (w_{Bj} - w_{Bi}) - \mu(w_{Ai} - w_{Aj})(1 - w_{Aj})
\]

\[
\bar{\pi}_{Rj}^* = \begin{cases} (1 - \mu)(1 - w_{Aj})^2 / 4 & \text{if } (1 + w_{Aj}) / 2 \leq w_{Ai} \\
(1 - \mu)(w_{Ai} - w_{Aj})(1 - w_{Ai}) & \text{otherwise} \end{cases}
\]

It is easy to see that candidate is indeed an equilibrium. \( Rj \) does not have “local” incentives to deviate given the maximization above, neither does she have “global” incentives since this would require pricing below cost. Since the exact same arguments apply also for \( Ri \), this concludes the analysis of possibility 2(a), which, as we discuss at the end, will serve as proof of Claim 1.1.2.

Consider next possibility 2(b), that is, when \( Rj \) sells good \( A \) and the bundle, and \( Ri \) sells only good \( B \). Proceeding as before, and noticing that now \( 1 - p_{ABj}^* + p_{Bi}^* > 0 \), we can rapidly establish that for this possibility to be an equilibrium it must be true that \( w_{Aj} < w_{Ai} \) and \( w_{Aj} + w_{Bj} < w_{Ai} + w_{Bi} \). We now use the fact that both \( Ri \) and \( Rj \) must be maximizing “locally” in any equilibrium given the rival’s prices, that is,

\[
p_{Bi}^* \in \arg\max_{p_{Bi}}\{(1 - \mu)(p_{Bi} - w_{Bi})D_{Bi}^{(1-\mu)}(p_{Bi}, p_{Bj}^*) + \mu(p_{Bi} - w_{Bi})D_{Bi}^{(\mu)}(p_{ABi}, p_{ABj}^*, p_{Bi}, p_{Bj}^*)\}
\]

which yields

\[
p_{Bi}^* = \min\left\{ p_{Bj}^* - \epsilon \{ w_{Bi} > w_{Bj} \}, \frac{p_{ABj}^* + w_{Bi}}{2} + \frac{1 - \mu}{2\mu}, b \right\}
\]

and

\[
(p_{Aj}^*, p_{ABj}^*) \in \arg\max_{p_{Aj}, p_{ABj}}\{(1 - \mu)(p_{Aj} - w_{Aj})D_{Aj}^{(1-\mu)}(p_{Aj}, p_{Ai}^*)\} + \mu(p_{ABj} - w_{Aj} - w_{Bj})D_{ABj}^{(1-\mu)}(p_{ABj}, p_{ABi}^*, p_{Bj}, p_{Bi}^*)
\]

Noticing that in equilibrium the restriction \( p_{Bj}^* \leq p_{Aj}^* + p_{Bi}^* \) cannot be strictly binding, as \( Rj \) can always costlessly relax it by deviating to a higher \( p_{Bj} \) (since she is not selling units of \( B \)), yields

\[
p_{Aj}^* = \min\left\{ \frac{1 + w_{Aj}}{2}, p_{Aj}^* \right\}
\]

\[
p_{ABj}^* = \min\left\{ p_{ABi}^*, \frac{1 + w_{Aj}}{2} + \frac{p_{Bi}^* + w_{Bj}}{2} \right\}
\]
We will now show, by contradiction, that \( p^*_{Bi} \leq w_{Bj} \), which in turn implies \( w_{Bi} \leq w_{Bj} \). Then, suppose that \( p^*_{Bi} > w_{Bj} \), which yields

\[
\begin{align*}
\bar{p}^*_{ABj} &= \min \left\{ p^*_{ABj}, \frac{1 + w_{Aj}}{2} +\right.\left. \frac{p^*_{Bi} + w_{Bj}}{2} \right\} \\
&< \min \left\{ p^*_{ABj}, \frac{1 + w_{Aj}}{2} + p^*_{Bi} \right\}
\end{align*}
\]

where the latter inequality follows from the fact that \( w_{Bj} < p^*_{Bi} \). From here we can establish that

\[
\begin{align*}
p^*_{Aj} + p^*_{Bi} - p^*_{ABj} &> p^*_{Aj} + p^*_{Bi} - \min \left\{ p^*_{ABj}, \frac{1 + w_{Aj}}{2} + p^*_{Bi} \right\} \\
&= \min \left\{ \frac{1 + w_{Aj}}{2} + p^*_{Bi}, p^*_{Aj} + p^*_{Bi} \right\} - \min \left\{ p^*_{ABj}, \frac{1 + w_{Aj}}{2} + p^*_{Bi} \right\} \\
&> 0
\end{align*}
\]

Hence \( p^*_{ABj} < p^*_{Aj} + p^*_{Bi} \). Now, \( Rj \) is obtaining on path

\[
\bar{\pi}^*_{Rj} = (1 - \mu)(p^*_{AJ} - w_{AJ})(1 - p^*_{ABj}) + \mu(p^*_{ABj} - w_{AJ} - w_{Bj})(1 - p^*_{ABj} + p^*_{Bi})
\]

But if \( w_{Bj} < p^*_{Bi} \leq p^*_{Bj} - \epsilon \{ w_{Bi} > w_{Bj} \} \), \( Rj \) would deviate to \( p'_{Aj} = p^*_{Aj} \), \( p'_{Bj} = p^*_{Bi} - \epsilon \)

\[
p^*_{ABj} = p^*_{ABj}' \]

which (given that \( p^*_{ABj} < p^*_{Aj} + p^*_{Bi} \)) satisfy \( p^*_{ABj} = p^*_{ABj}' \leq p^*_{Aj} + p^*_{Bi} = p^*_{Aj} + p^*_{Bi} - \epsilon \), and obtain

\[
\bar{\pi}^*_{Rj} = \bar{\pi}^*_{Rj} + (1 - \mu)(p^*_{Bi} - w_{Bj}) + \mu(p^*_{Bi} - w_{Bj})(p^*_{ABj} - p^*_{Bi})
\]

which is obviously greater than \( \bar{\pi}^*_{Rj} \); a contradiction. This shows that \( p^*_{Bi} \leq w_{Bj} \), and hence, that \( w_{Bi} < w_{Bj} \).

Therefore, an equilibrium candidate in which \( Rj \) sells \( A \) and \( AB \) and \( Ri \) sells only \( B \) can exist only when \( w_{Aj} < w_{Ai} \), \( w_{Bi} < w_{Bj} \), and \( w_{Aj} + w_{Bj} < w_{Ai} + w_{Bi} \). We now finish the characterization of this equilibrium candidate. We know that it must satisfy

\[
p^*_{Bi} = \min \left\{ p^*_{Bj}, \frac{p^*_{ABj} + w_{Bi}}{2} + \frac{1 (1 - \mu)}{2 \mu} \right\} \leq w_{Bj}, \; p^*_{Bj} \geq w_{Bj},
\]

\[
p^*_{Aj} = \min \left\{ p^*_{Aj}, \frac{1 + w_{Ai}}{2} \right\} \leq w_{Ai}, \; p^*_{Aj} \geq w_{Ai},
\]

\[
p^*_{ABj} = \min \left\{ p^*_{ABj}, \frac{1 + w_{Aj}}{2} + \frac{p^*_{Bi} + w_{Bj}}{2} \right\} \leq w_{Ai} + w_{Bj}, \; p^*_{ABj} \geq w_{Ai} + w_{Bi}
\]

and

\[
1 - p^*_{ABj} + p^*_{Bi} > 0, \; p^*_{ABj} \leq p^*_{Ai} + \min \{ b, p^*_{Bi} \}, \; p^*_{ABj} \leq p^*_{Aj} + \min \{ b, p^*_{Bj} \}
\]

Note that the conditions over \( p^*_{Aj} \) and \( p^*_{Alt} \) immediately imply that

\[
p^*_{Aj} = \min \left\{ w_{Ai}, \frac{1 + w_{Aj}}{2} \right\}, \; p^*_{Ai} = \begin{cases} [w_{Ai} + \infty) & \text{if } (1 + w_{Aj})/2 \leq w_{Ai} \\ w_{Ai} & \text{otherwise} \end{cases}
\]

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It remains to find \((p^*_B, p^*_B, p^*_A, p^*_A)\) using the remaining conditions. For this we will first conjecture four different equilibrium prices configurations, consistent with (1.1) and (1.3), and then show that each of these configurations is supported by four distinct regions of wholesale prices that cover the entire span of admissible wholesale prices, that is, wholesale prices satisfying \(w_A < w_B < w_B < w_A + w_B\).

Our first equilibrium configuration entails (1.4) and

\[
p^*_B = \min \{ p^*_B, b \} \quad \text{and} \quad p^*_A = \frac{1 + w_A + w_B}{2}
\]

From this and \(p^*_B \leq w_B\) and \(p^*_B \geq w_B\), we have that \(p^*_B = p^*_B = w_B\), so

\[
p^*_A = w_B + \left( \frac{1 + w_A}{2} \right)
\]

We also need \(p^*_A \leq w_A + w_B\) to hold, which requires

\[
w_B - \left[ w_A - \frac{1 + w_A}{2} \right] \leq w_B \quad (1.5)
\]

Notice that since \(w_B < w_B \leq b\) a necessary condition for the above to hold is \((1 + w_A)/2 < w_A\), which implies that \(p^*_A = (1 + w_A)/2\). On the other hand, we need

\[
\min \{ b, p^*_B \} = w_B \leq \frac{p^*_A + w_B}{2} + \frac{1 - \mu}{2\mu}
\]

to hold, which requires

\[
w_B - \left[ w_A - \frac{1 + w_A}{2} \right] + \left( w_A - w_A - \frac{1}{\mu} \right) \leq w_B \quad (1.6)
\]

Combining both (1.5) and (1.6) yields

\[
w_B - \left[ w_A - \frac{1 + w_A}{2} \right] + \max \left\{ 0, w_A - w_A - \frac{1}{\mu} \right\} \leq w_B < w_B \leq b \quad (1.7)
\]

Furthermore, we need the following to hold: (i) \(p^*_A \leq p^*_A + w_B\), which does because \(p^*_A\) is not exactly pinned down so it can be freely chosen; (ii) \(p^*_A \leq p^*_A + w_B\), which also does because

\[
w_B + \left( \frac{1 + w_A}{2} \right) \leq \left( \frac{1 + w_A}{2} \right) + w_B;
\]

and (iii) \(1 - p^*_A + p^*_B > 0\), which also does because

\[
w_B + \left( \frac{1 + w_A}{2} \right) < 1 + w_B
\]
as \(w_A < 1\). Hence, if (1.7) holds, an equilibrium candidate for this type of equilibria is

\[
p^*_A = \frac{1 + w_A}{2}, \quad p^*_B = p^*_B = w_B, \quad p^*_A = \left( \frac{1 + w_A}{2} \right) + w_B
\]

\[
p^*_A \in [w_A, +\infty), \quad \text{and} \quad p^*_A \geq w_A + w_B
\]
Moreover, using the previous arguments regarding global deviations, it is easy to see that this candidate is indeed an equilibrium, resulting in equilibrium profits equal to

\[
\bar{\pi}_R^* = (w_{Bj} - w_{Bi}) - \mu(w_{Bj} - w_{Bi})(1 - w_{Aj})/2
\]

\[
\bar{\pi}_R^* = (1 - w_{Aj})^2/4
\]

This concludes the analysis of the first price equilibrium configuration of possibility 2(b), which, as we discuss at the end, will serve as proof of Claim 1.1.3 parts (a)(i) and (b)(i).

Our second equilibrium price configuration within possibility 2(b) entails (1.4) and

\[p_{Bi}^* = \min \{p_{Bj}^*, b\} \quad \text{and} \quad p_{ABi}^* = p_{ABj}^*\]

Since \(p_{Bi}^* \leq w_{Bj} \leq p_{Bj}^*\) and \(p_{ABj}^* \leq w_{Ai} + w_{Bi} \leq p_{ABi}^*\), it is immediate that \(p_{Bi}^* = p_{Bj}^* = w_{Bj}\) and \(p_{ABi}^* = p_{ABj}^* = w_{Ai} + w_{Bi}\). For this to be indeed an equilibrium we need

\[
\min \{b, p_{Bj}^*\} = w_{Bj} \leq \left(\frac{p_{ABj}^* + w_{Bi}}{2}\right) + \frac{1}{2} \left(1 - \frac{1}{d}\right)
\]

which implies

\[
w_{Bj} - \left[w_{Ai} - \frac{1 + w_{Aj}}{2}\right] + \frac{1}{2} \left(w_{Ai} - w_{Aj} - \frac{1}{d}\right) \leq w_{Bi} \quad (1.8)
\]

and

\[
w_{Ai} + w_{Bi} \leq \frac{1 + w_{Aj}}{2} + \frac{p_{Bi}^* + w_{Bj}}{2}
\]

which implies

\[
w_{Bi} \leq w_{Bj} - \left[w_{Ai} - \frac{1 + w_{Aj}}{2}\right] \quad (1.9)
\]

Since both (1.8) and (1.9) can only be satisfied if \(w_{Ai} - w_{Aj} - 1/d < 0\), a necessary condition for this price-configuration candidate to be an equilibrium is

\[
w_{Ai} - w_{Aj} - \frac{1}{d} < 0
\]

Furthermore, we need the following to hold: (i) \(p_{ABi}^* = w_{Ai} + w_{Bi} \leq p_{Ai}^* + \min \{b, p_{Bi}^*\} = p_{Ai}^* + w_{Bj}\), which does because \(p_{Ai}^* \geq w_{Ai}\) and \(w_{Bi} \leq w_{Bj}\); (ii) \(p_{ABj}^* \leq p_{Aj}^* + \min \{b, p_{Bj}^*\}\), which holds if \(w_{Ai} + w_{Bi} \leq \min \{w_{Ai}, (1 + w_{Aj})/2\} + w_{Bj}\) does or, equivalently, if

\[
w_{Bi} \leq w_{Bj} - \left[w_{Ai} - \min \left\{w_{Ai}, \frac{1 + w_{Aj}}{2}\right\}\right]
\]

(1.10)

does; and (iii) \(1 - p_{ABj}^* + p_{Bi}^* > 0\), which holds if \(1 - w_{Ai} - w_{Bi} + w_{Bj} > 0\) does, or equivalently, if

\[
w_{Bi} < w_{Bj} - (w_{Ai} - 1)
\]

(1.11)

does.

It is not difficult to prove that the binding conditions that define this second region are (1.8) and (1.10). In this region the equilibrium candidate is then given by (1.4), \(p_{Bi}^* = p_{Bj}^* = w_{Bj}\),
and \( p'_{ABi} = p'_{ABj} = w_{Ai} + w_{Bi} \). Since it is easy to see that there are no global deviations, the candidate is indeed an equilibrium, resulting in retailer profits equal to
\[
\pi^*_{ri} = (w_{Bj} - w_{Bi}) - \mu(w_{Bj} - w_{Bi})(1 - w_{Ai} - w_{Bi} + w_{Bj})
\]
\[
\pi^*_{rj} = (1 - \mu)(1 - w_{Aj})^2 + \mu(w_{Ai} + w_{Bi} - w_{Aj} - w_{Bj})(1 - w_{Ai} - w_{Bi} - w_{Bj})
\]
This concludes the analysis of the second price equilibrium configuration of possibility 2(b), which, as we discuss at the end, will serve as proof of Claim 1.1.3, and

The third equilibrium price configuration within possibility 2(b) entails (1.4),
\[
p^*_r = \left( \frac{p'_{ABj} + w_{Bi}}{2} \right) + \frac{1}{2} \left( \frac{1 - \mu}{\mu} \right), \quad \text{and } p^*_{ABj} = \left( \frac{1 + w_{Aj}}{2} \right) + \left( \frac{p'^*_{Bj} + w_{Bj}}{2} \right)
\]
Solving, we get
\[
p^*_r = w_{Bi} + \frac{1}{3}(w_{Bj} - w_{Bi}) + \frac{2}{3} \left( 1 - \frac{\mu}{2}(1 - w_{Aj}) \right)
\]
\[
p^*_{ABj} = w_{Bi} + \frac{2}{3}(w_{Bj} - w_{Bi}) + \frac{1}{3} \left[ 1 + \mu(1 + 2w_{Aj}) \right]
\]
For this to be indeed an equilibrium we need \( p^*_r \leq \min\{b, p^*_r, w_{Bj}\} = w_{Bj} \), which implies
\[
w_{Bi} \leq w_{Bj} - \left[ w_{Ai} - \frac{1 + w_{Aj}}{2} \right] + \left( \frac{w_{Ai} - w_{Aj} - 1}{\mu} \right) \tag{1.12}
\]
and \( p^*_{ABj} \leq w_{Ai} + w_{Bi} \), which implies
\[
w_{Bj} - \left[ w_{Ai} - \frac{1 + w_{Aj}}{2} \right] + \left( \frac{w_{Ai} - w_{Aj} - 1}{\mu} \right) - \frac{2}{3} \left( w_{Ai} - w_{Aj} - \frac{1}{\mu} \right) \leq w_{Bi} \tag{1.13}
\]
Since both (1.12) and (1.13) can only be satisfied if \( w_{Ai} - w_{Aj} - 1/\mu > 0 \), a necessary condition for this third price-configuration candidate to be an equilibrium is
\[
w_{Ai} - w_{Aj} - \frac{1}{\mu} > 0
\]
Furthermore, we need the following to hold: (i) \( p^*_{ABi} \leq p^*_r + \min\{p^*_r, b\} \), which does because \( p^*_{ABi} \) is not pinned down, so it can be freely picked; (ii) \( p^*_{ABj} \leq p^*_r + \min\{p^*_r, b\} \), which holds if
\[
w_{Bi} \leq w_{Bj} - \left[ w_{Ai} - \frac{1 + w_{Aj}}{2} \right] + \left( \frac{w_{Ai} - w_{Aj} - 1}{\mu} \right) - \frac{3}{2} \left( \frac{1 + w_{Aj}}{2} - \min \left\{ w_{Ai}, \frac{1 + w_{Aj}}{2} \right\} \right) \tag{1.14}
\]
does; and (iii) \( 1 - p^*_{ABj} + p^*_r > 0 \), which holds if \( w_{Aj} + w_{Bj} - [1 + 1/\mu - w_{Ai}] < w_{Ai} + w_{Bi} \) does, which is indeed the case since \( w_{Aj} + w_{Bj} < w_{Ai} + w_{Bi} \) and \( 1 + 1/\mu - w_{Ai} > 0 \).

It is not difficult to prove that the binding conditions characterizing this third region are (1.13) and (1.13). In this region the equilibrium candidate is then given by (1.4),
\[
p^*_r = w_{Bi} + \frac{1}{3}(w_{Bj} - w_{Bi}) + \frac{2}{3\mu} \left[ 1 - \frac{\mu}{2}(1 - w_{Aj}) \right],
\]
\[ p_{ABj}^* \in [w_{BJ}, +\infty), \quad p_{ABi}^* \geq w_{Ai} + w_{Bi}, \quad \text{and} \]
\[ p_{ABj}^* = w_{Bi} + \frac{2}{3}(w_{BJ} - w_{Bi}) + \frac{1}{3\mu} \left[ 1 + \mu(1 - 2w_{Aj}) \right] \]

Since it is easy to see that there are no global deviations, the candidate is indeed an equilibrium, resulting in retailer profits equal to
\[ \pi_{Bi}^* = \frac{1}{9\mu} [2 - \mu(1 - w_{Aj}) + \mu(w_{BJ} - w_{Bi})]^2 \]
\[ \pi_{Bj}^* = (1 - \mu) \left( \frac{1 - w_{Aj}}{4} \right)^2 + \frac{1}{9\mu} \left[ 1 + \mu(1 - w_{Aj}) - \mu(w_{BJ} - w_{Bi}) \right]^2 \]

This concludes the analysis of the third price equilibrium configuration of possibility 2(b), which, as we discuss at the end, will serve as proof of Claim 1.1.3 part (a)(ii).

The fourth and last equilibrium price configuration within possibility 2(b) entails (1.4),
\[ p_{Bi}^* = \left( \frac{p_{Bj}^* + w_{Bi}}{2} \right) + \frac{1}{2} \left( \frac{1 - \mu}{\mu} \right), \]
and \( p_{ABj}^* = p_{ABi}^* \). This immediately implies that \( p_{ABi}^* = p_{ABj}^* = w_{Ai} + w_{Bi} \), and therefore
\[ p_{Bi}^* = w_{Bi} + \frac{w_{Ai}}{2} + \frac{1}{2} \left( \frac{1 - \mu}{\mu} \right) \]

Furthermore, we need the following to hold: (i) \( p_{Bi}^* \leq \min\{b, p_{Bj}^*, w_{Bi}\} = w_{BJ} \), which does if
\[ w_{Bi} \leq w_{BJ} - \left[ w_{Ai} - \frac{1 + w_{Aj}}{2} \right] + \frac{1}{2} \left[ w_{Ai} - w_{Aj} - \frac{1}{\mu} \right] \equiv c_1 \quad (1.15) \]
does; (ii) \( w_{Ai} + w_{Bi} \leq (1 + w_{Aj})/2 + (p_{Bi}^* + w_{BJ})/2 \), which holds if
\[ w_{Bi} \leq w_{BJ} - \left[ w_{Ai} - \frac{1 + w_{Aj}}{2} \right] + \frac{1}{2} \left[ w_{Ai} - w_{Aj} - \frac{1}{\mu} \right] - \left[ w_{Ai} - w_{Aj} - \frac{1}{\mu} \right] \equiv c_2 \quad (1.16) \]
does; (iii) \( p_{ABi}^* = w_{Ai} + w_{Bi} \leq p_{ABi}^* + \min\{b, p_{Bi}^*\} \), which holds because \( p_{ABi}^* \geq w_{Ai} \) and \( w_{Bi} \leq w_{BJ} \);
(iv) \( p_{ABj}^* = w_{Ai} + w_{Bi} \leq p_{ABj}^* + \min\{b, p_{Bj}^*\} \), which holds if
\[ w_{Bi} \leq w_{BJ} - \left[ w_{Ai} - \min \left\{ w_{Ai}, \frac{1 + w_{Aj}}{2} \right\} \right] \equiv c_3 \quad (1.17) \]
does; and (v) \( 1 - p_{ABj}^* + p_{Bi}^* > 0 \), which holds if \( 1 - w_{Ai}/2 + (1 - \mu)/2\mu > 0 \), which is indeed the case since \( w_{Ai} < 1 + 1/\mu \).

This fourth region is then characterized by
\[ w_{Bi} \leq \min\{c_1, c_2, c_3\} \]

In this region the equilibrium candidate is given by (1.4), \( p_{Bi}^* = w_{Bi} + w_{Ai}/2 + (1 - \mu)/2\mu \), \( p_{Bj}^* \in [w_{BJ}, +\infty) \), and \( p_{ABi}^* = p_{ABj}^* = w_{Ai} + w_{Bi} \). Since it is easy to see that there are no global deviations, the candidate is indeed an equilibrium, resulting in retailer profits equal to
\[ \pi_{Bi}^* = \frac{1}{4\mu} [1 - \mu(1 - w_{Ai})]^2 \]
\[ \pi_{Bj}^* = (1 - \mu) \left( \frac{1 - w_{Aj}}{4} \right)^2 + \frac{\mu}{2} (w_{Ai} + w_{Bi} - w_{Aj} - w_{BJ})[1 - \mu(1 - w_{Aj})] \]
This concludes the analysis of the last price equilibrium configuration of possibility 2(b), which, as we discuss at the end, will serve as proof of Claim 1.1.3 parts (a)(iii) and (b)(iii).

To conclude the analysis of possibility 2(b), we need to check that the four regions identified above do not overlap, and that they span the entire space of admissible wholesale prices (i.e., wholesale prices satisfying \( w_{Aj} < w_{Ai} \), \( w_{Bi} < w_{Bj} \) and \( w_{Ai} + w_{Bi} < w_{Aj} + w_{Bj} \)). This is easy to show. First, consider

\[
\frac{1 + w_{Aj}}{2} < w_{Aj} + \frac{1}{\mu} \leq w_{Ai}
\]

In this case region 2 vanishes, and regions 1, 3 and 4 reduce to

\[
w_{Bj} - \left[ w_{Ai} - \frac{1 + w_{Aj}}{2} \right] + \left( w_{Ai} - w_{Aj} - \frac{1}{\mu} \right) \leq w_{Bi} < w_{Bj},
\]

\[
w_{Bj} - \left[ w_{Ai} - \frac{1 + w_{Aj}}{2} \right] - \frac{1}{2} \left( w_{Ai} - w_{Aj} - \frac{1}{\mu} \right) \leq w_{Bi} \leq w_{Bj} - \left[ w_{Ai} - \frac{1 + w_{Aj}}{2} \right] + \left( w_{Ai} - w_{Aj} - \frac{1}{\mu} \right)
\]

and

\[
w_{Bi} \leq w_{Bj} - \left[ w_{Ai} - \frac{1 + w_{Aj}}{2} \right] - \frac{1}{2} \left( w_{Ai} - w_{Aj} - \frac{1}{\mu} \right)
\]

respectively. This covers the entire range of values that \( w_{Bi} < w_{Bj} \) can take without any overlap in case \( w_{Ai} \geq w_{Aj} + 1/\mu \).

Second, consider the next relevant range for values of \( w_{Ai} \) so that

\[
\frac{1 + w_{Aj}}{2} < w_{Ai} \leq w_{Aj} + \frac{1}{\mu}
\]

In this case we have that region 3 vanishes and regions 1, 2 and 4 reduce to

\[
w_{Bj} - \left[ w_{Ai} - \left( \frac{1 + w_{Aj}}{2} \right) \right] \leq w_{Bi} \leq w_{Bj}
\]

\[
w_{Bj} - \left[ w_{Ai} - \frac{1 + w_{Aj}}{2} \right] + \frac{1}{2} \left( w_{Ai} - w_{Aj} - \frac{1}{\mu} \right) \leq w_{Bi} \leq w_{Bj} - \left[ w_{Ai} - \left( \frac{1 + w_{Aj}}{2} \right) \right]
\]

and

\[
w_{Bi} \leq w_{Bj} - \left[ w_{Ai} - \left( \frac{1 + w_{Aj}}{2} \right) \right] + \frac{1}{2} \left( w_{Ai} - w_{Aj} - \frac{1}{\mu} \right)
\]

respectively. Again, this covers the entire range of values that \( w_{Bi} < w_{Bj} \) can take without any overlap.

Finally, consider

\[
w_{Aj} < w_{Ai} < \left( \frac{1 + w_{Aj}}{2} \right) < w_{Aj} + \frac{1}{\mu}
\]

so we cover the entire range of values that \( w_{Ai} > w_{Aj} \) can take without any overlap. In this case, regions 1 and 3 vanish, and regions 2 and 4 reduce to

\[
w_{Bj} - \left[ w_{Ai} - \frac{1 + w_{Aj}}{2} \right] + \frac{1}{2} \left( w_{Ai} - w_{Aj} - \frac{1}{\mu} \right) \leq w_{Bi} \leq w_{Bj}
\]

and

\[
w_{Bi} \leq w_{Bj} - \left[ w_{Ai} - \frac{1 + w_{Aj}}{2} \right] + \frac{1}{2} \left( w_{Ai} - w_{Aj} - \frac{1}{\mu} \right)
\]
respectively.

We now consider **Possibility 2(c):** $R_j$ sells good $A$, $R_i$ sells good $B$, and both sell the bundle $AB$. The only way for both retailers to be selling strictly positive units of the bundle is if $p_{ABi}^* = p_{ABj}^*$ and $w_{Ai} + w_{Bi} = w_{Aj} + w_{Bj}$. This immediately implies that

$$p_{ABi}^* = p_{ABj}^* = w_{Ai} + w_{Bi} = w_{Aj} + w_{Bj}$$

otherwise, at least one retailer would have an incentive to slightly undercut the bundle (the restriction $p_{ABi}^* \leq p_{Ai}^* + p_{Bi}^*$ is never binding in a deviation since each retailer, on-path, is making profits from at most one item on stand-alone basis).

But the above indicates that retailers are obtaining zero profits from selling the bundle. This has two implications. First, $p_{Ai}^* \leq w_{Ai}$, otherwise $R_i$ would deviate to $p_{Ai}^* = p_{Ai}^* - \epsilon$, $p_{Bi}^* = p_{Bi}^*$, and $p_{ABi}^* \to \infty$, and strictly increase her profits. And second, $p_{Bj}^* \leq w_{Bj}$, otherwise $R_j$ would deviate to $p_{Aj}^* = p_{Aj}^*$, $p_{Bj}^* = p_{Bj}^* - \epsilon$, and $p_{ABj}^* \to \infty$, and strictly increase her profits. These two conditions imply that an equilibrium of this sort can only exists if $w_{Aj} < w_{Ai}$, $w_{Bi} < w_{Bj}$ (and obviously $w_{Ai} + w_{Bi} = w_{Aj} + w_{Bj}$). It is then not difficult to see that the equilibrium must exhibit a Bertrand-like competition for product $A$

$$p_{Aj}^* = \min \left\{ w_{Ai}, \frac{1 + w_{Aj}}{2} \right\}, \quad p_{Ai}^* = \begin{cases} w_{Ai}, +\infty & \text{if } (1 + w_{Aj})/2 \leq w_{Ai} \\ w_{Ai} & \text{otherwise} \end{cases} \tag{1.18}$$

Regarding product $B$, $R_i$ must be maximizing locally, that is,

$$p_{Bi}^* \in \arg \max_{p_{Bi}} \{(1 - \mu)(p_{Bi} - w_{Bi})D^{(1-\mu)}_{Bi}(p_{Bi}, p_{Bj}^*) + \mu(p_{Bi} - w_{Bj})D_{Bi}^{(\mu)}(p_{ABi}^*, p_{Bi}, p_{Bj}^*)\}$$

which leads to

$$p_{Bi}^* = \min \left\{ \min \{p_{Bj}^*, b\}, w_{Bi} + \frac{w_{Ai}}{2} + \frac{1 - \mu}{2\mu} \right\}$$

But then it is immediate that this equilibrium is identical to the one found in possibility 2(b) evaluated at $w_{Ai} + w_{Bi} = w_{Aj} + w_{Bj}$.

It is now the turn of **Possibility 2(d):** $R_j$ sells good $A$, $R_i$ sells good $B$, and no retailer sells $AB$. This latter implies that $1 - p_{ABi}^* - p_{Bi}^* \leq 0$ and $1 - p_{ABj}^* + p_{Bj}^* \leq 0$. Since by assumption no bundles are being sold, the equilibrium follows a Bertrand-like competition, whereby the necessary conditions for $R_i$ to be selling $B$ and $R_j$ to be selling $A$ are

$$w_{Aj} < w_{Ai} \text{ and } w_{Bi} < w_{Bj}$$

Prices in this equilibrium candidate must then be given by \[1.18\], $p_{Bi}^* = p_{Bj}^* = w_{Bj}$, $p_{ABi}^* \geq w_{Ai} + w_{Bi}$, and $p_{ABj}^* \geq w_{Aj} + w_{Bj}$. But for this to be indeed an equilibrium, we need: (i) $p_{ABi}^* \leq p_{Ai}^* + \min\{p_{Bi}^*, b\} = p_{Ai}^* + w_{Bj}$; (ii) $p_{ABj}^* \leq p_{Aj}^* + \min\{p_{Bj}^*, b\} = p_{Aj}^* + w_{Bj}$; and (iii)
1 - p_{ABj}^* + p_{Bi}^* \geq 0. But plugging (ii) and p_{Bi}^* = w_{Bj} in (iii) yields 0 \geq 1 - p_{ABj}^* + p_{Bi}^* \geq 1 - p_{Aj}^*, which leads to p_{Aj}^* \geq 1; a contradiction since \min\{w_{Ai}, (1 + w_{Aj})/2\}. Hence an equilibrium as the one conjectured cannot exist.

**Case II:** \( p_{ABi}^* \leq p_{Ai}^* + \min\{b, p_{Bi}^*\} \) and \( p_{ABj}^* > p_{Aj}^* + \min\{b, p_{Bj}^*\} \). From \( p_{ABj}^* > p_{Aj}^* + \min\{b, p_{Bj}^*\} \) and \( p_{ABj}^* \leq p_{Aj}^* + p_{Bj}^* \) we have that \( p_{Bj}^* > b \). But if so, then \( p_{Bi}^* \leq b \), which implies that \( w_{Bi} \leq b \) (by the fact that \( \min\{p_{Bi}^*, p_{Bj}^*\} \leq b \)). There are two possibilities to consider:

1. When \( Ri \) sells both goods, and either
   
   (a) \( Ri \) sells \( AB \) to consumers that value both goods and have a high valuation for \( A \) (i.e., \( p_{ABi}^* \leq p_{Aj}^* + b \)) or
   
   (b) \( Rj \) sells \( AB \) to those consumers (i.e., \( p_{ABi}^* > p_{Aj}^* + b \))

2. When \( Rj \) sells good \( A \) and \( Ri \) sells good \( B \), and either

   (a) \( Ri \) sells \( AB \) to consumers that value both goods and have a high valuation for \( A \), or
   
   (b) \( Rj \) sells \( AB \) to those consumers

Before we start with the analysis, there three observations to make. First, notice that since we know that \( p_{Bj}^* > b \) and \( p_{Bi}^* \leq b \), only \( Ri \) can be selling product \( B \), so the above list of possibilities is exhaustive. Second, notice that the reason we have two subcases under each possibility is because we know that a consumer who is indifferent between buying the bundle and product \( A \) only always chooses the bundle. Then, it cannot be that both \( Ri \) and \( Rj \) are simultaneously serving consumers that value both goods and have a high valuation for \( A \). And third, possibility (b) can be discarded right away since it leads to a contradiction. Indeed from \( p_{Aj}^* + b < p_{ABi}^* \leq p_{Ai}^* + p_{Bi}^* \) we obtain that \( p_{Aj}^* + (b - p_{Bi}^*) < p_{Ai}^* \), which implies that \( p_{Ai}^* > p_{Aj}^* \); a contradiction.

Then, consider first **possibility 1(a)**. The assumptions under this possibility implies that \( w_{Ai} + w_{Bi} \leq p_{ABi}^* \leq p_{Aj}^* + b \) and \( \bar{\pi}_{Rj} = 0 \). It is easy to see that \( p_{Ai}^* \leq w_{Aj} \); otherwise \( Rj \) who is obtaining zero profits would deviate and undercut \( p_{Ai}^* \) to obtain strictly positive profits. But this implies \( w_{Ai} \leq p_{Ai}^* \leq w_{Aj} \), leading to \( w_{Ai} < w_{Aj} \). On the other hand, for any given triplet \((p_{Aj}^*, p_{Bj}^*, p_{ABj}^*)\), \( Ri \) must be maximizing locally:

\[
(p_{Ai}^*, p_{Bi}^*, p_{ABi}^*) \in \arg\max_{p_{Ai}^*, p_{Bi}^*, p_{ABi}^*} \{(1 - \mu)(p_{Ai} - w_{Ai})D_{Ai}^{(1-\mu)}(p_{Ai}^*, p_{Aj}^*) + (1 - \mu)(p_{Bi} - w_{Bi}) \mathbb{1}\{p_{Bi} \leq b\} + \mu(p_{Bi} - w_{Bi})\mathbb{1}\{p_{ABi} \leq b + p_{Aj}^*\} + \mu(p_{Bi} - w_{Bi})(p_{ABi} - p_{Bi})\}
\]

subject to \( p_{ABi} \leq p_{Ai}^* + p_{Bi}^* \). From this maximization we obtain

\[
p_{Ai}^* = \min\left\{p_{Aj}^*, \frac{1 + w_{Aj}}{2}\right\}
\]
And since $p^*_A \leq p^*_B$ and $p^*_B \leq b$, the binding condition is $p^*_{ABi} \leq p^*_A + p^*_B$ not $p^*_{ABi} \leq p^*_A + b$. Hence the solution to (1.19) is (1.20), $p^*_{ABi} = p^*_A + p^*_B$ and $p^*_B = b$. But if $p^*_B = b$, then a necessary condition for this to be an equilibrium is $w_{Bj} \geq b$; otherwise $Rj$ would deviate and undercut $p^*_B = b$. But this contradicts our assumption that $\max\{w_{Bi}, w_{Bj}\} \leq b$ except in the limiting case where $w_{Bj} = b$, in which case the equilibrium converges to the one in case I, possibility 1.

Consider now possibility 2: $Rj$ sells $A$ and $Ri$ sells $B$. From here we have, respectively, that $w_{Aj} \leq p^*_A \leq p^*_A - \epsilon 1\{w_{Aj} > w_{Ai}\}$ and $w_{Bi} \leq p^*_B \leq b$. On the one hand, subcase (a) involves $p^*_{ABi} \leq p^*_A + b$. For this to be an equilibrium $Rj$ and $Ri$ must be maximizing locally given the rival’s set of prices. In the case of $Rj$ this requires

$$p^*_A \in \arg \max_{p_A} \left\{ (1 - \mu)(p_A - w_{Aj})D_{Aj}^{1-\mu}(p^*_A, p_A) \right\}$$

which leads to

$$p^*_A = \begin{cases} 
[w_{Aj}, +\infty) & \text{if } p^*_A \leq w_{Aj} \\
\min\{(1 + w_{Aj})/2, p^*_A - \epsilon 1\{w_{Aj} > w_{Ai}\}\} & \text{otherwise}
\end{cases}$$

and in the case of $Ri$

$$(p^*_B, p^*_{ABi}) \in \arg \max_{p_A, p_B, p_{ABi}} \left\{ (1 - \mu)(p_B - w_{Bi})1\{p_B \leq b\} + \mu(p_{ABi} - w_{Bi})(1 - p_{ABi} + p_B)1\{p_{ABi} \leq b + p^*_A\} + p_{ABi}(p_{ABi} - p_B) \right\}$$

subject to $p_{ABi} \leq p_A + p_B$, which leads to $p^*_B = b$, and $p^*_{ABi} = \min\{p^*_A + b, p^*_A + b\}$.

From here we have that $p^*_{ABi} = p^*_A + b$, because $p^*_A \leq p^*_A$, and from that $p^*_A \leq w_{Ai}$; otherwise, $Ri$ would deviate to $p^*_A = p^*_A - \epsilon$, $p^*_B = p^*_B$ and $p^*_{ABi} = p^*_{ABi} - \epsilon \leq p^*_A + p^*_B$ (where $\epsilon \to 0$) and strictly increase her profits from selling units of $A$. But if $p^*_A \leq w_{Ai}$, $w_{Aj} < w_{Ai}$ becomes a necessary equilibrium condition as well. We will now show that this latter implies that no pure-strategy equilibrium can exist. Suppose otherwise, that is, provided that $w_{Aj} < w_{Ai}$, there is a pure strategy equilibrium in which $p^*_A > w_{Aj}, p^*_B = b$ and $p^*_{ABi} = b + p^*_A$, resulting in $\pi^*_{Rj} = (1 - \mu)(p^*_A - w_{Aj})(1 - p^*_A)$. Suppose that $Rj$ deviates to $p^*_A = p^*_A - \epsilon$ (where $\epsilon \to 0$), which results in a payoff of $\tilde{\pi}^*_{Rj} = (1 - \mu)(p^*_A - w_{Aj})(1 - p^*_A) + p^*_A - \epsilon (\mu(p^*_A - w_{Aj})(1 - p^*_A - b + p^*_B)) = (p^*_A - w_{Aj})(1 - p^*_A)$. But this is greater than $(1 - \mu)(p^*_A - w_{Aj})(1 - p^*_A) = \pi^*_{Rj} \forall \mu \in (0, 1]$; a contradiction.

On the other hand, subcase (b) involves $p^*_{ABj} > p^*_A + b$, which implies that on-path $Rj$ is getting $\pi^*_{Rj} = (1 - \mu)(p^*_A - w_{Aj})(1 - p^*_A) + p^*_A - \epsilon (\mu(p^*_A - w_{Aj})(1 - p^*_A - b + p^*_B))$. But if $w_{Bj} < b$, then $Rj$ has a profitable deviation. In fact, if $Rj$ deviates to $p^*_A = p^*_A$ and $p^*_{ABj} = p^*_A + b - \epsilon$ (where $\epsilon \to 0$), then $\tilde{\pi}^*_{Rj} = (1 - \mu)(p^*_A - w_{Aj})(1 - p^*_A) + p^*_A - \epsilon (\mu(p^*_A - w_{Aj})(1 - p^*_A - b + p^*_B)) = \pi^*_{Rj} + \mu(b - w_{Bj})(1 - p^*_A - b + p^*_B)$, which is strictly greater than $\pi^*_{Rj}$. From here we conclude that equilibrium under this possibility requires $w_{Bj} \geq b$. But this contradicts the assumption $\min\{w_{Bi}, w_{Bj}\} \leq b$ except in the limiting case of $w_{Bj} = b$. But in that case, the equilibrium proposed converges to the one outlined in case I, possibility 2(b).
Case III: $p_{A_i}^* > p_{A_i}^* + \min\{b, p_{B_i}^*\}$ and $p_{A Bj}^* \leq p_{A j}^* + \min\{b, p_{B j}^*\}$ This case is analogous to case II, so it is omitted.

Case IV: $p_{A Bi}^* > p_{A i}^* + \min\{b, p_{B i}^*\}$ and $p_{A Bj}^* > p_{A j}^* + \min\{b, p_{B j}^*\}$ It is easy to see that this case leads to a contradiction that $\min\{p_{B i}^*, p_{B j}^*\} > b$, so it can be readily discarded. In fact, from $p_{A Bi}^* > p_{A i}^* + \min\{b, p_{B i}^*\}$, we have that $p_{B i}^* > b$, and from $p_{A Bj}^* > p_{A j}^* + \min\{b, p_{B j}^*\}$ that $p_{B j}^* > b$.

This case concludes our first step of the characterization of the equilibrium set $E^*$. We have learned the set $E^*$ is fully characterized by case I (all possibilities in case II cannot be equilibrium, except in the limiting case $w_{B j} = b$, in which case the equilibria in case II are also present in case I). We now move to the second step of the characterization, which involves fixing a set of wholesale prices and asking whether there are elements in $E^*$ associated to those wholesale prices.

Provided that we are considering only combinations of wholesale prices where $\min\{w_{A i}, w_{A j}\} < 1$ and $\max\{w_{B i}, w_{B j}\} \leq b$ we need to consider the following cases:

1. $w_{A i} < w_{A j}$ and $w_{B i} < w_{B j} \leq b$

2. $w_{A i} > w_{A j}$ and $w_{B i} < w_{B j} \leq b$

   (a) $w_{A i} + w_{B i} < w_{A j} + w_{B j}$

   (b) $w_{A i} + w_{B i} \geq w_{A j} + w_{B j}$

So, consider the first case: $w_{A i} < w_{A j}$ and $w_{B i} < w_{B j} \leq b$. There is a single element in $E^*$ satisfying those constraints, the one characterized in case I, possibility 1. This contributes to the proof of Claim 1.1.1 as far as $w_{A i} \neq w_{B j}$ and $w_{B i} \neq w_{B j}$. Consider now case 2(a): $w_{A i} > w_{A j}$, $w_{B i} < w_{B j} \leq b$ and $w_{A i} + w_{B i} < w_{A j} + w_{B j}$. Again, there is a single element in $E^*$ satisfying those constraints, the one characterized in case I, possibility 2(a). This contributes to the proof of Claim 1.1.2 as far as $w_{A i} \neq w_{B j}$ and $w_{B i} \neq w_{B j}$. Finally, consider $w_{A i} > w_{A j}$, $w_{B i} < w_{B i} \leq b$ and $w_{A i} + w_{B i} \geq w_{A j} + w_{B j}$. Also here, there is a single element in $E^*$ satisfying those constraints, the one in case I, either possibility 2(b) and 2(c) depending on whether $w_{A i} + w_{B i} > w_{A j} + w_{B j}$ or $w_{A i} + w_{B i} = w_{A j} + w_{B j}$. This contributes to the proof of Claim 1.1.3 as far as $w_{A i} \neq w_{B j}$ and $w_{B i} \neq w_{B j}$.

To complete the proofs of Claims 1.1.1 1.1.2 and 1.1.3 we argue that our results hold as we let $w_{A i} \rightarrow w_{A j}$ and $w_{B i} \rightarrow w_{B j}$. This is easy to see, since the limit of our equilibrium coincides with the equilibrium of the game when $w_{A i} = w_{A j}$ and/or $w_{B i} = w_{B j}$, and that such equilibrium is also unique.
Proof of Claim 1.1.4

This claim considers the remaining combination of wholesale prices, that is, \( \min \{w_{B1}, w_{B2}\} \leq b < \max \{w_{B1}, w_{B2}\} \). Since the characterization of the equilibrium is not needed for any of the proofs in the text, we only prove its existence. But this latter is guaranteed from Reny’s (1999) Proposition 5.1 and Corollary 5.2, since the game is payoff secured and the sum of retailers’ profits is upper-semicontinuous.

Convergence to \( \mu = 0 \) and \( \mu = 1 \)

We now check whether the equilibrium characterization in Claims 1.1.1–1.1.3 converge to the equilibrium solutions for the extreme cases of \( \mu = 0 \) and \( \mu = 1 \), when these latter are obtained separately. Consider first the case of \( \mu = 0 \). We know that the (essentially) unique equilibrium outcome is the standard Bertrand outcome. So, assuming that \( \min \{w_{A1}, w_{A2}\} < 1, w_{Ai} \leq w_{Aj} \) and \( \max \{w_{B1}, w_{B2}\} \leq b \), it is easy to see that there exists an essentially unique pure-strategy equilibrium in the retail pricing subgame characterized by: \( p^*_A = \min \{w_{Aj}, (1 + w_{Ai})/2\} \), \( p^*_A = w_{Aj} \) and \( p^*_B = p^*_B = \max \{w_{B1}, w_{B2}\} \). Hence, retailers’ equilibrium profits are

\[
\pi^*_Ri = \begin{cases} 
(1 - w_{Ai})^2/4 + (\max\{w_{B1}, w_{B2}\} - w_{Bi}) \\
(w_{Aj} - w_{Ai})(1 - w_{Aj}) + (\max\{w_{B1}, w_{B2}\} - w_{Bi}) 
\end{cases}
\]

if \( (1 + w_{Ai})/2 \leq w_{Aj} \)

and \( \pi^*_Rj = \max \{w_{B1}, w_{B2}\} - w_{Bj} \). Note that this equilibrium is not different from the one in Claim 1.1.1 or Claim 1.1.2 as \( \mu \to 0 \).

Consider now the case of \( \mu = 1 \). This case is also simple because competition reduces to two products, \( B \) and \( AB \). Thus, either (i) one retailer, say \( Ri \), sells both \( B \) and the bundle \( AB \) or (ii) \( Ri \) sells \( B \) and \( Rj \) sells the bundle \( AB \). Take (i) first. For \( Ri \) to be selling both \( B \) and \( AB \) in equilibrium it must be true that \( w_{Bi} < w_{Bj} \leq b \) and \( w_{Ai} + w_{Bi} < w_{Aj} + w_{Bj} \) (the cases in which \( w_{Bi} = w_{Bj} \) and/or \( w_{Ai} + w_{Bi} = w_{Aj} + w_{Bj} \) follow straightforwardly, so they are omitted); otherwise \( Ri \) would deviate and price either \( B \) or \( AB \) slightly below \( Rj \)’s prices (or \( Rj \)’s prices if any higher) and make a profit. For such wholesale prices, it is easy to see that the equilibrium candidate is essentially unique and given by

\[
p^*_A = \min \left\{ \frac{1 + w_{Ai}}{2}, w_{Aj} \right\}, \quad p^*_A = \begin{cases} \left[ w_{Aj}, +\infty \right) \\
w_{Aj} 
\end{cases}
\]

if \( (1 + w_{Ai})/2 \leq w_{Aj} \)

\[
p^*_B = w_{Bj} = p^*_B, \quad p^*_{AB} = p^*_A + p^*_B, \quad p^*_{AB} \geq p^*_A + p^*_B
\]

leading to \( \pi^*_Rj = 0 \) and

\[
\pi^*_Ri = (w_{Bj} - w_{Bi}) + \left( \min \left\{ \frac{1 + w_{Ai}}{2}, w_{Aj} \right\} - w_{Ai} \right) \left( 1 - \min \left\{ \frac{1 + w_{Ai}}{2}, w_{Aj} \right\} \right)
\]

Note again that this equilibrium is not different from the one in Claim 1.1.1 (which considers the case of \( w_{Ai} \leq w_{Aj} \)) or Claim 1.1.2 (which accepts \( w_{Ai} > w_{Aj} \), implying \( (1 + w_{Ai})/2 > w_{Aj} \)) as \( \mu \to 1 \).
Take now case (ii). For \( R_i \) to be selling \( B \) and \( R_j \) the bundle, it must be true that \( w_{B_i} < w_{B_j} \leq b \) and \( w_{A_i} + w_{B_i} > w_{A_j} + w_{B_j} \). If \( R_i \) is not selling \( B \), so \( R_j \) is the one selling both \( B \) and \( AB \), \( R_i \) would deviate and price \( B \) slightly below \( R_j \)'s cost \( w_{B_j} \) (or \( p^*_{B_j} \) if any higher) and make a profit. Similarly, if \( R_j \) is not selling \( AB \), so \( R_i \) is the one selling both \( B \) and \( AB \), \( R_j \) would deviate and price \( AB \) slightly below \( R_i \)'s cost \( w_{A_i} + w_{B_i} \) (or \( p^*_{AB_i} \) if any higher) and make a profit. As done before, it can be shown that the equilibrium is essentially unique but that vary depending on the set of wholesale prices considered. For example, if we assume that wholesale prices satisfy (1) \( w_{A_j} + 1 \leq w_{A_i} \), (2) \( w_{B_i} \leq w_{B_j} - (1 + w_{A_j})/2 \), and (3) \( w_{B_i} \geq 1 + w_{B_j} - w_{A_j} - 3w_{A_i}/2 \), the equilibrium candidate is given by

\[
P^*_{B_i} = \frac{1 + w_{A_j} + 2w_{B_i}}{3}
\]

\[
P^*_{AB_j} = \frac{2(1 + w_{A_j} + w_{B_j}) + w_{B_i}}{3}
\]

and \( p^*_{B_j} \in [w_{B_j}, +\infty) \), \( p^*_{AB_i} \geq p^*_{A_i} + p^*_{B_i} \), leading to

\[
\bar{\pi}^*_{R_i} = \frac{(1 + w_{A_j} + w_{B_j} - w_{B_i})^2}{9}
\]

\[
\bar{\pi}^*_{R_j} = \frac{(2 + w_{B_i} - w_{A_j} - w_{B_j})^2}{9}
\]

It is easy to see that the candidate is indeed an equilibrium. \( R_j \) does not have “local” incentives to deviate given that \( p^*_{AB_j} \) is the local best response to \( p^*_{B_i} \) and \( p^*_{AB_j} \leq w_{A_i} + w_{B_i} \) from condition (3). Neither does \( R_j \) have “global” incentives to deviate since this would require pricing below cost. As for \( R_i \), it can be shown that neither she has “local” incentives to deviate given that \( p^*_{B_i} \) is the local best response to \( p^*_{AB_j} \) and \( p^*_{B_i} \leq w_{B_j} \) from condition (2) nor “global” incentives since this would require pricing below cost. Condition (1) ensures that (2) and (3) can hold simultaneously. To conclude the example, note that this equilibrium (with all the conditions imposed upon wholesale prices) is not different from the one in Claim 1.1.3(a)(ii) as \( \mu \to 1 \).

Proceeding similarly, we can find equilibria for any combination of wholesale prices that satisfy \( w_{B_i} < w_{B_j} \leq b \) and \( w_{A_i} + w_{B_i} > w_{A_j} + w_{B_j} \), and show that they all can be obtained from Claim 1.1.3 by letting \( \mu \to 1 \). For instance, if \( w_{B_j} \) is only barely above \( w_{B_i} \) and so is \( w_{A_i} + w_{B_i} \) above \( w_{A_j} + w_{B_j} \), the sections of Claim 1.1.3 relevant to this case are, depending on the value of \( w_{A_j} \), either (b)(ii) or (c)(i).\(^1\)

### 1.2 Proofs of Lemmas A.1–A.3

The proof of Lemma A.1 follows directly from Claim 1.1.1 after making \( w_{B_i} = w_{B_j} \) in the claim. The proof of Lemma A.2 follows directly from Claim 1.1.2 after making \( w_{A_i} = w_{A_j} \) in the claim. The proof of Lemma A.3 is bit more involved. If \( w_{B_2} \leq w_{B_1} \), it is only relevant the case in which \( w_{B_1} \geq w_B \) since \( 1/\mu > (1 - w_{A_2})/2 \). In this case the proof follows directly from

\(^1\)Note that here the value of \( w_{A_j} \) is only relevant through its impact on the cost \( w_{A_j} + w_{B_j} \).
Claim 1.1.1 since $R2$ has equal or lower costs in both products (strictly lower in product $A$). If, on the other hand, $w_{B2} > w_{B1}$, the proof follows Claim 1.1.3 it follows part (a)(i) whenever $w_{B1} \geq w_B$ and part (a)(ii) whenever $w_{B1} < w_B = w_{B2} - 1/\mu + (1 - w_A)/2$.

### 1.3 Standalone Proof of Lemma A.3.

Lemma A.3 is an important result that we invoke in several passages of the paper, so we provide an alternative, standalone proof of it, which does not require going over the lemmas of the retail market equilibrium as we did above. So suppose that $R1$ carries only $B$ (i.e., $w_{A1} \to +\infty$ and $w_{B1} \leq b$) and $R2$ carries both $A$ and $B$ (i.e., $w_{A2} < 1$ and $w_{B2} \leq b$). We prove the following:

(i) If $w_{B1} \in [w_{B2}, b]$, then $p_{A2}^* = (1 + w_{A2})/2$, $p_{B1}^* = p_{B2}^* = w_{B1}$, and $p_{AB2}^* \geq p_{A2}^* + p_{B2}^*$. Therefore $\bar{\pi}_{R1}^* = 0$ and $\bar{\pi}_{R2}^* = (1 - w_{A2})^2/4 + (w_{B1} - w_{B2})$.

(ii) If $w_{B1} \in [w_B, w_{B2}]$, where

\[
\bar{w}_B = \max \left\{ 0, w_{B2} - \frac{1}{\mu} + \frac{1 - w_{A2}}{2} \right\},
\]

then $p_{A2}^* = (1 + w_{A2})/2$, $p_{B1}^* = p_{B2}^* = w_{B2}$, and $p_{AB2}^* \geq p_{A2}^* + p_{B2}^*$. Therefore $\bar{\pi}_{R1}^* = (w_{B2} - w_{B1})[1 - \mu(1 - w_{A2})/2]$ and $\bar{\pi}_{R2}^* = (1 - w_{A2})^2/4$.

(iii) If $w_{B1} \in [0, w_B]$, then $p_{A2}^* = (1 + w_{A2})/2$, $p_{B1}^* = w_{B2} - 2(w_B - w_{B1})/3$, $p_{B2}^* \in [w_{B2}, +\infty)$, $p_{AB2}^* = p_{A2}^* + w_{B2} - (w_B - w_{B1})/3$. Therefore

\[
\bar{\pi}_{R1}^* = \left[ (w_{B2} - w_{B1}) - \frac{2}{3}(w_B - w_{B1}) \right] \left[ 1 - \frac{\mu}{2}(1 - w_{A2}) + \frac{\mu}{3}(w_B - w_{B1}) \right]
\]

\[
\bar{\pi}_{R2}^* = \frac{1}{4}(1 - w_{A2})^2 - \frac{\mu}{3}(w_B - w_{B1}) \left[ 1 - w_{A2} - \frac{1}{3}(w_B - w_{B1}) \right]
\]

**Proof.** The proof consists of 4 parts.

**Part 1: $R2$’s best response.** Begin by setting $p_{B1} > b$. Then $R2$’s problem is

\[
\max_{p_{A2}^*, p_{B2}^*} \bar{\pi}_{R2}^* = (1 - \mu)(p_{A2} - w_{A2})(1 - p_{A2}) + (1 - \mu)(p_{B2} - w_{B2})
\]

\[
+ \mu(p_{AB2} - w_{A2} - w_{B2})(1 - p_{AB2} + p_{B2}) + \mu(p_{B2} - w_{B2})(p_{AB2} - p_{B2})
\]

Solving, we get $p_{A2}^{BR}(p_{B1}) = (1 + w_{A2})/2$, $p_{B2}^{BR}(p_{B1}) = b$, and $p_{AB2}^{BR}(p_{B1}) = p_{A2}^{BR}(p_{B1}) + p_{B2}^{BR}(p_{B1})$ (so bundling is not strictly necessary).

On the other hand, suppose $p_{B1} \in (w_{B2}, b]$. Then $R2$’s problem is

\[
\max_{p_{A2}^*, p_{B2}^*} \bar{\pi}_{R2}^* = (1 - \mu)(p_{A2} - w_{A2})(1 - p_{A2}) + (1 - \mu)(p_{B2} - w_{B2})D_{B2}^{(1-\mu)}(p_B)
\]

\[
+ \mu(p_{AB2} - w_{A2} - w_{B2})(1 - p_{AB2} + \min\{p_{B1}, p_{B2}\}) + \mu(p_{B2} - w_{B2})D_{B2}^{(\mu)}(p_{AB2}, p_B)
\]
The solution then is $\epsilon_p$ plus the usual refinement implies that solving, we get $p = p_{B2}(p_{B1}) = p_{AB2}(p_{B1}) = p_{A2}(p_{B1}) + p_{B2}(p_{B1})$ (so, again, bundling is not strictly necessary).

Finally, suppose $p_{B1} \leq w_{B2}$, then it is clear that $p_{B2} \leq p_{B1}$ is not optimal. This, plus the refinement that retailers do not price below cost if they expect to sell nothing of the product implies that $p_{B2} = [w_{B2}, +\infty)$. Therefore, $R2$’s problem is

$$\max_{p_{A2}, p_{B2}, p_{AB2} \geq p_{A2} + p_{B2}} \pi_{R2} = (1 - \mu)(p_{A2} - w_{A2})(1 - p_{A2}) + \mu(p_{AB2} - w_{A2} - w_{B2})(1 - p_{AB2} + p_{B1})$$

Solving, we get $p_{A2}(p_{B1}) = (1 + w_{A2})/2$, and

$$p_{AB2}(p_{B1}) = p_{A2}(p_{B1}) + w_{B2} + \frac{1}{2}(p_{B1} - w_{B2})$$

Notice that since $p_{AB2}(p_{B1}) < p_{A2}(p_{B1}) + w_{B2} \leq p_{A2}(p_{B1}) + p_{B2}(p_{B1})$, this is a case in which bundling is indeed necessary.

**Part 2: R1’s best response.** Retailer $R1$’s best response, on the other hand, is obtained from the program:

$$\max_{p_{B1}} \pi_{R1} = (1 - \mu)(p_{B1} - w_{B1}) D_{B1}^{(1-\mu)}(p_B) + \mu(p_{B1} - w_{B1}) D_{B1}^{(\mu)}(p_{AB2}, p_B)$$

First, consider the case $p_{B2} \leq w_{B1}$, then it is clear that $p_{B1} \leq p_{B2}$ is not optimal. This, plus the usual refinement implies that $p_{B1} = [w_{B1}, +\infty)$; recall that $p_2 = (p_{A2}, p_{B2}, p_{AB2})$. Consider, on the other hand, the case $p_{B2} > w_{B1}$. It is then clear that $p_{B1} = p_{B2} - \epsilon \{w_{B1} \leq w_{B2}\}$, hence we can restrict the optimization domain to all $p_{B1} \leq p_{B2} - \epsilon \{w_{B1} \geq w_{B2}\}$ and solve instead

$$\max_{p_{B1} \leq p_{B2} - \epsilon \{w_{B1} \geq w_{B2}\}} \pi_{R1} = (1 - \mu)(p_{B1} - w_{B1}) + \mu(p_{B1} - w_{B1})(p_{AB2} - p_{B1})$$

The solution then is

$$p_{B1}^*(p_2) = \min \left\{ p_{B2} - \epsilon \{w_{B1} \geq w_{B2}\}, \frac{1}{2}(p_{AB2} + w_{B1}) + \frac{1}{2} \left( \frac{1 - \mu}{\mu} \right) \right\}$$

**Part 3: Equilibrium.** We obtain the equilibrium set by intersecting the best response correspondences. Begin by considering the case in which $w_{B1} \in [w_{B2}, b]$. First, it is clear that $p_{B1}^* < w_{B1}$ cannot happen in equilibrium: either $R1$ is obtaining negative profits (if he is selling strictly positive units of $B$) and therefore he would deviate; or he is obtaining zero profits, and $p_{B1}^* < w_{B1}$ is weakly dominated by $p_{B1}$’s greater than, or equal to $w_{B1}$. On the other hand, it is also clear that in equilibrium cannot happen $p_{B1}^* > b$, since if it does, then $p_{B2}^* = b$, but then $R1$’s best response would be $p_{B1}^*(p_2) = b - \epsilon \{w_{B1} \geq w_{B2}\} \leq b < p_{B1}^*$, a contradiction.

Hence, the only remaining possibility is for $p_{B1}^* \in [w_{B1}, b]$. If so, $R2$’s optimal response is $p_{B2}^*(p_{B1}) = p_{B1}^*$, $p_{A2}^*(p_{B1}) = (1 + w_{A2})/2$, and $p_{AB2}^*(p_{B1}) \geq (1 + w_{A2})/2 + p_{B1}^*$. But then, $p_{B1}^*(p_2) = p_{B1}^*$ if and only if $p_{B1}^* = p_{B2}^* = w_{B1}$; any other price induces $R1$ to slightly undercut
of 

\[ p_{B2}^{BR}(p_{B1}) = p_{B1}^{*} \]. Hence, the unique equilibrium if \( w_{B1} \in [w_{B2},b] \) is \( p_{A2}^{*} = (1 + w_{A2})/2, p_{B1}^{*} = p_{B2}^{*} = w_{B1}, p_{AB2}^{*} \geq p_{A2}^{*} + p_{B2}^{*}, \) and therefore \( \pi_{R1}^{*} = 0 \) and \( \pi_{R2}^{*} = (1 - w_{A2})^2/4 + (w_{B1} - w_{B2}) \).

Consider then the case \( w_{B1} < w_{B2} \). Again, by the same argument as before, it cannot happen in equilibrium that \( p_{B1}^{*} > b \). Moreover, it cannot be equilibrium for \( p_{B1}^{*} \in (w_{B2},b] \) either; if it were \( R2 \)'s best response would be \( p_{A2}^{BR}(p_{B1}^{*}) = (1 + w_{A2})/2, p_{B2}^{BR}(p_{B1}^{*}) = p_{B1}^{*} - \epsilon, \) and \( p_{AB2}^{BR}(p_{B1}^{*}) \geq (1 + w_{A2})/2 + p_{B1}^{*} - \epsilon \). But if so, \( p_{B1}^{BR}(p_{B1}^{*}) = p_{B1}^{*} - \epsilon \neq p_{B1}^{*} \), leading to a contradiction.

Hence, it must be that \( p_{B1}^{*} \leq w_{B2} \). Then \( R2 \)'s best response correspondence is given by

\[ p_{A2}^{BR}(p_{B1}^{*}) = (1 + w_{A2})/2, p_{B2}^{BR}(p_{B1}^{*}) \in [w_{B2},+\infty), \]

and \( w \) and \( \bar{w} \). Because for any combination of wholesale prices the intersection of the Nash equilibrium. Evaluating to obtain the manufacturers’ profit:

\[ p_{B1}^{BR}(p_{B1}^{*}) = \min \left\{ p_{B2}, \frac{1}{2}(p_{AB2} + w_{B1}) + \frac{1}{2}\left(\frac{1 - \mu}{\mu}\right) \right\} \]

(1.22)

So let us conjecture that \( p_{B1}^{*} = p_{B2}^{*} \). This immediately implies that \( p_{B2}^{*} = p_{B1}^{*} = w_{B2} \). Plugging in (1.21) we get \( p_{A2}^{BR} = p_{A2}^{*} + w_{B2}, \) as \( w_{A2} - p_{A2}^{BR}(p_{B1}^{*}) < (p_{B1}^{*} - w_{B2})/2 = 0. \)

Therefore, our equilibrium candidate would be \( p_{A2}^{*} = (1 + w_{A2})/2, p_{B2}^{*} = p_{B1}^{*} = w_{B2} \), and \( p_{AB2}^{*} = p_{A2}^{*} + w_{B2} \). But such candidate must also satisfy [1.22], hence we need

\[ p_{B2}^{*} = w_{B2} \leq \frac{1}{2}(p_{AB2}^{*} + w_{B1}) + \frac{1}{2}\left(\frac{1 - \mu}{\mu}\right) \iff w_{B1} \geq w_{B2} - \frac{1}{\mu}\left[1 - \frac{\mu}{2}(1 - w_{A2})\right] \equiv w_{B} \]

(1.23)

Hence, if \( w_{B1} \in [w_{B},w_{B2}] \), then the following is a pure-strategy Nash equilibrium: \( p_{A2}^{*} = (1 + w_{A2})/2, p_{B2}^{*} = p_{B1}^{*} = w_{B2}, \) and \( p_{AB2}^{*} \geq p_{A2}^{*} + p_{B2}^{*} \); and \( \pi_{R1}^{*} = (w_{B2} - w_{B1})[1 - (1/2)(1 - w_{A1})] \) and \( \pi_{R2}^{*} = (1 - w_{A2})^2/4. \)

The remaining possibility is for \( p_{B1}^{*} = (p_{AB2}^{*} + w_{B1})/2 + (1 - \mu)/2\mu. \) Then (1.21) requires

\[ p_{AB2}^{*} = p_{A2}^{*} + w_{B2} + \frac{1}{2}(p_{B1}^{*} - w_{B2}) \]

But for such candidate to be feasible, it must be that \( p_{B1}^{*} \leq w_{B2}, \) that is

\[ p_{B2}^{*} \geq \frac{1}{2}(p_{AB2}^{*} + w_{B1}) + \frac{1}{2}\left(\frac{1 - \mu}{\mu}\right) \iff w_{B1} \leq w_{B} \]

which coincidentally satisfies condition (1.22). Hence if \( w_{B1} \leq w_{B}, \) then this is a pure-strategy Nash equilibrium. Evaluating to obtain the manufacturers’ profit:

\[ \bar{\pi}_{R1}^{*} = \left[w_{B2} - w_{B1}\right] - \frac{2}{3}(w_{B} - w_{B1}) \left[1 - \frac{\mu}{2}(1 - w_{A2}) + \frac{\mu}{3}(w_{B} - w_{B1})\right] \]

\[ \bar{\pi}_{R2}^{*} = \frac{1}{4}(1 - w_{A2})^2 - \frac{\mu}{3}(w_{B} - w_{B1}) \left[1 - w_{A2} - \frac{1}{3}(w_{B} - w_{B1})\right] \]

Part 4: Uniqueness. Because for any combination of wholesale prices the intersection of the best responses is a singleton, uniqueness immediately ensues. ■

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Chapter 2

Baseline Model - Remaining Proofs

2.1 Proofs of Lemmas D.3–D.5

2.1.1 Proof of Lemma D.3

Suppose
\[ w_{A1}^M w_{A2}^M + w_{A2}^M + T_{A1}^M + T_{A2}^M > 1/4 \]  
(2.1)
and let \((w_{B1}^*, T_{B1}^*), (w_{B2}^*, T_{B2}^*)\) be the contracts for \(B\) accepted on-path by retailers. By Lemma D.2 we know
\[ [(w_{B1}^*, T_{B1}^*) = (w_{B1}^S, T_{B1}^S)] \land [(w_{B2}^*, T_{B2}^*) = (w_{B2}^S, T_{B2}^S)] \]
or
\[ [(w_{B1}^*, T_{B1}^*) = (w_{B1}^M, T_{B1}^M)] \land [(w_{B2}^*, T_{B2}^*) = (w_{B2}^M, T_{B2}^M)] \]
To further simplify notation, let \(\pi_{Ri}(x, y)\), where \(x \in \{A_M, \emptyset\}\) and \(y \in \{B_*, B_S, B_M, \emptyset\}\), be \(Ri\)'s profits after fixed fees when she accepts contract \(x\) for \(A\) and \(y\) for \(B\), while \(Rj\) continues to play on-path (that is, accepting \(A_M\) and \(B_*\)), and where \(\emptyset\) denotes “no contract accepted”. Obviously, either \(B_* = B_S\), or \(B_* = B_M\).

This proof is somewhat involved, though the main idea is simple. The first step is to show that if \(2.1\) holds, then \(w_{B1}^* \neq w_{B2}^*\). Without loss of generality assume then \(w_{B1}^* < w_{B2}^*\). We then have 3 cases to consider, (a) \(w_{A1}^M \leq w_{A2}^M\); (b) \(w_{A1}^M > w_{A2}^M\) and \(w_{A1}^M + w_{B1}^* \leq w_{A2}^M + w_{B2}^*\); or (c) \(w_{A1}^M > w_{A2}^M\) and \(w_{A1}^M + w_{B1}^* > w_{A2}^M + w_{B2}^*\). We begin by showing that (c) is unfeasible, irrespective on whether \(B_* = B_S\) or \(B_* = B_M\) as it would mean at least one retailer is obtaining strictly negative profits. Then we move to cases (a) and (b), and show that irrespective on whether \(B_* = B_S\) or \(B_* = B_M\), \(S\) has a strictly profitable deviation in which he overcomes \(M\)’s cross-subsidization strategy by making the retailer who is receiving the worst offer for \(B\), more efficient and therefore aggressive in the downstream market. Thus, there cannot be an equilibrium satisfying \(2.1\).

So, we first establish that \(w_{B1}^* \neq w_{B2}^*\). Suppose \(w_{B1}^* = w_{B2}^*\), then \(T_{B1}^* = T_{B2}^* = 0\) necessarily. Now, without loss of generality let \(w_{A1}^M \leq w_{A2}^M\). Then by Lemma A.1 we know \(\bar{\pi}_i(A_M, B_*) = \)
\[ \varphi(w_{A1}^M, w_{Aj}^M), \] where

\[
\varphi(w_{A1}^M, w_{Aj}^M) = \begin{cases} 
(1 - w_{A1}^M)^2/4 & \text{if } (1 + w_{A1}^M)/2 \leq w_{Aj}^M \\
(w_{Aj}^M - w_{A1}^M)(1 - w_{A1}^M) & \text{otherwise}
\end{cases}
\]

Hence \( T_{A1}^M \leq \varphi(w_{A1}^M, w_{Aj}^M) \), and \( q_{Aj}^M = 0 \), so \( T_{A1}^M = 0 \). Thus \( w_{A1}^M q_{A1}^M + w_{A2}^M q_{A2}^M + T_{A1}^M + T_{A2}^M \)
\[ = w_{A1}^M q_{A1}^M + T_{A1}^M \leq \rho(w_{A1}^M, w_{Aj}^M), \]
where

\[
\rho(w_{A1}^M, w_{Aj}^M) = \begin{cases} 
1/4 - (w_{A1}^M)^2/4 & \text{if } (1 + w_{A1}^M)/2 \leq w_{Aj}^M \\
w_{Aj}^M(1 - w_{A1}^M) & \text{otherwise}
\end{cases}
\]

Consequently \( w_{A1}^M q_{A1}^M + w_{A2}^M q_{A2}^M + T_{A1}^M + T_{A2}^M \leq 1/4 \), as \( \rho(w_{A1}^M, w_{Aj}^M) \leq 1/4 \), contradicting our premise. Thus \( w_{B1}^* \neq w_{B2}^* \). Without loss of generality assume then \( w_{B1}^* < w_{B2}^* \).

Now, suppose (c). That is, \( w_{A1}^M > w_{A2}^M \) and \( w_{A1}^M + w_{B1}^* > w_{A2}^M + w_{B2}^* \). Using Claim 1.1.3 it is easy to verify that, if manufacturers are restricted to separate pricing contracts, then \( R1 \) would never accept \( M \)'s contract for \( A \). Indeed, carrying \( A \) does not increase \( R1 \)'s profit on-path, as \( R2 \) will be more efficient on both \( A \) and on the bundle. Moreover, by carrying \( A \), \( R1 \) increases the competition for the bundle downstream, which erodes here profits for product \( B \).\(^1\) Hence, this case is equivalent to one in which \( R1 \) does not carry \( A \). Then, because \( \pi_A^* \geq 1/4 \), it must be that \( R2 \) is accepting \( M \)'s contract for \( A \). Hence, by Lemma A.3 on Appendix A of the text we have

\[
\pi_{R1}^* = \pi_{R1}(\emptyset, B_*) = (w_{B2}^* - w_{B1}^*) \left[ 1 - \mu(1 + w_{A2}^M)/2 \right] - T_{B1}^*
\]
\[
\pi_{R2}^* = \pi_{R2}(A_M, B_*) = (1 - w_{A2}^M)^2/4 - T_{A2}^* - T_{B2}^*
\]

But then (2.1) is equivalent to \( w_{A2}^M q_{A2}^M + T_{A2}^M > 1/4 \), which can only happen if \( w_{A2}^M = 0 \) and \( T_{A2}^M > 1/4 \), which would imply that \( R2 \) is obtaining strictly negative profits. We therefore move to cases (a) and (b).

Case (a): \( w_{A1}^M \leq w_{A2}^M \)

(i) \( B_* = B_S \). By Claim 1 we know \( \pi_{R1}^* = \pi_{R1}(A_M, B_S) = \varphi(w_{A1}^M, w_{A2}^M) + (w_{B2}^S - w_{B1}^S) - T_{A1}^M - T_{B1}^S \), that \( q_{A2}^M = 0 \) (and therefore \( T_{A2}^M = 0 \)), and that \( q_{B2}^S = 0 \) (so \( T_{B2}^S = 0 \)). This implies, \( \pi_S^* = w_{B1}^S + T_{B1}^S - F_S \). On the other hand, since \( R1 \) is on-path accepting \( (A_M, B_S) \), then it must be true that

\[
\pi_{R1}(A_M, B_S) \geq \max \{ 0, \pi_{R1}(A_M, B_M), \pi_{R1}(\emptyset, B_S), \pi_{R1}(\emptyset, B_M), \pi_{R1}(A_M, \emptyset) \}
\]

Now, a necessary condition for equilibrium is

\[
\pi_{R1}(A_M, B_S) = \max \{ 0, \pi_{R1}(\emptyset, B_S), \pi_{R1}(\emptyset, B_M) \}
\]

\(^1\)This only can occur as retailers can accept contracts on a piecewise fashion, meaning that this is not an issue for full-line forcing contracts for instance.
where
\[
\pi_{R1}(\emptyset, B_S) = (w_{B2}^S - w_{B1}^S)(1 - \mu(1 - w_{A2}^M)/2) - T_{B1}^S \\
\pi_{R1}(\emptyset, B_M) = (\max\{w_{B2}, w_{B1}^M\} - w_{B1}^M)(1 - \mu(1 - w_{A2}^M)/2) - T_{B1}^M
\]

That is, R1 must be indifferent between accepting or not accepting M’s contract for A; otherwise M could unilaterally increase \(T_{A1}^M\) without affecting R1’s decision, and increase his profits. Hence \(T_{A1}^M = \varphi(w_{A1}^M, w_{A2}^M) + (w_{B2}^S - w_{B1}^S) - T_{B1}^S - \max\{0, \pi_{R1}(\emptyset, B_S), \pi_{R1}(\emptyset, B_M)\}\). Furthermore, this implies that in equilibrium \(\pi_{R1}(\emptyset, B_M) \leq \pi_{R1}(\emptyset, B_S)\) whenever \(\pi_{R1}(\emptyset, B_S) \geq 0\); otherwise M can increase \(T_{A1}^M\) (which enters into his profits function) at no cost by increasing \(T_{B1}^M\) (which is off-path, as \(B_4 = B_S\)). Finally, notice that
\[
w_{A1}^M q_{A1}^{M^*} + w_{A2}^M q_{A2}^{M^*} + T_{A1}^M + T_{A2}^M = \rho(w_{A1}^M, w_{A1}^M) + (w_{B2}^S - w_{B1}^S) - T_{B1}^S - \max\{0, \pi_{R1}(\emptyset, B_S)\}
\]
Hence for (2.1) to hold we need
\[
(w_{B2}^S - w_{B1}^S) - T_{B1}^S - \max\{0, \pi_{R1}(\emptyset, B_S)\} > 0
\]
Or equivalently \(T_{A1}^M > \varphi(w_{A1}^M, w_{A1}^M)\).

Now, consider the following deviation by S
\[
C_{B1}' = (w_{B1}^S = w_{B1}^S, T_{B1}' = T_{B1}^S - 3\epsilon/4), \quad C_{B2}' = (w_{B2}^S = w_{B2}^S - \epsilon, T_{B2}' = 0)
\]
For \(\epsilon > 0\) small. After such deviation, it is easy to see that R2 will continue accepting M’s contract for A, and that she will accept S’s new offering regarding B. Define then \(\pi_{R1}(x, y)\) as R1’s profits after fixed fees, when she accepts contracts \(x \in \{A_M, \emptyset\}\) and \(y \in \{B_S', B_M, \emptyset\}\), and R2 accepts M’s contract for A, and S’s new contract for B.

It is easy to verify that \(\pi_{R1}'(A_M, \emptyset) \leq \varphi(w_{A1}^M, w_{A1}^M) - T_{A1}^M < 0\) (hence R1 will never accept only \(A_M\) following that deviation), and that \(\pi_{R1}'(A_M, B_S) = \pi_{R1}(A_M, B_S) - \epsilon/4\), and \(\pi_{R1}'(\emptyset, B_S) = \pi_{R1}(\emptyset, B_S) - \epsilon/4 + \epsilon \mu(1 - w_{A2}^M)/2\). But because \(\pi_{R1}(A_M, B_S) = \max\{0, \pi_{R1}(\emptyset, B_S)\}\), then either \(\pi_{R1}'(A_M, B_S) < 0\) (when \(\pi_{R1}(\emptyset, B_S) \leq 0\)), or \(\pi_{R1}'(A_M, B_S) < \pi_{R1}'(\emptyset, B_S)\) (when \(\pi_{R1}(\emptyset, B_S) > 0\)). That is, R1 either strictly prefers accepting only \(B_S\), or not accepting anything at all, than to accept \(B_S\) in combination with \(A_M\). Hence R1’s choice must be between \((\emptyset, B_S)\), \((\emptyset, B_M)\), \((A_M, B_M)\), and \((\emptyset, \emptyset)\). Now, if \(w_{B1}^M < w_{B2}^S\). Then,
\[
\pi_{R1}'(A_M, B_M) = \varphi(w_{A1}^M, w_{A1}^M) + (w_{B2}^S - w_{B1}^S) - T_{A1}^M - T_{B1}^M - \epsilon \\
\pi_{R1}'(\emptyset, B_M) = (w_{B2}^S - w_{B1}^M)(1 - \mu(1 - w_{A2}^M)/2) - T_{B1}^M - \epsilon + \epsilon \mu(1 - w_{A2}^M)/2
\]
That is, \(\pi_{R1}'(A_M, B_M) = \pi_{R1}(A_M, B_M) - \epsilon\), and \(\pi_{R1}'(\emptyset, B_M) = \pi_{R1}(\emptyset, B_M) - \epsilon + \epsilon \mu(1 - w_{A2}^M)/2\). But since \(\pi_{R1}(A_M, B_M) \leq \pi_{R1}(A_M, B_S)\), then \(\pi_{R1}'(A_M, B_M) \leq \pi_{R1}'(A_M, B_S)\) (so \(A_M, B_M\) is never a choice, as \((A_M, B_S)\) never is); and since \(\pi_{R1}(\emptyset, B_M) \leq \pi_{R1}(\emptyset, B_S)\), then \(\pi_{R1}'(\emptyset, B_M) < \pi_{R1}'(\emptyset, B_S)\) (and therefore \((\emptyset, B_M)\) is strictly dominated by \((\emptyset, B_S)\)). Therefore, following S’s deviation, R1 either plays \((\emptyset, B_S)\), or \((\emptyset, \emptyset)\).
But, if $R1$ plays $(\emptyset, B_S)$, then using Claim $1.1.2$ we have

$$
\pi_S^* = w_{B1}^S(1 - \mu) + \mu(1 + w_{A2}^M)/2 + (w_{B2}^S - \epsilon)\mu(1 - w_{A2}^M)/2 + T_{B1}^S - 3\epsilon/4 - F_S
$$

$$
= w_{B1}^S + T_{B1}^S + \mu(w_{B2}^S - w_{B1}^S)(1 - w_{A2}^M)/2 - \epsilon[3/4 + \mu(1 - w_{A2}^M)/2] - F_S
$$

which is strictly greater than $\pi_S^* = w_{B1}^S + T_{B1}^S - F_S$, for $\epsilon$ close enough to zero. Contradiction.

Now if $R1$ plays $(\emptyset, \emptyset)$, then $\pi'_S = w_{B2}^S - F_S$. But we know,

$$
(w_{B2}^S - w_{B1}^S) - T_{B1}^S - \max\{0, \pi_R1(\emptyset, B_S) \} > 0 \implies w_{B2}^S > w_{B1}^S + T_{B1}^S
$$

Hence, again $\pi_S^* = w_{B1}^S + T_{B1}^S - F_S < \pi'_S$. Hence, if $w_{B1}^M < w_{B2}^S$, $S$ has a strictly positive deviation.

On the other hand, suppose $w_{B1}^M \geq w_{B2}^S$. If $R1$ chooses $(A_M, B_M)$, then $\pi_{R1}^*(A_M, B_M) \leq \varphi(w_{A1}^M, w_{A2}^M) - T_{A1}^M < 0$, hence this is not an option. Notice then, that $R1$’s choice is between $(\emptyset, B_S)$, $(\emptyset, B_M)$, and $(\emptyset, \emptyset)$. We already showed that if either $(\emptyset, B_S)$ or $(\emptyset, \emptyset)$, $S$ has a strictly profitable deviation. Finally, notice that if $R1$ chooses $(\emptyset, B_M)$, because $w_{B1}^M \geq w_{B2}^S$, this is equivalent from $S$’s perspective as to $(\emptyset, \emptyset)$, given that $q_{B2}^M = 1$, and therefore $\pi_S^* = w_{B2}^S - F_S$.

So again, there is a strictly profitable deviation. Thus, there cannot be an equilibrium satisfying (2.1), $w_{A2}^M \leq w_{A2}^S$, and $B_* = B_S$.

(ii) $B_* = B_M$. First of all, $\pi_S^* = 0$. Now, by Claim 1 $\pi_{R1}^* = \pi_{R1}(A_M, B_M) = \varphi(w_{A1}^M, w_{A2}^M) + (w_{B2}^M - w_{B1}^M) - T_{A1}^M - T_{B1}^M$, that $q_{A2}^M = 0$ (and therefore $T_{A2}^M = 0$), and that $q_{B2}^M = 0$ (so $T_{B2}^M = 0$).

On the other hand, since $R1$ is on-path accepting $(A_M, B_M)$, then it must be true that

$$
\pi_{R1}(A_M, B_M) \geq \max\{0, \pi_{R1}(A_M, B_S), \pi_{R1}(\emptyset, B_S), \pi_{R1}(\emptyset, B_M), \pi_{R1}(A_M, \emptyset) \}
$$

Now, a necessary condition for equilibrium is

$$
\pi_{R1}(A_M, B_M) = \max\{0, \pi_{R1}(\emptyset, B_S), \pi_{R1}(\emptyset, B_M) \}
$$

where

$$
\pi_{R1}(\emptyset, B_S) = (\max\{w_{B2}^M, w_{B1}^S\} - w_{B1}^S)[1 - \mu(1 - w_{A2}^M)/2] - T_{B1}^S
$$

$$
\pi_{R1}(\emptyset, B_M) = (w_{B2}^S - w_{B1}^S)[1 - \mu(1 - w_{A2}^M)/2] - T_{B1}^M
$$

That is, $R1$ must be indifferent between accepting or not accepting $M$’s contract for $A$; otherwise $M$ could unilaterally increase $T_{A1}^M$ without affecting $R1$’s decision, and increase his profits. Hence $T_{A1}^M = \varphi(w_{A1}^M, w_{A2}^M) + (w_{B2}^M - w_{B1}^M) - T_{B1}^M - \max\{0, \pi_{R1}(\emptyset, B_S), \pi_{R1}(\emptyset, B_M) \}$. Notice that

$$
w_{A1}^M q_{A1}^M + w_{A2}^M q_{A2}^M + T_{A1}^M + T_{A2}^M = \rho(w_{A1}^M, w_{A2}^M) + (w_{B2}^M - w_{B1}^M) - T_{B1}^M - \max\{0, \pi_{R1}(\emptyset, B_S), \pi_{R1}(\emptyset, B_M) \}
$$

Hence for (2.1) to hold we need

$$
(w_{B2}^M - w_{B1}^M) - T_{B1}^M - \max\{0, \pi_{R1}(\emptyset, B_S), \pi_{R1}(\emptyset, B_M) \} > 0
$$
or equivalently \( T_{A1}^M > \varphi(w_{A1}^M, w_{A2}^M). \) Finally,

\[
\pi_M^* = w_{A1}^{M*} + w_{A2}^{M*} + T_{A1}^M + T_{A2}^M + w_{B1}' + w_{B2}' - \max\{0, \pi_{R1}(\emptyset, B_S), \pi_{R1}(\emptyset, B_M)\} - F_M
\]

which, given that \( \pi_M^* \geq 1/4, \) and \( \rho(w_{A1}^M, w_{A2}^M) \leq 1/4, \) implies

\[
w_{B2}' \geq F_M + \max\{0, \pi_{R1}(\emptyset, B_S), \pi_{R1}(\emptyset, B_M)\} \quad (2.2)
\]

Now, consider the following deviation by \( S \)

\[
C_S^{S'} = \begin{cases} 
(w_{B1}' = w_{B1}' - \epsilon, T_{B1}' = T_{B1}^S) & \text{if } \pi_{R1}(\emptyset, B_M) < \pi_{R1}(\emptyset, B_S) \\
(w_{B1}' = w_{B1}, T_{B1}' = T_{B1} - 3\epsilon/4) & \text{otherwise}
\end{cases}
\]

\[
C_{B2}' = (w_{B2}' = w_{B2}' - \epsilon, T_{B2}' = 0)
\]

For \( \epsilon > 0 \) small. After such deviation, it is easy to see that \( R2 \) will continue accepting \( M \)'s contract for \( A, \) and that she will accept \( S \)'s new offering regarding \( B. \) Define then \( \pi_{R1}'(x, y) \) as \( R1 \)'s profits after fixed fees, when she accepts contracts \( x \in \{A, \emptyset\} \) and \( y \in \{B'_S, B_M, \emptyset\}, \) and \( R2 \) accepts \( M \)'s contract for \( A, \) and \( S \)'s new contract for \( B. \)

**Subcase 1:** \( \pi_{R1}(\emptyset, B_M) \geq \pi_{R1}(\emptyset, B_S) \)

So suppose \( \pi_{R1}(\emptyset, B_M) \geq \pi_{R1}(\emptyset, B_S), \) so that \( C_S^{S'} = (w_{B1}' = w_{B1}, T_{B1}' = T_{B1} - 3\epsilon/4). \) It is clear that \( S \)'s new contract is strictly better than \( (w_{B1}, T_{B1}), \) and thus the latter will never be accepted. Hence \( y \in \{B'_S, \emptyset\}. \) Now, using Claim 1.1.1 and Lemma A.3, we have

\[
\pi_{R1}'(A, B'_S) = \varphi(w_{A1}^M, w_{A2}^M) + (w_{B2}' - w_{B1}') - T_{B1}' - \epsilon/4
\]

\[
\pi_{R1}(\emptyset, B'_S) = (w_{B2}' - w_{B1}')[1 - \mu(1 - w_{A2}^M)/2] - T_{B1}' - \epsilon/4 + \mu(1 - w_{A2}^M)/2
\]

That is, \( \pi_{R1}'(A, B'_S) = \pi_{R1}(A, B_M) - \epsilon, \) and \( \pi_{R1}'(\emptyset, B'_S) = \pi_{R1}(\emptyset, B_M) - \epsilon/4 + \mu(1 - w_{A2}^M)/2. \)

But

\[
\pi_{R1}(A, B_M) = \max\{0, \pi_{R1}(\emptyset, B_S), \pi_{R1}(\emptyset, B_M)\} = \max\{0, \pi_{R1}(\emptyset, B_M)\}
\]

Hence either \( \pi_{R1}'(A, B'_S) < 0 \) (when \( \pi_{R1}(\emptyset, B_M) \leq 0), \) or \( \pi_{R1}'(A, B'_S) < \pi_{R1}'(\emptyset, B'_S) \) (when \( \pi_{R1}(\emptyset, B_M) > 0). \) That is, \( R1 \) either strictly prefers accepting only \( B'_S, \) or not accepting anything at all, than to accept \( B'_S \) in combination with \( A. \) Hence \( R1 \)'s choice is between \( (\emptyset, B'_S), \) or \( (\emptyset, \emptyset). \)

Now, suppose \( \pi_{R1}(\emptyset, B_M) \leq 0, \) then \( \pi_{R1}'(\emptyset, B'_S) = \pi_{R1}(\emptyset, B_M) - \epsilon/4 + \mu(1 - w_{A2}^M)/2 < 0. \) Thus \( R1 \) plays \( (\emptyset, 0). \) But if so, then \( \pi_S = w_{B2}' - \epsilon - F_S \geq F_M - F_S - \epsilon > 0, \) where we are using \( \Box \) to conclude \( w_{B2}' \geq F_M. \) Hence \( S \) has a strictly profitable deviation. On the other hand, if \( \pi_{R1}(\emptyset, B_M) > 0, \) then \( R1 \) plays \( (\emptyset, B'_S). \) Using Claim 1.1.2 we have

\[
\pi_S = w_{B1}'[1 - \mu(1 + w_{A2}^M)/2] + (w_{B2}' - \epsilon)[1 - w_{A2}^M]/2 + T_{B1}' - 3\epsilon/4 - F_S
\]

\[
= w_{B1}' + T_{B1}' + \mu(w_{B2}' - w_{B1})[1 - w_{A2}^M]/2 - \epsilon[3/4 + \mu(1 - w_{A2}^M)/2] - F_S
\]
But then using (2.2) plus \( \pi_{R1}(\emptyset, B_M) > 0 \) (and \( \pi_{R1}(\emptyset, B_M) \geq \pi_{R1}(\emptyset, B_S) \)), we get

\[
F_M \leq w_{B1}^M + T_{B1}^M + \mu(w_{B2}^M - w_{B1}^M)(1 - w_{A2}^M)/2
\]

Hence \( \pi_S' \geq F_M - F_S - \epsilon[3/4 + \mu(1 - w_{A2}^M)/2] > 0 \), for an \( \epsilon \) sufficiently small. Again a contradiction.

**Subcase 2:** \( \pi_{R1}(\emptyset, B_M) < \pi_{R1}(\emptyset, B_S) \)

So suppose \( \pi_{R1}(\emptyset, B_M) < \pi_{R1}(\emptyset, B_S) \), so that \( C_{B1}^S = (w_{B1}^S = w_{B1}^S - \epsilon, T_{B1}^S = T_{B1}^S) \). Now, using Claim 1.1.1 and Lemma A.3, we have

\[
\pi_{R1}'(A_M, B_M) = \varphi(w_{A1}^M, w_{A2}^M) + (w_{B2}^M - w_{B1}^M) - T_{A1}^M - T_{B1}^M - \epsilon
\]

\[
\pi_{R1}'(\emptyset, B_M) = (w_{B2}^M - w_{B1}^M)[1 - \mu(1 - w_{A2}^M)/2] - T_{B1}^M - \epsilon[1 - \mu(1 - w_{A2}^M)/2]
\]

And

\[
\pi_{R1}'(\emptyset, B_S') = \max\{w_{B1}^S - \epsilon, w_{B2}^M - \epsilon\} - w_{B1}^S + \epsilon[1 - \mu(1 - w_{A2}^M)/2] - T_{B1}^S
\]

Hence \( \pi_{R1}'(A_M, B_M) = \pi_{R1}(A_M, B_M) - \epsilon, \pi_{R1}'(\emptyset, B_M) = \pi_{R1}(\emptyset, B_M) - \epsilon[1 - \mu(1 - w_{A2}^M)/2] \), and \( \pi_{R1}'(\emptyset, B_S') = \pi_{R1}(\emptyset, B_S) \). Clearly, \( \pi_{R1}'(\emptyset, B_M) < \pi_{R1}'(\emptyset, B_S') \). Moreover, since \( \pi_{R1}(A_M, B_M) = \max\{0, \pi_{R1}(\emptyset, B_S)\} \), then either \( \pi_{R1}(A_M, B_M) < 0 \) (when \( \pi_{R1}(\emptyset, B_S) \leq 0 \)), or \( \pi_{R1}'(A_M, B_M) < \pi_{R1}'(\emptyset, B_S') \) (when \( \pi_{R1}(\emptyset, B_S) > 0 \)). That is, R1 either strictly prefers accepting only B’S, or not accepting anything at all, than to accept B’S in combination with A_M. Hence R1’s choice must be between (\( \emptyset, B_S \)), and (\( \emptyset, \emptyset \)).

Now, if \( \pi_{R1}(\emptyset, B_S) \leq 0 \), then R1 will choose (\( \emptyset, \emptyset \)). But if so, then \( \pi_S' = w_{B2}^M - \epsilon - F_S \geq F_M - F_S - \epsilon > 0 \), for \( \epsilon \) small enough, and where we are using (2.2) to conclude \( w_{B2}^M \geq F_M \).

Hence S has a strictly profitable deviation. On the other hand, if \( \pi_{R1}(\emptyset, B_S) > 0 \), then R1 will choose (\( \emptyset, B_S \)). Notice that \( \pi_{R1}(\emptyset, B_S) > 0 \) if and only if \( w_{B1}^S < w_{B2}^M \). Using Claim 1.1.2, we then have

\[
\pi_S' = (w_{B1}^S - \epsilon)[(1 - \mu) + (1 + w_{A2}^M)/2] + (w_{B2}^M - \epsilon)\mu(1 - w_{A2}^M)/2 + T_{B1}^S - F_S
\]

\[
= w_{B1}^S + T_{B1}^S + \mu(w_{B2}^M - w_{B1}^S)(1 - w_{A2}^M)/2 - \epsilon - F_S
\]

But then using (2.2) plus \( \pi_{R1}(\emptyset, B_S) > 0 \) (and \( \pi_{R1}(\emptyset, B_M) < \pi_{R1}(\emptyset, B_S) \)), we get

\[
F_M \leq w_{B1}^S + T_{B1}^S + \mu(w_{B2}^M - w_{B1}^S)(1 - w_{A2}^M)/2
\]

Hence \( \pi_S' \geq F_M - F_S - \epsilon > 0 \), for an \( \epsilon \) sufficiently small. Again a contradiction.

We conclude there that when \( w_{A1}^M \leq w_{A2}^M \), then irrespective on whether \( B_\ast = B_M \) or \( B_\ast = B_S \), S always has a strictly positive deviation. Hence an equilibrium with \( w_{A1}^M \leq w_{A2}^M \), and where (2.1) holds, cannot exists.

**Case (b):** \( w_{A1}^M > w_{A2}^M \) and \( w_{A1}^M + w_{B1}^* \leq w_{A2}^M + w_{B2}^* \)
This case follows a similar logic than (a) using the exact same deviations for $S$ (which, as we show above, depend on whether $B_* = B_S$ or $B_* = B_M$), and is therefore omitted.

Hence, there cannot exists an equilibrium in which

$$w_{A1}^{M^*} + w_{A2}^{M^*} + T_{A1}^M + T_{A2}^M > 1/4$$

### 2.1.2 Proof of Lemma D.4

Suppose $\pi^*_S < F_M - F_S$. Let $q^*_i = (q_{A1}^{M^*}, 0, q_{B1}^{S^*})$ be the equilibrium quantities sold by retailer $R_i$ on path, that is, after retailers accept $(C_{A1}^M, C_{A2}^M, C_{B1}^S, C_{B2}^S)$, and let $q'_i = (q_{A1}^{M'}, q_{B1}^{S'}, 0)$ be the equilibrium quantities when retailers accept $(C_{A1}^M, C_{A2}^M, C_{B1}^S, C_{B2}^S)$ instead.

Notice that $S$ always has the option to simultaneously undercut $C_{B1}^M$ and $C_{B2}^M$, to obtain profits $\pi'_S = w_{B1}^{M'} q_{B1}^{M'} + w_{B2}^{M'} q_{B2}^{M'} + T_{B1}^M + T_{B2}^M - F_S - \epsilon$. Since $\pi'_S \leq \pi^*_S < F_M - F_S$, then

$$w_{B1}^{M'} q_{B1}^{M'} + w_{B2}^{M'} q_{B2}^{M'} + T_{B1}^M + T_{B2}^M < F_M \tag{2.3}$$

On the other hand, since $\pi^*_M = 1/4$ and $M$ is only selling $A$ on-path, then $\pi^*_M = w_{A1}^{M^*} + w_{A2}^{M^*} + T_{A1}^M + T_{A2}^M = 1/4$. But $1/4$ is the maximum industry profit in market $A$, hence

$$w_{A1}^{M} q_{A1}^{M'} + w_{A2}^{M} q_{A2}^{M'} + T_{A1}^M I\{q_{A1}^{M'} > 0\} + T_{A2}^M I\{q_{A2}^{M'} > 0\} \leq 1/4 \tag{2.4}$$

But then, in the off-path event that both manufacturers chose to accept $M$’s offer for $B$ instead of $S$’s offer, $M$’s profits would be

$$\pi'_M = w_{A1}^{M'} q_{A1}^{M'} + w_{A2}^{M'} q_{A2}^{M'} + w_{B1}^{M'} q_{B1}^{M'} + w_{B2}^{M'} q_{B2}^{M'} + T_{A1}^M I\{q_{A1}^{M'} > 0\} + T_{A2}^M I\{q_{A2}^{M'} > 0\} + T_{B1}^M + T_{B2}^M - F_M$$

$$< w_{A1}^{M} q_{A1}^{M'} + w_{A2}^{M} q_{A2}^{M'} + T_{A1}^M I\{q_{A1}^{M'} > 0\} + T_{A2}^M I\{q_{A2}^{M'} > 0\} \leq 1/4$$

Thus, $\pi'_M < 1/4$, which implies that $M$ must be offering below-cost contracts that he expect not to be accepted on-path.

### 2.1.3 Proof of Lemma D.5

We prove this lemma using the following four claims.

**Claim 2.1.1.** $p_{B1}^S \neq p_{B2}^S$.

**Proof.** From the claims of the retail-pricing subgame, $p_{B1}^S \neq p_{B2}^S$ only when $w_{A1}^{M} < w_{A1}^{M'}$, $w_{B1}^{S} \leq w_{B1}^{S'}$, and $w_{A1}^{M} + w_{B1}^{S} < w_{A1}^{M'} + w_{B1}^{S'}$ (Claim 3). However, it is easy to verify using that claim that, if manufacturers are restricted to separate pricing contracts, then $R1$ would never accept $M$’s contract for $A$. Indeed, carrying $A$ does not increase $R1$’s profit on-path, as $R2$ will be more efficient on both $A$ and on the bundle. Moreover, by carrying $A$, $R1$ increases the competition for the bundle downstream, which erodes here profits for product $B$.\footnote{This only can occur as retailers can accept contracts on a piecewisefashion, meaning that this is not an issue for full-line forcing contracts for instance.}

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Therefore, as \( \pi \to \infty \), to see that after such deviation the continuation play involves both retailers procuring goods where \( T \). Claim 2.1.3. know \( \epsilon > M \). 

**Proof.** We know \( \pi_M^* = 1/4 \), and \( \pi_S^* = F_M - F_S \). Therefore, total equilibrium industry revenues for products \( A \) and \( B \) combined, \( \hat{\Pi}^* \), must be of at least \( 1/4 + F_M \). We claim that if \( p_B^* < F_M \), then total industry revenues are strictly less than \( 1/4 + F_M \). To see this, notice \( \hat{\Pi}^* \) is bounded above by the maximum industry revenues achievable when retailers and manufacturers are horizontally and vertically integrated. Hence, if \( p_B^* < F_M \), we have

\[
\hat{\Pi}^* \leq \max_{p_A, p_B, p_{AB}} (1 - \mu)p_A(1 - p_A) + (1 - \mu)p_B + \mu p_{AB}(1 - p_{AB} + p_B) + \mu p_B(p_{AB} - p_B)
\]

subject to \( p_B \leq F_M - \epsilon \) and \( p_{AB} \leq p_A + p_B \), for \( \epsilon > 0 \). But the solution to the right-hand side entails \( p_A = 1/2, p_B = F_M - \epsilon \), and \( p_{AB} = 1/2 + F_M - \epsilon \). Evaluating,

\[
\hat{\Pi}^* \leq 1/4 + F_M - \epsilon < 1/4 + F_M
\]

Hence \( p_B^* < F_M \) leads to a contradiction. ■

Claim 2.1.3. \( p_{B1}^* = p_{B2}^* \equiv p_B^* \leq F_M \).

**Proof.** Suppose \( p_{B1}^* = p_{B2}^* \equiv p_B^* > F_M \). First, notice \( \min\{w_{B1}^S, w_{B2}^S\} \leq F_M \); otherwise \( M \) could deviate to \( C_{A1}' = C_{A2}' = (w_{A1}' = 1/2, T_{A1}' = 0) \), and \( C_{B1}' = C_{B2}' = (w_{B1}' = \min\{w_{B1}^S, w_{B2}^S\} - \epsilon, T_{B1}' = 0) \), to obtain \( \pi_M' = 1/4 + \min\{w_{B1}^S, w_{B2}^S\} - F_M - \epsilon > 1/4 = \pi_M^* \), for \( \epsilon > 0 \) but small enough. On the other hand, by the Claims of the Retail pricing subgame, we know \( p_B^* = \max\{w_{B1}^S, w_{B2}^S\} \). Hence \( p_B^* > F_M \), implies \( \min\{w_{B1}^S, w_{B2}^S\} \leq F_M < \max\{w_{B1}^S, w_{B2}^S\} \), or \( w_{B1}^S \neq w_{B2}^S \). Without loss of generality then, let \( w_{B1}^S < w_{B2}^S \), so \( w_{B1}^S \leq F_M < w_{B2}^S \).

We now claim that this implies that \( w_{B2}^S - \max\{0, (w_{B2}^S - w_{B1}^S)\{1 - \mu/2\} - T_{B1}^S\} \leq F_M \) is necessary for equilibrium. Notice that if \( w_{B1}^S < w_{B2}^S \), then \( M \) can always offer the contracts

\[
C_{A1}' = (w_{A1}' = \infty, T_{A1}' = \infty) \quad C_{A2}' = (w_{A2}' = 0, T_{A2}' = 1/4) \quad C_{B1}' = (w_{B1}' = w_{B1}^S - \epsilon, T_{B1}' = 0) \quad C_{B2}' = (w_{B2}' = w_{B2}^S - \epsilon, T_{B2}' = 0)
\]

where \( T_{B1}' = (w_{B2}^S - w_{B1}^S)\{1 - \mu/2\} - \max\{0, (w_{B2}^S - w_{B1}^S)\{1 - \mu/2\} - T_{B1}^S\} - \epsilon \). It is then easy to see that after such deviation the continuation play involves both retailers procuring goods \( A \) and \( B \) from \( M \), and that the latter’s profits are

\[
\pi_M' = w_{B1}'\{1 - \mu/2\} + \mu w_{B2}'/2 + T_{A2}' + T_{B1}' - F_M
\]

\[
= 1/4 + w_{B2}^S - \max\{0, (w_{B2}^S - w_{B1}^S)\{1 - \mu/2\} - T_{B1}^S\} - F_M - 2\epsilon
\]

Therefore, as \( \pi_M^* \leq \pi_M^* = 1/4 \), we obtain that a necessary condition for equilibrium is \( w_{B2}^S - \max\{0, (w_{B2}^S - w_{B1}^S)\{1 - \mu/2\} - T_{B1}^S\} \leq F_M \), as stated.
Now, if \((w^S_{B_2} - w^S_{B_1})(1 - \mu/2) - T^S_{B_1} \leq 0\), then
\[
w^S_{B_2} - \max\{0, (w^S_{B_2} - w^S_{B_1})(1 - \mu/2) - T^S_{B_1}\} = w^S_{B_2} \leq F_M
\]
immediately contradicting \(w^S_{B_2} > F_M\). Hence, suppose \((w^S_{B_2} - w^S_{B_1})(1 - \mu/2) - T^S_{B_1} \geq 0\), so
\[
w^S_{B_2} - \max\{0, (w^S_{B_2} - w^S_{B_1})(1 - \mu/2) - T^S_{B_1}\} = w^S_{B_1} + T^S_{B_1} + \mu(w^S_{B_2} - w^S_{B_1})/2
\]
We have three cases to consider depending on the equilibrium offers made by \(M\) regarding \(A\). Either \(w^M_{A_1} \leq w^M_{A_2}\), (b) \(w^M_{A_1} > w^M_{A_2}\) and \(w^M_{A_1} + w^S_{B_1} \leq w^M_{A_2} + w^S_{B_2}\); or (c) \(w^M_{A_1} > w^M_{A_2}\) and \(w^M_{A_1} + w^S_{B_1} > w^M_{A_2} + w^S_{B_2}\).

Take cases (a) and (b). Then using Claims 1.1.1 and 1.1.2, respectively, we have
\[q^S_{B_2} = 0\] (and therefore \(T^S_{B_2} = 0\), and that \(q^S_{B_1} = 1\). Hence \(\pi^*_s = w^S_{B_1} + T^S_{B_1} - F_S = F_M - F_S\). Thus, \(w^S_{B_1} + T^S_{B_1} = F_M\). But if so, then
\[
w^S_{B_1} + T^S_{B_1} + \mu(w^S_{B_2} - w^S_{B_1})/2 = F_M + \mu(w^S_{B_2} - w^S_{B_1})/2
\]
which is strictly greater than \(F_M\), as \(w^S_{B_1} < w^S_{B_2}\), and \(\mu > 0\), contradicting our necessary condition for equilibrium.

So consider case (c). As we already argued, this case is equivalent to one in which \(R1\) does not carry \(A\). Then, according to Lemma A.3 on Appendix A of the text, we have
\[
\pi^*_R_1 = (w^S_{B_2} - w^S_{B_1})\left[1 - \frac{\mu}{2}(1 - w^M_{A_2})\right] - T^S_{B_1}
\]
\[
\pi^*_R_2 = (1 - w^M_{A_2})^2/4 - T^M_{A_2} - T^S_{B_2}
\]
\[
\pi^*_M = w^M_{A_2}(1 - w^M_{A_2})/2 + T^M_{A_2}
\]
But since \(\pi^*_M = 1/4\), then \(w^M_{A_2} = 0\) and \(T^M_{A_2} = 1/4\) necessarily, if this were to be an equilibrium. And therefore \(T^S_{B_2} = 0\), and that \(T^S_{B_1} \leq (w^S_{B_2} - w^S_{B_1})(1 - \mu/2)\), as \(\pi^*_R_1\) and \(\pi^*_R_2\) must be greater than or equal to zero.

But consider then, the following deviation by \(M\):
\[
C^M_{A_1} = (w^M_{A_1} = 1/2, T^M_{A_1} = \mu(w^S_{B_2} - w^S_{B_1})/(4 - \epsilon)) \quad C^M_{A_2} = (w^M_{A_2} = 1/2, T^M_{A_2} = 0)
\]
\[
C^M_{B_1} = C^M_{B_2} = (w^M_B, T^M_B) \rightarrow (+\infty, +\infty)
\]
It is clear that \(R2\) will then accept \(M\’s offer \(A\), and \(S\’s offer for \(B\). Moreover, using Lemmas A.2 and A.3, we can also verify that \(R1\) will do the same. If she accepts nothing she gets zero, while if she accepts only \(A\), she gets strictly negative profits due to the fixed fee. On the other hand, if she accepts only \(B\), she gets \((w^S_{B_2} - w^S_{B_1})(1 - \mu/4) - T_{B_1} > 0\). While if she accepts both offers she gets \((w^S_{B_2} - w^S_{B_1}) - T^M_{A_1} - T^S_{B_1} = (w^S_{B_2} - w^S_{B_1})(1 - \mu/4) - T_{B_1} + \epsilon\). But then, \(M\’s profits after this deviation are \(\pi^*_M = 1/4 + T^M_{A_1} = 1/4 + \mu(w^S_{B_2} - w^S_{B_1})/4 > 1/4 = \pi^*_M\), a contradiction.

Hence \(p^*_B = p^*_{B_2} \equiv p^*_B > F_M\), implies \(w^S_{B_1} \leq F_M < w^S_{B_2}\), which in turn always leads to a contradiction. Thus, \(p^*_{B_1} = p^*_{B_2} \equiv p^*_B \leq F_M\). ■
Hence, combining all the claims we get $p_{B1}^* = p_{B2}^* = F_M$. The final claim then shows that there is no retail bundling on-path.

**Claim 2.1.4.** $p_{ABi}^* \geq p_{Ai}^* + p_{Bi}^*$, for $i = 1, 2$.

**Proof.** Suppose $p_{ABi}^* < p_{Ai}^* + p_{Bi}^*$. We know $p_{Ai}^* = 1/2$, and that $p_{Bi}^* = F_M$. Furthermore, $\hat{\Pi} \geq 1/4 + F_M$. But,

$$\hat{\Pi}^* \leq \max \frac{(1 - \mu)}{4} + \frac{(1 - \mu)}{2} F_M + \mu p_{AB}(1 - p_{AB} + F_M) + \mu F_M(p_{AB} - F_M)$$

subject to $p_{AB} \leq p_A + p_B - \epsilon$, for $\epsilon > 0$. But the solution to the right-hand side entails $p_{AB} = 1/2 + F_M - \epsilon$. Evaluating,

$$\hat{\Pi}^* \leq 1/4 + F_M - \mu \epsilon^2 < 1/4 + F_M$$

Hence $p_{ABi}^* < p_{Ai}^* + p_{Bi}^*$ leads to a contradiction. ■

**2.2 Proof of Lemma 3**

Given $S$’s offers, $M$’s optimal foreclosure response is either:

1. to approach only one retailer, say $R_i$, with an offer (to be accepted in equilibrium) with wholesale prices $(w_{M,Ai}, w_{M,Bi})$ and $w_{M,Bi} \leq \omega$ from Lemma A.3, or

2. to approach both retailers with offers (to be accepted in equilibrium) with wholesale prices $(w_{M,Ai}, w_{M,Bi})$ and $(w_{M,Aj}, w_{M,Bj})$, which w.l.o.g. we assume to satisfy $w_{M,Ai} + w_{M,Bi} \leq w_{M,Aj} + w_{M,Bj}$ and either

   - (a) $w_{M,Ai} \leq w_{M,Aj}$, $w_{M,Bi} \leq w_{M,Bj}$, or
   - (b) $w_{M,Ai} \geq w_{M,Aj}$, $w_{M,Bi} \leq w_{M,Bj}$, or
   - (c) $w_{M,Ai} \leq w_{M,Aj}$, $w_{M,Bi} \geq w_{M,Bj}$.

From Claims 1-4 above, we know that for each combination of wholesale prices, a pure-strategy equilibrium exists and is unique. $M$’s profit from following any of these foreclosure strategies can then be written as

$$\pi_M^f = \Pi(w_{M,Ai}, w_{M,Aj}, w_{M,Bi}, w_{M,Bj}) - \hat{\Pi}_{Ri}(\infty, w_{M,Aj}, \omega, w_{M,Bj}) - \hat{\Pi}_{Rj}(w_{M,Ai}, \infty, w_{M,Bi}, \omega) - F_M$$

where $\Pi(w_{M,Ai}, w_{M,Aj}, w_{M,Bi}, w_{M,Bj})$ is total (equilibrium) industry profit for $(w_{M,Ai}, w_{M,Aj}, w_{M,Bi}, w_{M,Bj})$ and $\hat{\Pi}_{Ri}()$ is $R_i$’s outside option, i.e., the payoff she receives if she take $S$’s offer while her rival remains exclusively with $M$.

An expression for $\Pi()$ can be written as a function of the retail equilibrium prices that result from $(w_{M,Ai}, w_{M,Aj}, w_{M,Bi}, w_{M,Bj})$ as follows

$$\Pi() = (1 - \mu)p_{A}^*(1 - p_{A}^*) + (1 - \mu)p_{B}^* + \mu p_{AB}^*(1 - p_{AB}^* + p_{B}^*) + \mu p_{B}^*(p_{AB}^* - p_{B}^*)$$
where $p_A^*, p_A^*$, and $p_{AB}^*$ are the (effective) retail equilibrium prices at which final consumers are effectively purchasing the different items.

Similarly, an expression for $\bar{\pi}_{Ri}(\cdot)$ can be obtained from applying Lemma A.3 as follows

\[
\bar{\pi}_{Ri}(\infty, w_{Aj}, \omega, w_{Bj}) = \begin{cases} 
0 & \text{if } w_{Bj} \leq \omega \\
(w_{Bj} - \omega) \left[1 - \frac{\mu}{2}(1 - w_{Aj})\right] & \text{if } \frac{w_{Bi}}{\omega} \leq \omega \leq w_{Bj} \\
\left[(w_{Bj} - \omega) - \frac{2}{3}(w_{Bi} - \omega)\right] \left[1 - \frac{\mu}{2}(1 - w_{Aj}) + \frac{\mu}{3}(w_{Bi} - \omega)\right] & \text{if } \omega \leq \frac{w_{Bi}}{\omega} 
\end{cases}
\]

where $w_{Bi} = \max\{0, w_{Bj} - 1/\mu + (1 - w_{AJ})/2\}$. The following claim will be used extensively in the rest of the proof.

**Claim 2.2.1.** $\bar{\pi}_{Ri}(\infty, w_{Aj}, \omega, w_{Bj})$ is weakly increasing in $w_{Aj}$ and in $w_{Bj}$ (and strictly increasing if $w_{Bj} > \omega$). Furthermore, $\partial \bar{\pi}_{Ri}(\cdot)/\partial w_{Bj} \leq 1$ (almost) everywhere, with strict inequality if simultaneously $\mu \in (0,1]$ and $w_{Aj} < 1$.

**Proof.** Start by noticing that $\bar{\pi}_{Ri}(\cdot)$ is a continuous function. Moreover,

\[
\frac{\partial \bar{\pi}_{Ri}(\cdot)}{\partial w_{Aj}} = \begin{cases} 
0 & \text{if } w_{Bj} \leq \omega \\
\frac{\mu}{2}(w_{Bj} - \omega) & \text{if } \frac{w_{Bi}}{\omega} \leq \omega \leq w_{Bj} \\
\frac{2}{3} \left[2 - \mu(1 + \omega - w_{Aj} - w_{Bj})\right] & \text{if } \omega \leq \frac{w_{Bi}}{\omega}
\end{cases}
\]

indicating that $\bar{\pi}_{Ri}(\cdot)$ is indeed weakly increasing in $w_{Aj}$ (and strictly when $w_{Bj} > \omega$). Similarly, we have that

\[
\frac{\partial \bar{\pi}_{Ri}(\cdot)}{\partial w_{Bj}} = \begin{cases} 
0 & \text{if } w_{Bj} \leq \omega \\
1 - \frac{\mu}{2}(1 - w_{Aj}) & \text{if } \frac{w_{Bi}}{\omega} \leq \omega \leq w_{Bj} \\
\frac{2}{3} \left[2 - \mu(1 + \omega - w_{Aj} - w_{Bj})\right] & \text{if } \omega \leq \frac{w_{Bi}}{\omega}
\end{cases}
\]

and

\[
\lim_{{w_{Bj} \to \omega^+}} \frac{\bar{\pi}_{Ri}(\cdot, w_{Aj}, w_{Bj}) - \bar{\pi}_{Ri}(\cdot, w_{Aj}, \omega)}{w_{Bj} - \omega} = 1 - \frac{\mu}{2}(1 - w_{Aj})
\]

\[
\lim_{{w_{Bj} \to \omega^-}} \frac{\bar{\pi}_{Ri}(\cdot, w_{Aj}, w_{Bj}) - \bar{\pi}_{Ri}(\cdot, w_{Aj}, \omega)}{w_{Bj} - \omega} = 0
\]

\[
\lim_{{w_{Bi}(w_{Aj}, w_{Bj}) \to \omega^+}} \frac{\bar{\pi}_{Ri}(\cdot, w_{Aj}, w_{Bj}) - \mu \left[\frac{1}{\mu^2} - \frac{1}{2}(1 - w_{Aj})\right]^2}{w_{Bi}(w_{Aj}, w_{Bj}) - \omega} = 1 - \frac{\mu}{2}(1 - w_{Aj})
\]

\[
\lim_{{w_{Bi}(w_{Aj}, w_{Bj}) \to \omega^-}} \frac{\bar{\pi}_{Ri}(\cdot, w_{Aj}, w_{Bj}) - \mu \left[\frac{1}{\mu} - \frac{1}{2}(1 - w_{Aj})\right]^2}{w_{Bi}(w_{Aj}, w_{Bj}) - \omega} = \frac{2}{3} - \frac{\mu}{3}(1 - w_{Aj})
\]

indicating as well that $\bar{\pi}_{Ri}(\cdot)$ is weakly increasing in $w_{Bj}$ (and strictly when $w_{Bj} > \omega$). Finally, notice that for all $\mu \in (0,1]$ and $w_{Aj} < 1$ the left and right derivatives of $\bar{\pi}_{Ri}(\cdot)$ with respect to to $w_{Bj}$ are strictly less than 1.

Consider first, foreclosure strategy 1, that is, to approach only $Ri$ with the full-line forcing offer $(w_{Ai}^M, w_{Bi}^M \leq \omega)$. According to Lemma A.3, the effective retail equilibrium prices are
$p_A^* = (1 + w_{A1}^M)/2$, $p_B^* = \omega$, and $p_{AB}^* = \omega + (1 + w_{A1}^M)/2$. Since $\bar{\pi}_{Ri}(\infty, \infty, \omega, \omega) = 0$, $M$ can include a fixed fee in his offer arbitrarily close to $\bar{\pi}_{Ri}^*(w_{A1}^M, \infty, w_{B1}^M, \omega)$, so his profit from following this “first” foreclosure strategy is

$$\pi_{f1}^M = \bar{\pi}_{f1}^M - F_M = \frac{1}{4} + \omega - F_M$$  \hspace{1cm} (2.5)

Consider now, foreclosure strategy 2 in its different cases. Take case 2(a). Provided that $M$’s offers are such that both retailers accept them, the effective retail equilibrium prices are $p_A^* = \min\{(1 + w_{A1}^M)/2, w_{A2}^M\}$, $p_B^* = w_{B1}^M$, and $p^*_{AB} = p_A^* + p_B^*$. $M$’s problem is to choose the set of wholesale prices that maximize

$$\bar{\pi}_{M} = \begin{cases} 
(1 - w_{A1}^M)^2/4 + w_{B1}^M - \bar{\pi}_{R1}(\infty, w_{A1}^M, \omega, w_{B1}^M) - \\
\bar{\pi}_{R1}(w_{A1}^M, \omega, w_{B1}^M) & \text{if } (1 + w_{A1}^M)/2 \leq w_{A1}^M \\
\frac{1}{2}M(1 - w_{A1}^M) + w_{B1}^M - \bar{\pi}_{R1}(\infty, w_{A1}^M, \omega, w_{B1}^M) - \\
\bar{\pi}_{R1}(w_{A1}^M, \omega, w_{B1}^M) & \text{otherwise}
\end{cases}$$

Since $w_{B1}^M$ is only in $\bar{\pi}_{R1}(w_{A1}^M, \omega, w_{B1}^M)$, it is optimal to set $w_{B1}^M \leq \omega$, so that $\bar{\pi}_{R1}(w_{A1}^M, \omega, w_{B1}^M)$ is equal to 0. Furthermore, since $\partial\bar{\pi}_{R1}(\cdot)/\partial w_{B1}^M \leq 1$, it is optimal to set $w_{B1}^M = b$, which yields

$$\bar{\pi}_{R1}(\infty, w_{A1}^M, \omega, b) = \begin{cases} 
(b - \omega)\left[1 - \frac{1}{2}(1 - w_{A1}^M)\right] & \text{if } \omega \geq b - \frac{1}{\mu}[1 - \frac{1}{2}(1 - w_{A1}^M)] \\
\frac{\mu}{2} \left[2 + \mu(b - \omega) - \mu(1 - w_{A1}^M)\right]^2 & \text{otherwise}
\end{cases}$$  \hspace{1cm} (2.6)

Suppose $M$ chooses wholesale prices so that $(1 + w_{A1}^M)/2 \leq w_{A1}^M$, then

$$\max \bar{\pi}_{M} = (1 - w_{A1}^M)^2/4 + b - \bar{\pi}_{R1}(\infty, w_{A1}^M, \omega, w_{B1}^M)$$

which implies that it is optimal to set $w_{A1}^M$ as low as possible, that is, $w_{A1}^M = (1 + w_{A1}^M)/2$. Given that it is optimal to set $w_{A1}^M \leq (1 + w_{A1}^M)/2$, the maximum that $M$ can obtain following strategy 2(a) is

$$\bar{\pi}_{f2a}^M = \max_{w_{A1}^M} \bar{\pi}_{M} = w_{A1}^M(1 - w_{A1}^M) + b - \bar{\pi}_{R1}(\infty, w_{A1}^M, \omega, b)$$  \hspace{1cm} (2.7)

where $\bar{\pi}_{R1}(\infty, w_{A1}^M, \omega, b)$ is given by (2.6).

Take now case 2(b), that is, $w_{A1}^M \geq w_{A1}^M$ and $w_{B1}^M \leq w_{B1}^M$. The effective retail equilibrium prices are now $p_A^* = \min\{(1 + w_{A1}^M)/2, w_{A2}^M\}$, $p_B^* = w_{B1}^M$, and $p^*_{AB} = w_{A1}^M + w_{B1}^M$. Again, $M$’s problem is to choose the set of wholesale prices that maximize

$$\bar{\pi}_{M} = \begin{cases} 
(1 - \mu)(1 - w_{A1}^M)^2/4 + w_{B1}^M + \mu w_{A1}^M(1 - w_{A1}^M) - \\
\bar{\pi}_{R1}(\infty, w_{A1}^M, \omega, w_{B1}^M) - \bar{\pi}_{R1}(w_{A1}^M, \infty, w_{B1}^M, \omega) & \text{if } (1 + w_{A1}^M)/2 \leq w_{A1}^M \\
(1 - \mu)w_{A1}^M(1 - w_{A1}^M) + w_{B1}^M + \mu w_{A1}^M(1 - w_{A1}^M) - \\
\bar{\pi}_{R1}(\infty, w_{A1}^M, \omega, w_{B1}^M) - \bar{\pi}_{R1}(w_{A1}^M, \infty, w_{B1}^M, \omega) & \text{if } (1 + w_{A1}^M)/2 > w_{A1}^M
\end{cases}$$

Since $w_{B1}^M$ is only in $\bar{\pi}_{R1}(w_{A1}^M, \infty, w_{B1}^M, \omega)$, it is optimal to set $w_{B1}^M \leq \omega$, so that $\bar{\pi}_{R1}(w_{A1}^M, \omega, w_{B1}^M, \omega)$ is equal to 0. Furthermore, since $\partial\bar{\pi}_{R1}(\cdot)/\partial w_{B1}^M \leq 1$, it is optimal to set $w_{B1}^M = b$. Then, the
maximum that $M$ can obtain following strategy 2(b) is

$$
\pi_{M}^{2b} = \max_{w_{M}^{A}, w_{M}^{B}} \pi_{M} = \begin{cases} 
(1 - \mu)(1 - w_{M}^{A})^{2}/4 + b^{+} & \text{if } (1 + w_{M}^{A})/2 \leq w_{Ai} \\
\mu w_{M}^{A}(1 - w_{M}^{A}) - \pi_{Ri}(\infty, w_{M}^{A}, \omega, w_{M}^{B}) & \\
(1 - \mu)w_{M}^{A}(1 - w_{M}^{A}) + b^{+} & \text{if } (1 + w_{M}^{A})/2 > w_{Ai} \\
\mu w_{M}^{A}(1 - w_{M}^{A}) - \pi_{Ri}(\infty, w_{M}^{A}, \omega, w_{M}^{B}) & 
\end{cases}
$$

(2.8)

subject to $w_{M}^{A} \leq w_{M}^{A}$ and where $\pi_{Ri}(\infty, w_{M}^{A}, \omega, w_{M}^{B})$ is given by (2.6).

Finally, let us take case 2(c), that is, when $w_{Ai} \leq w_{M}^{A}$ and $w_{Bi} \geq w_{M}^{B}$. In this final case we have three possibilities to consider depending on the relative values of the wholesale prices.

- Possibility (i): If $w_{M}^{B} \leq w_{M}^{B} \leq w_{M}^{B} + \max\{0, w_{M}^{A} - (1 + w_{M}^{A})/2\}$, then the effective retail equilibrium prices are $p_{A}^{*} = (1 + w_{M}^{A})/2$, $p_{B}^{*} = w_{M}^{B}$, and $p_{AB}^{*} = p_{A}^{*} + p_{B}^{*}$, which yields total industry profit equal to $\Pi = (1 - w_{M}^{A})^{2}/4 + w_{M}^{B}$.

- Possibility (ii): If $w_{M}^{B} + \max\{0, w_{M}^{A} - (1 + w_{M}^{A})/2\} \leq w_{M}^{B} \leq \min\{b_{A}, w_{M}^{A} + \frac{1}{2\mu} - \frac{1}{2}(1 - w_{M}^{A})\}$

$$
\Pi = \begin{cases} 
(1 - \mu)(1 - w_{M}^{A})^{2}/4 + w_{M}^{B} + \\
\mu(w_{M}^{B} + w_{M}^{B} - w_{M}^{B})(1 - w_{M}^{A} - w_{M}^{B} + w_{M}^{B}) & \text{if } (1 + w_{M}^{A})/2 \leq w_{Ai} \\
(1 - \mu)w_{M}^{A}(1 - w_{M}^{A}) + w_{M}^{B} + \\
\mu(w_{M}^{A} + w_{M}^{B} - w_{M}^{B})(1 - w_{M}^{A} - w_{M}^{B} + w_{M}^{B}) & \text{otherwise}
\end{cases}
$$

(2.10)

- Possibility (iii): If $\min\{b_{A}, w_{M}^{A} + \frac{1}{2\mu} - \frac{1}{2}(1 - w_{M}^{A})\} \leq w_{M}^{B} \leq b$ and $p_{AB}^{*} = w_{A} + w_{B}$, which yields total industry profits equal to

$$
\Pi = \begin{cases} 
(1 - \mu)(1 - w_{M}^{A})^{2}/4 + (1 - \mu)p_{B}^{*} + \\
\mu p_{AB}^{*}(1 - p_{AB}^{*} + p_{B}^{*}) + \mu p_{B}^{*}(p_{AB}^{*} - p_{B}^{*}) & \text{if } (1 + w_{M}^{A})/2 \leq w_{Ai} \\
(1 - \mu)w_{M}^{A}(1 - w_{M}^{A}) + (1 - \mu)p_{B}^{*} + \\
\mu p_{AB}^{*}(1 - p_{AB}^{*} + p_{B}^{*}) + \mu p_{B}^{*}(p_{AB}^{*} - p_{B}^{*}) & \text{if } (1 + w_{M}^{A})/2 > w_{Ai}
\end{cases}
$$

(2.12)

To find $M$’s optimal solution under possibility (i), notice that $\pi_{M} = \Pi - \bar{\pi}_{Ri}(\infty, w_{M}^{A}, \omega, w_{M}^{B}) - \bar{\pi}_{Rj}(w_{M}^{A}, \infty, w_{M}^{B}, \omega)$ is decreasing in $w_{M}^{A}$ and the constraint $w_{M}^{A} \leq w_{M}^{B} + w_{M}^{A} - (1 + w_{M}^{A})/2$ is relaxed when decreasing $w_{M}^{A}$, so it is optimal to set $w_{M}^{A} = 0$, which implies $w_{M}^{A} \geq 1/2$. Moreover, since $\pi_{M}$ is increasing in $w_{M}^{B}$, because $\partial \bar{\pi}_{Rj}(\cdot)/\partial w_{M}^{B} < 1$, it is also optimal to set
\[ w_{BI}^M = \min\{b, w_{Bj}^M + w_{Aj}^M - 1/2\} \]. And to complete the solution, we want to argue that it is also optimal to set \( w_{Bj}^M = \omega \) and \( w_{Aj}^M = 1/2 + (b - \omega) \). For this latter, notice that \( w_{Aj}^M \) and \( w_{Bj}^M \) only enter into \( \pi_M \) through \( \bar{\pi}_{RI}^{\infty}(\omega, w_{Aj}^M, \omega, w_{Bj}^M) \) and the constraint \( w_{Bj}^M \leq w_{Bj}^M + w_{Aj}^M - (1 + w_{Aj}^M)/2 \). However, letting \( w_{Bj}^M \leq \omega \) yields \( \bar{\pi}_{RI}(\cdot) = 0 \), so \( M \)'s only concern for choosing \( w_{Aj}^M \) is to relaxed the constraint and let \( b = w_{Bj}^M + w_{Aj}^M - 1/2 \). Thus, \( M \)'s maximum profit from following possibility (i) within strategy 2(c) would be

\[
\pi_{f^{2c}(i)}^M = \frac{1}{4} + b - \bar{\pi}_{Rj}(0, \infty, b, \omega) - F_M
\] (2.13)

where

\[
\bar{\pi}_{Rj}(0, \infty, b, \omega) = \begin{cases} 
(b - \omega)[1 - \frac{1}{2}] & \text{if } \omega \geq b - \frac{1}{\mu}(1 - \frac{\mu}{2}) \\
\mu \frac{1}{2}[2 + \mu(b - \omega - 1)]^2 & \text{if } \omega < b - \frac{1}{\mu}(1 - \frac{\mu}{2})
\end{cases}
\]

Based on what we have so far, we are ready to establish that the best foreclosure strategy involves approaching both retailers and setting \( w_{AI}^M \leq w_{Aj}^M \) and \( w_{AI}^M \geq w_{Bj}^M \); in other words, we can establish that \( \pi_{f^{2c}(i)}^M \geq \max\{\pi_{f^1}^M, \pi_{f^{2a}}^M, \pi_{f^{2b}}^M\} \), where \( \pi_{f^1}^M, \pi_{f^{2a}}^M \) and \( \pi_{f^{2b}}^M \) are given by, respectively, (2.5), (2.7) and (2.8). Indeed, using the fact that \( \bar{\pi}_{Rj}(0, \infty, b, \omega) \leq b - \omega \) we can show that \( \pi_{f^{2c}(i)}^M \geq \pi_{f^1}^M \); using \( w_{Aj}^M(1 - w_{AI}^M) \leq 1/4 \) and \( \bar{\pi}_{RI}(\infty, w_{AI}^M, \omega, b) \geq \bar{\pi}_{Rj}(0, \infty, b, \omega) \) that \( \pi_{f^{2c}(i)}^M \geq \pi_{f^{2a}}^M \); and using \( \bar{\pi}_{RI}(\infty, w_{AI}^M, \omega, b) \geq \bar{\pi}_{Rj}(0, \infty, b, \omega), (1 - \mu)(1 - w_{AI}^M)^2/4 + \mu w_{AI}^M(1 - w_{AI}^M) \leq 1/4 \) and \( (1 - \mu)w_{AI}^M(1 - w_{AI}^M) + \mu w_{AI}^M(1 - w_{AI}^M) \leq 1/4 \) that \( \pi_{f^{2c}(i)}^M \geq \pi_{f^{2b}}^M \).

Therefore, from now on we can concentrate on case (2c) as the optimal foreclosure strategy. Therefore, to conclude the proof of the lemma, it remains to check whether possibilities (ii) and (iii) can improve upon (i). Recall that \( M \)'s problem in (ii) is to maximize \( \bar{\pi}_M = \Pi - \bar{\pi}_{RI}(\infty, w_{Ai}^M, \omega, w_{Bj}^M) - \bar{\pi}_{Rj}(w_{AI}^M, \infty, w_{Bj}^M, \omega) \), where \( \Pi \) is given by (2.10), subject to \( w_{AI}^M \leq w_{Aj}^M \) and (2.9). Consider the following auxiliary problem: everything is as before except that constraint (2.9) is not longer considered. In this auxiliary problem, it is optimal to let \( w_{AI}^M = 0 \), since \( \pi_M \) is strictly decreasing in \( w_{AI}^M \). We will now show that it is also optimal to set \( w_{Bj}^M = \omega \) and \( w_{Aj}^M = 1/2 + (b - \omega) \). First, notice that since \((1 - \mu)/4 \geq (1 - \mu)w_{AI}^M(1 - w_{AI}^M)\) for any \( w_{AI}^M \), it is optimal to let \( w_{AI}^M \geq (1 + w_{AI}^M)/2 = 1/2 \). Second, setting \( w_{Bj}^M = \omega \) implies that \( \bar{\pi}_{RI}(\infty, w_{Aj}^M, \omega, w_{Bj}^M) = 0 \), so \( M \) can now freely choose \( w_{Aj}^M \) and \( w_{Bj}^M \) so as to solve

\[
\max_{w_{Aj}^M, w_{Bj}^M} \frac{1 - \mu}{4} + w_{Bj}^M + \mu(w_{Aj}^M + \omega - w_{Bj}^M)(1 - w_{Aj}^M - \omega + w_{Bj}^M) - \bar{\pi}_{Rj}(0, \infty, w_{Bj}^M, \omega)
\] (2.14)

The FOC for \( w_{Aj}^M \) yields \( w_{Aj}^M = 1/2 + (w_{Bj}^M - \omega) \), which plugged back into (2.12) reduces \( M \)'s problem to

\[
\max_{w_{Bj}^M} \frac{1}{4} + w_{Bj}^M - \bar{\pi}_{Rj}(0, \infty, w_{Bj}^M, \omega)
\]

Given that \( \partial \bar{\pi}_{Rj}(0, \infty, w_{Bj}^M, \omega)/\partial w_{Bj}^M < 1 \), it is clearly optimal to set \( w_{Bj}^M = b \), implying that \( M \)'s payoff under this auxiliary problem, \( 1/4 + b - \bar{\pi}_{Rj}(0, \infty, w_{Bj}^M, \omega) - F_M \), is not different from \( \pi_{f^{2c}(i)}^M \). Since the auxiliary problem is a relaxed version of the true problem, we necessarily have that \( \pi_{f^{2c}(i)}^M \leq \pi_{f^{2c}(i)}^M \).
For the third and last possibility, observe from (2.11) that since $\pi_M$ is strictly decreasing in $w_{bi}^M$, then it is optimal to set

$$w_{bi}^M = w_{bj}^M + w_{bj}^M + \frac{1}{2\mu} - \frac{1}{2}(1 - w_{aj}^M)$$

But if so, this possibility has converged to possibility (ii) since in essence we have $p_A^* = \min\{w_{aj}^M, (1 + w_{aj}^M)/2\}$, $p_B^* = w_{bi}^M$, and $p_{AB}^* = w_{aj} + w_{bj}$.

### 2.3 Completion of proof of Proposition 4

From Proposition 3b and its proof we know that when manufacturers are restricted to separate pricing the offers

$$C_{Aj}^M = C_{Aj}^M = (w_A^M = 1/2, T_A^M = 0), C_{Bj}^M = C_{Bj}^M = (w_B^M = F_M, T_B^M = 0)$$

$$C_{Ai}^S = (w_{bi}^S = F_M - \Delta, T_{bi}^S = \Delta),$$

with $\epsilon \to 0$ and $\Delta \in (0, F_M]$, constitute an equilibrium of the game. Now, to demonstrate that this equilibrium does not survive the introduction of wholesale bundling consider the same (foreclosure) response as in Lemma 3, that is

$$\tilde{C}_{Ai}^M = (\tilde{w}_{Ai}^M = 1/2 + b - F_M, \tilde{T}_{Ai}^M = 0), \tilde{C}_{Bj}^M = (\tilde{w}_{Bj}^M = F_M, \tilde{T}_{Bj}^M = 0)$$

$$\tilde{C}_{Aj}^M = (\tilde{w}_{Aj}^M = 0, \tilde{T}_{Aj}^M = 1/4 - \epsilon), \tilde{C}_{Bj}^S = (\tilde{w}_{Bj}^S = b, \tilde{T}_{Bj}^S = 0)$$

It is easy to see that both retailers will take $M$’s offer. No matter what $R_i$ does, $R_j$ obtains $\epsilon$ as opposed to zero when taking $M$’s offer. Anticipating this, $R_i$ is also ready to take $M$’s offer and obtain $\pi_{Ri}^f(M)$ as opposed to max$\{0, \pi_{Ri}^0(F_M - \Delta) - \Delta\}$. Furthermore, since $\pi_{Ri}^0(F_M)$ is strictly greater than max$\{0, \pi_{Ri}^0(F_M - \Delta) - \Delta\}$ for all $\Delta > 0$, $M$ can increase his foreclosure payoff above $\pi_{M}^f$ (see Lemma 3) by increasing $\hat{w}_{Bi}^M$ (and reduce $\hat{w}_{Ai}^M$ accordingly as to keep $\hat{w}_{Ai}^M + \hat{w}_{Bi}^M = 1/2 + b$) to the point where $\pi_{Ri}^0(\hat{w}_{Bi}^M) = \max\{0, \pi_{Ri}^0(F_M - \Delta) - \Delta\} + \epsilon$.

### 2.4 Single-Product Exclusivity Discounts

In the text we argue that in principle we could think of stand-alone contracts as the menu

$$\{(w_{ki}^{he}, T_{ki}^{he}), (w_{ki}^{he}, T_{ki}^{he})\},$$

where $(w_{ki}^{he}, T_{ki}^{he})$ governs the terms of trade for product $k$ if $R_i$ commits to obtaining $k$ exclusively from $h$, and $(w_{ki}^{he}, T_{ki}^{he})$ determines the terms of trade otherwise. The difference between $(w_{ki}^{he}, T_{ki}^{he})$ and $(w_{ki}^{he}, T_{ki}^{he})$ is a single-product loyalty discount. We will now go through different results of the paper to explain that allowing for these single-product discounts is irrelevant in our setting, so there is no loss of generality in collapsing the menu

$$\{(w_{ki}^{he}, T_{ki}^{he}), (w_{ki}^{he}, T_{ki}^{he})\}$$

into a single two-part tariff $(w_{ki}^{he}, T_{ki}^{he})$ as done in the text.
Propositions 1, 4, and 5: Since in both cases $M$’s full-line forcing offers leave retailers no option but to sign with either $M$ or $S$, it becomes irrelevant for $S$ whether to offer $\{(w^S_{Bi}, T^S_{Bi}), (w^S_{Bi}, T^S_{Bi})\}$ or $\{(w^S_{Bi}, T^S_{Bi})\}$.

Propositions 2a and 2b: In the proof of both propositions (see also Appendix H in the text) we allow $M$ to operate under general schedules. And given that $M$ cannot foreclose when $S$ is restricted to two part-tariffs, enlarging $S$’s action space only reinforces the latter result.

Propositions 3a and 3b: Under the assumption of perfect substitutability of input $B$ a retailer will never purchase $B$ from more than one manufacturer and pay two fixed fees, hence $(w^c_{ki}, T^c_{ki}) = (w^c_{ki}, T^c_{ki})$.

2.5 Relaxing One-Stop Shopping

Let $p_{Ai}, p_{Bi},$ and $p_{ABi} \leq p_{Ai} + p_{Bi}$ be the prices charged by retailer $i = 1, 2$, where $\min\{p_{A1}, p_{A2}\} < 1$ and $\min\{p_{B1}, p_{B2}\} \leq b$. In addition, let $p_k = \min\{p_{k1}, p_{k2}\}$, where $k = A, B, AB$, $R_k$ be the retailer offering the lowest price for choice $k$ (or in case of a tie, the retailer with the lowest wholesale price for that choice), and $\tilde{p}_{AB} = \min\{p_A + p_B, p_{AB}\}$. Note that $\tilde{p}_{AB}$ is how much multi-stop shoppers spend when purchasing both goods. Those consumers who value only $A$ will buy a total of $(1 - \mu)(1 - p_A)$ units from retailer $R_A$, while those who value only $B$ will buy a total of $(1 - \mu)(1 - p_B)$ units from $R_B$. The decision of consumers who value both goods is more involved now because we need to distinguish between consumers that face no shopping costs from those that do. On the one hand, the fraction $\mu z$ of consumers that value both goods but must one-stop shop decide between whether to buy both products from $R_{AB}$ or just $B$ from $R_B$. Since the indifferent consumer in this group will have valuation $p_{AB} - p_B$ for good $A$, consumers with $v_A \geq p_{AB} - p_B$ will buy a total of $\mu z(1 - p_{AB} + p_B)$ units of each good from $R_{AB}$, while the remaining consumers will buy a total of $\mu z(p_{AB} - p_B)$ units of $B$ from $R_B$. On the other hand, the fraction $\mu(1 - z)$ of consumers that value both goods and multi-stop shop decides between whether to buy both products from either $R_{AB}$ or $R_A$ and $R_B$, whatever is cheaper, or just $B$ from $R_B$. Since the indifferent consumer in this group will have valuation $\tilde{p}_{AB} - p_B$ for good $A$, consumers with $v_A \geq \tilde{p}_{AB} - p_B$ will buy a total of $\mu (1 - z)(1 - \tilde{p}_{AB} + p_B)$ units of each good from either $R_{AB}$ or $R_A$ and $R_B$, while the remaining consumers of this group will buy a total of $\mu (1 - z)(\tilde{p}_{AB} - p_B)$ units of $B$ from $R_B$. Consider first the case in which $\tilde{p}_{AB} = p_{AB}$. In this case, the two groups of consumers that value both goods behave no different. Regardless of their shopping costs, those with $v_A \geq p_{AB} - p_B$ will buy both goods from $R_{AB}$ and those with $v_A < p_{AB} - p_B$ will buy $B$ from
So, in this case the total demand for \( A \), \( B \), and \( AB \) reduces, respectively, to

\[
D_A = (1 - \mu)(1 - p_A) \\
D_B = 1 - \mu + \mu(p_{AB} - p_B) \\
D_{AB} = \mu(1 - p_{AB} + p_B)
\]

Note that these demands are not different from those when \( z = 1 \).

Consider now the case in which \( \hat{p}_{AB} = p_A + p_B \). In this case, the two groups of consumers that value both goods behave differently, so the total demand for \( A \), \( B \), and \( AB \) are now

\[
D_A = (1 - \mu)(1 - p_A) + \mu(1 - z)[1 - (p_A + p_B) + p_B] \\
D_B = 1 - \mu + \mu(1 - z) + \mu z(p_{AB} - p_B) \\
D_{AB} = \mu z(1 - p_{AB} + p_B)
\]

respectively. Rearranging leads to

\[
D_A = (1 - \mu z)(1 - p_A) \\
D_B = 1 - \mu z + \mu z(1 - p_B) \\
D_{AB} = \mu z(1 - p_{AB} + p_B)
\]

Since \( z \) enters multiplying \( \mu \) in all these demands, the analysis that follow from this case is not different from the analysis in the text that considers just \( \mu \). It only requires replacing \( \mu \) by \( \mu z \) and not the other. But this is easy. Take for instance Lemma 2 in the text. This lemma would now read as follows: If \( M \) approaches the two Bertrand retailers with the full-line forcing offer

\[
C^M_{Ai} = (w^M_{Ai} = 1, T^M_{Ai} = 0), C^M_{Bi} = (w^M_{Bi} = b, T^M_{Bi} = 0) \\
\hat{C}^M_{Ai} = (\hat{w}^M_{Ai} = 1/2, \hat{T}^M_{Ai} = 0), \hat{C}^M_{Bi} = (\hat{w}^M_{Bi} = b, \hat{T}^M_{Bi} = 0)
\]

for \( i = 1, 2 \), then, the most that \( S \) can obtain is \( b(1 - \mu z/4) \), where \( z \) is the probability that a given consumer is a one-stop shopper.

An important step in proof of the lemma is to compute the continuation play when \( S \) signs one of the retailers, say \( R_1 \), with the offer \( (w^S_{B1} = b - \epsilon, T^S_{B1} = 0) \), where \( \epsilon \) is positive but relatively small so that the first bullet in Lemma A.3 applies (i.e., \( w^S_{B1} \geq w_B \)). Since in the continuation play \( p_B = p_{B1} \leq p_{B2} = b \), it is clear that all consumers that value both goods and face no shopping costs will buy \( B \) from \( R_1 \) (i.e., \( \hat{p}_{AB} = p_A + p_B \)), unless, of course, \( p_{AB} < p_{A2} + p_{B1} \leq p_{A2} + p_{B2} \). But this latter can never be part of \( R_2 \)’s continuation play, as doing so necessarily entails a loss on consumers that value both goods. This loss can be interpreted as either pricing good \( A \) below its monopoly price \( (1 + \hat{w}^M_{A2})/2 \) or pricing good \( B \) below its cost \( \hat{w}^M_{B2} \).
Chapter 3

Extensions and Robustness

As explained in the text, the baseline version of the model has the simplest possible structure that allows us to distill the essential market conditions for foreclosure to emerge. Because such a simple structure necessarily omits features that may be present in actual cases, we now extend our model to accommodate these features and show that the policy implications summarized in Figure 1 are robust to such extensions.

We organize this final chapter as follows. In the first section (section 3.1) we provide a general analysis of why foreclosure fails to emerge in equilibrium under monopoly retailers. We not only extend the analysis to general schedules, as done in the proof of Proposition 2a, but also to a general formulation of the market. This will allow us to generalize and better communicate our contribution to the common-agency literature (see, e.g., Bernheim and Whinston 1998) in an environment of multiple products and (monopoly) retailers.

In section 3.2 we extend the baseline model to account for other features found in actual cases: (i) efficiency differences not in fixed costs but in variable costs or in both, (ii) horizontal differentiation in product $B$, (iii) vertical differentiation in product $B$, (iv) heterogeneous valuations for product $B$, and (v) price discrimination at the retail level. For each of these extensions, we show that the policy implications summarized in Figure 1 remain unchanged. Essentially, this requires to demonstrate the validity of Proposition 1 (possibility of foreclosure under Bertrand retailers), the validity of Proposition 2b (impossibility of foreclosure when no consumer is interested in both goods), and, building on the analysis of the previous section, the validity of Proposition 2a (impossibility of foreclosure under monopoly retailers).

The next two sections of the chapter can also be viewed as extensions of the baseline model, but they involve more fundamental departures, both methodological and conceptual. In section 3.3, we depart from the extreme modes of retail competition assumed so far — monopoly and Bertrand. We examine whether our results are robust to small amounts of retail homogeneity (almost retail monopolies), on the one hand, and to small amounts of retail heterogeneity (almost Bertrand competitors), on the other.

Finally, in section 3.4 we present an exercise where wholesale bundling emerges but for...
a reason other than foreclosure. This exercise serves to illustrate the use of our theory to
distinguish between situations that may share the existence of bundling and foreclosure but
have radically different antitrust implications. In one bundling is employed to foreclose a more
efficient rival and monopolize an otherwise competitive market, while in the other, to eliminate a
distortion in the retail market, benefiting both the multi-product manufacturer and consumers.

3.1 General treatment of monopoly retailers

In this section we provide a more general treatment of wholesale competition with local monopoly
retailers, a treatment that encompasses, among others, all the different models covered in the
text and in the next section (section 3.2) of this online Appendix. This analysis is not only
useful to show the robustness of Proposition 2a, but also to better connect our results with the
existing literature on common agency.

The analysis is organized as follows. To develop intuition, we set the stage by revisiting
the single-product, single monopoly retailer “bidding” model of Bernheim & Whinston (1998).
Keeping the single retailer setting, we then analyze the more novel case of a multi-product
firm competing against a single-product rival. In the third section we return to a single-
product environment, but we consider multiple monopoly retailers and scale economies. This
subsection draws heavily on the analysis of Ide, Montero and Figueroa (2016). In the fourth
section we consider the case of a multi-product firm competing against a single-product rival in
the presence of multiple monopoly retailers and scale economies. Finally, in the fifth and last
section, we show how this simple “bidding game” can be applied to more realistic settings (in
which retailers’ payoffs are not reduced form), by giving an alternative proof of Proposition 2a
for our baseline model.

3.1.1 One Retailer, a Single Product

This subsection revisits Bernheim & Whinston’s (1998) “simplest possible setting”. The game
is as follows:

\[ t = 1 \] Two single-product manufacturers \((h = M, S)\) simultaneously bid for representation by a
retailer \(R\). A bid is a pair \((t^{he}, t^{hc})\), where \(t^{he}\) is the required payment from \(R\) to \(h\) if \(R\)
decides to only represent \(h\); and \(t^{hc}\) is the required payment from \(R\) to \(h\) if \(R\) decides to
represent both manufacturers. Manufacturers incur in no costs whatsoever.

\[ t = 2 \] \(R\) chooses whether to represent one, both, or neither manufacturer. If he chooses the
latter, the game ends and all parties receive zero profits.

\[ t = 3 \] The retailer enters into a contract or contracts, and receives reduced form payoffs \(\Pi^M, \Pi^S, \text{ and } \Pi^{MS}\) if she chooses to represent only \(M\), only \(S\), or both \(M\) and \(S\), respectively.
That is, $R$’s payoffs is given by

$$
\pi_R = \begin{cases} 
\Pi^{MS} - t^{Mc} - t^{Sc} & \text{if } R \text{ represents both} \\
\Pi^h - t^{he} & \text{if } R \text{ only represents } h 
\end{cases}
$$

The following assumptions are made:

- **A1** (Products are valuable). $\Pi^{MS} > \max\{\Pi^M, \Pi^S\}$ and $\Pi^h > 0$, for $h = M, S$
- **A2** (Products are substitutes). $\Pi^M + \Pi^S > \Pi^{MS}$
- **A3** ($S$ is more efficient). $\Pi^S > \Pi^M$

Equilibria can the be generally classified as “common” (in which $R$ represents both manufacturers); and “exclusive” (where $R$ represents a single manufacturer).

**Lemma 3.1.1.** *(Bernheim & Whinston 1998)*

- **Lemma 3.1.1.** *(Bernheim & Whinston 1998)*

  - **When** $\Pi^{MS} < \Pi^S$ in equilibrium $R$ represents only $S$.

  - **When** $\Pi^S < \Pi^{MS}$ there are both exclusive (in which $R$ represents only $S$) and common equilibria. However, there is a unique equilibrium that Pareto dominates all other equilibria, and entails common representation.

  - **When** $\Pi^S = \Pi^{MS}$ there is a unique Pareto-dominant (for the manufacturers) payoff vector, but it is achievable through either exclusive or common equilibria.

**Proof.** For the formal proof see Bernheim & Whinston (1998). ■

Here we just provide a sketch of the proof. First, consider exclusive equilibria. When $t^{hc} \to +\infty$ for $h = M, S$, wholesale competition can be seen as a competition for exclusive representation. The following conditions characterize the set of exclusive equilibria:

$$
(1e) \quad \Pi^S - t^{Sc} = \Pi^M - t^{Me} > 0 \\
(2e) \quad t^{Sc} \geq 0 \geq t^{Me}
$$

On the one hand, condition (1e) states that (i) $R$ is obtaining strictly positive profits (otherwise the “losing” manufacturer could change $R$’s decision and profitably operate by requiring a payment of $\Pi^h - \epsilon$); and (ii) that $R$ must be indifferent between the manufacturers’ offers (otherwise, the “winning” manufacturer can increase the required payment, and obtain a higher surplus while still winning the competition for exclusive representation). On the other hand, condition (2e) states that (i) the winning manufacturer must be obtaining non-negative profits; and (ii) the losing manufacturer is unable to profitably undercut the rivals’ bid.

From here, it is clear that in any exclusive equilibrium only $S$ operates (he has “more surplus to offer” as $\Pi^S > \Pi^M$, and therefore necessarily wins the competition for exclusive
representation). And that this type of equilibria always exists \( t^{Sc} = \Pi^S - \Pi^M \), and \( t^{Me} = 0 \) always satisfies (1e) and (2e)).

Now let’s consider common equilibria. The set of common equilibria is characterized by the following set of conditions:

\[
\begin{align*}
(1c) \quad & \Pi^{MS} - t^{Mc} - t^{Sc} = \Pi^h - t^{he} > 0 \\
(2c) \quad & t^{hc} \geq 0, \quad t^{hc} \geq t^{he}, \quad \text{for } h = M, S
\end{align*}
\]

In any common equilibrium the retailer must be obtaining strictly positive profits (for the same reason as above). Moreover, she must be indifferent between accepting both offers, or accepting \( h \)'s exclusive offer (otherwise \( h \) could deviate and increase his profits slightly increasing \( t^{hc} \) while keeping \( t^{he} \) constant). Furthermore, manufacturers must be obtaining non-negative profits, (i.e. \( t^{hc} \geq 0 \)), and they should not find it profitable to unilaterally lower \( t^{he} \) to induce \( S \) to take their exclusive offer instead (i.e. \( t^{hc} \geq t^{he} \)).

Notice that a common equilibria exists if and only if \( \Pi^{MS} > \Pi^S \). Furthermore, when \( \Pi^{MS} > \Pi^S \) there is a continuum of common equilibria, but there is a unique equilibrium that Pareto dominates all other equilibria. Such equilibrium is found by setting \( t^{he} = t^{hc} \) for \( h = M, S \), and solving the system of two equations and two unknowns:

\[
\begin{align*}
\Pi^{MS} - t^{Mc} - t^{Sc} &= \Pi^M - t^{Mc} \\
\Pi^{MS} - t^{Mc} - t^{Sc} &= \Pi^S - t^{Sc}
\end{align*}
\]

to obtain \( t^{Mc} = \Pi^{MS} - \Pi^S \) and \( t^{Sc} = \Pi^{MS} - \Pi^M \), so the manufacturers’ and the retailer’s profits are then \( \pi_M = \Pi^{MS} - \Pi^S \), \( \pi_S = \Pi^{MS} - \Pi^M \), and \( \pi_R = \Pi^M + \Pi^S - \Pi^{MS} \). It is then easy to prove that this equilibrium dominates all other common equilibria. It also dominates all exclusive equilibria, since in the latter \( \pi_S \leq \Pi^S - \Pi^M < \Pi^{MS} - \Pi^M \), and \( \pi_M = 0 < \Pi^{MS} - \Pi^M \).

### 3.1.2 One Retailer, Multiple Products

We now introduce multiple products. We assume that both \( M \) and \( S \) produce different varieties of \( B \), lets call them \( B_M \) and \( B_S \), respectively, but that \( M \) is also capable of producing product \( A \). The timing of the game is as in the previous section, and we keep the assumption that manufacturers do not have any costs of production whatsoever.

\[\text{For sufficiency, notice that the following is a common equilibrium: } t^{hc} = t^{hc} = \Pi^{MS} - \Pi^h, \text{ for } h = M, S.\]

\[\text{For necessity, we prove the contrapositive statement } \Pi^{MS} < \Pi^S \implies \# \text{ common equilibrium. Suppose by contradiction that } \Pi^{MS} < \Pi^S, \text{ but that a common equilibrium exists. Then}\]

\[
\begin{align*}
\Pi^{MS} - t^{Mc} - t^{Sc} &= \Pi^S - t^{Sc} > 0 \\
\implies t^{Sc} - t^{Mc} - t^{Sc} &= \Pi^S - \Pi^{MS} > 0 \\
\implies t^{Sc} > t^{Mc} + t^{Sc}
\end{align*}
\]

But this contradicts \( t^{Sc} \geq t^{Sc} \), given that \( t^{Mc} \geq 0 \).
On Payoffs. As before, we assume that the retailer has a reduced form payoff depending on the representation decisions she makes. In particular, we denote by \( \Pi^{(x)(y)} \) the retailer’s realized payoff after accepting contract \( x \in \{0, A\} \) for \( A \), and \( y \in \{0, M, S, MS\} \) for product-line \( B \). We make the following assumptions in the spirit of the previous subsection:

\[
\begin{align*}
A1 & \quad (A \text{ is valuable}). & \quad \Pi^{(A)(y)} > \Pi^{(0)(y)}, \text{ for } y \in \{0, M, S, MS\} \\
A2 & \quad (B_h \text{ is valuable}). & \quad \Pi^{(x)(M)} > \Pi^{(x)(0)} \text{ and } \Pi^{(x)(S)} > \Pi^{(x)(0)}, \text{ for } x \in \{0, A\} \\
A3 & \quad \text{(Substitutability).} & \quad \Pi^{(A)(S)} + \Pi^{(A)(M)} + \Pi^{(0)(MS)} - 2\Pi^{(A)(MS)} > 0 \\
A4 & \quad (S \text{ is more efficient}). & \quad \Pi^{(x)(S)} > \Pi^{(x)(M)}, \text{ for } x \in \{0, A\}
\end{align*}
\]

Of these, the only assumption that is not straightforward is assumption \( A3 \). Notice that this can be rewritten as

\[
\frac{\Pi^{(A)(S)} + \Pi^{(A)(M)} - \Pi^{(A)(MS)} - \Pi^{(A)(0)}}{\Pi^{(A)(0)} + \Pi^{(0)(MS)} - \Pi^{(A)(MS)}} > 0
\]

Hence \( A3 \) is demanding \( A, B_M, \) and \( B_S \) to be “substitutes” in a “broad” sense. This condition would be satisfied when

1. \( A, B_M, \) and \( B_S \) are all substitutes among them.
2. \( A \) is unrelated to \( (B_M, B_S) \) (i.e. \( \Pi^{(x)(y)} = \Pi^{(x)(0)} + \Pi^{(0)(y)} \)), and \( B_M \) and \( B_S \) are substitutes (i.e. \( \Pi^{(0)(S)} + \Pi^{(0)(M)} > \Pi^{(0)(MS)} \)), as in the main text.
3. \( A \) and \( (B_M, B_S) \) are complements (for instance due to price discrimination motives at the retail market), but \( B_M \) and \( B_S \) are sufficiently close substitutes between each other.

On Bids. Under separate pricing a bid for representation in product-line \( k \) is a pair \( (t_{k}^{hc}, t_{k}^{hc}) \), where \( t_{k}^{hc} \) is the required payment from \( R \) to \( h \) for \( k \), if \( R \) decides to carry only \( h \)'s products in the \( k \)-line. While \( t_{k}^{hc} \) is the required payment from \( R \) to \( h \) for \( k \), if \( R \) sells different varieties of \( k \).

Hence a separate pricing bid for \( M \) is \( \{(t_{A}^{Me}, t_{A}^{Me}), (t_{B}^{Me}, t_{B}^{Me})\} \), while a separate pricing bid for \( S \) is \( (t_{B}^{Se}, t_{B}^{Se}) \). Notice, though, that since \( M \) is the only manufacturer selling \( A \), then \( t_{A}^{Me} = t_{A}^{Me} = t_{A}^{M} \), so we can collapse

\[\{(t_{A}^{Me}, t_{A}^{Mc}), (t_{B}^{Me}, t_{B}^{Mc})\} = \{(t_{A}^{M}), (t_{B}^{Me}, t_{B}^{Mc})\}\]

However, the space of bids for \( M \) is larger than \( \{(t_{A}^{M}), (t_{B}^{Me}, t_{B}^{Mc})\} \). The reason is that he can also offer “wholesale bundling” by making the required payment contingent on the exclusive representation in both product lines: \( \{(t_{A}^{M}), (t_{B}^{Mc}, t_{AB}^{Mc})\} \). This is equivalent to a triplet \( (t_{A}^{M}, t_{B}^{Mc}, t_{AB}^{Mc}) \), since for \( t_{AB}^{Mc} \) to have bite, it must be \( t_{AB}^{Mc} \leq t_{A}^{M} + t_{B}^{Mc} \). Summarizing, \( M \) offers \( (t_{A}^{M}, t_{B}^{Mc}, t_{AB}^{Mc}) \), while \( S \) offers \( (t_{B}^{Se}, t_{B}^{Se}) \).
Finally, the equilibria can be classified as follows:

![Equilibria Diagram](image)

**Figure 3.1: Classification of Equilibria**

**Characterization of Equilibria.** So let’s first consider the possibility of a fully exclusive equilibrium in which $R$ only carries products from $M$. An initial (rather naive, as we will see) approach following the characterization of exclusive equilibria in the single-product environment would be to set $(t^M_A, t^M_B) \rightarrow (+\infty, +\infty)$, and $p^S_B \rightarrow +\infty$, and claim that fully exclusive equilibria is given by the following set of conditions:

\[ (1e') \quad \Pi^{(0)(S)} - t^S_B = \Pi^{(A)(M)} - t^M_{AB} > 0 \]
\[ (2e') \quad t^M_{AB} \geq 0 \geq t^S_B \]

On the one hand, condition (1e’) states that $R$ must be getting strictly positive profits, and that he must be indifferent between $M$’s and $S$’s “fully exclusive” representation offers. On the other hand, (2e’) states that $M$ must be getting non-negative payoffs, and that $S$ in unable to profitably undercut $M$’s bid.

Clearly, conditions (1e’) and (2e’) can only be satisfied when $\Pi^{(A)(M)} > \Pi^{(0)(S)}$. But it is perfectly possible to satisfy this condition, even if $S$ is more “efficient” (i.e. $B_S$ creates more surplus than $B_M$). In fact, we would usually expect $\Pi^{(A)(M)} > \Pi^{(0)(S)}$ to hold when product $A$ is very valuable and/or $B_M$ and $B_S$ are close substitutes between each other. This discussion appears to imply that a fully exclusive equilibrium exists, even if $S$ is more efficient, since market $A$ gives $M$ additional “leverage” in winning the competition for exclusive representation.

The problem, however, is that conditions (1e’) and (2e’) are only necessary, though not sufficient, for a fully exclusive equilibrium to exist, since there is a new possible deviation for $M$ that we have not accounted for. In particular, notice that if a fully exclusive equilibrium existed, then (1e’) and (2e’) would imply that

\[ \pi^*_M = t^M_{AB} = \Pi^{(A)(M)} - \Pi^{(0)(S)} + t^S_B \leq \Pi^{(A)(M)} - \Pi^{(0)(S)} \]

where we are using the fact that $t^S_B \leq 0$. However, if $M$ then unilaterally deviates to $t^M_{AB}' \rightarrow +\infty$, $t^M_A = \Pi^{(A)(S)} - \Pi^{(0)(S)}$, $t^M_B = 0$, then $R$’s optimal response is to accept $M$’s contract for...
A, and S’s contract for $B_S$,² so $\pi'_M = \Pi^{(A)}(S) - \Pi^{(\emptyset)}(S)$, which is strictly greater than $\pi^*_M$, as

$$\pi^*_M \leq \Pi^{(A)}(M) - \Pi^{(\emptyset)}(S) < \Pi^{(A)}(S) - \Pi^{(\emptyset)}(S)$$

given that S is more efficient, i.e., $\Pi^{(A)}(S) > \Pi^{(A)}(M)$.

Hence, there cannot exists a fully exclusive equilibrium, since the necessary and sufficient conditions for its existence:

$$(1e') \quad \Pi^{(\emptyset)}(S) - t^{Se}_B = \Pi^{(A)}(M) - t^M_{AB} > 0$$
$$(2e') \quad t^{Se}_B \leq 0$$
$$(3e') \quad t^M_{AB} \geq \Pi^{(A)}(S) - \Pi^{(\emptyset)}(S)$$

with $(t^M_M, t^M_B) \to (+\infty, +\infty)$, and $t^{Se}_B \to +\infty$, leads to an empty set given assumption A4.

What explains the difference? The key is that in the multi-product setting, since $t^{Se}_B$ is only demanding exclusivity for product-line $B$, $M$ always has the option to “unbundle” his offer, an option that $M$ always finds valuable when $S$ is more efficient. In the single-product case, in contrast, once $S$ starts demanding exclusive representation $(t^{Se}_B < \infty, t^{Sc}_B \to +\infty)$, $M$ does not have such an option, as $S$ exclusive offers has made competition for product $B$ an all-or-nothing affair.

Having ruled out the possibility of fully exclusive equilibria, the characterization of the equilibrium is not that different from the single-product case:

**Lemma 3.1.2.** Under assumptions A1. to A4. then:

- There does not exists a fully exclusive equilibrium.
- When $\Pi^{(A)(MS)} < \Pi^{(A)}(S)$ there exists only $B$-exclusive equilibria.
- When $\Pi^{(A)}(S) < \Pi^{(A)(MS)}$ there are both $B$-exclusive and $B$-common equilibria. However, there is a unique $B$-common equilibrium that Pareto dominates all other equilibria. This equilibrium is characterized by: $t^M_A = \Pi^{(A)(MS)} - \Pi^{(\emptyset)(MS)}$, $t^{Mc}_B = \Pi^{(A)}(MS) - \Pi^{(A)(S)}$, $t^M_{AB} = t^M_A + t^{Mc}_B$, and $t^{Sc}_B = t^{Se}_B = \Pi^{(A)(MS)} - \Pi^{(A)(M)}$.
- When $\Pi^S = \Pi^{MS}$ there is a unique Pareto-dominant (for the manufacturers) payoff vector, but it is achievable through either $B$-exclusive or $B$-common equilibria.

**Proof.** We have already proven the first point. For the second and third points, notice that the set of $B$-exclusive equilibria is characterized by setting $t^{hc}_B \to +\infty$, for $h = M, S$, plus the following conditions:

²Following this deviation, on-path $R$ would be getting $\Pi^{(A)}(S) - t^{M'}_A - t^{Se}_B = \Pi^{(\emptyset)}(S) - t^{Se}_B$ (so she is indifferent between accepting $A$ from $M$ and $B_S$ from $S$, or only the latter), and this is strictly greater than $\max\{0, \Pi^{(A)(\emptyset)} - t^{M'}_A - \Pi^{(A)(\emptyset)}(S) - \Pi^{(A)(S)}\}$ (her profits if she declines all contracts, or accepts only $A$ from $M$)
1. $\Pi^{(A)}(S) - t_A^M - t_B^{Se} = \Pi^{(A)}(M) - t_{AB}^M = \Pi^{(\emptyset)}(S) - t_B^{Se} > 0$

2. $t_B^{Se} \leq 0, t_A^M \geq 0, t_A^M \geq t_{AB}^M$

The existence of this equilibrium is guaranteed, since the following bids satisfy the required conditions $t_A^M = t_{AB}^M = \Pi^{(A)}(S) - \Pi^{(\emptyset)}(S) > 0$, and $t_B^{Se} = \Pi^{(A)}(S) - \Pi^{(A)}(M) > 0$.

The set of $B$-common equilibria, is given by

1. $\Pi^{(A)}(MS) - t_A^M - t_B^{Mc} - t_B^{Sc} = \Pi^{(\emptyset)}(MS) - t_B^{Mc} - t_B^{Sc} > 0$
2. $\Pi^{(A)}(MS) - t_A^M - t_B^{Mc} - t_B^{Se} = \Pi^{(A)}(S) - t_A^M - t_B^{Mc} - t_B^{Se} > 0$
3. $\Pi^{(A)}(MS) - t_A^M - t_B^{Mc} - t_B^{Sc} = \Pi^{(A)}(M) - t_{AB}^M > 0$
4. $t_A^M, t_B^{Mc} \geq 0, t_A^M + t_B^{Mc} \geq t_{AB}^M$, and $t_B^{Se} \geq 0, t_B^{Sc} \geq t_B^{Se}$

It is then easy to establish that such equilibria exists iff $\Pi^{(A)}(MS) > \Pi^{(A)}(S)$. Furthermore, the unique Pareto undominated equilibrium is found by setting $t_B^{Sc} = t_B^{Se}$, and $t_{AB}^M = t_A^M + t_B^{Mc}$, and solving the system of three equations and three unknowns. The solution is the one given in the statement.

### 3.1.3 Multiple Retailers, a Single Product

We now return to the single-product environment, but consider two retailers and scale economies at the manufacturer level. In particular, we are interested in studying the possibility of an exclusive equilibrium in which $M$ operates despite being more inefficient, if $S$’s production technology involves important scale economies, as this could potentially allow the emergence of bilateral contracting externalities (Rasmusen, Ramseyer & Wiley 1991; Segal and Whinston 2000). The game is as follows:

$t = 1$ Two single-product manufacturers ($h = M, S$) simultaneously bid for representation by two retailers $R_1$ and $R_2$. A bid from $h$ to $i$ is a pair $(i^h_1, i^h_i)$, where $i^h_1$ is the required payment from $R_i$ to $h$ if $R_i$ decides to only represent $h$; and $i^h_i$ is the required payment from $R_i$ to $h$ if $R_i$ decides to represent both manufacturers.

$t = 2$ $R_1$ and $R_2$ simultaneously make contract decisions.

$t = 3$ $R_1$ and $R_2$ enter into a contract or contracts. $R_i$ receives reduced form payoffs $\Pi_i^M$, $\Pi_i^S$, and $\Pi_i^{MS}$ if she chooses to represent only $M$, only $S$, or both $M$ and $S$, respectively.

When a retailer enters into a contract with a manufacturer, the latter must incur in a fixed cost of $F_h > 0$.

For simplicity we are going to assume that retailers are symmetric: $\Pi_i^x = \Pi_{-i}^x \equiv \Pi^x$ for $x \in \{\emptyset, M, S, MS\}$, but this assumption can be easily relaxed without affecting any of the
results that follow. We make the following assumptions in the spirit of the previous sections:

A1 (Products are valuable). Π^MS > max{Π^M, Π^S}
A2 (Products are substitutes). Π^M + Π^S > Π^MS
A3 (S is more efficient). 2Π^S - F_S > 2Π^M - F_M > 0

Is foreclosure of a more efficient rival an equilibrium? In this section we study whether the exclusion of S can arise as an equilibrium outcome. To develop intuition, we begin by assuming

A4. Π^S - F_S < 0, and F_M → 0

The idea is to study the feasibility of foreclosure in the extreme case in which S brings more surplus to either relationship with retailers but faces important scale economies, while M brings less surplus but his scale economies are negligible.

First of all, given Assumption A4, it is clear that there cannot exist an equilibrium in which S operates in a single retail market. Hence in equilibrium, either S operates in both or in none. Consider the latter. For that to be the case, M must win the competition for exclusive representation in both markets, hence we need (letting \( p^h_i \rightarrow +\infty \) for \( h = M, S \))

\[
(1e') \quad \Pi^S - t^{Se}_i = \Pi^M - t^{Me}_i > 0 \text{ for } i = 1, 2
\]

\[
(2e') \quad t^{Me}_1 + t^{Me}_2 \geq F_M \text{ and } t^{Se}_1 + t^{Se}_2 \leq F_S
\]

(1e”) states the usual condition that the retailer must be indifferent among offers, and obtaining strictly positive profits. Condition (2e”) states that M must be obtaining non-negative profits, and that S must be unable to profitably undercut M’s bid. It is very important to notice that S’s undercutting condition involves the sum of the bids (i.e. \( t^{Se}_1 + t^{Se}_2 \leq F_S \)) not each bid individually (i.e. \( t^{Se}_1 \leq F_S \)). The reason is that S can always multilaterally deviate, and undercut M’s bids on both markets.

From here it is clear that if S is more efficient (i.e. Assumption A3), then the above conditions cannot be satisfied, even if scale economies are important for S (Assumption A4). Indeed suppose 2Π^S - F_S > 2Π^M - F_M, but that M is the only manufacturer operating. Then Π^S - t^{Se}_i = Π^M - t^{Me}_i > 0 for i = 1, 2. Summing across retailers we get

\[
2\Pi^S - t^{Se}_1 - t^{Se}_2 = 2\Pi^M - t^{Me}_1 - t^{Me}_2 > 0
\]

\[
\implies (2\Pi^S - F_S) - (2\Pi^M - F_M) = (t^{Se}_1 + t^{Se}_2) - (t^{Me}_1 + t^{Me}_2) + (F_M - F_S) > 0
\]

\[
\implies (t^{Se}_1 + t^{Se}_2 - F_S) > (t^{Me}_1 + t^{Me}_2 - F_M)
\]

But \( t^{Me}_1 + t^{Me}_2 - F_M \geq 0 \), so \( t^{Se}_1 + t^{Se}_2 - F_S > 0 \), contradicting the second condition of (2e”). Hence, even under the (rather extreme) assumption A4, there is no equilibrium in which S does not operate in any market.
We now show that this results holds more generally.

Claim 3.1.1. Under assumptions A1. to A3. there is no equilibrium in which $S$ operates in no market.

Proof. First, suppose that there is an equilibrium in which $S$ operates in no market. Then following the exact same proof as above we obtain a contradiction (notice that we never used Assumption A4. on such derivation.)

Now suppose there exists an equilibrium in which $S$ operates in only one market, say $-i$. Now, since $S$ is not operating in $i$, then such retail market must be exhibiting exclusive representation (and $M$ must be winning such representation). Hence it must be $\Pi^S - t^S_i = \Pi^M - t^Me_i > 0$. Furthermore, it must be $t^Me_i \geq 0$ (otherwise $M$ could set $t^Me_i$ to $+\infty$ and increase his profits). But if so, then $S$ can deviate and bid $t^S_i - \epsilon = \Pi^S - \Pi^M + t^Me_i - \epsilon > 0$, winning the competition for exclusive in such market, and obtaining additional profits of $t^S_i > 0$. Contradiction. ■

(Partial) Characterization of Equilibria. We now provide a partial characterization of the equilibrium set.

Lemma 3.1.3. Under assumptions A1. to A3. then:

- In any equilibrium, $S$ operates in both retail markets.
- When $2(\Pi^MS - \Pi^S) < F_M$ in equilibrium both retailers only represent $S$.
- When $F_M \leq 2(\Pi^MS - \Pi^S)$, there is an equilibrium in which both retailers represent both manufacturers. This equilibrium Pareto dominates any equilibrium in which one or both retailers only represent $S$.

Proof. Consider retail market $i$. If manufacturers are competing for exclusive representation, then (letting $p^{hc}_i \to +\infty$ for $h = M, S$) a necessary condition is

$$\Pi^S - t^S_i = \Pi^M - t^Me_i > 0$$

. While if they are offering common representation, then we need

$$\Pi^MS - t^Mc_i - t^Sc_i = \Pi^S - t^S_i = \Pi^M - t^Me_i > 0$$

Now, given the symmetry of the retail markets, we have three possible equilibrium candidates:

- If both markets operate under exclusive representation, then we further need $t^S_i + t^S_j \geq F_S$, and $t^Me_i + t^Me_j \leq F_M$

- If one market operates under exclusive representation, while the other under common representation, we need

$$t^S_j \geq t^S_i , \ t^S_i + t^Sc_j \geq F_S , \ t^Mc_j \geq F_M$$

$$t^Me_j \geq \max\{t^Me_i , t^Me_j \}$$
• While if both markets operate under common representation, then we need
\[ t_1^{Sc} + t_2^{Sc} \geq F_S, \quad t_1^{Mc} + t_2^{Mc} \geq F_M \]
\[ t_1^{Sc} + t_2^{Sc} \geq \max\{t_1^{Sc} + t_2^{Sc}, t_2^{Sc}, t_1^{Sc} + t_2^{Sc}\} \]
\[ t_1^{Mc} + t_2^{Mc} \geq \max\{t_1^{Mc} + t_2^{Mc}, t_1^{Mc} + t_2^{Mc}\} \]

Now, we already showed (Claim 3.1.1) that there cannot be an equilibrium in which S operates in no market. Hence, the only thing left in order to show the first point of the lemma is that the equilibrium set is non-empty. But this is easy, since the following sets of bids constitutes an equilibrium exhibiting exclusive representation by S on each retail market
\[ t_i^{Sc} = \Pi^S - \Pi^M + F_M/2, \quad t_i^{Mc} = F_M/2. \]

We now prove the second point. Suppose \( 2(\Pi^{MS} - \Pi^S) < F_M \), but that there exists an equilibrium in which at least one retailer is carrying M’s product B. Since in any equilibrium S must be operating in both market, a retailer carrying B_M must be a common retailer in B. First, consider the case in which a single retailer, say i, carries B_M. Then \( \pi_M^* = t_i^{Mc} - F_M = \Pi^{MS} - \Pi^S + (t_i^{Se} - t_i^{Sc}) - F_M \leq \Pi^{MS} - \Pi^S - F_M < 0 \). Contradiction. So consider the case in which both retailers are common retailers, then
\[ \pi_M^* = t_1^{Mc} + t_2^{Mc} - F_M = 2(\Pi^{MS} - \Pi^S) - F_M + (t_1^{Se} - t_1^{Sc}) + (t_2^{Se} - t_2^{Sc}) \leq 2(\Pi^{MS} - \Pi^S) - F_M < 0 \]

For the third, an final point, notice that the following is an equilibrium that exhibits common representation in each retail market: \( t_i^{Sc} = \Pi^{MS} - \Pi^M \), and \( t_i^{Mc} = \Pi^{MS} - \Pi^S \). In such equilibria \( \pi_S = 2(\Pi^{MS} - \Pi^M) - F_S > 0 \), \( \pi_M = 2(\Pi^{MS} - \Pi^S) - F_M > 0 \), which clearly Pareto dominates dominates any equilibrium in which one or both retailers only represent S, as in any of the latter cases \( \pi_S \leq \Pi^{MS} + \Pi_S - 2\Pi^M - F_S < 2(\Pi^{MS} - \Pi^M) - F_S \), given that \( F_M \leq 2(\Pi^{MS} - \Pi^S) \) implies that \( \Pi^{MS} > \Pi^S \). ■

3.1.4 Multiple Retailers, Multiple Products

Finally, we tackle the more general case of two retailers and multiple products. As in section 3.1.2 we assume that both M and S produce different varieties of B, B_M and B_S, but that M is also capable of manufacturing product A. The game is as follows:

\( t = 1 \) M and S simultaneously bid for representation by two retailers R1 and R2 on each product-line. A bid of M to Ri is a pair \( (t_i^{M}, t_i^{Mc}, t_i^{Me}) \), and a bid of S to Ri a pair \( (t_i^{Se}, t_i^{Sc}) \).

\( t = 2 \) R1 and R2 simultaneously make contract decisions.

\(^3\)Notice that \( 2(\Pi^{MS} - \Pi^M) - F_S > 2(\Pi^S - \Pi^M) + F_M - F_S \), since \( F_M \leq 2(\Pi^{MS} - \Pi^S) \). And that \( 2(\Pi^S - \Pi^M) + F_M - F_S > 0 \) by A3.
$t = 3$ \( R1 \) and \( R2 \) enter into a contract or contracts. \( R_i \) receives a reduced form payoff \( \Pi_{i}^{(x)(y)} \) after accepting contract \( x \in \{ \emptyset, A \} \) for \( A \), and \( y \in \{ \emptyset, M, S, MS \} \) for product-line \( B \).

For simplicity we keep assuming symmetry of the retail markets (i.e. \( \Pi_{i}^{(x)(y)} = \Pi_{-i}^{(x)(y)} \equiv \Pi^{(x)(y)} \)). We assume that \( M \) faces no fixed cost for producing \( A \). However, when a retailer enters into a contract for product \( B \) with \( M \) (\( S \)), the latter must incur in a fixed cost of \( F_M > 0 \) (\( F_S > 0 \)).

We make the following assumptions:

- **A1** (\( A \) is valuable at retail level): \( \Pi^{(A)(y)} > \Pi^{(0)(y)} \), for \( y \in \{ \emptyset, M, S, MS \} \)
- **A2** (\( B_i \) is valuable at the retail): \( \Pi^{(x)(M)} > \Pi^{(x)(\emptyset)} \) and \( \Pi^{(x)(S)} > \Pi^{(x)(\emptyset)} \), for \( x \in \{ \emptyset, A \} \)
- **A3** (Substitutability): \( \Pi^{(A)(S)} + \Pi^{(A)(M)} + \Pi^{(0)(MS)} - 2\Pi^{(A)(MS)} > 0 \)
- **A4** (\( S \) is more efficient): \( 2\Pi^{(x)(S)} - F_S > 2\Pi^{(x)(M)} - F_M \), for \( x \in \{ \emptyset, A \} \)

**Lemma 3.1.4.** Under assumptions A1* to A4* then:

- In any equilibrium, \( S \) sells \( B_S \) in both retail markets, and \( M \) sells \( A \) in both retail markets.
- When \( 2(\Pi^{(A)(MS)} - \Pi^{(A)(S)}) < F_M \) in equilibrium both retailers only represent \( S \) in product-line \( B \).
- When \( F_M \leq 2(\Pi^{(A)(MS)} - \Pi^{(A)(S)}) \), there is an equilibrium in which both retailers represent both manufacturers in product-line \( B \). This equilibrium Pareto dominates any equilibrium in which one or both retailers only represent \( S \) in product-line \( B \).

That \( M \) sells \( A \) in both retail market is straightforward, as product \( A \) is valuable at the retail level (Assumption A1*), and \( M \) does not incur in any fixed costs in producing it. So we focus on the remaining statements of the proof. We will prove these in several claims that, in some sense, summarize all we have covered in this section.

**Claim 3.1.2.** Under assumptions A1* to A4* there is no equilibrium in which \( S \) operates in no market. There always exists, however, and equilibrium in which \( S \) sells \( B_S \) in both retail markets.

**Proof.** First, suppose that there is an equilibrium in which \( S \) does not operate in any market. Then \( S \) must be losing the competition for exclusive in each of them, \( \Pi^{(A)(M)} - t_{ABi}^M = \Pi^{(0)(S)} - t_{Bi}^{Se} > 0 \) for \( i = 1, 2 \), with \( t_{Ai}^M, t_{Bi}^{Me}, t_{Bi}^{Sc} \rightarrow +\infty \), and \( t_{B1}^{Se} + t_{B2}^{Se} \leq F_S \). On-path \( M \) is then obtaining

\[
\pi_M^* = t_{AB1}^M + t_{AB1}^M - F_M = 2(\Pi^{(A)(M)} - \Pi^{(0)(S)}) + t_{B1}^{Sc} + t_{B2}^{Se} - F_M \\
\leq 2(\Pi^{(A)(M)} - \Pi^{(0)(S)}) + F_S - F_M
\]

But then, consider the following deviation by \( M \): \( t_{ABi}^M \rightarrow +\infty \), \( t_{Ai}^M = \Pi^{(A)(S)} - \Pi^{(0)(S)} \), \( t_{Bi}^{Sc} = 0 \), for \( i = 1, 2 \). Then, \( R_i \)'s optimal response it to accept \( M \)'s contract for \( A \), and \( S \)'s contract for
\( B_2 \), so \( \pi'_M = 2(\Pi^{(A)}(S) - \Pi^{(0)}(S)) \), which is strictly greater than \( 2(\Pi^{(A)}(M) - \Pi^{(0)}(S)) + F_S - F_M \) by A4*, and therefore strictly greater than \( \pi'_M \). Contradiction.

Now suppose there is an equilibrium in which \( S \) operates in a single market, say \(-i\). This implies that a competition for exclusives must be ensuing in market \( i \), so \( \Pi^{(A)}(M) - t_{ABi}^{M} = \Pi^{(0)}(S) - t_{Bi}^{Se} > 0 \) with \( t_{Ai}^{M}, t_{Bi}^{Mc}, t_{Bi}^{Sc} \rightarrow +\infty \). Furthermore, since \( S \) is already incurring \( F_S \), then he is able to bid all the way down to zero, i.e. \( t_{Bi}^{Se} \leq 0 \). Hence the profit \( M \) is obtaining from market \( i \) is less than or equal to \( \Pi^{(A)}(M) - \Pi^{(0)}(S) \). But if so, then \( M \) could deviate to \( t_{Ai}^{M} \rightarrow +\infty \), \( t_{Ai}^{M} = \Pi^{(A)}(S) - \Pi^{(0)}(S) \), \( t_{Bi}^{Mc} = 0 \) in market \( i \), an obtain instead \( \Pi^{(A)}(S) - \Pi^{(0)}(S) > \Pi^{(A)}(M) - \Pi^{(0)}(S) \) in such market. Contradiction.

For the final part of the proof, notice that the following is an equilibrium where \( S \) operates in both markets: \( t_{Ai}^{M} = \Pi^{(A)}(S) - \Pi^{(0)}(S), t_{Bi}^{Mc} \rightarrow +\infty, t_{Ai}^{M} = t_{Ai}^{M} + F_M / 2 \) and \( t_{Bi}^{Sc} = \Pi^{(A)}(S) - \Pi^{(A)}(M) + F_M / 2 \). ■

Claim 3.1.3. When \( 2(\Pi^{(A)}(MS) - \Pi^{(A)}(S)) < F_M \) in equilibrium both retailers only represent \( S \) in product-line \( B \).

Proof. Suppose \( 2(\Pi^{(A)}(MS) - \Pi^{(A)}(S)) < F_M \), but that there exists an equilibrium in which at least one retailer is carrying \( B_M \). Since in any equilibrium \( S \) must be operating in both market, a retailer carrying \( M \) must be a common retailer. First, consider the case in which a single retailer, say \( i \), carries \( M \). Then \( M \)'s profits stemming for product \( B_M \) are \( t_{Bi}^{Mc} = \Pi^{(A)(MS)} - \Pi^{(A)}(S) + (t_{Bi}^{Sc} - t_{Bi}^{Sc}) - F_M \leq \Pi^{(A)(MS)} - \Pi^{(A)}(S) - F_M < 0 \). But if so, \( M \) is better off deviating to \( t_{Bi}^{Mc} \rightarrow +\infty \), not producing \( B_M \) at all, as this gives him zero profits. A similar argument can be used to argue that there cannot be an equilibrium in which both retailers are common retailer for \( B \). ■

Claim 3.1.4. When \( F_M \leq 2(\Pi^{MS} - \Pi^{S}) \), there is an equilibrium in which both retailers represent both manufacturers. This equilibrium Pareto dominates any equilibrium in which one or both retailers only represent \( S \).

Proof. The following is an equilibrium that exhibits common representation in each retail market for product \( B \): \( t_{Ai}^{M} = \Pi^{(A)(MS)} - \Pi^{(0)(S)}, t_{Bi}^{Mc} = \Pi^{(A)(MS)} - \Pi^{(A)}(S), t_{Ai}^{M} = t_{Ai}^{M} + F_M / 2 \), and \( t_{Bi}^{Sc} = t_{Bi}^{Sc} = \Pi^{(A)(MS)} - \Pi^{(A)(M)} \), for \( i = 1, 2 \). In such equilibria \( \pi_S = 2(\Pi^{(A)(MS)} - \Pi^{(A)(M)}) - F_S > 0 \), and \( \pi_M = 2(\Pi^{(A)(MS)} - \Pi^{(A)(S)}) - F_M > 0 \). It is the easy to prove that this equilibrium Pareto dominates any equilibrium in which one or both retailers only represent \( S \). ■

3.1.5 Back to Proposition 2a

Now let's return to the baseline model found in the text with two monopoly retailers. Notice that, since retailers are local monopolies, in any equilibrium the manufacturers are going to set marginal wholesale prices equal to their true cost, in order for retailers to internalize any
potential double-marginalization, and will then simply compete on lump-sum transfers. But if so, it is as if retailers receive reduced form payoffs equal to the optimal industry profits created in that market, and manufacturers simply “bid” for representation via the transfers they are demanding. Hence all the results previously given hold in this more realistic setting.

Now, to prove Proposition 2a, first notice optimal industry profits in each retail location satisfy symmetry \( \Pi_i^{(x)(y)} = \Pi_{i-1}^{(x)(y)} \equiv \Pi^{(x)(y)} \), and separability \( \Pi^{(x)(y)} = \Pi^{(x)(\emptyset)} + \Pi^{(\emptyset)(y)} \) (the latter is explained since \( A \) is unrelated to \( B \), and there is no retail price discrimination motive downstream). Furthermore, we have \( \Pi^{(A)(\emptyset)} = 1/8 > 0 \) (satisfying \( A1^* \)), \( \Pi^{(\emptyset)(M)} = \Pi^{(\emptyset)(S)} = \Pi^{(\emptyset)(MS)} = b/2 > 0 \) (satisfying \( A2^* \)). Furthermore, \( \Pi^{(A)(S)} + \Pi^{(A)(M)} + \Pi^{(\emptyset)(MS)} - 2\Pi^{(A)(MS)} = b/2 > 0 \) (satisfying \( A3^* \)); and \( 2\Pi^{(x)(S)} - F_S = b - F_S > 2\Pi^{(x)(M)} - F_M = b - F_M \) (satisfying \( A4^* \)). Hence, since the baseline model satisfies assumptions \( A1^* \) to \( A4^* \) so we can use Lemma 3.1.4 Given that \( 2(\Pi^{(A)(MS)} - \Pi^{(A)(S)}) = 0 < F_M \) we conclude then that in equilibrium both retailers only represent \( S \) in product-line \( B \). That is, there is no foreclosure in equilibrium and we are back to Proposition 2a of the main text.

### 3.2 Extending the baseline model

These are the five (independent) extensions we consider:

1. Differences in marginal costs: we allow manufacturers to differ in both fixed and marginal costs of production of good \( B \).
2. Horizontal differentiation in product \( B \): we allow a fraction of consumers to see the two versions of product \( B \) not as perfect substitutes.
3. Vertical differentiation in product \( B \): we allow all consumers to have a higher valuation for \( S \)'s version of product \( B \)
4. Heterogenous valuations for product \( B \): we allow valuations for good \( B \) to vary across consumers, but without inducing price discrimination at retail level.
5. Price discrimination at the retail level: we allow valuations for good \( B \) to vary across consumers, inducing price discrimination at the retail level.

#### 3.2.1 Differences in marginal costs

In this first extension of the baseline model, we allow manufacturers to differ in both fixed and marginal costs of production of good \( B \). We maintain the assumption that \( M \) produces \( A \) at no cost. The production of \( B \), however, requires manufacturers to incur in both fixed (\( F_h \)) and

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Footnote:

\(^4\)Strictly speaking, this is only true for market \( A \), since demand for \( B \) is inelastic. However, setting marginal wholesale prices equal to the manufacturers’ costs in product \( B \) is without loss of generality, as the inelastic demand implies that wholesale prices and transfers are equivalent.
variable costs \((c_h)\), where \(h \in \{M, S\}\). \(S\) is assumed to have a lower overall cost of production, i.e.,
\[
c_S + F_S < c_M + F_M \leq b
\]
We are flexible as to what may explain this difference in production efficiency, whether differences in fixed cost, variable costs or both.

**Revisiting Proposition 1**

We start by computing the full-monopolization outcome. As in the baseline model, the retail prices that implement this outcome are \(p_A^* = 1/2, p_B^* = b\), resulting in \(\Pi_M^* = 1/4 + b - c_M - F_M\).

Suppose now that \(M\) approaches the two Bertrand retailers with the full-line forcing offer
\[
\begin{align*}
\mathcal{C}_M^i &= (u_{Ai}^M = 1, T_{Ai}^M = 0), \quad \mathcal{C}_{Bi}^M = (u_{Bi}^M = b, T_{Bi}^M = 0) \quad (3.1) \\
\tilde{\mathcal{C}}_M^i &= (\hat{u}_{Ai}^M = 1/2, \hat{T}_{Ai}^M = 0), \quad \tilde{\mathcal{C}}_{Bi}^M = (\hat{u}_{Bi}^M = b, \hat{T}_{Bi}^M = 0)
\end{align*}
\]
for \(i = 1, 2\). Following the proof of Lemma 2 in the text, one can show that \(S\) can at best persuade one retailer to stock his product, in which case the most \(S\) can obtain in the market is \((b - c_S)(1 - \mu/4)\). Hence, when \(F_S \geq (b - c_S)(1 - \mu/4)\), or alternatively, when
\[
b - \frac{\mu}{4}(b - c_S) \leq c_S + F_S \leq c_M + F_M \leq b
\]
there exists an equilibrium in which \(M\) fully monopolizes the market with the wholesale bundling contracts \((3.1)\).

**Revisiting Proposition 2a**

In this model, optimal industry profits in each retail location clearly satisfy symmetry (i.e. \(\Pi_i^{(x)(y)} = \Pi_i^{(x)(y)} \equiv \Pi^{(x)(y)}\)), and separability (i.e. \(\Pi^{(x)(y)} = \Pi^{(x)(0)} + \Pi^{(0)(y)}\)). Furthermore \(\Pi^{(A)(0)} = 1/8 > 0\) (satisfying A1*), \(\Pi^{(0)(M)} = (1/2)(b - c_M) > 0\), \(\Pi^{(0)(S)} = \Pi^{(0)(MS)} = (1/2)(b - c_M) > 0\) (satisfying A2*). Furthermore, \(\Pi^{(A)(S)} + \Pi^{(A)(M)} + \Pi^{(0)(MS)} = 2\Pi^{(A)(MS)} = (1/2)(b - c_M) > 0\) (satisfying A3*); \(2\Pi^{(x)(S)} - F_S = b - c_S - F_S > 2\Pi^{(x)(M)} - F_M = b - c_M - F_M\) (satisfying A4*). Since the model satisfies assumptions A1* to A4* we can use Lemma 3.1.4. Given that \(2(\Pi^{(A)(MS)} - \Pi^{(A)(S)}) = 0 < F_M\) we conclude then that in equilibrium both retailers only represent \(S\) in product-line \(B\).

**Revisiting Proposition 2b**

Following the proof of Proposition 2b in the text, we restrict attention to full-line forcing contracts (see also Appendix H in the text). Suppose the following pair of contracts constitute a foreclosure equilibrium: \(\{(u_{Ai}^M = 1, T_{Ai}^M = 0), (u_{Bi}^M = b, T_{Bi}^M = 0)\}\) and \(\{(\hat{u}_{Ai}^M, \hat{T}_{Ai}^M), (\hat{u}_{Bi}^M, \hat{T}_{Bi}^M)\}\) for \(i = 1, 2\). Suppose further that wholesale prices are such that \(p_{Ai}^* = \max\{\hat{u}_{Ai}^M, \hat{u}_{Aj}^M\}\) and
Suppose that a fraction $\gamma/3.2$. Horizontal differentiation in product $B$

Consider first the case in which equilibrium offers are such that $\hat{w}_{M_{Bi}} \leq \hat{w}_{M_{Bj}}$, which implies (for $\mu = 0$) that $\hat{T}_{M_{Ai}} = \hat{T}_{M_{Bi}} = \pi_{Ri} = 0$ and $\pi_{Ri}^* = (\hat{w}_{M_{Ai}} - \hat{w}_{M_{Bi}})(1 - \hat{w}_{M_{Bi}}) + (\hat{w}_{M_{Bj}} - \hat{w}_{M_{Bi}}) - \hat{T}_{M_{Ai}} - \hat{T}_{M_{Bi}}$. Using (3.2) to obtain $\hat{T}_{M_{Ai}} + \hat{T}_{M_{Bi}} \geq 1/4 + F_i + c_M - \hat{w}_{M_{Ai}}(1 - \hat{w}_{M_{Bi}}) - \hat{w}_{M_{Bi}}$ and the fact that $\pi_{Ri} \geq 0$ and $\hat{w}_{M_{Ai}} - \hat{w}_{M_{Bi}} \leq 1/4$, we arrive that in equilibrium $\pi_{Ri}^* \leq \hat{w}_{M_{Bj}} - F_i - c_M$ and $\hat{w}_{M_{Bj}} \geq F_i + c_M$. But if so, $S$ would profitably deviate by approaching $Ri$ with the offer $(w_{M_{Bi}}^* = F_i + c_M - \epsilon, T_{M_{Bi}}^* = 0)$ with $\epsilon \to 0$, which $Ri$ would be ready to take since $\hat{w}_{M_{Bj}}^* = F_i - c_M + \epsilon > \hat{w}_{M_{Bj}}^* - F_i - c_M$ (recall that $p_{M_{Bi}}$ remains at $\hat{w}_{M_{Bj}}^*$); a contradiction.

Consider now the case in which equilibrium offers are such that $\hat{w}_{M_{Bi}}^* > \hat{w}_{M_{Bj}}^*$, which implies (for $\mu = 0$) that $\pi_{Ri}^* = (\hat{w}_{M_{Ai}} - \hat{w}_{M_{Bi}})(1 - \hat{w}_{M_{Bi}}) - \hat{T}_{M_{Ai}} - \hat{T}_{M_{Bi}}$ and $\pi_{Rj}^* = (\hat{w}_{M_{Bi}} - \hat{w}_{M_{Bj}}) - \hat{T}_{M_{Ai}} - \hat{T}_{M_{Bi}}$. Again, using (3.2) to obtain $\sum_{k,i} \hat{T}_{M_{ki}} \geq 1/4 + F_i + c_M - \hat{w}_{M_{Ai}}(1 - \hat{w}_{M_{Bj}}) - \hat{w}_{M_{Bj}}$ and the fact that $\pi_{Ri}^* \geq 0, \pi_{Rj}^* \geq 0$ and $\hat{w}_{M_{Ai}}(1 - \hat{w}_{M_{Bj}}) \leq 1/4$, we arrive that in equilibrium $\pi_{Rj}^* \leq \hat{w}_{M_{Bi}}^* - F_i - c_M$ and $\hat{w}_{M_{Bi}}^* \geq F_i + c_M$. But if so, $S$ would profitably deviate by approaching $Rj$ with the offer $(w_{M_{Bi}}^* = F_i + c_M - \epsilon, T_{M_{Bi}}^* = 0)$ with $\epsilon \to 0$, which $Rj$ would be ready to take; a contradiction.

3.2.2 Horizontal differentiation in product $B$

Suppose that a fraction $\gamma/2$ of consumers who value $B$ (regardless of their valuation for $A$) have a strong preference for $M$’s variety of product $B$, so much that they value $S$’s variety of product $B$ in zero. Another $\gamma/2$ of consumers who value $B$ have reverse preferences, that is, they only value $S$’s variety of product $B$. The remaining $1 - \gamma$ fraction of these consumers see no difference between the two varieties of $B$. We this formulation we can cover the entire range of horizontal differentiation in relatively simple fashion, from no differentiation ($\gamma = 0$) to complete differentiation ($\gamma = 1$).

Assume further that neither manufacturer needs the rival’s “captive consumers” to operate profitably in the market 

$$F_S < F_M \leq b \left(1 - \frac{\gamma}{2}\right)$$

Note that it may be efficient to sell both versions of $B$ if $F_M$ is sufficiently small, that is, if $F_M \leq \gamma/2$. This will not be relevant when looking at foreclosure under Bertrand retailers but it will be when looking at foreclosure under monopoly retailers.
Revisiting Proposition 1

We start by computing the full-monopolization outcome. As in the baseline model, the retail prices that implement this outcome are \( p_A^* = 1/2, p_B^* = b, p_{AB}^* = 1/2 + b \), resulting in \( \Pi_M^* = 1/4 + b(1 - \gamma/2) - F_M \).

Suppose now that \( M \) approaches the two Bertrand retailers with the full-line forcing offer

\[
\begin{align*}
C_{Ai}^M &= (u_{Ai}^M = 1, T_{Ai}^M = 0), \quad C_{Bi}^M = (u_{Bi}^M = b, T_{Bi}^M = 0) \\
\hat{c}_{Ai}^M &= (\hat{u}_{Ai}^M = 1/2, \hat{T}_{Ai}^M = 0), \quad \hat{c}_{Bi}^M = (\hat{u}_{Bi}^M = b, \hat{T}_{Bi}^M = 0)
\end{align*}
\] 

\( i = 1, 2 \).

Following the proof of Lemma 2 in the text, one can show that \( S \) can at best persuade one retailer, say \( R_1 \), to stock his product. This implies

**Lemma 3.2.1.** Suppose that \( M \) approaches each retailer with the full-line forcing contract \( (3.3) \). The largest revenue \( S \) can obtain from the retail market is

\[
b \left( 1 - \frac{\gamma}{2} \right) - \frac{\mu b}{8} (2 - \gamma)
\]

**Proof.** When \( R_2 \) carries products \( A \) and \( B \) from \( M \), while \( R_1 \) only carries \( B \) from \( S \), then for an arbitrary pair of (feasible) prices \( \{p_{B1}, (p_{A2}, p_{B2}, p_{AB2})\} \), with \( p_{AB2} \leq p_{A2} + p_{B2} \) we have that retailers’ demands are given by

\[
D_{A1}^{(1-\mu)}(\cdot) = 0 \quad \quad D_{A2}^{(1-\mu)}(\cdot) = 1 - p_{A2}
\]

\[
D_{B1}^{(1-\mu),M} = 0 \quad \quad D_{B1}^{(1-\mu),M/S} = 1 \{p_{B1} < p_{B2}\} \quad \quad D_{B1}^{(1-\mu),S} = 1 \{p_{B1} \leq b\}
\]

\[
D_{B2}^{(1-\mu),M} = 1 \{p_{B2} \leq b\} \quad \quad D_{B2}^{(1-\mu),M/S} = 1 \{p_{B2} < p_{B1}\} \quad \quad D_{B2}^{(1-\mu),S} = 0
\]

\[
D_{AB1}^{(1-\mu),M} = 0 \quad \quad D_{AB1}^{(1-\mu),M/S} = 0 \quad \quad D_{AB1}^{(1-\mu),S} = 0
\]

\[
D_{AB2}^{(1-\mu),M} = 1 - p_{AB2} + p_{B2} \quad \quad D_{AB2}^{(1-\mu),M/S} = 1 - p_{AB2} + \min\{p_{B1}, p_{B2}\} \quad \quad D_{AB2}^{(1-\mu),S} = 1 - (p_{A2} + b) + p_{B1}
\]

\[
D_{B1}^{(\mu),M} = 0 \quad \quad D_{B1}^{(\mu),M/S} = (p_{AB2} - p_{B1}) 1 \{p_{B1} < p_{B2}\} \quad \quad D_{B1}^{(\mu),S} = b + p_{A2} - p_{B1}
\]

\[
D_{B2}^{(\mu),M} = p_{AB2} - p_{B2} \quad \quad D_{B2}^{(\mu),M/S} = (p_{AB2} - p_{B2}) 1 \{p_{B2} < p_{B1}\} \quad \quad D_{B2}^{(\mu),S} = 0
\]

where \( D_{k1}^{(g),h} \) is retailer’s \( i = 1, 2 \) demand for choice \( k \in \{B, AB\} \) from six distinct groups of consumers, namely, those that value both products but only \( M \)’s version of product \( B \) \( (g = \mu, h = M) \), those that value both products and see no difference between the two versions of product \( B \) \( (g = \mu, h = M/S) \), those that value both products but only \( S \)’s version of product \( B \) \( (g = \mu, h = S) \), those that value only \( M \)’s version of product \( B \) \( (g = 1 - \mu, h = M) \), those that value only product \( B \) in either of its two versions \( (g = 1 - \mu, h = M/S) \), and those that value only \( S \)’s version of product \( B \) \( (g = 1 - \mu, h = S) \).

Now, suppose \( M \) offers \( (3.3) \), while \( S \) approaches a single retailer, say \( R_1 \), with the offer \( (w_{B1}^S < b, T_{B1}^S) \). Notice that \( R_1 \)’s outside option (i.e., payoff from taking \( M \)’s contract) is zero, so \( S \) can leave \( R_1 \) with no profit. Now, to calculate \( T_{B1}^S \), consider the continuation play
after $R1$ signs with $S$ (so $w_{B1} = w_{B1}^S < b$), while $R2$ signs with $M$ ($w_{A2} = w_{A2}^M = 1/2$ and $w_{B2} = w_{B2}^M = b$). Using the previously derived demands, we now compute the retail-pricing equilibrium.

Since $w_{B2}^M = b$, it is clear that $p_{B2}^* = b$. And since $w_{B1}^S < b$, it is also clear that $p_{B1}^* \leq b - \epsilon < p_{B2}^*$, with $\epsilon \to 0$. This implies that in equilibrium $R1$ must be solving
\[
\max_{p_{B1} \leq b - \epsilon} (1 - \mu)(p_{B1} - w_{B1}^S)(1 - \gamma/2) + \mu(p_{B1} - w_{B1}^S)[(1 - \gamma)(p_{AB2} - p_{B1}) + \gamma(p_{A2} + b - p_{B1})/2]
\]
which leads to
\[
p_{B1}^{BR}(p_{A2}, p_{AB2}) = \min \left\{ b - \epsilon, \frac{1}{2}(p_{AB2} + w_{B1}^S) + \frac{1 - \mu}{2\mu} + \frac{\gamma}{2(2 - \gamma)}(b + p_{A2} - p_{AB2}) \right\}
\]
On the other hand, $R2$ must be solving
\[
\max_{p_{A2}, p_{AB2} \leq p_{A2} + b} (1 - \mu)(p_{A2} - 1/2)(1 - p_{A2}) + \mu(p_{A2} - 1/2)\gamma(1 - p_{A2} - b + p_{B1})/2 + \\
\mu(p_{AB2} - 1/2 - b)(1 - \gamma)(1 - p_{AB2} + p_{B1}) + \gamma(1 - p_{AB2} + b)/2
\]
which leads to
\[
p_{A2}^{BR}(p_{B1}) = \frac{3}{4} - \frac{1}{2} \left( \frac{\mu\gamma}{2(1 - \mu) + \mu\gamma} \right) (b - p_{B1})
\]
and
\[
p_{A2}^{BR}(p_{B1}) = b + \frac{3}{4} - \left( \frac{1 - \gamma}{2 - \gamma} \right) (b - p_{B1})
\]
_intersectioning these best-responses, the retail pricing equilibrium is characterized by:

- If $w_{B1}^S \geq b + 1/4 - 1/\mu \equiv w_B$, then
  
  \[ p_{A2}^* = 3/4, p_{B2}^* = b, p_{AB2}^* = b + 3/4, \text{ and } p_{B1}^* = b - \epsilon \]

  with $\epsilon \to 0$.

- If $w_{B1}^S < w_B$, then $p_{B2}^* = b$ and
  
  \[ p_{A2}^* = \frac{3}{4} - \left[ \frac{\gamma\mu(2 - \gamma)^2}{24 - 16\gamma - \mu(6 - \gamma)(2 - \gamma)^2} \right] (w_B - w_{B1}^S); \]

  \[ p_{AB2}^* = p_{A2}^* + b - \frac{(2 - \gamma)[4(1 - \gamma) - \mu(2 - \gamma)]}{26 - 16\gamma - \mu(6 - \gamma)(2 - \gamma)^2} (w_B - w_{B1}^S); \]

  \[ p_{B1}^* = b - \frac{2(2 - \gamma)^2[2 - \mu(2 - \gamma)]}{24 - 16\gamma - \mu(6 - \gamma)(2 - \gamma)^2} (w_B - w_{B1}^S) \]

From the above expressions, it is not difficult to prove that $S$’s optimal contract involves $w_{B1}^S \geq w_B$ and $T_{B1}^S = \pi_{R1}^*$ (for instance $w_{B1}^S = b - \epsilon$, and $T_{B1}^S = 0$). Thus, for all $w_{B1}^S \in [w_B, b)$ we have that

\[ \pi_{R1}^* = \frac{1}{3}(2 - \gamma)(4 - \mu)(b - w_{B1}^S) \]

so

\[ \pi_S^* = b \left( 1 - \frac{\gamma}{2} \right) - \frac{\mu b}{8} (2 - \gamma) \]

which concludes the proof. ■
Lemma 3.2.2. If
\[ b \left( 1 - \frac{\gamma}{2} \right) - \frac{\mu b}{8} (2 - \gamma) \leq F_S < F_M \leq b \left( 1 - \frac{\gamma}{2} \right), \]
then there exists an equilibrium in which S leaves the market and M fully monopolizes the market with the wholesale bundling contracts \( \text{[3.3]} \).

Proof. Immediate from the previous lemma. ■

Revisiting Proposition 2a

We use the same approach as in subsection 3.1.5. In this model, optimal industry profits in each retail location clearly satisfy symmetry (i.e. \( \Pi^{(x)}_1 = \Pi^{(x)}_2 \equiv \Pi^{(x)}(y) \)), and separability (i.e. \( \Pi^{(x)}(y) = \Pi^{(x)} + \Pi^{(0)}(y) \)). Furthermore \( \Pi^{(A)(y)} = 1/8 > 0 \) (satisfying A1*), \( \Pi^{(S)}(M) = \Pi^{(S)} = (b/2)(1 - \gamma/2) \) and \( \Pi^{(M)} = b/2 > 0 \) (satisfying A2*). Furthermore, \( \Pi^{(A)}(S) + \Pi^{(M)}(M) = \Pi^{(M)}(S) - 2\Pi^{(A)}(M) = (b/2)(1 - \gamma) > 0 \) (satisfying A3*); \( 2\Pi^{(S)} - F_S = b(1 - \gamma/2) > F_S > 2\Pi^{(M)} - F_M = b(1 - \gamma/2) - F_M \) (satisfying A4*). Since the model satisfies assumptions A1 to A4 we can use Lemma 3.1.4 and conclude that in equilibrium S is never excluded from the market.

Furthermore, notice that \( 2\Pi^{(A)}(M) - \Pi^{(A)}(S) = b\gamma/2 \). Hence using the same lemma we conclude that if \( F_M > b\gamma/2 \) in any equilibrium both retailers only represent S in product-line B. While if \( F_M \leq b\gamma/2 \), there is an equilibrium in which both retailers represent both manufacturers in product-line B, and that this equilibrium Pareto dominates any equilibrium in which one or both retailers only represent S in product-line B.

Revisiting Proposition 2b

Following the proof of Proposition 2b in the text, we restrict attention to full-line forcing contracts (see also Appendix H in the text). Suppose the following pair of contracts constitute a foreclosure equilibrium: \( \{(w^M_{A1} = 1, T^M_{A1} = 0), (w^M_{B1} = b, T^M_{B1} = 0)\} \) and \( \{(\hat{w}^M_{A1}, \hat{T}^M_{A1}, (\hat{w}^M_{B1}, \hat{T}^M_{B1})\} \) for \( i = 1, 2 \). Suppose further that wholesale prices are such that \( p^*_{A1} = \max\{\hat{w}^M_{A1}, \hat{w}^M_{B1}\} \) and \( p^*_{B1} = \max\{\hat{w}^M_{B1}, \hat{w}^M_{B1}\} \) (the logic of the proof prevails for other cases), so M’s equilibrium payoff can be written as

\[
\pi^*_M = \min\{\hat{w}^M_{A1}, \hat{w}^M_{B1}\}(1 - \max\{\hat{w}^M_{A1}, \hat{w}^M_{B1}\}) + \hat{T}^M_{A1} + \hat{T}^M_{B1} + (1 - \gamma/2)\min\{\hat{w}^M_{B1}, \hat{w}^M_{B1}\} + \hat{T}^M_{B1} - F_M \tag{3.4}
\]

which must be at least 1/4.

Without loss of generality let \( \hat{w}^M_{A1} \leq \hat{w}^M_{B1} \). Consider first the case in which equilibrium offers are such that \( \hat{w}^M_{B1} \leq \hat{w}^M_{B1} \), which implies (for \( \mu = 0 \) that \( \hat{T}^M_{A1} = \hat{T}^M_{B1} = \pi^*_R \geq 0 \) and \( \pi^*_R = (\hat{w}^M_{A1} - \hat{w}^M_{B1})(1 - \hat{w}^M_{A1}) + (1 - \gamma/2)(\hat{w}^M_{B1} - \hat{w}^M_{A1}) - \hat{T}^M_{A1} - \hat{T}^M_{B1} \). Using (3.4) to obtain \( \hat{T}^M_{A1} + \hat{T}^M_{B1} \leq 1/4 + F_M - \hat{w}^M_{A1}(1 - \hat{w}^M_{A1}) - (1 - \gamma/2)\hat{w}^M_{B1} \) and the fact that \( \pi^*_R \geq 0 \) and \( \hat{w}^M_{A1}(1 - \hat{w}^M_{A1}) \leq 1/4 \), we arrive that in equilibrium \( \pi^*_R \leq (1 - \gamma/2)\hat{w}^M_{B1} - F_M \) and \( (1 - \gamma/2)\hat{w}^M_{B1} \geq F_M \). But if so, S
would profitably deviate by approaching \( R_i \) with the offer \((w_{Bi}^{St} = F_M - \epsilon, T_{Bi}^{St} = 0)\) with \( \epsilon \to 0 \), which \( R_i \) would be ready to take since \((1 - \gamma/2)\hat{w}_{Bj}^M - F_M + \epsilon > (1 - \gamma/2)\hat{w}_{Bj}^M - F_M\) (recall that \( p_{Bi} \) remains at \( \hat{w}_{Bj}^M \)); a contradiction.

Consider now the case in which equilibrium offers are such that \( \hat{w}_{Bi}^M > \hat{w}_{Bj}^M \), which implies (for \( \mu = 0 \)) that \( \pi_{R_i}^* = (\hat{w}_{Aj}^M - \hat{w}_{Ai}^M)(1 - \hat{w}_{Aj}^M) - \hat{T}_{Ai}^M - \hat{T}_{Bi}^M \) and \( \pi_{R_j}^* = (1 - \gamma/2)(\hat{w}_{Bi}^M - \hat{w}_{Bj}^M) - \hat{T}_{Aj}^M - \hat{T}_{Bj}^M \). Again, using (3.4) to obtain \( \sum_{k,i} \hat{T}_{ki}^M \geq 1/4 + F_M - \hat{w}_{Ai}^M(1 - \hat{w}_{Aj}^M) - (1 - \gamma/2)\hat{w}_{Bj}^M \) and the fact that \( \pi_{R_i}^* \geq 0 \), \( \pi_{R_i}^* \geq 0 \) and \( \hat{w}_{Aj}^M(1 - \hat{w}_{Aj}^M) \leq 1/4 \), we arrive that in equilibrium \( \pi_{R_j}^* \leq (1 - \gamma/2)\hat{w}_{Bi}^M - F_M \) and \( (1 - \gamma/2)\hat{w}_{Bi}^M \geq F_M \). But if so, \( S \) would profitably deviate by approaching \( R_j \) with the offer \((w_{Bj}^{St} = F_M - \epsilon, T_{Bj}^{St} = 0)\) with \( \epsilon \to 0 \), which \( R_j \) would be ready to take; a contradiction.

3.2.3 Vertical differentiation in product \( B \)

Consumers who value product \( B \) (regardless of their valuation for product \( A \)) see its available versions as vertically distinct products. They value \( M \)'s version in \( b \in (0,1) \) while \( S \)'s in \( b + \Delta \), where \( \Delta \in [0,1/2] \). The rest of the model is as in the baseline model.

Revisiting Proposition 1

We start by computing the full-monopolization outcome. Since up to here there is no difference with the baseline model, the retail prices that implement the full-monopolization outcome are again \( p_A^* = 1/2 \), \( p_B^* = b \), \( p_{AB}^* = 1/2 + b \), resulting in \( \Pi_M^* = 1/4 + b - F_M \).

Suppose now that \( M \) approaches the two Bertrand retailers with the full-line forcing offer

\[
\begin{align*}
C_{Ai}^M &= (w_{Ai}^M = 1, T_{Ai}^M = 0), \quad C_{Bi}^M = (w_{Bi}^M = b, T_{Bi}^M = 0) \\
\hat{C}_{Ai}^M &= (w_{Ai}^M = 1/2, T_{Ai}^M = 0), \quad \hat{C}_{Bi}^M = (\hat{w}_{Bi}^M = b, \hat{T}_{Bi}^M = 0)
\end{align*}
\]

for \( i = 1,2 \). Following the proof of Lemma 2 in the text, one can show that \( S \) can at best persuade one retailer, say \( R1 \), to stock his product.

**Lemma 3.2.3.** Suppose that \( M \) approaches each retailer with the full-line forcing contract \((3.5)\). The largest revenue \( S \) can obtain from the retail market is

\[
(b + \Delta) \left( 1 - \frac{H}{4} \right)
\]

**Proof.** When \( R2 \) carries products \( A \) and \( B \) from \( M \), while \( R1 \) only carries \( B \) from \( S \), then for an arbitrary pair of (feasible) prices \( \{p_{B1}, (p_{A2}, p_{B2}, p_{AB2})\} \), with \( p_{AB2} \leq p_{A2} + p_{B2} \) we have that retailers’ demands are given by

\[
\begin{align*}
D_{A1}^{(1-\mu)} &= 0 \\
D_{B1}^{(1-\mu)} &= 1 \{p_{B1} \leq \Delta + p_{B2}\} \\
D_{AB1}^{(\mu)} &= 0 \\
D_{A2}^{(1-\mu)} &= 1 - p_{A2} \\
D_{B2}^{(1-\mu)} &= 1 \{p_{B2} \leq p_{B1} - \Delta\} \\
D_{AB2}^{(\mu)} &= \max\{0, 1 - p_{AB2} + \min\{p_{B1} - \Delta, p_{B2}\}\}
\end{align*}
\]

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And

\[
D_{B1}^{(\mu)} = \min \{1, p_{AB2} - p_{B1} + \Delta \} \{p_{B1} \leq \Delta + p_{B2}\}
\]

\[
D_{B2}^{(\mu)} = \min \{1, p_{AB2} - p_{B2}\} \{p_{B2} \leq p_{B1} - \Delta\}
\]

Now, suppose \(M\) offers \([3,5]\), while \(S\) approaches a single retailer, say \(R1\), with the offer \((w_{B1}^S < b, T_{B1}^S)\). Notice that \(R1\)'s outside option (i.e., payoff from taking \(M\)'s contract) is zero, so \(S\) can leave \(R1\) with no profit. Now, to calculate \(T_{B1}^S\), consider the continuation play after \(R1\) signs with \(S\) (so \(w_{B1} = w_{B1}^S < b\), while \(R2\) signs with \(M\) (\(w_{A2} = w_{A2}^M = \frac{1}{2}\) and \(w_{B2} = w_{B2}^M = b\)). Using the previously derived demands, we now compute the retail-pricing equilibrium.

Since \(w_{B2}^M = b\), it is clear that \(p_{B2}^* = b\). And since \(w_{B1}^S < b\), it is also clear that \(p_{B1}^* \leq p_{B2}^* + \Delta - \epsilon = b + \Delta - \epsilon\), with \(\epsilon \to 0\). This implies that in equilibrium \(R1\) must be solving

\[
\max_{p_{B1} \leq b + \Delta - \epsilon} (1 - \mu)(p_{B1} - w_{B1}^S) + \mu(p_{B1} - w_{B1}^S) \min \{1, p_{AB2} - p_{B1} + \Delta\}
\]

while \(R2\) must be solving

\[
\max_{p_{A2}, p_{AB2} \leq b + p_{A2}} \{1 - \mu\}(p_{A2} - \frac{1}{2})(1 - p_{A2}) + \mu(p_{AB2} - 1/2 - b) \max \{0, 1 - p_{AB2} - \Delta + p_{B1}\}
\]

It is then possible to prove that the equilibrium of the retail-pricing subgame takes the following form (as it will be apparent shortly, note that the assumption \(\Delta \leq 1/2\) is to ensure that \(1 - p_{AB2}^* - \Delta + p_{B1}^* > 0\) for all \(w_{B1}^S\); letting \(\Delta > 1/2\) would require going over additional cases without altering the main message):

- If \(w_{B1}^S \geq \max \{0, b + \Delta + 1/4 - 1/\mu\} \equiv w_B\), then
  \[p_{B1}^* = b + \Delta - \epsilon, p_{A2}^* = \frac{3}{4}, p_{B2}^* = b, \text{ and } p_{AB2}^* = \frac{3}{4} + b\]
  (note that \(1 - p_{AB2}^* - \Delta + p_{B1}^* = 1/4\))

- If \(w_{B1}^S < w_B\), then
  \[p_{B1}^* = b + \Delta - \frac{2}{3}(w_B - w_{B1}^S), p_{A2}^* = \frac{3}{4}, p_{B2}^* = b, \text{ and } p_{AB2}^* = \frac{3}{4} + b - \frac{1}{3}(w_B - w_{B1}^S)\]
  (note that \(1 - p_{AB2}^* - \Delta + p_{B1}^* = 1/6 - (b + \Delta - w_{B1}^S)/3 + 1/3\mu \geq 1/2 - (b + \Delta)/3\), which is greater than zero provided that \(b \leq 1\) and \(\Delta \leq 1/2\))

Finally, from the above expressions, it is not difficult to prove that \(S\)'s optimal contract involves setting \(w_{B1}^S \in [w_B, b - \epsilon]\) and \(T_{B1}^S\) accordingly to obtain (recall that \(R1\)'s outside option is zero):

\[
\pi_S^* = (b + \Delta) \left(1 - \frac{\mu}{4}\right)
\]

which concludes the proof. ■

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Lemma 3.2.4. If
\[(b + \Delta) \left(1 - \frac{\mu}{4}\right) \leq F_S < F_M \leq b,\]
then there exists an equilibrium in which $S$ leaves the market and $M$ fully monopolizes the market with the wholesale bundling contracts (3.5).

Proof. Immediate from the previous lemma. ■

Revisiting Proposition 2a

We use the same approach as in subsection 3.1.5. In this model, optimal industry profits in each retail location clearly satisfy symmetry (i.e. $\Pi^{(x)}_S = \Pi^{(x)}_i \equiv \Pi^{(x)}(y)$), and separability (i.e. $\Pi^{(x)}(y) = \Pi^{(x)}(\emptyset) + \Pi^{(0)}(y)$). Furthermore $\Pi^{(A)}(\emptyset) = 1/8 > 0$ (satisfying A1*), $\Pi^{(0)}(M) = b/2 + 0$, $\Pi^{(0)}(S) = \Pi^{(0)}(M) = (1/2)(b + Delta) > 0$ (satisfying A2*). Furthermore, $\Pi^{(A)}(S) + \Pi^{(A)}(M) + \Pi^{(0)}(M) - 2\Pi^{(A)}(M) = b/2 > 0$ (satisfying A3*); $2\Pi^{(x)}(S) - F_S = b + \Delta - F_S > 2\Pi^{(x)}(M) - F_M = b - F_M$ (satisfying A4*). Since the model satisfies assumptions A1* to A4* we can use Lemma 3.1.4. Given that $2(\Pi^{(A)}(M) - \Pi^{(A)}(S)) = 0 < F_M$ we conclude then that in equilibrium both retailers only represent $S$ in product-line $B$.

Revisiting Proposition 2b

Following the proof of Proposition 2b in the text, we restrict attention to full-line forcing contracts (see also Appendix H in the text). Suppose the following pair of contracts constitute a foreclosure equilibrium: $\{(w^{M}_{A_i} = 1, T^{M}_{A_i} = 0), (w^{M}_{B_i} = b, T^{M}_{B_i} = 0)\}$ and $\{(\hat{w}^{M}_{A_i}, \hat{T}^{M}_{A_i}, (\hat{w}^{M}_{B_i}, \hat{T}^{M}_{B_i})\}$ for $i = 1, 2$. Suppose further that wholesale prices are such that $p^{*}_{A_i} = \max\{\hat{w}^{M}_{A_i}, \hat{w}^{M}_{A_j}\}$ and $p^{*}_{B_i} = \max\{\hat{w}^{M}_{B_i}, \hat{w}^{M}_{B_j}\}$ (the logic of the proof prevails for other cases), so $M$’s equilibrium payoff can be written as

$$
\pi^{*}_{M} = \min\{\hat{w}^{M}_{A_i}, \hat{w}^{M}_{A_j}\}(1 - \max\{\hat{w}^{M}_{A_i}, \hat{w}^{M}_{A_j}\}) + \hat{T}^{M}_{A_i} + \hat{T}^{M}_{A_j} + \min\{\hat{w}^{M}_{B_i}, \hat{w}^{M}_{B_j}\} + \hat{T}^{M}_{B_i} + \hat{T}^{M}_{B_j} - F_M,
$$

which must be at least $1/4$.

Without loss of generality let $\hat{w}^{M}_{A_i} \leq \hat{w}^{M}_{A_j}$. Consider first the case in which equilibrium offers are such that $\hat{w}^{M}_{B_i} \leq \hat{w}^{M}_{B_j}$, which implies (for $\mu = 0$) that $\hat{T}^{M}_{A_j} = \hat{T}^{M}_{B_j} = \pi^{*}_{R_j} = 0$ and $\pi^{*}_{R_i} = (\hat{w}^{M}_{A_j} - \hat{w}^{M}_{A_j})(1 - \hat{w}^{M}_{B_j}) + (\hat{w}^{M}_{B_j} - \hat{w}^{M}_{B_j}) - \hat{T}^{M}_{A_i} - \hat{T}^{M}_{B_i}$. Using (3.6) to obtain $\hat{T}^{M}_{A_i} + \hat{T}^{M}_{B_i} \geq 1/4 + F_M - \hat{w}^{M}_{A_j}(1 - \hat{w}^{M}_{B_j}) - F_M$ and the fact that $\pi^{*}_{R_i} \geq 0$ and $\hat{w}^{M}_{A_j}(1 - \hat{w}^{M}_{A_j}) \leq 1/4$, we arrive that in equilibrium $\pi^{*}_{R_i} \leq \hat{w}^{M}_{B_j} - F_M$ and $\hat{w}^{M}_{B_j} \geq F_M$. But if so, $S$ would profitably deviate by approaching $R_i$ with the offer $(w^{S}_{B_i} = F_M + \epsilon, T^{S}_{B_i} = 0)$ with $\epsilon \to 0$, which $R_i$ would be ready to take since $\hat{w}^{M}_{B_j} - F_M + \epsilon > \hat{w}^{M}_{B_j} - F_M$ (recall that $p^{*}_{Bi}$ remains at $\hat{w}^{M}_{B_j}$); a contradiction.

Consider now the case in which equilibrium offers are such that $\hat{w}^{M}_{B_i} > \hat{w}^{M}_{B_j}$, which implies (for $\mu = 0$) that $\pi^{*}_{R_i} = (\hat{w}^{M}_{A_j} - \hat{w}^{M}_{A_j})(1 - \hat{w}^{M}_{A_j}) - \hat{T}^{M}_{A_j} - \hat{T}^{M}_{B_j}$ and $\pi^{*}_{R_j} = (\hat{w}^{M}_{B_j} - \hat{w}^{M}_{B_j}) - \hat{T}^{M}_{A_j} - \hat{T}^{M}_{B_j}$. Again, using (3.6) to obtain $\sum_{k, i} \hat{T}^{M}_{ki} \geq 1/4 + F_M - \hat{w}^{M}_{A_j}(1 - \hat{w}^{M}_{A_j}) - \hat{w}^{M}_{B_j}$ and the fact that $\pi^{*}_{R_i} \geq 0$,
\[ \pi_{Ri}^* \geq 0 \text{ and } \hat{w}_{M}^{M}(1 - \hat{w}_{M}^{M}) \leq 1/4, \text{ we arrive that in equilibrium } \pi_{Rj}^* \leq \hat{w}_{M}^{M} - F_{M} \text{ and } \hat{w}_{M}^{M} \geq F_{M}. \]

But if so, \( S \) would profitably deviate by approaching \( Rj \) with the offer \( (w_{Bj}^{St} = F_{M} - \epsilon, T_{Bj}^{St} = 0) \) with \( \epsilon \to 0 \), which \( Rj \) would be ready to take; a contradiction.

### 3.2.4 Heterogenous valuations for good \( B \)

Consider the case in which \( v_B \) also distributes uniformly over the unit interval. And to continue ruling out any price discrimination at the retail level, further assume a perfectly positive correlation in valuations, that is, \( v_B = v_A \) for any consumer that have preferences for both goods. Since the maximum profit to be made in market \( B \) is \( 1/4 \) now, we also assume that \( F_S < F_M < 1/4 \).

#### Revisiting Proposition 1

Consider the case of Bertrand retailers. We first offer some preliminaries of the retail subgame and then, following Proposition 1 in the text, conditions under which \( M \) can fully monopolize the market while excluding \( S \).

**Retailers’ demand** Given retail prices \( p_{Ai}, p_{Bi} \) and \( p_{ABi} \leq p_{Ai} + p_{Bi} \) charged by retailer \( i = 1, 2 \), demands from consumers with preferences for only one good \( k = A, B \) are rather simple to obtain:

\[
D_{k}^{(1-\mu)}(p_{k}) = [1 - \min(p_{ki}, p_{kj})]T_{k}(p_{ki}, p_{kj}, w_{ki}, w_{kj})
\]

where \( p_{k} = (p_{ki}, p_{kj}) \), and \( T_{k}(p_{ki}, p_{kj}, w_{ki}, w_{kj}) \) is an indicator that can take three values (this also applies for \( k = AB \)): 1 if \([p_{ki} < p_{kj}] \cup [(p_{ki} = p_{kj}] \cap [w_{ki} < w_{kj}]], 1/2 \text{ if } [p_{ki} = p_{kj}] \cap [w_{ki} = w_{kj}], \text{ and 0 otherwise. Note that superscript } “(1-\mu)” \text{ is used to denote demand from consumers that value only one good, either } A \text{ or } B, \text{ and } “(\mu)” \text{ will be used to denote demand from consumers that value both goods.}

Obtaining demands from consumers with preferences for both products is more involved since their option set expands to \( k = A, B, AB \). Because of the existence of one-stop shopping, these individuals must make two decisions: (1) which retailer to visit; and (2) whether to purchase one or multiple products. Of course both decisions are interrelated, since a particular consumer decides which store to visit comparing the maximum utility he can obtain at each one of them, which in turn depends on whether he plans to purchase one or multiple products.

Now, consider a consumer who is planning on visiting \( Ri \). It is easy to see that because of perfect correlation in valuations, if a consumer purchases only a single product then he will always purchase the one with the lowest standalone price. Hence, the consumer’s decision after arriving to \( Ri \) can be simplified into (a) purchasing both goods to get \( u_X = 2v - \min(p_{ABi}, p_{Ai} + p_{Bi}) \); or (b) purchase a single good to obtain \( u_Y = v - \min(p_{Ai}, p_{Bi}) \).
But then, it is easy to see that competition for these type of consumers is not really in terms of products \( A \) and \( B \) individually, but rather in two different retail “bundles”, \( X \) and \( Y \) that differ in size (i.e. the number of products the bundle contains), so denote \( p_{X_i} \equiv \min\{p_{ABi}, p_{Ai} + p_{Bi}\} \), and \( p_{Y_i} \equiv \min\{p_{Ai}, p_{Bi}\} \) as the prices \( Ri \) charges for \( X \) and \( Y \), respectively. Bertrand competition then implies that all consumers buying \( X \) will purchase from the same lowest price retailer, and the same goes to the disjoint set of consumers purchasing \( Y \). The final step is determining the fraction of consumers purchasing each product: all consumers for whom \( 2v - \min\{p_{X_1}, p_{X_2}\} \geq v - \min\{p_{Y_1}, p_{Y_2}\} \), that is, \( v \geq \min\{p_{X_1}, p_{X_2}\} - \min\{p_{Y_1}, p_{Y_2}\} \) will purchase \( X \); while all consumers with valuations \( \min\{p_{Y_1}, p_{Y_2}\} \leq v \leq \min\{p_{X_1}, p_{X_2}\} - \min\{p_{Y_1}, p_{Y_2}\} \) will buy \( Y \).

Therefore, letting \( p_X \equiv (p_{X_1}, p_{X_2}) \), and \( p_Y \equiv (p_{Y_1}, p_{Y_2}) \), \( Ri \)'s demand stemming from consumers that value both goods is given by:

\[
D_{X_i}^{(\mu)}(p_X, p_Y) = \left[ 1 - \min\{p_{X_i}, p_{Y_i}\} + \min\{p_{Y_i}, p_{X_i}\} \right] T_X(p_{X_i}, p_{X_j}, w_{Xi}, w_{Xj})
\]

\[
D_{Y_i}^{(\mu)}(p_X, p_Y) = \left[ \max\{0, \min\{p_{X_i}, p_{X_j}\} - 2 \min\{p_{Y_i}, p_{Y_j}\}\} \right] T_Y(p_{Y_i}, p_{Y_j}, w_{Yi}, w_{Yj})
\]

where \( w_{X_i} = w_{Ai} + w_{Bi} \) and

\[
 w_{Y_i} \equiv w_{Ai}1\{p_{Ai} < p_{Bi}\} + w_{Bi}[1 - 1\{p_{Ai} < p_{Bi}\}] + \min\{w_{Ai}, w_{Bi}\}1\{p_{Ai} = p_{Bi}\}
\]

**Retailers’ profit** In the retail pricing subgame \( Ri \)'s action space is given by the triple \((p_{Ai}, p_{Bi}, p_{ABi})\) which, using the definitions \( p_{X_i} \equiv \min\{p_{ABi}, p_{Ai} + p_{Bi}\} \), and \( p_{Y_i} \equiv \min\{p_{Ai}, p_{Bi}\} \), induces a tuple \((p_{Ai}, p_{Bi}, p_{Xi}, p_{Yi})\). By convention we say that \( w_{ki} = +\infty \) (and therefore \( p_{ki} = p_{Xi} = +\infty \) necessarily) if \( Ri \) does not carry product \( k \), and that at infinite prices demand drops to zero. \( Ri \)'s profit, before any fixed fees, can then be written as:

\[
\bar{\pi}_{Ri} = (1 - \mu)(p_{Ai} - w_{Ai})D_{Ai}^{(1-\mu)}(p_A) + (1 - \mu)(p_{Bi} - w_{Bi})D_{Bi}^{(1-\mu)}(p_B) + \mu(p_{Xi} - w_{Ai} - w_{Bi})D_{Xi}^{(\mu)}(p_X, p_Y) + \mu(p_{Yi} - w_{Yi})D_{Yi}^{(\mu)}(p_X, p_Y)
\]

Unfortunately, a pure-strategy Nash equilibrium might fail to exist for some combinations of wholesale prices \((w_{Ai}, w_{Aj}, w_{Bi}, w_{Bj})\). It is easily established however, that a mixed strategy equilibrium always exists.\(^5\)

**Equilibrium at selected retail-pricing subgames** We now provide two lemmas that characterize the equilibrium at selected retail-pricing subgames to be used in the next section.

**Lemma 3.2.5.** If \( w_{A1} = w_{A2} = w_A \), and \( w_{B1} = w_{B2} = w_B \), there is a pure-strategy equilibrium in the retail-pricing subgame which entails \( p^*_{A1} = p^*_{A2} = w_A, p^*_{B1} = p^*_{B2} = w_B, \) and \( p^*_{AB1}, p^*_{AB2} \geq w_A + w_B. \)

\(^5\)For instance use Reny’s (1999) Proposition 5.1 and Corollary 5.2, as the game is payoff secure and \( \bar{\pi}_{R1} + \bar{\pi}_{R2} \) is upper semicontinuous.
Proof. Any deviation from marginal-cost pricing, whether for a single product or the bundle, results in either selling nothing or selling below cost, neither of which is profitable. □

Lemma 3.2.6. In the retail pricing subgame in which (1) Ri carries products A and B; (2) Rj carries only B (i.e. \( w_{Aj} = +\infty \)); (3) \( \max\{w_{Bi}, w_{Bj}\} \leq w_{Ai} \); and (4) \( w_{Ai}, w_{Bi}, w_{Bj} \leq 1/2 \), there is a unique pure-strategy Nash equilibrium in weakly undominated strategies\(^6\) characterized by \( p_{Ai}^* = (1 + w_{Ai})/2 \), and \( p_{Bi}^*, p_{Bj}^* \) and \( p_{ABi}^* \) such that:

(i) If \( w_{Bj} \geq w_{Bi} \), then \( p_{Bi}^* = p_{Bj}^* = w_{Bj} \) and \( p_{ABi}^* \geq p_{Ai}^* + p_{Bi}^* \). Moreover \( \bar{\pi}_{Ri}^* > 0 \), and \( \bar{\pi}_{Rj}^* = 0 \).

(ii) If \( w_B \leq w_{Bj} < w_{Bi} \), where

\[
\overline{w}_B = \frac{2 + \mu}{1 + \mu} w_{Bi} - \frac{w_{Ai}}{2} \left( \frac{\mu}{1 + \mu} \right) - \frac{1}{2} \left( \frac{2 - \mu}{1 + \mu} \right),
\]

then \( p_{Bi}^* = p_{Bj}^* = w_{Bj} \), and \( p_{ABi}^* \geq p_{Ai}^* + p_{Bi}^* \). Moreover \( \bar{\pi}_{Ri}^* > 0 \), and \( \bar{\pi}_{Rj}^* > 0 \).

(iii) If \( w_{Bj} < \overline{w}_B \), then \( p_{Bi}^* = w_{Bi} \) and

\[
\begin{align*}
p_{ABi}^* &= \left( \frac{1 + \mu}{4 + 3\mu} \right) (w_{Bj} + 2w_{Bi}) + 2 \left( \frac{1 + \mu}{4 + 3\mu} \right) w_{Ai} + \left( \frac{3 + \mu}{4 + 3\mu} \right), \\
p_{Bj}^* &= \left( \frac{\mu}{4 + 3\mu} \right) (w_{Ai} + w_{Bi}) + 2w_{Bj} \left( \frac{1 + \mu}{4 + 3\mu} \right) + \left( \frac{2 - \mu}{4 + 3\mu} \right).
\end{align*}
\]

Moreover \( \bar{\pi}_{Ri}^* > 0 \), and \( \bar{\pi}_{Rj}^* > 0 \).

Proof. First, as \( w_{Aj} = +\infty \), then \( p_{Aj} = +\infty \) and \( p_{ABj} = +\infty \), so \( \overline{p}_{ABj} = +\infty \). Hence to simplify notation, instead of writing \( \underline{p}_{AB} = (\overline{p}_{ABi}, +\infty) \), we denote \( \underline{p}_{AB} = \overline{p}_{ABi} \). Moreover, as \( \overline{p}_{ABi} = \min\{p_{Ai} + p_{Bi}, p_{ABi}\} \), without loss of generality we can work directly with \( p_{ABi} \) by imposing the restriction \( p_{ABi} \leq p_{Ai} + p_{Bi} \), therefore \( \underline{p}_{AB} = p_{ABi} \). The rest of the proof consists of four parts.

Part 1: Ri’s Best Response. Begin by setting \( p_{Bj} > (1 + w_{Bi})/2 \). Ri’s problem can then be written as

\[
\max_{\substack{p_{Ai}, p_{Bi} \geq p_{Ai}, p_{Bj} \geq \overline{p}_{ABi}, p_{ABi} \leq p_{Ai}, p_{Bi}, p_{ABi} \geq \overline{p}_{ABi}}} \quad \bar{\pi}_{Ri} = (1 - \mu)(p_{Ai} - w_{Ai})(1 - p_{Ai}) + (1 - \mu)(p_{Bi} - w_{Bi})(1 - p_{Bi}) + \\
\mu \left( \min\{p_{Ai}, p_{Bi}\} - w_{Ai} \mathbb{1}_{p_{Ai} \leq p_{Bi}} - w_{Bi} \mathbb{1}_{p_{Ai} > p_{Bi}} \right) \max\{0, p_{ABi} - 2 \min\{p_{Ai}, p_{Bi}\}\} - \mu(p_{ABi} - w_{Ai} - w_{Bi})(1 - p_{ABi} + \min\{p_{Ai}, p_{Bi}\})
\]

\(^6\)We use this refinement to rule out equilibria in which a higher cost retailer prices below cost without selling any units of that product. For instance, in case (i) a continuum of equilibria with form \( p_{Ai}^* = (1 + w_{Ai})/2, p_{ABi}^* \geq p_{Ai}^* + p_{Bj}^*, \) and \( p_{Bi}^* = p_{Bj}^* = p_{Bj}^* \in [w_{Bi}, w_{Bj}] \) exists. However, for \( Rj \) all prices except \( p_{Bj}^* = w_{Bj} \) are weakly dominated, and therefore after eliminating equilibrium actions which are weakly dominated, uniqueness ensues.
Solving, we get \( p_{ABi}(p_{Bj}) = \frac{1 + w_{Ai}}{2} \), \( p_{Bi}^{BR}(p_{Bj}) = \frac{1 + w_{Bi}}{2} \), and \( p_{ABi}^{BR}(p_{Bj}) = p_{Ai}^{BR}(p_{Bj}) + p_{Bi}^{BR}(p_{Bj}) \) (so bundling is not strictly necessary). Recall that \( w_{Bi} \leq w_{Ai} \), so \( Ri \) will always be in equilibrium use good \( B \), if any, to serve those consumers with preferences for both products but that end up buying only one (i.e., \( p_{Bi} = \min\{p_{Ai}, p_{Bi}\} \)).

Suppose next \( p_{Bj} \in (w_{Bi}, (1 + w_{Bi})/2) \). Then \( Ri \)'s problem is

\[
\max_{p_{Ai}, p_{Bi}, p_{ABi} \leq p_{Ai} + p_{Bi}} \pi_{Ri} = (1 - \mu)(p_{Ai} - w_{Ai})(1 - p_{Ai}) + (1 - \mu)(p_{Bi} - w_{Bi})D_{Bi}^{(1 - \mu)(p_{Bj})} + \mu(p_{ABi} - w_{Ai} - w_{Bi})(1 - p_{ABi} + \min\{p_{Bi}, p_{Bj}\}) + \mu(p_{Bi} - w_{Bi})D_{Bi}^{(\mu)}(p_{ABi}, p_{Bj})
\]

Solving, we get \( p_{Ai}^{BR}(p_{Bj}) = (1 + w_{Ai})/2 \), \( p_{Bi}^{BR}(p_{Bj}) = p_{Bj} - \epsilon 1\{w_{Bi} \geq w_{Bj}\} \), and \( p_{ABi}^{BR}(p_{Bj}) = p_{Ai}^{BR}(p_{Bj}) + p_{Bi}^{BR}(p_{Bj}) \) (so again bundling is not strictly necessary).

Finally, suppose \( p_{Bj} \leq w_{Bi} \), then it is clear that \( p_{Bj} \leq p_{Bi} \) is not optimal. This, plus the assumption that players do not play weakly dominated strategies implies that \( p_{Bi}^{BR}(p_{Bj}) = |w_{Bi}, +\infty) \). Therefore, \( Ri \)'s problem is

\[
\max_{p_{Ai}, p_{Bi}, p_{ABi} \leq p_{Ai} + p_{Bi}} \pi_{Ri} = (1 - \mu)(p_{Ai} - w_{Ai})(1 - p_{Ai}) + \mu(p_{ABi} - w_{Ai} - w_{Bi})(1 - p_{ABi} + p_{Bi})
\]

Solving, we get \( p_{Ai}^{BR}(p_{Bj}) = (1 + w_{Ai})/2 \), and

\[
p_{ABi}^{BR}(p_{Bj}) = p_{Ai}^{BR}(p_{Bj}) + w_{Bi} + \max\left\{ w_{Ai} - p_{Ai}^{BR}(p_{Bj}), \frac{1}{2}(p_{Bj} - w_{Bi}) \right\}
\]

where the first term inside the “max” expression ensures that \( p_{ABi}^{BR}(p_{Bj}) \geq w_{Ai} + w_{Bi} \) for all \( p_{Bj} \leq w_{Bi} \) (so that the non-negativity constraints in the demand for \( AB \) is not violated). Notice that since \( p_{ABi}^{BR}(p_{Bj}) < p_{Ai}^{BR}(p_{Bj}) + w_{Bi} \leq p_{Ai}^{BR}(p_{Bj}) + p_{Bi}^{BR}(p_{Bj}) \), in this case bundling is indeed necessary.

**Part 2: Rj’s Best Response.** Rj’s best response, on the other hand, comes from the program:

\[
\max_{p_{Bj}} \pi_{Rj} = (1 - \mu)(p_{Bj} - w_{Bj})D_{Bj}^{(1 - \mu)(p_{Bj})} + \mu(p_{Bj} - w_{Bj})D_{Bj}^{(\mu)}(p_{ABi}, p_{Bj})
\]

First, consider the case \( p_{Bi} \leq w_{Bj} \), then it is clear that \( p_{Bj} \leq p_{Bi} \) is not optimal. This, plus the assumption that players do not play weakly dominated strategies implies that \( p_{Bj}^{BR}(p_{Bj}) = [w_{Bj}, +\infty) \), where \( p_{Bj} \equiv (p_{Ai}, p_{Bi}, p_{ABi}) \). Consider, on the other hand, the case \( p_{Bi} > w_{Bj} \). It is then clear that \( p_{Bj}^{BR}(p_{Bj}) \leq p_{Bi} - \epsilon 1\{w_{Bj} \geq w_{Bi}\} \), hence we can restrict the optimization domain to all \( p_{Bj} \leq p_{Bi} - \epsilon 1\{w_{Bj} \geq w_{Bi}\} \) and solve instead

\[
\max_{p_{Bj} \leq p_{Bi} - \epsilon 1\{w_{Bj} \geq w_{Bi}\}} \pi_{Rj} = (1 - \mu)(p_{Bj} - w_{Bj})(1 - p_{Bj}) + \mu(p_{Bj} - w_{Bj})(p_{ABi} - 2p_{Bj})
\]

The solution then is

\[
p_{Bj}^{BR}(p_{Bj}) = \min\left\{ p_{Bi} - \epsilon 1\{w_{Bj} \geq w_{Bi}\}, \frac{1 + w_{Bj}}{2} - \frac{1}{2} \left( \frac{\mu}{1 + \mu} \right) (2 - p_{ABi}) \right\}
\]

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Part 3: Equilibrium. Finally, we obtain the equilibrium set by intersecting the best response correspondences.

Begin by considering the case in which $w_{Bj} \geq w_{Bi}$. First, it is clear that $p_{Bj}^* < w_{Bj}$ cannot be equilibrium: either $Rj$ is obtaining negative profits (if he is selling strictly positive units of $B$) and therefore deviate; or he is obtaining zero profits, and $p_{Bj}^* < w_{Bj}$ is weakly dominated by $p_{Bj}$’s greater than, or equal to $w_{Bj}$. On the other hand, it is also clear that the equilibrium cannot be $p_{Bj}^* > w_{Bj}$, since if it were, then $p_{Bi}^* = p_{Bj}^* - \epsilon$, but then $Rj$’s best response would demand $p_{Bj}^{BR}(p_i^*) = p_{Bi}^* < \epsilon < p_{Bj}^*$; a contradiction.

Hence, the only remaining possibility is for $p_{Bj}^* = w_{Bj}$. If so, $Ri$’s optimal response is $p_{Bi}^{BR}(p_{Bj}^*) = p_{Bi}^*$, $p_{Ai}^{BR}(p_{Bj}^*) = (1 + w_{Ai})/2$, and $p_{ABi}^{BR}(p_{Bj}^*) \geq (1 + w_{Ai})/2 + p_{Bj}^*$. But then, $p_{Bj}^{BR}(p_i^*) = p_{Bj}^*$ if and only if $p_{Bj}^* = p_{Bi}^* = w_{Bj}$; any other price induces $Rj$ to slightly undercut $p_{Bj}^{BR}(p_{Bj}^*) = p_{Bj}^*$. Hence, the unique equilibrium if $w_{Bj} \geq w_{Bi}$ is $p_{Ai}^* = (1 + w_{Ai})/2$, $p_{Bi}^* = p_{Bj}^* = w_{Bj}$, $p_{ABi}^* \geq p_{Ai}^* + p_{Bi}^*$, and therefore $\pi_{Ri}^* = (1 - w_{Ai})^2/4 + (w_{Bj} - w_{Bi})(1 - w_{Bj})$, and $\pi_{Rj}^* = 0$.

Consider then the case $w_{Bj} < w_{Bi}$. Again, by the same argument as before, it cannot be equilibrium $p_{Bj}^* > (1 + w_{Bi})/2$. Moreover, it cannot be equilibrium for $p_{Bj}^* \in (w_{Bi}, (1 + w_{Bi})/2]$ either; if it were $Ri$’s best response would be $p_{Bi}^{BR}(p_{Bj}^*) = (1 + w_{Ai})/2$, $p_{Bj}^{BR}(p_{Bj}^*) = p_{Bj}^* - \epsilon$, and $p_{ABi}^{BR}(p_{Bj}^*) \geq (1 + w_{Ai})/2 + p_{Bj}^* - \epsilon$. But if so, $p_{Bj}^{BR}(p_i^*) = p_{Bj}^* - \epsilon \neq p_{Bj}^*$, leading to a contradiction.

Hence, it must be that $p_{Bj}^* \leq w_{Bi}$. Then $Ri$’s best response correspondence is given by $p_{Ai}^{BR}(p_{Bj}^*) = (1 + w_{Ai})/2$, $p_{Bi}^{BR}(p_{Bj}^*) \in [w_{Bi}, +\infty)$, and

$$p_{ABi}^{BR}(p_{Bj}^*) = p_{Ai}^{BR}(p_{Bj}^*) + w_{Bi} + \max \left\{ w_{Ai} - p_{Ai}^{BR}(p_{Bj}^*), \frac{1}{2}(p_{Bj}^* - w_{Bi}) \right\} \quad (3.7)$$

But since $p_{Bi}^{BR}(p_{Bj}^*) \in [w_{Bi}, +\infty)$, then, $Rj$’s best response is

$$p_{Bj}^{BR}(p_i^*) = \min \left\{ p_{Bi}^*, \frac{1 + w_{Bj}}{2} - \frac{1}{2} \left( \frac{\mu}{1 + \mu} \right) (2 - p_{ABi}) \right\} \quad (3.8)$$

So let’s conjecture that $p_{Bi}^* = p_{Bj}^*$. This immediately implies that $p_{Bi}^* = p_{Bj}^* = w_{Bi}$. Plugging in (3.7) we get $p_{ABi}^* = p_{Ai}^* + w_{Bi}$, as $w_{Ai} - p_{Ai}^{BR}(p_{Bj}^*) < (p_{Bj}^* - w_{Bi})/2 = 0$. Therefore, our equilibrium candidate would be $p_{Ai}^* = (1 + w_{Ai})/2$, $p_{Bi}^* = p_{Bj}^* = w_{Bi}$, and $p_{ABi}^* = p_{Ai}^* + w_{Bi}$. But such candidate must also satisfy (3.8), hence we need

$$p_{Bi}^* = w_{Bi} \leq \frac{1 + w_{Bj}}{2} - \frac{1}{2} \left( \frac{\mu}{1 + \mu} \right) (2 - p_{ABi}) \iff w_{Bj} \geq w_{B} \quad (3.9)$$

Hence, if $w_{Bj} \in [w_{B}, w_{Bi}]$, then the following is a pure-strategy Nash equilibrium in weakly undominated strategies: $p_{Ai}^* = (1 + w_{Ai})/2$, $p_{Bi}^* = p_{Bj}^* = w_{Bi}$, and $p_{ABi}^* \geq p_{Ai}^* + p_{Bi}^*$, and therefore $\pi_{Ri}^* = (1 - w_{Ai})^2/4$, and $\pi_{Rj}^* = (w_{Bi} - w_{Bj})(1 - w_{Bi} - \mu(1 - w_{Ai})/2]$.

The remaining possibility is for

$$p_{Bj} = \frac{1 + w_{Bj}}{2} - \frac{1}{2} \left( \frac{\mu}{1 + \mu} \right) (2 - p_{ABi})$$
Then (3.7) requires
\[
p^*_ABi = p^*_Ai + w_{Bi} + \frac{1}{2}(p^*_Bj - w_{Bi})
\]
Solving after replacing \(p^*_Ai = (1 + w_{Ai})/2\) we obtain
\[
p^*_ABi = \left(\frac{1 + \mu}{4 + 3\mu}\right) (w_{Bj} + 2w_{Bi}) + 2 \left(\frac{1 + \mu}{4 + 3\mu}\right) w_{Ai} + \left(\frac{3 + \mu}{4 + 3\mu}\right)
p^*_Bj = \left(\frac{\mu}{4 + 3\mu}\right) (w_{Ai} + w_{Bi}) + 2w_{Bj} \left(\frac{1 + \mu}{4 + 3\mu}\right) + \left(\frac{2 - \mu}{4 + 3\mu}\right)
\]
But for such candidate to be feasible, it must be that \(p^*_Bj \leq w_{Bi}\), that is
\[
p^*_Bj = w_{Bi} \geq \frac{1 + w_{Bj}}{2} \frac{\mu}{1 + \mu} \Rightarrow w_{Bj} \leq w_B
\]
which coincidentally satisfies condition (3.8). Hence if \(w_{Bj} \leq w_B\), then this is a pure-strategy Nash equilibrium. Evaluating to obtain the manufacturers’ profit:
\[
\pi^*_Ri = (1 - \mu)(p^*_Ai - w_{Ai})(1 - p^*_Ai) + \mu(p^*_ABi - w_{Ai} - w_{Bi})(1 - p^*_ABi + p^*_Bj)
\]
\[
\pi^*_Rj = (1 - \mu)(p^*_Bj - w_{Bj})(1 - p^*_Bj) + \mu(p^*_Bj - w_{Bj})(p^*_ABi - 2p^*_Bj)
\]

**Part 4: Uniqueness.** Because for any combination of wholesale prices the intersection of the best responses is a singleton, uniqueness immediately ensues. ■

**The full-monopolization outcome** We are now ready to investigate the conditions under which \(M\) can implement the full-monopolization outcome. Proceeding as in Section 3.1 of the paper, the full-monopolization outcome is obtained from maximizing
\[
\Pi_M = (1 - \mu)p_A(1 - p_A) + (1 - \mu)p_B(1 - p_B) +
\mu p_{AB}(1 - p_{AB} + \min\{p_A, p_B\}) + \mu \min\{p_A, p_B\}(p_{AB} - 2\min\{p_B, p_A\}) - F_M
\]
subject to \(p_{AB} \leq p_A + p_B\). The first line captures the profit made on consumers that value only one good while the second the profit made on consumers that have preferences for both goods, including those that end up buying both goods and those that just buy one of the two, the cheapest one. The solution to this problem is \(p^*_A = p^*_B = 1/2\), and \(p^*_{AB} = 1\), resulting in a total payoff of \(\Pi^*_M \equiv 1/2 - F_M\). As in section 3.1, notice that retail bundling does not emerge.

**Foreclosure** Suppose now that \(M\) approaches each retailer with the same full-line forcing contract \(\{C^{M}_{Ai}, C^{M}_{Bi}, (\tilde{C}^M_{Ai}, \tilde{C}^M_{Bi})\}\):
\[
C^{M}_{Ai} = (w^M_{Ai} = 1, T^M_{Ai} = 0) \quad C^{M}_{Bi} = (w^M_{Bi} = 1/2, T^M_{Bi} = 0) \\
\tilde{C}^M_{Ai} = (\hat{w}^M_{Ai} = 1/2, \hat{T}^M_{Ai} = 0) \quad \tilde{C}^M_{Bi} = (\hat{w}^M_{Bi} = 1/2, \hat{T}^M_{Bi} = 0)
\]
for \(i = 1, 2\). A retailer must decide then, whether to sign a contract allowing her the flexibility to obtain inputs from both manufacturers, or to sign a contract committing her to obtain both
inputs exclusively from \( M \). A retailer opting for the latter, gets in return a discount of \( 1/2 \) off the otherwise list price of \( 1 \) in each unit of product \( A \). According to Lemma 3.2.5 if both retailers decide to sign exclusively with \( M \), the above contracts ensure the implementation of the full-monopolization outcome with a payoff of \( \Pi^*_M \).

But to evaluate retailers’ decisions, we first need to understand \( S \)'s optimal reaction if he anticipates retailers are being approached by \( M \) with the offer [3.10]. \( S \) can try to persuade both retailers to carry his product, one of them, or none. Following the discussion in the text, \( S \) can at best aim for one retailer to carry his product in equilibrium. Suppose then that \( S \) approaches one of the retailers, say \( R2 \), with the schedule \( (w_{B2}^S = 1/2 - \epsilon, T_{B2}^S = 0) \), where \( \epsilon \) is positive but relatively small. It is easy to see from Lemma 3.2.6 that the continuation play then involves \( R1 \) accepting \( M \)'s full-line forcing discount, and \( R2 \) accepting \( S \)'s schedule for \( B \).

According to the lemma, \( R1 \) will set \( p^*_{A1} = 3/4 \) and \( p^*_{B1} = 1/2 \) (with \( p^*_{AB1} \geq p^*_{A1} + p^*_{B1} \)), while \( R2 \) will set \( p^*_{B2} = 1/2 \). \( R1 \) will then serve all consumers that value \( A \) more than \( 3/4 \), whether those that value just \( A \) or those that value \( A \) and \( B \), for a total profit of \( \Pi_{R1} = (1-1/2)^2/4 = 1/16 \). And \( R2 \) will attract all \( 1 - \mu \) consumers that only value \( B \), and all of those \( \mu \) consumers that value both goods, but either \( A \) and \( B \) in less than \( 3/4 \) and more than \( 1/2 \) for a payoff of \( \Pi_{R2} = \epsilon(1-1/2 - \mu(1-1/2))/2 = \epsilon(2-\mu)/4 \). As far as \( \epsilon \) is positive, \( S \) can induce \( R2 \) to sign with him (while \( R1 \) will inevitable go with \( M \)). And since \( R2 \) will be selling \( 1/2 - \mu/4 \) units of product \( B \), \( S \)'s revenues can approach as close to \( (2-\mu)/8 \) as \( \epsilon \to 0 \), that is, only half the monopoly profits when \( \mu = 1 \).

As the next lemma establishes, \( S \) can obtain a bit more if he approaches \( R2 \) with a contract offer that would make that retailer more aggressive in the retail market, but always short of \( 1/4 \).

**Lemma 3.2.7.** Suppose that \( M \) approaches each retailer with the full-line forcing contract [3.10]. The largest revenue \( S \) can obtain from the retail market is \( 1/(4 + 2\mu) < 1/4 \).

**Proof.** We will show that \( S \) can obtain more than \( (2-\mu)/8 \) by setting \( w_{Bj} < w_B = 3\mu/4(1+\mu) \), where this latter is obtained from the definition of \( w_B \) in Lemma 3.2.6. Since \( S \) can always include a fixed fee in \( R2 \)'s offer extracting almost her entire profit, \( S \)'s payoff, before fixed costs, from setting \( w_{Bj} \in [0, w_B] \) is equal to \( \bar{\Pi}_S(w_{Bj}) = \bar{\Pi}_{Rj}^*(w_{Bj}) + w_{Bj}[(1-\mu)(1-p_{Bj}^*(w_{Bj}))+\mu(p_{ABj}^*(w_{Bj})-2p_{Bj}^*(w_{Bj}))] \). Using Lemma 3.2.6 to obtain expressions for \( \bar{\Pi}_{Rj}^*(w_{Bj}), p_{Bj}^*(w_{Bj}) \) and \( p_{ABj}^*(w_{Bj}) \) yields

\[
\bar{\Pi}_S(w_{Bj}) = \frac{2(\mu + 1)(1 + w_{Bj} + \mu w_{Bj})(2 - 2w_{Bj} - \mu w_{Bj})}{(3\mu + 4)^2}
\]

which has a maximum at

\[
w_B^* = \frac{\mu}{2(\mu + 1)(\mu + 2)} < w_B
\]

so that

\[
\bar{\Pi}_S(w_{Bj} = w_B^*) = \frac{1}{4 + 2\mu}
\]

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which concludes the proof. ■

The main implication of Lemma 3.2.7 is that whenever economies of scale are important, i.e. \( F_S \geq 1/(4 + 2\mu) \), \( S \)'s best response to \((3.10)\) is to approach no retailer and exit the market. Moreover, since \((3.10)\) maximizes \( M \)'s profits in the absence of a competitive threat (see Lemma 3.2.5), then \((3.10)\) is also a best response to \( S \)'s reaction. Hence, when \( 1/4 > F_M > F_S \geq 1/(4 + 2\mu) \), it is clearly an equilibrium for \( M \) to offer the full-line forcing contracts in \((3.10)\) and for \( S \) to exit the market. That is, \( M \) has fully monopolized the market.

Revisiting Proposition 2a

We use the same approach as in subsection 3.1.5. In this model, optimal industry profits in each retail location clearly satisfy symmetry (i.e. \( \Pi_i(x) = \Pi_i^{(2)}(y) = \Pi_i(x(y)) \)), and separability (i.e. \( \Pi_i(x) = \Pi_i^{(2)}(y) + \Pi_i^{(0)}(y) \)). Furthermore \( \Pi_i^{(2)}(\emptyset) = 1/8 > 0 \) (satisfying A1*), \( \Pi_i^{(0)}(M) = \Pi_i^{(0)}(S) = \Pi_i^{(0)(MS)} = 1/8 > 0 \) (satisfying A2*). Furthermore, \( \Pi_i^{(A)(S)} + \Pi_i^{(A)(M)} + \Pi_i^{(0)(MS)} - 2\Pi_i^{(A)(MS)} = 1/8 > 0 \) (satisfying A3*); \( 2\Pi_i^{(S)}(S) - F_S = 1/4 - F_S > 2\Pi_i^{(M)}(M) - F_M = 1/4 - F_M \) (satisfying A4*). Since the model satisfies assumptions A1* to A4* we can use Lemma 3.1.4. Given that \( 2(\Pi_i^{(A)(MS)} - \Pi_i^{(A)(S)}) = 0 < F_M \) we conclude then that in equilibrium both retailers only represent \( S \) in product-line \( B \).

Revisiting Proposition 2b

Following the proof of Proposition 2b in the text, we restrict attention to full-line forcing contracts (see also Appendix H in the text). Suppose the following pair of contracts constitute a foreclosure equilibrium: \( \{(w_{Ai}^*, T_{Ai}^*) = 0, (w_{Bi}^*, T_{Bi}^*) = 0\} \) and \( \{(w_{Ai}, T_{Ai}), (w_{Bi}, T_{Bi})\} \) for \( i = 1, 2 \). Suppose further that wholesale prices are such that \( p_{Ai}^* = \max\{w_{Ai}^*, w_{Aj}^*\} \) and \( p_{Bi}^* = \max\{w_{Bi}^*, w_{Bj}^*\} \leq 1/2 \) (the case \( \max\{w_{Bi}^*, w_{Bj}^*\} > 1/2 \) is automatically ruled out for profitability and foreclosure reasons), so \( M \)'s equilibrium payoff can be written as

\[
\pi^*_M = \min\{w_{Ai}^*, w_{Aj}^*\} (1 - \max\{w_{Ai}^*, w_{Aj}^*\}) + \hat{T}_{Ai}^* + \hat{T}_{Aj}^* + \min\{w_{Bi}^*, w_{Bj}^*\} (1 - \max\{w_{Bi}^*, w_{Bj}^*\}) + \hat{T}_{Bi}^* + \hat{T}_{Bj}^* - F_M
\]

which must be at least 1/4.

Without loss of generality let \( \hat{w}_{Ai}^* \leq \hat{w}_{Aj}^* \). Consider first the case in which equilibrium offers are such that \( \hat{w}_{Bi}^* \leq \hat{w}_{Bj}^* \), which implies (for \( \mu = 0 \)) that \( \hat{T}_{Aj}^* = \hat{T}_{Bj}^* = \pi_{Bj}^* = 0 \) and \( \pi_{Ai}^* = (\hat{w}_{Aj}^* - \hat{w}_{Ai}^*) (1 - \hat{w}_{Aj}^*) + (\hat{w}_{Bj}^* - \hat{w}_{Bi}^*) (1 - \hat{w}_{Bj}^*), \hat{T}_{Ai}^* - \hat{T}_{Bi}^* = -\hat{T}_{Bi}^* \). Using \((3.11)\) to obtain \( \hat{T}_{Ai}^* + \hat{T}_{Bi}^* \geq 1/4 + F_M - \hat{w}_{Ai}^* (1 - \hat{w}_{Ai}^*) - \hat{w}_{Bi}^* (1 - \hat{w}_{Bi}^*) \) and the fact that \( \pi_{Bi}^* \geq 0 \) and \( \hat{w}_{Ai}^* (1 - \hat{w}_{Aj}^*) \leq 1/4 \), we arrive that in equilibrium \( \pi_{Bi}^* \leq \hat{w}_{Bj}^* (1 - \hat{w}_{Bj}^*) - F_M \) and \( \hat{w}_{Bj}^* (1 - \hat{w}_{Bj}^*) \leq F_M \). But if so, \( S \) would profitably deviate by approaching \( R_i \) with the offer \( (w_{Bi}^S = 0, T_{Bi}^S = F_M - \epsilon) \) with \( \epsilon \to 0 \), which \( R_i \) would be ready to take since \( \hat{w}_{Bj}^* (1 - \hat{w}_{Bj}^*) - F_M - \epsilon > \hat{w}_{Bj}^* (1 - \hat{w}_{Bj}^*) - F_M \) (recall that \( p_{Bi}^* \) remains at \( \hat{w}_{Bj}^* \)); a contradiction.
Consider now the case in which equilibrium offers are such that \( \hat{w}_{Bj}^M > \hat{w}_{Bj}' \), which implies (for \( \mu = 0 \)) that \( \pi_{Ri}^* = (\hat{w}_{Aj}^M - \hat{w}_{Ai}^M)(1 - \hat{w}_{Aj}^M) - \hat{T}_{Aj}^M - B_{Bi}^M \) and \( \pi_{Rj}^* = (\hat{w}_{Bi}^M - \hat{w}_{Bj}^M)(1 - \hat{w}_{Bi}^M) - \hat{T}_{Bj}^M - B_{Bi}^M \). Again, using (3.11) to obtain \( \sum_{k,i} \hat{T}_{ki}^M \geq 1/4 + F_M - \hat{w}_{Ai}^M(1 - \hat{w}_{Aj}^M) - \hat{w}_{Bj}^M(1 - \hat{w}_{Bi}^M) \) and the fact that \( \pi_{Ri}^* \geq 0, \pi_{Rj}^* \geq 0 \) and \( \hat{w}_{Aj}^M(1 - \hat{w}_{Aj}^M) \leq 1/4 \), we arrive that in equilibrium \( \pi_{Rj}^* \leq \hat{w}_{Bj}^M(1 - \hat{w}_{Bi}^M) - F_M \) and \( \hat{w}_{Bj}^M(1 - \hat{w}_{Bi}^M) \geq F_M \). But if so, \( S \) would profitably deviate by approaching \( R_j \) with the offer \( (w_{Sj}^M = 0, T_{Bj}^M = F_M - \epsilon) \) with \( \epsilon \to 0 \), which \( R_j \) would be ready to take; a contradiction.

### 3.2.5 Retail Price Discrimination

We now extend the previous analysis to the case of consumer valuations that give rise to retail price discrimination. We adopt the standard unit-square set up (e.g., McAfee et al. 1989; Matutes and Regibeau 1992; Nalebuff 2004; Armstrong and Vickers 2010). Both \( v_A \) and \( v_B \) distribute uniformly over the unit interval and exhibit no correlation for consumers that have preferences for both goods. We continue assuming that \( F_S < F_M < 1/4 \).

#### Revisiting Proposition 1

Consider the case of Bertrand retailers and to simplify the exposition focus on the case of \( \mu = 1 \). Following Proposition 1 in the text, we look for conditions under which \( M \) can fully monopolize the market while excluding \( S \).

**The full-monopolization outcome** The demand for the case of \( \mu = 1 \) is well know (see, e.g., McAfee et al. 1989). For any given triplet of prices \( p_A \leq 1, p_B \leq 1 \) and \( p_{AB} \leq p_A + p_B \), \( D_A(p_A,p_B,p_{AB}) = (p_{AB} - p_A)(1 - p_A) \), \( D_A(p_A,p_B,p_{AB}) = (p_{AB} - p_B)(1 - p_B) \), and \( D_{AB}(p_A,p_B,p_{AB}) = (1 - p_{AB} + p_A)(1 - p_{AB} + p_B) - (p_A + p_B - p_{AB})^2/2 \). Thus, the full-monopolization outcome is obtained from maximizing

\[
\Pi_M = p_A D_A(p_A,p_B,p_{AB}) + p_B D_B(p_A,p_B,p_{AB}) + p_{AB} D_{AB}(p_A,p_B,p_{AB}) - F_M
\]

The solution to this problem is \( p_A^* = p_B^* = 2/3 \), and \( p_{AB}^* = p_A^* + p_B^* - \sqrt{2}/3 = (4 - \sqrt{2})/3 = 0.862 \), resulting in a total payoff of \( \Pi_M^* = 4/9 + 2\sqrt{2}/27 - F_M = 0.549 - F_M \).

Unlike in the baseline model and in the previous model, notice that now retail bundling does emerge. This raises the question as to whether \( M \) can still implement the full-monopolization outcome with contract offers that impose no restrictions on how retailers sell products downstream. We will consider both cases. First, the case in which \( M \) can closely monitor retailers’ sales, as in Proposition 6 in the text, and then, the case in which \( M \) cannot monitor retailers’s sales, that is, the case in which he cannot condition wholesale prices to the form in which products are sold downstream, whether as a bundle or in stand-alone fashion.
Foreclosure with monitoring of sales  If $M$ is alone in the market and can closely monitor retailers’ sales, it is easy to see that he can implement the full-monopolization outcome by approaching retailers with the non-discriminatory offers $C_{ki}^M = (w_{ki}^M = p_{ki}^*, T_{ki}^M = 0)$ where $k = A,B,AB$ and $i = 1,2$. Following the same logic of Lemma 3.2.5 above, retail (Bertrand) competition will drive prices of the three products down to marginal costs, that is, $p_{A1}^* = p_{A2}^* = w_{A1}^M, p_{B1}^* = p_{B2}^* = w_{B1}^M$, and $p_{AB1}^* = p_{AB2}^* = w_{AB1}^M$.

The question now is under what conditions $M$ can implement this same outcome under $S$’s presence, that is, under what conditions is optimal for $S$ to make no offer. Suppose then that $M$ approaches both retailers with the full-line forcing contract $\{C_{A1}^M, C_{B1}^M, (\hat{C}_{A1}^M, \hat{C}_{B1}^M)\}$:

$$
(w_{A1}^M = 1, T_{A1}^M = 0), (w_{B1}^M = p_{B1}^*, T_{B1}^M = 0)
$$

$$
(w_{A1}^M = p_{A1}^*, \hat{T}_{A1}^M = 0), (w_{B1}^M = p_{B1}^*, \hat{T}_{B1}^M = 0), (w_{AB1}^M = p_{AB1}^*, \hat{T}_{AB1}^M = 0)
$$

Since we know that $S$ can at best persuade one retailer to take his offer in equilibrium, suppose that $S$ approaches $R2$ with the offer $C_{B2}^M = (w_{B2}^S = p_{B2}^* - \Delta, T_{B2}^M > 0)$, with $\Delta$ small. Following the same logic of Lemma 3.2.6 above, in particular, points (ii) and (iii), it can be shown for $T_{B2}^M$ sufficiently small that $R2$ will sign with $S$ and $R1$ with $M$, and the equilibrium in the retail-pricing subgame is characterized as follows: $p_{A1}^* = 0.863, p_{B1}^* = p_{B2}^* = 0.667$, and $p_{AB1}^* = 1.209$. Since $S$ will set $T_{B2}^M$ to leave $R2$ with just enough to take his offer, $S$’s retail payoff reduces to $\bar{\pi}_S = (1 - p_{B2}^*)(p_{AB1}^* - p_{B2}^*)p_{B2}^* = 0.121$, which is insufficient to cover his fixed costs whenever $F_S \in (0.121, 0.25)$.

It remains to explore whether $S$ can obtain more in the retail market by making $R2$ more aggressive, that is, by offering her $C_{B2}^M = (w_{B2}^S = p_{B2}^* - \Delta, T_{B2}^M > 0)$, with $\Delta$ sufficiently large so that $R2$ will find it optimal to price below $p_{B2}^*$. To find this solution, we proceed first by finding the price $p_{B2}^R < p_{B2}^*$, if any, that maximizes $\bar{\pi}_S = (1 - p_{B2}^*)(p_{AB1}^*(p_{B2}^*) - p_{B2}^*)p_{B2}^*$, where $p_{AB1}^*(p_{B2}^*)$ is $R1$’s best bundling response to $R2$ when this latter is pricing good $B$ at $p_{B2}^* < p_{B2}^*$. Solving we obtain $p_{B2}^R = 0.4 < 2/3, p_{AB1}^*(p_{B2}^*) = 1.118$ and $p_{AB1}^*(p_{B2}^*) = 0.853$, which leads to $\bar{\pi}_S = (1 - p_{B2}^*)(p_{AB1}^*(p_{B2}^*) - p_{B2}^*)p_{B2}^* = 0.172 > 0.121$. And to implement this outcome, $S$ needs to approach $R2$ with $C_{B2}^M = (w_{B2}^S = 0.073, T_{B2}^M = \bar{\pi}_{R2} - \epsilon)$ with $\epsilon \to 0$ and where $\bar{\pi}_{R2} = (1 - p_{B2}^R)(p_{AB1}^*(p_{B2}^R) - p_{B2}^R)p_{B2}^R = 0.141$. Consequently, whenever $F_S \in (0.172, 0.25)$, $S$ will stay out of the market and $M$ will implement the full-monopolization outcome.

Foreclosure without monitoring of sales  If $M$ is alone in the market but cannot monitor retailers’ sales, so he can only approach retailers with contracts of the form $C_{ki}^M = (w_{ki}^M, T_{ki}^M)$, where $k = A,B$ and $i = 1,2$, he must resort to highly discriminatory offers in order to implement the full-monopolization outcome. He would need to approach one retailer, say $R2$, with no offer and the other with an offer that leaves her as the “residual claimant” in the retail market, that is, with the offer $C_{1}^M = (w_{A1}^M = w_{B1}^M = 0, T_{1}^M = \Pi_{M}^* + F_{M})$.

Clearly these offers can never be part of a foreclosure equilibrium. If so, $S$ would approach both retailers with the non-discriminatory offers $C_{Bi}^S = (w_{Bi}^S = 1/2, T_{Bi}^S = 0)$ for $i = 1,2$.  

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Since \( R2 \) would anticipate that it is a dominant strategy for \( R2 \) to take \( S \)'s offer, she would rather sign with \( S \) as well; otherwise she would make a loss. Thus, \( S \)'s entry is ensured for any \( F_S < 1/4 \). Finding the conditions and the kind of offers that \( M \) would need to make in order to foreclose \( S \) and implement the full-monopolization outcome in equilibrium would require to proceed as in Proposition 5 in the text. That is, finding offers that maximize industry profits while reducing retailers’ outside options to zero, if possible.

We leave this problem to the interested reader and rather focus, following Proposition 6, on the simpler problem of finding the conditions under which \( M \) can foreclose \( S \) provided he is restricted to the use of non-discriminatory offers (possibly because of antitrust considerations). Suppose then that \( M \) approaches both retailers with the full-line forcing contract \( \{\mathcal{C}_M^{A_i}, \mathcal{C}_M^{B_1}, (\hat{\mathcal{C}}_M^{A_i}, \hat{\mathcal{C}}_M^{B_1})\} \):

\[
\begin{align*}
(w_{A_i}^M &= 1, T_{A_i}^M = 0) \quad , \quad (w_{B_i}^M = w_B^M, T_{B_i}^M = 0) \\
(\hat{w}_{A_i}^M &= w_A^M, \hat{T}_{A_i}^M = 0) \quad , \quad (\hat{w}_{B_i}^M = w_B^M, \hat{T}_{B_i}^M = 0)
\end{align*}
\]

If both retailers sign exclusively with \( M \), the same Bertrand logic of Lemma 3.2.5 leads to the equilibrium retail prices: \( p_{A_i}^* = p_{A_j}^* = w_A^M \), \( p_{B_i}^* = p_{B_j}^* = w_B^M \), and \( p_{AB_i}^* \geq w_A^M + w_B^M \) for both \( i = 1, 2 \), so retail bundling does not emerge in equilibrium. Since regardless of \( S \)'s presence the most that \( M \) can obtain with these non-discriminatory offers is \( 1/2 - F_M \), after setting \( w_A^M = w_B^M = 1/2 \), we concentrate on this case in what follows. Given the monitoring constraint, we can view the outcome of this case as the constraint full-monopolization outcome.

Consider now what is \( S \)'s optimal response to \( M \)'s offers when \( w_A^M = w_B^M = 1/2 \). We know that \( S \) can at most persuade one retailer to take his offer in equilibrium, so suppose that \( S \) approaches \( R2 \) with the offer \( \mathcal{C}_B^M = (w_{B_2}^S = 1/2 - \Delta, T_{B_2}^M > 0) \), with \( \Delta \) small. Following the same logic of Lemma 3.2.6 in particular, points (ii) and (iii), it can be shown for \( T_{B_2}^M \) sufficiently small that \( R2 \) will sign with \( S \) and \( R1 \) with \( M \), and the equilibrium in retail-pricing subgame is characterized as follows: \( p_{A_1}^* = 3/4, p_{B_1}^* = p_{B_2}^* = 1/2 \), and \( p_{AB_1}^* = 5/4 \). Since \( S \) will set \( T_{B_2}^M \) to leave \( R2 \) with just enough to take his offer, \( S \)'s retail payoff reduces to \( \bar{\pi}_S = (1 - p_{B_2}^S)(p_{AB_1}^S - p_{B_2}^S)p_{B_2}^S = 3/16 = 0.188 \), which is insufficient to cover his fixed costs whenever \( F_S \in (3/16, 1/2) \).

As before, it remains to explore whether \( S \) can obtain more in the retail market by making \( R2 \) more aggressive, that is, by offering her \( \mathcal{C}_B^M = (w_{B_2}^S = 1/2 - \Delta, T_{B_2}^M > 0) \), with \( \Delta \) sufficiently large so that \( R2 \) will find it optimal to price below \( 1/2 \). To find this solution, we proceed first by finding the price \( p_{B_2}^S < 1/2 \), if any, that maximizes \( \bar{\pi}'_S = (1 - p_{B_2}^S)(p_{AB_1}^S(p_{B_2}^S) - p_{B_2}^S)p_{B_2}^S \), where \( p_{AB_1}^S(p_{B_2}^S) \) is \( R1 \)'s best bundling response to \( R2 \) when this latter is pricing good \( B \) at \( p_{B_2}^S < 1/2 \). Solving we obtain \( p_{B_2}^S = 0.42 < 1/2 \), \( p_{AB_1}^S(p_{B_2}^S) = 1.215 \) and \( p_{AB_1}^S(p_{B_2}^S) = 0.743 \), which leads to \( \bar{\pi}'_S = (1 - p_{B_2}^S)(p_{AB_1}^S(p_{B_2}^S) - p_{B_2}^S)p_{B_2}^S = 0.194 > 0.188 \). And to implement this outcome, \( S \) needs to approach \( R2 \) with \( \mathcal{C}_B^S = (w_{B_2}^S = 0.085, T_{B_2}^M = \bar{\pi}'_{R2} - \epsilon) \) with \( \epsilon \to 0 \) and where \( \bar{\pi}'_{R2} = (1 - p_{B_2}^S)(p_{AB_1}^S - p_{B_2}^S)(p_{B_2}^S - w_{B_2}^S) = 0.155 \). Consequently, whenever \( F_S \in (0.194, 0.25) \), \( S \) will stay out of the market and \( M \) will implement the constraint full-monopolization outcome.
Revisiting Proposition 2a

We use the same approach as in subsection 3.1.5. In this model, optimal industry profits in each retail location clearly satisfy symmetry (i.e. $\Pi_i(x) = \Pi_i(y) \equiv \Pi_i(x,y)$). Notice however, that due to price discrimination at the retail level they do not satisfy separability.

Now, $\Pi_i(A) = 1/8 > 0$, $\Pi_i(M) = \Pi_i(S) = \Pi_i(MS) = 1/8 > 0$, and $\Pi_i(A) = \Pi_i(M) = \Pi_i(S) = \Pi_i(MS) = (1/2)(4/9 + 2\sqrt{2}/27) = 0.2745 > 0$. Hence A1* and A2* are satisfied. Furthermore $\Pi_i(S) + \Pi_i(M) + \Pi_i(MS) - 2\Pi_i(MS) = 1/8 > 0$ so A3* is satisfied. Finally, $2\Pi_i(S) - F_S = 1/4 - F_S > 2\Pi_i(M) - F_M = 1/4 - F_M$

2\Pi_i(S) - F_S = (4/9 + 2\sqrt{2}/27) - F_S > 2\Pi_i(M) - F_M = (4/9 + 2\sqrt{2}/27) - F_M

So condition A4* is also satisfied. Since the model satisfies assumptions A1* to A4* we can use Lemma 3.1.4. Given that $2(\Pi_i(MS) - \Pi_i(S)) = 0 < F_M$ we conclude then that in equilibrium both retailers only represent S in product-line B.

Revisiting Proposition 2b

Since $\mu = 0$, the proof is identical to that of the previous case (see “Revisiting Proposition 2” in Section 3.2.4) regardless of whether sales can be monitored or not.

3.3 Relaxing modes of competition

So far we have assumed two polar modes of competition: monopoly retailers and Bertrand competitors. In this section we examine whether our results —in particular Propositions 1 and 2b— are robust to small amounts of retail homogeneity (almost retail monopolies), on the one hand, and to small amounts of retail heterogeneity (almost Bertrand retailers), on the other.

Final consumer preferences over the goods of the retailers are assumed to be given by the Shubik and Levitan (1980) utility function, which leads to simple linear downstream demands (see also Simpson and Wickelgren (2007) for a similar approach). In the limit these demands converge to the demands that are in the model of section 3.2.4, which come from consumer valuations that are uniformly distributed over the unit interval and perfectly correlated for those consumers who have preferences for both goods. We adopt this (symmetric) formulation because it is relatively tractable to take these “almost” results to the limit of either monopoly or Bertrand and formally check that they indeed converge to our results in section 3.2.4.

3.3.1 Retailers’ demand

Let $\lambda \in [0, +\infty)$ be a parameter that measures the extent of downstream competition or retail differentiation in the demand function of final consumers. When $\lambda = 0$, retailers are independent, and as $\lambda \to \infty$, retailers become completely homogeneous (downstream competition approaches Bertrand).
Demand from consumers that value only one product

The demand from consumers that have preferences for only one product \( k = A, B \) is given by

\[
D_{ki}^{(1-\mu)}(p_k; \lambda) = \max \left\{ 0, \min \left\{ \left( \frac{1 + \lambda}{2 + \lambda} \right) (1 - p_{ki}) + \frac{\lambda (p_{kj} - p_{ki})}{4}, 1 - p_{ki} \right\} \right\}
\]

Note that from this demand we arrive in the limit at the demands in section 3.2.4:

- \[ \lim_{\lambda \to 0} D_{ki}^{(1-\mu)}(p_k; \lambda) = \max \left\{ 0, \frac{1 - p_{ki}}{2} \right\} \]
- If \( p_{ki} - p_{kj} > \epsilon \), for \( \epsilon > 0 \) fixed, then
  \[ \lim_{\lambda \to \infty} D_{ki}^{(1-\mu)}(p_k; \lambda) = 0 \]
  and
  \[ \lim_{\lambda \to \infty} D_{kj}^{(1-\mu)}(p_k; \lambda) = 1 - p_{kj} \]
- If \( p_{ki} = p_{kj} \leq 1 \), then
  \[ D_{ki}^{(1-\mu)}(p_k; \lambda) = \frac{1 - p_{ki}}{2} \]
  for all \( \lambda \in [0, +\infty) \)

As mentioned before, these demands can be readily derived from the utility-maximization problem of a representative consumer with quadratic utility \textit{a la} Shubik and Levitan (1980):

\[
U(q_{ki}, q_{kj}, I; \lambda) = (q_{ki} + q_{kj}) - \left( \frac{1}{1 + \lambda} \right) \left( q_{ki}^2 + q_{kj}^2 + \frac{\lambda}{2} (q_{ki} + q_{kj})^2 \right) + I \quad (3.12)
\]

where \( I \) is income.

Demand from consumers that value both products With perfect correlation in valuation, the decision of final consumers can be recasted between buying an item of two units (item/product \( x \)) for which retailer \( i = 1, 2 \) charges

\[ p_{xi} = \min\{p_{ABi}, p_{Ai} + p_{Bi}\} \]

and buying an item (item/product \( y \)) of one unit for which retailer \( i = 1, 2 \) charges

\[ p_{yi} = \min\{p_{Ai}, p_{Bi}\} \]

Each store then changes prices \((p_{xi}, p_{yi})\) for these items, as defined in our framework with perfect correlation.

Again, demands are derived from the utility-maximization problem of a representative consumer with quadratic utility \textit{a la} Shubik and Levitan (1980) but instead of having preferences
for goods $A$ and $B$, this consumer is assumed to have preferences for items $x$ (of both $A + B$) and $y$ (of either $A$ or $B$) as follows:

$$U(q_{xi}, q_{xj}, q_{yi}, q_{yj}, I; \lambda) = 2(q_{xi} + q_{xj}) + (q_{yi} + q_{yj}) - \left( \frac{1}{1+\lambda} \right) \left[ q_{xi}^2 + q_{xj}^2 + \frac{\lambda}{2} (q_{xi} + q_{xj})^2 \right] + (q_{xi} + q_{yi})^2 + (q_{xj} + q_{yj})^2 + \frac{\lambda}{2} (q_{xi} + q_{yi} + q_{xj} + q_{yj})^2 + I$$

We will now derive the demands coming from this representative consumer, and show that in the monopoly and Bertrand limits converge to the ones in our model in section 3.2.4 above.

There are seven cases to consider:

1. $q_{xi}, q_{yi} \geq 0$ for $i = 1, 2$ (both retailers sell $x$ and $y$)

$$D_{xi}^{(\mu)}(P_x, P_y; \lambda) = \frac{1}{2} (1 - p_{xi} + p_{yi}) + \frac{\lambda}{4} (p_{xj} - p_{xi} + p_{yi} - p_{yj})$$

$$D_{yi}^{(\mu)}(P_x, P_y; \lambda) = \frac{1}{2} (p_{xj} - 2p_{yi}) + \frac{\lambda}{4} \left( [p_{xi} - 2p_{yi}] - [p_{xj} - 2p_{yj}] \right)$$

2. $q_{xi}, q_{yi} \geq 0, q_{xj} = q_{yj} = 0$

$$D_{xi}^{(\mu)}(P_x, P_y; \lambda) = \left( \frac{1 + \lambda}{2 + \lambda} \right) (1 - p_{xi} + p_{yi})$$

$$D_{yi}^{(\mu)}(P_x, P_y; \lambda) = \frac{p_{xj} - 2p_{yi}}{2} + \left( \frac{\lambda}{2 + \lambda} \right) \left[ \frac{(p_{xi} - 2p_{yi})}{2} - \frac{(1 - p_{yj})}{2} + \frac{\lambda}{4} (p_{yj} - p_{yi}) \right]$$

$$D_{xj}^{(\mu)}(P_x, P_y; \lambda) = 0$$

$$D_{yj}^{(\mu)}(P_x, P_y; \lambda) = \frac{1 - p_{yi}}{2} + \frac{\lambda}{4} (p_{yj} - p_{yi})$$

3. $q_{xi}, q_{yi}, q_{yj} \geq 0, q_{xj} = 0$

$$D_{xi}^{(\mu)}(P_x, P_y; \lambda) = \frac{1 - p_{xi} + p_{yi}}{2} + \frac{1}{2} \left( \frac{\lambda}{2 + \lambda} \right) \left[ \frac{p_{xj} - 2p_{yi}}{2} + \frac{\lambda}{4} (p_{xj} - p_{xi}) \right]$$

$$D_{yi}^{(\mu)}(P_x, P_y; \lambda) = \left( \frac{1 + \lambda}{2 + \lambda} \right) (p_{xj} - 2p_{yi})$$

$$D_{xj}^{(\mu)}(P_x, P_y; \lambda) = \frac{1}{2} \left( 1 - \frac{p_{xj}}{2} \right) + \frac{\lambda}{8} (p_{xj} - p_{xi})$$

$$D_{yj}^{(\mu)}(P_x, P_y; \lambda) = 0$$

4. $q_{xi}, q_{yi}, q_{xj} \geq 0, q_{yj} = 0$

$$D_{xi}^{(\mu)}(P_x, P_y; \lambda) = \left( \frac{1 - p_{xi} + p_{yi}}{2} \right) + \frac{1}{2} \left( \frac{\lambda}{2 + \lambda} \right) \left[ \frac{p_{xj} - 2p_{yi}}{2} + \frac{\lambda}{4} (p_{xj} - p_{xi}) \right]$$

$$D_{yi}^{(\mu)}(P_x, P_y; \lambda) = \left( \frac{1 + \lambda}{2 + \lambda} \right) (p_{xj} - 2p_{yi})$$

$$D_{xj}^{(\mu)}(P_x, P_y; \lambda) = \frac{1}{2} \left( 1 - \frac{p_{xj}}{2} \right) + \frac{\lambda}{8} (p_{xj} - p_{xi})$$

$$D_{yj}^{(\mu)}(P_x, P_y; \lambda) = 0$$

5. $q_{xi}, q_{xj} \geq 0, q_{yi} = q_{yj} = 0$

$$D_{xi}^{(\mu)}(P_x, P_y; \lambda) = \frac{1}{2} \left( 1 - \frac{p_{xi}}{2} \right) + \frac{\lambda}{8} (p_{xj} - p_{xi})$$

$$D_{yi}^{(\mu)}(P_x, P_y; \lambda) = 0$$
6. \( q_{yi}, q_{yj} \geq 0, q_{xi} = q_{xj} = 0 \)

\[
D_{xi}^{(\mu)}(p_x, p_y; \lambda) = 0 \\
D_{yi}^{(\mu)}(p_x, p_y; \lambda) = \frac{(1 - p_{yi})}{2} + \frac{\lambda}{4} (p_{yj} - p_{yi})
\]

7. \( q_{xi}, q_{yj} \geq 0, q_{xj} = q_{yi} = 0 \)

\[
D_{xi}^{(\mu)}(p_x, p_y; \lambda) = 4 \left( \frac{1 + \lambda}{\lambda^2 + 8\lambda + 8} \right) \left[ 1 - \frac{p_{xi}}{2} + \frac{\lambda}{4} (1 - p_{xi} + p_{yj}) \right] \\
D_{yi}^{(\mu)}(p_x, p_y; \lambda) = 0 \\
D_{xj}^{(\mu)}(p_x, p_y; \lambda) = 0 \\
D_{yj}^{(\mu)}(p_x, p_y; \lambda) = 4 \left( \frac{1 + \lambda}{\lambda^2 + 8\lambda + 8} \right) \left[ 1 - p_{yj} + \frac{\lambda}{4} (p_{xi} - 2p_{yj}) \right]
\]

Showing that these demands converge to the local monopoly setting as \( \lambda \to 0 \) is straightforward, so it is omitted. Showing that the converge to the Bertrand setting as \( \lambda \to \infty \) is more involved, as it requires to consider the following five cases:

1. \( p_{xi} > p_{xj}, p_{yi} > p_{yj} \)
2. \( p_{xi} > p_{xj}, p_{yi} < p_{yj} \)
3. \( p_{xi} = p_{xj}, p_{yi} = p_{yj} \)
4. \( p_{xi} = p_{xj}, p_{yi} > p_{yj} \)
5. \( p_{xi} > p_{xj}, p_{yi} = p_{yj} \)

Holding any price difference fixed and letting \( \lambda \to \infty \) we obtain in each case, respectively:

1. For a sufficiently high \( \lambda \) we have that \( p_{xi} > p_{xj} \Rightarrow q_{xi} = 0 \) and \( p_{yi} > p_{yj} \Rightarrow q_{yi} = 0 \); hence

\[
\lim_{\lambda \to \infty} D_{xi}^{(\mu)}(p_x, p_y; \lambda) = 0 = \lim_{\lambda \to \infty} D_{yi}^{(\mu)}(p_x, p_y; \lambda) \\
\lim_{\lambda \to \infty} D_{xj}^{(\mu)}(p_x, p_y; \lambda) = \lim_{\lambda \to \infty} \left( \frac{1 + \lambda}{2 + \lambda} \right) (1 - p_{xi} + p_{yj}) \\
\lim_{\lambda \to \infty} D_{yj}^{(\mu)}(p_x, p_y; \lambda) = \lim_{\lambda \to \infty} \left( \frac{1 + \lambda}{2 + \lambda} \right) (p_{xj} - 2p_{yj}) = p_{xj} - 2p_{yj}
\]

2. For a sufficiently high \( \lambda \) we have that \( p_{xi} > p_{xj} \Rightarrow q_{xi} = 0 \) and \( p_{yi} < p_{yj} \Rightarrow q_{yi} = 0 \); hence

\[
\lim_{\lambda \to \infty} D_{xi}^{(\mu)}(p_x, p_y; \lambda) = \lim_{\lambda \to \infty} D_{yi}^{(\mu)}(p_x, p_y; \lambda) = 0 \\
\lim_{\lambda \to \infty} D_{xj}^{(\mu)}(p_x, p_y; \lambda) = \lim_{\lambda \to \infty} 4 \left( \frac{1 + \lambda}{\lambda^2 + 8\lambda + 8} \right) \left[ 1 - \frac{p_{xj}}{2} + \frac{\lambda}{4} (1 - p_{xj} + p_{yi}) \right] \\
= 1 - p_{xj} + p_{yi} \\
\lim_{\lambda \to \infty} D_{yj}^{(\mu)}(p_x, p_y; \lambda) = \lim_{\lambda \to \infty} 4 \left( \frac{1 + \lambda}{\lambda^2 + 8\lambda + 8} \right) \left[ 1 - p_{yi} + \frac{\lambda}{4} (p_{xj} - 2p_{yi}) \right] \\
= p_{xj} - 2p_{yi}
\]

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3. When $p_{xi} = p_{xj}$ and $p_{yi} = p_{yj}$

$$D_{xi}^{(\mu)}(p_x, p_y; \lambda) = \frac{1 - p_{xi} + p_{yi}}{2}$$
$$D_{yi}^{(\mu)}(p_x, p_y; \lambda) = \frac{p_{xi} - 2p_{yi}}{2}$$

for all $\lambda \in [0, +\infty)$; hence

$$\lim_{\lambda \to \infty} D_{xi}^{(\mu)}(p_x, p_y; \lambda) = \frac{1 - p_{xi} + p_{yi}}{2}$$
$$\lim_{\lambda \to \infty} D_{yi}^{(\mu)}(p_x, p_y; \lambda) = \frac{p_{xi} - 2p_{yi}}{2}$$

4. For a sufficiently high $\lambda$ we have that $p_{yi} > p_{yj}$ $\Rightarrow$ $q_{yi} = 0$, therefore

$$D_{xi}^{(\mu)}(p_x, p_y; \lambda) = \frac{1}{2} \left( 1 - \frac{p_{xi}}{2} \right) \forall \lambda \in [0, +\infty)$$
$$D_{xj}^{(\mu)}(p_x, p_y; \lambda) = \frac{1}{2} - \frac{3}{4} p_{xj} + p_{yj}$$
$$D_{yj}^{(\mu)}(p_x, p_y; \lambda) = \left( \frac{1 + \lambda}{2 + \lambda} \right) (p_{xj} - 2p_{yj})$$

As $p_{xi} = p_{xj}$, we obtain

$$\lim_{\lambda \to \infty} D_{xi}^{(\mu)}(p_x, p_y; \lambda) = \frac{1}{2} \left( 1 - \frac{p_{xi}}{2} \right) = \frac{1 - p_{xi} + p_{yi}}{2} + \frac{1}{4} (p_{xi} - 2p_{yj})$$
$$\lim_{\lambda \to \infty} D_{xj}^{(\mu)}(p_x, p_y; \lambda) = \frac{1}{2} - \frac{3}{4} p_{xj} + p_{yj}$$
$$\lim_{\lambda \to \infty} D_{yj}^{(\mu)}(p_x, p_y; \lambda) = p_{xj} - 2p_{yj}$$

Note: There is a small difference with the perfect-correlation model. Now the sharing rule when retailers price the same for an item is “endogenous” to the portfolio of products they carry; so it cannot be exogenously imposed to be equal to 1/2.

5. For a sufficiently high $\lambda$, we have that $p_{xi} > p_{xj}$ $\Rightarrow$ $q_{xi} = 0$

$$\lim_{\lambda \to \infty} D_{xi}^{(\mu)}(p_x, p_y) = 0$$
$$\lim_{\lambda \to \infty} D_{xj}^{(\mu)}(p_x, p_y) = \lim_{\lambda \to \infty} \left( \frac{1 + \lambda}{2 + \lambda} \right) \left( 1 - p_{xj} + p_{yj} \right) = 1 - p_{xj} + p_{yj}$$
$$\lim_{\lambda \to \infty} D_{yi}^{(\mu)}(p_x, p_y) = \lim_{\lambda \to \infty} \left( \frac{1 - p_{yi}}{2} \right) = \frac{1 - p_{yi}}{2} = \frac{p_{xj} - 2p_{yj} + \frac{1 - p_{xj} + p_{yj}}{2}}{2}$$
$$\lim_{\lambda \to \infty} D_{yj}^{(\mu)}(p_x, p_y) = \lim_{\lambda \to \infty} \left[ \frac{p_{xj} - 2p_{yj}}{2} + \left( \frac{\lambda}{2 + \lambda} \right) \left( \frac{p_{xj} - 2p_{yj}}{2} - \frac{1 - p_{yj}}{2} \right) \right]$$
$$= \frac{p_{xj} - 2p_{yj}}{2} - \frac{1 - p_{xj} + p_{yj}}{2}$$

Note: Here also applies the above note about the “endogenous” of the sharing rule.
3.3.2 The full-monopolization outcome

We now characterize the vertical and horizontal integrated outcome, show that there are no size effects despite imperfect competition, and prove that standalone contracts are sufficient for $M$ to implement it in the absence of any competitive threat.

So suppose $S$ is not present and that $M-R1-R2$ are vertically and horizontally integrated. It is not difficult to prove that in the optimum both retailers must be selling positive units of all goods/items, i.e., $q_{ki} > 0$ for $k \in \{A, B, x, y\}$ and $i = 1, 2$. Hence this integrated structure solves

$$\max_{p_{Ai}, p_{Bi}, p_{ABi}} \sum_{i=1,2} \left( 1 - \mu \right) \left\{ p_{Ai} \left( \frac{1 - p_{Ai}}{2} + \frac{\lambda}{4} (p_{Aj} - p_{Ai}) \right) + p_{Bi} \left( \frac{1 - p_{Bi}}{2} + \frac{\lambda}{4} (p_{Bj} + p_{Bi}) \right) \right\}$$

$$+ \mu \left\{ p_{xi} \left( \frac{1 - p_{xi} + p_{yi}}{2} + \frac{\lambda}{4} (p_{xj} - p_{xi} + p_{yi} - p_{yj}) \right) + p_{yi} \left( \frac{p_{xi} - 2p_{yi}}{2} + \frac{\lambda}{4} (p_{xj} - 2p_{yi} - [p_{xj} - 2p_{yj}]) \right) \right\}$$

where $p_{xi} \equiv \min\{p_{ABi}, p_{Ai} + p_{Bi}\}$ and $p_{yi} \equiv \min\{p_{Ai}, p_{Bi}\}$.

In solving this maximization problem, abstract momentarily from the fact that $p_{xi} = \min\{\cdot, \cdot\}$ and $p_{yi} = \min\{\cdot, \cdot\}$, and proceed with an unconstrained maximizing. We then get:

$$p_{Ai}^* = 1/2, \quad p_{Bi}^* = 1/2, \quad p_{ABi}^* = 1$$

But if so, the following is the solution to the “original” problem:

$$p_{Ai}^* = 1/2, \quad p_{Bi}^* = 1/2, \quad p_{ABi}^* = 1$$

for $i = 1, 2$ since

$$p_{xi}^* \equiv \min\{p_{ABi}^*, p_{Ai}^* + p_{Bi}^*\} = 1, \quad p_{yi}^* \equiv \min\{p_{Ai}^*, p_{Bi}^*\} = 1/2$$

Evaluating, we get that $\Pi^*_M = 1/2 - F_M$.

Before we move on, two things are important to notice. First, there is no size effect (i.e., $\Pi^*_M$ is independent of $\lambda$). This is because of the utility functions adopted (see Shubik and Levitan 1980 and Simpson and Wickelgren 2007). And second, there is no retail building in the fully-integrated outcome. This is because we are extending the model with perfectly positive correlation.

We now explain how $M$ can implement the full-integrated outcome and show that standalone contracts suffice (in the absence of $S$). In particular, we claim that the following pair of non-discriminatory standalone schedules allow $M$ to implement such outcome while appropriating the entire surplus:

$$C^M_{Ai} = \left\{ w^M_{Ai} = \frac{1}{2} \left( \frac{\lambda}{2 + \lambda} \right), T^M_{Ai} = \frac{1}{4} \left( \frac{1}{2 + \lambda} \right) - \epsilon \right\}$$

$$C^M_{Bi} = \left\{ w^M_{Bi} = \frac{1}{2} \left( \frac{\lambda}{2 + \lambda} \right), T^M_{Bi} = \frac{1}{4} \left( \frac{1}{2 + \lambda} \right) - \epsilon \right\}$$
with $\epsilon \to 0$ for $i = 1, 2$.

First, suppose that both retailers accept both contracts (shortly we will come back acceptance/rejection decisions). In the Nash equilibrium of the retail pricing subgame, retailer $R_i$ solves:

$$\max_{p_{A_i}, p_{B_i}, p_{AB_i}} \sum_{i=1,2} \left[ (1-\mu) \left\{ (p_{A_i} - w) \left( \frac{1-p_{A_i}}{2} + \frac{\lambda}{4} (p_{A_j} - p_{A_i}) \right) 
+ (p_{B_i} - w) \left( \frac{1-p_{B_i}}{2} + \frac{\lambda}{4} (p_{B_j} + p_{B_i}) \right) \right\} 
+ \mu \left\{ (p_{x_i} - 2w) \left( \frac{1-p_{x_i} + p_{y_i}}{2} + \frac{\lambda}{4} (p_{x_j} - p_{x_i} + p_{y_i} - p_{y_j}) \right) 
+ (p_{y_i} - w) \left( \frac{p_{x_i} - 2p_{y_i}}{2} + \frac{\lambda}{4} ([p_{x_i} - 2p_{y_i} - [p_{x_j} - 2p_{y_j}]] \right) \right\} \right]$$

where $p_{x_i} = \min\{p_{A_i} + p_{B_i}, p_{AB_i}\}$, $p_{y_i} = \min\{p_{A_i}, p_{B_i}\}$ and $w = \frac{1}{2} \left( \frac{2\lambda}{x^2} \right) = w_A^M = w_B^M$.

To obtain best responses, we abstract momentarily from the restrictions imposed by the definitions of $p_{x_i}$ and $p_{y_i}$ and solve the relaxed problem. Imposing symmetry, we obtain unique equilibrium of the “relaxed” game:

$$p^*_{A_i} = 1/2, \quad p^*_{B_i} = 1/2, \quad p^*_{x_i} = 1 \quad p^*_{y_i} = 1/2$$

for $i = 1, 2$. But then

$$p^*_{A_i} = 1/2, \quad p^*_{B_i} = 1/2, \quad p^*_{AB_i} = p^*_{A_i} + p^*_{B_i}$$

for $i = 1, 2$, is an equilibrium of the original game. It is then possible to prove as well that this is actually the unique equilibrium.

Evaluating, we get that retailer’s equilibrium profit before fixes fees is equal to

$$\bar{\pi}^*_R = \frac{1}{2(2+\lambda)}$$

so $\pi^*_R = \bar{\pi}^*_R - T_{A_i}^M - T_{B_i}^M = 2\epsilon$, with $\epsilon \to 0$. Hence if $M$ approaches retailers with the above contracts and they accept them, then $M$ implements the fully-integrated outcome appropriating the entire surplus.

We now deal with retailers’ contract decisions. On path, each of them is getting $\epsilon \to 0$. Therefore accepting both clearly dominates rejecting both. We will now argue that there cannot be an equilibrium in which one retailer accepts a single contract. Suppose the contrary. In particular, suppose $R_j$ decides to accept only $B$ (this is w.l.o.g given that demands and contracts are fully symmetric). It is then not difficult to prove that $R_i$’s best response is to accept both. Doing so makes her the only supplier of $A$, that is, the only on serving consumers that value $A$ only and the best positioned to compete for those consumers that value both goods but $A$ highly.
So consider the retail pricing subgame where \( R_j \) carries only \( B \) and \( R_i \) carries \( A \) and \( B \). Then, it is possible to prove that the pricing equilibrium is given by:

\[
\begin{align*}
p^{*}_{Ai} &= \frac{1}{4} \left( \frac{4 + 3\lambda}{2 + \lambda} \right), \quad p^{*}_{Bi} = p^{*}_{Bj} = \frac{1}{2}, \quad p^{*}_{ABi} = p^{*}_{Ai} + p^{*}_{Bi}
\end{align*}
\]

which implies

\[
\begin{align*}
q^{*}_{Ai} &= \frac{1}{4} \frac{(1 + \lambda)(4 + \lambda)}{(2 + \lambda)^2}, \quad q^{*}_{Bi} = q^{*}_{Bj} = \frac{1}{4}, \quad q^{*}_{xi} = \frac{1}{4} \frac{(1 + \lambda)(4 + \lambda)^2}{(2 + \lambda)(\lambda^2 + 8\lambda + 8)}
\end{align*}
\]

But that cannot be equilibrium given that

\[
\begin{align*}
\pi^{*}_{Rj} &= \pi^{*}_{Rj} - T_{Bj} = \frac{-\mu\lambda^2}{4(2 + \lambda)^2(\lambda^2 + 8\lambda + 8)} < 0
\end{align*}
\]

and

\[
\begin{align*}
\pi^{*}_{Ri} &= \pi^{*}_{Ri} - T_{Ai} - T_{Bi} = \frac{\lambda}{16} \frac{(13\lambda^3 + 56\lambda^2 + 104\lambda - 4\mu\lambda + 64)}{(2 + \lambda)^3(\lambda^2 + 8\lambda + 8)} > \epsilon > 0
\end{align*}
\]

Clearly, \( R_j \) would be better-off accepting both contracts and get \( 2\epsilon > 0 \); a contradiction. Therefore, the only equilibrium is for both retailers to accept both offers, which leads \( M \) to implement the fully-integrated outcome, that is, the full-monopolization outcome.

### 3.3.3 Almost Bertrand competitors

Suppose \( M \) approaches both retailers with the following non-discriminatory full-line pricing offers:

\[
\begin{align*}
C^M_{Ai} &= (w^M_{Ai} = 1, T_{Ai} = 0), \quad C^M_{Bi} = \left( w^M_{Bi} = \frac{1}{2} \left( \frac{\lambda}{2 + \lambda} \right), T^M_{Bi} = \frac{1}{4} \left( \frac{1}{2 + \lambda} \right) \right)
\end{align*}
\]

for \( i = 1, 2 \). What is \( S \)'s optimal reaction? The next lemma states that for a \( \lambda \) sufficiently large, \( S \) anticipates that at most one retailer will accept his offer, never both, independently of the offers he makes.

**Lemma 3.3.1.** Suppose \( M \) approaches retailers with offers (3.13) and \( S \) with the (arbitrary) offers \((w^S_{Bi}, T^S_{Bi})\) to \( i = 1, 2 \). Then, \( \exists \tilde{\lambda} \in (0, +\infty) \) such that \( \forall \lambda > \tilde{\lambda} \) there is no equilibrium in the subgame starting at date 2, in which both retailers accept \( S \)'s offers.

In the proof of Lemma 3.3.1, the following claim will prove useful.
Claim 3.3.1. Suppose $M$ approaches retailers with offers (3.13) and $S$ with the (arbitrary) offers $(w^S_{Bi}, T^S_{Bi})$ to $i = 1, 2$. Also, suppose there exists an equilibrium at the subgame beginning at date 2 in which both retailers accept $S$’s offer. Then

$$\pi^*_Ri > \frac{2 + \lambda}{(4 + \lambda)^2} \quad \text{for } i = 1, 2 \text{ and } \forall \lambda > \bar{\lambda} \in (0, +\infty)$$

Proof. For the claim to hold, it must be true that a retailer does not find it optimal to deviate and choose $M$’s full-line forcing contract instead. Let $\pi'_Ri$ be $Ri$’s profit from such deviation. We argue that

$$\pi'_Ri > \frac{2 + \lambda}{(4 + \lambda)^2} \quad \forall \lambda > \bar{\lambda} \in (0, +\infty)$$

Indeed, suppose $Ri$ deviates and signs with $M$. Then, the ensuing game has $Ri$ carrying both $A$ and $B$ from $M$, while $Rj$ carrying $B$ from $S$. Notice that the worst possible scenario for $Ri$ following this deviation is when $Rj$ has $w^B_{Bj} = 0$ (i.e., $w^B_{Bj} = 0$ results in the lowest $\pi'_Ri$).

So, consider the retail pricing equilibrium when $Ri$ procures $A$ and $B$ at $w^A_{Ai} = w^B_{Bi} = w \equiv (1/2)(\lambda/2 + \lambda)$ and $Rj$ procures $B$ at $w^B_{Bj} = 0$. It is then not difficult to show that $\exists \bar{\lambda}_1 \in (0, +\infty)$ such that for all $\lambda > \bar{\lambda}_1$, the retail pricing equilibrium is characterized by

$$p^*_A = \frac{1}{4} \left( \frac{4 + 3\lambda}{2 + \lambda} \right), \quad p^*_B = \frac{1}{4} \left( \frac{\lambda}{2 + \lambda} \right), \quad p^*_B = 2 \left( \frac{\lambda^2 + 8\lambda(1 - \mu/4) + 8}{\lambda^2(4 + 3\mu) + 32(1 + \lambda)} \right)$$

$$p^*_{xi} = p^*_A + p^*_B - \frac{1}{4} \left( \frac{1}{2 + \lambda} \right) \left[ \frac{3\mu\lambda^2 - 4\lambda^2(4 + \mu) - 128(1 + \lambda)}{\lambda^2(4 + 3\mu) + 32(1 + \lambda)} \right]$$

Therefore, in equilibrium $Ri$ sells $A$ and $B$ (i.e., item $x$), and $Rj$ sells $B$ to consumers who value $B$ only and to consumers who value both goods but not as much.

We can now evaluate $Ri$’s deviation profit to obtain:

$$\pi'_Ri = \frac{(1 - \mu)}{16} \frac{(1 + \lambda)(4 + \lambda)^2}{(2 + \lambda)^3} + \left( \frac{\mu}{\lambda^2 + 8\lambda + 8} \right) \left( \frac{1 + \lambda}{2 + \lambda} \right) \left( \frac{\lambda^2 + 4\lambda(4 + \mu) + 72\lambda + 64}{\lambda^2(4 + 3\mu) + 32(1 + \lambda)} \right)^2 - \frac{1}{2} \left( \frac{1}{2 + \lambda} \right)$$

But then

$$\lim_{\lambda \to \infty} \left( \frac{\pi'_Ri}{\pi^*_Ri} - \frac{(2 + \lambda)}{(4 + \lambda)^2} \right) = \frac{16 + 9\mu(1 - \mu^2) + 15\mu(1 - \mu)}{16(4 + 3\mu)^2} > 0$$

Hence, by continuity, $\exists \bar{\lambda}_2 \in (0, +\infty)$ such that for all $\lambda > \bar{\lambda}_2$

$$\pi'_Ri > \frac{2 + \lambda}{(4 + \lambda)^2}$$

But if so, $\forall \lambda > \max\{\bar{\lambda}_1, \bar{\lambda}_2\} \equiv \bar{\lambda}$ we have that

$$\pi^*_Ri \geq \pi'_Ri > \frac{2 + \lambda}{(4 + \lambda)^2}$$

which implies that $\pi^*_Ri > (2 + \lambda)/(4 + \lambda)^2$, which concludes the proof of the claim. ■
We are now in a position to prove Lemma 3.3.1. Suppose that $S$ makes arbitrary offers that both retailers accept, and w.l.o.g assume $w_{B1}^S \leq w_{B2}^S$. By Claim 3.3.1 both retailers must be making strictly positive profits and therefore selling strictly positive units of $B$. Hence the equilibrium of the ensuing retail pricing subgame comes from intersecting the best responses derived from the following maximization:

$$\max_{p_{Bi}} (p_{Bi} - w_{Bi}^S) \left( \frac{1 - p_{Bi}}{2} + \frac{\lambda}{4} (p_{Bj} - p_{Bi}) \right)$$

which yields

$$p_{Bi}^* = \left( \frac{2 + w_{Bi}(2 + \lambda)}{4 + \lambda} \right) - \left( \frac{\lambda}{4 + \lambda} \right) \left( \frac{2 + \lambda}{4 + 3\lambda} \right) (w_{Bi}^S - w_{Bj}^S)$$

$$\bar{\pi}_{Ri}^* = \frac{1}{4} \frac{(2 + \lambda)}{(4 + \lambda)^2(4 + 3\lambda)^2} \frac{2(4 + 3\lambda)(1 - w_{B1}^S) + \lambda(2 + \lambda)(w_{Bj} - w_{Bi})}{(2 + \lambda)(4 + \lambda)^2}$$

Hence

$$\bar{\pi}_{R2}^* \leq \frac{1}{4} \frac{2 + \lambda}{(4 + \lambda)^2(4 + 3\lambda)^2} \frac{2(4 + 3\lambda)(1 - w_{B1}^S)^2}{(2 + \lambda)(4 + \lambda)^2} \leq \frac{2 + \lambda}{(4 + \lambda)^2}$$

But this contradicts the fact that

$$\pi_{R2}^* > \frac{2 + \lambda}{(4 + \lambda)^2}$$

which concludes the proof of Lemma 3.3.1.

**Lemma 3.3.2.** Suppose $M$ approaches retailers with offers \(3.13\). Then, $\forall \lambda > \bar{\lambda} \in (0, +\infty)$, the largest revenue that $S$ can obtain in the market is:

$$1 \left( \frac{1 + \lambda}{2 + \lambda} \right) \frac{(\lambda^2 + 8\lambda(1 - \mu/4) + 8)^2}{(\lambda^2 + 8\lambda + 8)(\lambda^2(2 + \mu) + 16(1 + \lambda))}$$

**Proof.** Take $\lambda > \bar{\lambda}$. From Lemma 3.3.1 it is w.l.o.g to assume that $S$ approaches a single retailer. Suppose $R2$ is such retailer and that $S$’s offer to her is $(w_{B2}^S, T_{B2}^S)$. Although cumbersome, it is possible to show that for all $\lambda > \bar{\lambda} \in (0, +\infty)$, $S$’s optimal contract to $R2$ is given by:

$$w_{B2}^S = \frac{\mu \lambda^2}{2 \left( \lambda^2 + 8\lambda + 8 \right) \left( \lambda^2(2 + \mu) + 16(1 + \lambda) \right)}$$

$$T_{B2}^S = \frac{1}{4} \left( \frac{\mu \lambda^2}{\lambda^2 + 8\lambda + 8} \right) \left( \frac{1 + \lambda}{2 + \lambda} \right) \frac{(\lambda^2 + 8\lambda(1 - \mu/4) + 8)^2}{(\lambda^2 + 8\lambda + 8)(\lambda^2(2 + \mu) + 16(1 + \lambda))}$$

which results in the following equilibrium prices:

$$p_{A1}^* = \frac{1}{4} \left( \frac{4 + 3\lambda}{2 + \lambda} \right), \quad p_{B1}^* = \frac{1}{2} \left( \frac{\lambda}{2 + \lambda} \right), \quad p_{B2}^* = \frac{\lambda^2 + 8\lambda(1 - \mu/4) + 8}{\lambda^2(2 + \mu) + 16(1 + \lambda)}$$

$$p_{AB1}^* = p_{A1}^* + p_{B1}^* = \frac{1}{4} \left[ \frac{\mu \lambda^3 - 8\lambda^2 - 64(1 + \lambda) - 4}{(2 + \lambda)(\lambda^2(2 + \mu) + 16(1 + \lambda))} \right]$$

This implies that $R1$ sells $A$ and $AB$ while $R2$ is the only retailer selling only $B$. 

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$S$’s revenue is then equal to

$$\pi^*_S = \frac{1}{2} \left( \frac{1 + \lambda}{2 + \lambda} \right) \frac{(\lambda^2 + 8\lambda(1 - \mu/4) + 8)^2}{(\lambda^2 + 8\lambda + 8)(\lambda^2(2 + \mu) + 16(1 + \lambda))}$$

Finally, defining $\lambda = \max\{\check{\lambda}, \hat{\lambda}\}$ we obtain the desired result. □

**Proposition 3.3.1.** For all $\lambda > \check{\lambda} \in (0, +\infty)$, if

$$\frac{1}{2} \left( \frac{1 + \lambda}{2 + \lambda} \right) \frac{(\lambda^2 + 8\lambda(1 - \mu/4) + 8)^2}{(\lambda^2 + 8\lambda + 8)(\lambda^2(2 + \mu) + 16(1 + \lambda))} < F_S < F_M \leq \frac{1}{4}$$

then, there exists a foreclosure equilibrium in which $M$ fully monopolizes the market with offers $\{3.13\}$.

**Proof.** From Lemma 3.3.1 and

$$\lim_{\lambda \to \infty} \frac{1}{2} \left( \frac{1 + \lambda}{2 + \lambda} \right) \frac{(\lambda^2 + 8\lambda(1 - \mu/4) + 8)^2}{(\lambda^2 + 8\lambda + 8)(\lambda^2(2 + \mu) + 16(1 + \lambda))} = \frac{1}{4 + 2\mu} < \frac{1}{4}$$

continuity ensures that the foreclosure region is non-empty for $\lambda < \infty$. □

### 3.3.4 Almost monopoly retailers

In the main text we already proved that there does not exist a foreclosure equilibrium when retailers are local monopolies. We will now extend that proof to the case of $\lambda > 0$, but sufficiently close to zero. In essence, the argument is one of continuity, but we will still provide a detailed and formal proof for completeness.

**Lemma 3.3.3.** $\exists \lambda \in (0, +\infty)$ such that $\forall \lambda \in [0, \check{\lambda})$ there is no foreclosure equilibrium (and the equilibrium set is non-empty).

**Proof.** The proof follows a similar logic than the proof of Proposition 2b. For simplicity, we will focus on the case that $M$ offers full-line pricing contracts, but extending it for more general schedules is straightforward. First, as in Proposition 2a, it is easy to see that in all equilibria either $M$ is the only manufacturer selling $B$ or $S$ is the only manufacturer selling $B$. Moreover, using the same argument as in that proposition it must be that $\pi^*_M \geq 1/4$. We will now show that there cannot be an equilibrium in which $M$ is the sole provider of $B$. Suppose otherwise, that there exists a foreclosure equilibrium where $M$ approaches retailers with offers $\{(w^M_{Ai}, T^M_{Ai}), (w^M_{Bi}, T^M_{Bi}), (\hat{w}^M_{Ai}, \hat{T}^M_{Ai}, \hat{w}^M_{Bi}, \hat{T}^M_{Bi})\}$ for $i = 1, 2$ and $S$ with offers $(w^M_{Bi}, T^M_{Bi})$ for $i = 1, 2$.

Define $\pi^R_i$ to be $R_i$’s before-fixed-fees profit when $R_i$ accepts $h$’s contract and $R_j$ accepts $l$’s contract, where $h, l \in \{M, S\}$, $V^M_i(h, l)$ to be $M$’s before-fixed-fees profit coming from retailer $R_i$ when $R_i$ accepts $h$’s contract and $R_j$ accepts $l$’s, and $V^S_i(h, l)$ to be likewise but for manufacturer $S$. Then

$$\pi^*_M = V^M_1(M, M) + V^M_2(M, M) + \hat{T}^M_{A1} + \hat{T}^M_{B1} + \hat{T}^M_{A2} + \hat{T}^M_{B2} - F_M$$
Clearly \( \hat{T}_{ABi}^M \equiv \hat{T}_{AI}^M + \hat{T}_{Bi}^M \leq \hat{\pi}_{Ri}^{MM} \) due to \( R_i \)'s participation constraint.

Consider then the following deviation by \( S \)

\[
\left( u_{Bi}^S = \frac{1}{2} \left( \frac{\lambda}{2 + \lambda} \right), T_{Bi}^S \right)
\]  

(3.14)

for \( i = 1, 2 \) and where \( T_{Bi}^S \) is chosen so that both retailers end up accepting \( S \)'s offers. Define \( \hat{\pi}_{Ri}^{hl} \), \( V_i^{Si}(h, l) \) and \( V_i^{Mi}(h, l) \) as above with \( S \)'s new contract/deviation (while keeping \( M \)'s equilibrium offers fixed). Clearly \( \hat{\pi}_{Ri}^{MM} = \hat{\pi}_{Ri}^{MM} \) and \( V_i^{Mi}(M, M) = V_i^{M}(M, M) \). Retailers' acceptance decision game can then be summarized as follows:

<table>
<thead>
<tr>
<th>( R1 )</th>
<th>( R2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( M )</td>
<td>( S )</td>
</tr>
<tr>
<td>( (\hat{\pi}<em>{R1}^{MM} - \hat{T}</em>{AB1}^M, \hat{\pi}<em>{R2}^{MM} - \hat{T}</em>{AB2}^M) )</td>
<td>( (\max{0, \hat{\pi}<em>{R1}^{Si} - \hat{T}</em>{AB1}^M}, \hat{\pi}<em>{R2}^{Si} - T</em>{B2}^S) )</td>
</tr>
<tr>
<td>( (\hat{\pi}<em>{R1}^{Si} - T</em>{B1}^S, \max{0, \hat{\pi}<em>{R2}^{Si} - \hat{T}</em>{AB2}^M}) )</td>
<td>( (\hat{\pi}<em>{R1}^{Si} - T</em>{B1}^S, \hat{\pi}<em>{R2}^{Si} - T</em>{B2}^S) )</td>
</tr>
</tbody>
</table>

Hence for both retailers to take \( S \)'s contract we need for \( i = 1, 2 \): (i) \( \hat{\pi}_{Ri}^{Si} - T_{B1}^S \geq \hat{\pi}_{Ri}^{MM} - \hat{T}_{ABi}^M \) and (ii) \( \hat{\pi}_{Ri}^{Si} - T_{B1}^S \geq \max\{0, \hat{\pi}_{Ri}^{Si} - \hat{T}_{ABi}^M\} \), which implies that

\[
T_{Bi}^S = \min\{\hat{\pi}_{Ri}^{Si} - \hat{\pi}_{Ri}^{MM} + \hat{T}_{ABi}^M, \hat{\pi}_{Ri}^{Si} - \max\{0, \hat{\pi}_{Ri}^{Si} - \hat{T}_{ABi}^M\} - \epsilon\}
\]  

(3.15)

And using (3.14) we can obtain \( S \)'s deviation payoff:

\[
\pi_S' = \left[ V_1^{Si}(S, S) + V_2^{Si}(S, S) + \hat{T}_{B1}^S + \hat{T}_{B2}^S - F_S \right]
\]

\[
= \left[ \max\{\hat{\pi}_{R1}^{Si} - \hat{\pi}_{R1}^{MM} + \hat{T}_{AB1}^M, \hat{\pi}_{R2}^{Si} - \max\{0, \hat{\pi}_{R2}^{Si} - \hat{T}_{AB2}^M\} - \epsilon\} \right]
\]

But if \( S \) offers (3.14), we have that irrespective of \( \lambda \):

\[
V_1^{Si}(S, S) + V_2^{Si}(S, S) + \hat{\pi}_{R1}^{Si} + \hat{\pi}_{R2}^{Si} = \frac{1}{4}
\]

Hence

\[
\pi_S' = \frac{1}{4} - F_S + \max\{\hat{\pi}_{R1}^{MM} + \hat{\pi}_{R1}^{Si} - \hat{\pi}_{R1}^{MM} + \hat{T}_{AB1}^M, \hat{\pi}_{R2}^{Si} - \max\{0, \hat{\pi}_{R2}^{Si} - \hat{T}_{AB2}^M\} \}
\]

which can be rewritten as:

\[
\pi_S' = \frac{1}{4} - F_S + \hat{T}_{AB1}^M + \hat{T}_{AB2}^M - \max\{\hat{\pi}_{R1}^{MM} + \hat{\pi}_{R1}^{Si} - \hat{\pi}_{R1}^{MM}, \max\{\hat{T}_{AB1}^M, \hat{\pi}_{R1}^{Si}\} \}
\]

\[
- \max\{\hat{\pi}_{R2}^{MM} + \hat{\pi}_{R2}^{Si} - \hat{\pi}_{R2}^{MM}, \max\{\hat{T}_{AB2}^M, \hat{\pi}_{R2}^{Si}\} \} - 2\epsilon
\]
Now, let \( K_i = \max\{\bar{\pi}_{R_i}^{MM} + \bar{\pi}_{R_i}^{SS}, \bar{\pi}_{R_i}^{SM}, \max\{\bar{\pi}_{R_i}^{MM}, \bar{\pi}_{R_i}^{MS}\}\} \). Since \( T_{AB_i}^M \leq \bar{\pi}_{R_i}^{MM} \), we have that
\[
\pi' \geq \frac{1}{4} - F_S + \hat{T}_{AB_1}^M + \hat{T}_{AB_2}^M - K_1 - K_2 - 2\epsilon \equiv \pi_S'
\]
Therefore, a necessary condition for exclusion is that \( \pi_S' \leq 0 \), that is,
\[
\hat{T}_{AB_1}^M + \hat{T}_{AB_2}^M \leq F_S - \frac{1}{4} + K_1 + K_2
\]
But if so, then
\[
\pi_{AB}^* = V_{1}^{MM}(M, M) + V_{2}^{MM}(M, M) + \hat{T}_{AB_1}^M + \hat{T}_{AB_2}^M - F_M
\]
\[
\leq [V_{1}^{MM}(M, M) + V_{2}^{MM}(M, M) + \bar{\pi}_{R_1}^{MM} + \bar{\pi}_{R_2}^{MM} - F_M]
\]
\[
+ (K_1 - \bar{\pi}_{R_1}^{MM}) + (K_2 - \bar{\pi}_{R_2}^{MM}) - 1/4 + F_S
\]
\[
\leq 1/4 - F_M + (K_1 - \bar{\pi}_{R_1}^{MM}) + (K_2 - \bar{\pi}_{R_2}^{MM}) + F_S
\]
where the last inequality comes from the fact that
\[
V_{1}^{MM}(M, M) + V_{2}^{MM}(M, M) + \bar{\pi}_{R_1}^{MM} + \bar{\pi}_{R_2}^{MM} \leq 1/2
\]
which is the maximum industry revenue.

We now want to evaluate \( \pi_{AB}^* \) as \( \lambda \to 0 \). We know that
\[
\lim_{\lambda \to 0} \pi_{AB}^* \leq \frac{1}{4} - F_M + \lim_{\lambda \to 0}(K_1 - \bar{\pi}_{R_1}^{MM}) + \lim_{\lambda \to 0}(K_2 - \bar{\pi}_{R_2}^{MM}) + F_S
\]
where
\[
K_i - \bar{\pi}_{R_i}^{MM} = \max\{\bar{\pi}_{R_i}^{SS} - \bar{\pi}_{R_i}^{SM}, \max\{0, \bar{\pi}_{R_i}^{MS} - \bar{\pi}_{R_i}^{MM}\}\}
\]
Using the fact that \( \lim_{\lambda \to 0} \bar{\pi}_{R_i}^{SS} = \lim_{\lambda \to 0} \bar{\pi}_{R_i}^{SM} \) (as in the limit \( Ri \)'s profit from taking \( S \)'s offer is independent of \( Rj \)'s choice) and \( \lim_{\lambda \to 0} \bar{\pi}_{R_i}^{MS} = \lim_{\lambda \to 0} \bar{\pi}_{R_i}^{MM} \) (as in the limit \( Ri \)'s profit from taking \( M \)'s offer is independent of \( Rj \) choice), it is immediate that \( \lim_{\lambda \to 0}(K_i - \bar{\pi}_{R_i}^{MM}) = 0 \) for \( i = 1, 2 \) and hence that
\[
\lim_{\lambda \to 0} \pi_{AB}^* \leq \frac{1}{4} - (F_M - F_S) < \frac{1}{4}
\]
leading to a contradiction, as desired. 

### 3.4 Efficiency-Enhancing Wholesale Bundling Contracts

While in our baseline model the only reason for wholesale bundling to emerge is to foreclose a more efficient rival, this does not need always be the case (e.g., Shaffer 1991, Vergé 2001, Jeon and Menicucci 2012). In this section we present an exercise where wholesale bundling emerges but for a very different reason. It does to eliminate a distortion in the retail market, resulting in an increase not only of \( M \)'s profit but also of consumer surplus. The problem for a competition authority, however, is that this “pro-competitive” bundling might be coupled with the exclusion of more inefficient rival. Consistent with our opening quote, this poses a
challenge to any competition authority: how to distinguish between two situations that share many similarities—existence of bundling and foreclosure—but at the same time have radically different antitrust implications. This exercise will serve to illustrate how our theory can be of great help on this task.

Consider the case of retail monopolies, which for simplicity we will assume there is only one, which we denote by \( R \). We will assume some diseconomies of carrying multiple products at the retail level, that is,

\[ A1: \Pi(q_A, q_B) < \Pi(q_A, 0) + \Pi(0, q_B) \]

for any pair \((q_A, q_B)\) and where \( \Pi(x, y) \) is total industry profits from selling \( x \) units of good \( A \) and \( y \) units of product \( B \). We will also assume that it is industry optimal to sell strictly positive units of both products, that is

\[ A2: (q^*_A, q^*_B) \in \arg \max_{q_A, q_B} \Pi(q_A, q_B) \]

where \( q^*_A, q^*_B > 0 \). Let \( \Pi^{AB} \equiv \Pi(q^*_A, q^*_B) \). Note that \( q^*_A < q^{fb}_A \) and \( q^*_B < q^{fb}_B \), where \( q^{fb}_k \) is the welfare-maximizing output for \( k = A, B \).

A setting that fits well to our assumptions is one in which \( A \) and \( B \) are completely unrelated at the consumer level (as in our baseline model) but not at the retail level so that industry profits can be written as

\[ \Pi(q_A, q_B) = Q^A(q_A) + Q^B(q_B) - E(q_A, q_B) \]

where \( E(q_A, q_B) \) corresponds to retail effort and \( \partial E(q_A, q_B)/\partial q_A > 0, \partial E(q_A, q_B)/\partial q_B > 0, \) and \( \partial^2 E(q_A, q_B)/\partial q_A \partial q_B > 0 \). A function such as \( E(q_A, q_B) \) captures the idea that any additional effort allocated to selling \( A \) makes any effort allocated to \( B \) less effective. For example, we could extend our baseline model so that

\[ \Pi^{AB} = \max_{s_A, s_B, q_A, q_B} Q^A(q_A(s_A)) + Q^B(q_B(s_B)) - E(s_A, s_B) \]

where \( E(s_A, s_B) = \kappa s^2_A + \kappa s^2_B + (1 - \kappa)(s_A + s_B)^2, v_B = b(1 + s_B) \) and \( v_A \sim U[0, 1 + s_A] \). The parameter \( \kappa \in [0, 1] \) would then capture the diseconomies in retail effort of carrying multiple products.

Going back to our more general formulation ask now what would be \( M \)'s optimal contract offer to \( R \) in \( S \)'s absence.

**Lemma 3.4.1.** Suppose that \( M \) is the only manufacturer in the market. If \( M \) is restricted to separate pricing, \( M \)'s optimal (separate) contract fails to implement the industry profit-maximizing outcome.

**Proof.** The most general separate pricing schedule \( M \) can offer \( R \) is the “direct” pair \((q_A, t_A) \) and \((q_B, t_B) \), where \( t_k \) is the transfer to \( M \) that \( R \) agrees to make in exchange of \( q_k \).
units of good \( k = A, B \). \( M \)'s problem then is to maximize \( t_A + t_B \) subject to \( R \)'s participation and incentive compatibility constraints

\[
\begin{align*}
\Pi(q_A, q_B) - t_A - t_B & \geq 0 \quad (3.16) \\
\Pi(q_A, q_B) - t_A - t_B & \geq \Pi(q_A, 0) - t_A \quad (3.17) \\
\Pi(q_A, q_B) - t_A - t_B & \geq \Pi(0, q_B) - t_B \quad (3.18)
\end{align*}
\]

From (3.17) and (3.18) we obtain that (3.16) is not binding, so at the optimum must hold that 
\( t_A = \Pi(q_A, q_B) - \Pi(0, q_B) \) and 
\( t_B = \Pi(q_A, q_B) - \Pi(q_A, 0) \). This reduces \( M \)'s ”separate” problem to

\[
\max_{q_A, q_B} 2\Pi(q_A, q_B) - \Pi(q_A, 0) - \Pi(0, q_B) 
\]

whose solution is denoted by \((q^*_A, q^*_B)\). It is clear from A1 that \((q^*_A, q^*_B) \neq (q^*_A, q^*_B)\). Finally, for \((q^*_A, q^*_B)\) to be indeed optimal, it must also hold that

\[
2\Pi(q^*_A, q^*_B) - \Pi(q^*_A, 0) - \Pi(0, q^*_B) \geq \max\{\Pi(q^*_A, 0), \Pi(0, q^*_B)\}
\]

where \( q^*_A = \arg \max_{q_A} \Pi(q_A, 0) \) and \( q^*_B = \arg \max_{q_B} \Pi(0, q_B) \); otherwise \( M \) either sells \( q^*_A \) units of \( A \) or \( q^*_B \) units of \( B \), whatever is more profitable. \( \blacksquare \)

When is optimal for \( M \) to have \( R \) stocking both goods, which is the case we focus from now on, \( R \)'s profit is \( \Pi(q^*_A, 0) + \Pi(0, q^*_B) - \Pi(q^*_A, q^*_B) > 0 \). This positive payoff is what Shaffer (1991) terms “strategic rent”. This rent arises because from the retailer’s perspective there is an opportunity cost of stocking one product: the forgone profit from the reduced sales of the other product. In our model, this opportunity cost is not due to limited shelf space or product substitutability but rather by a diseconomy in retail effort. Therefore, \( M \)'s must leave \( R \) with some rents to induce her to carry both products. And to keep these rents at a minimum, \( M \) distorts output away from the industry profit-maximizing level.

As in Shaffer (1991), however, these distortions are eliminated if \( M \) is free to make full-line forcing offers.

**Lemma 3.4.2.** Suppose that \( M \) is the only manufacturer in the market. If \( M \) is allowed to make full-line forcing offers, the industry profit-maximizing outcome is restored, resulting in more profit for \( M \) and more surplus for consumers relative to the situation in which \( M \) is restricted to separate pricing.

**Proof.** The first two parts are immediate: \( M \) approaches \( R \) with the “direct” bundling offer \((q^*_A, q^*_B, t_{AB})\), where \( t_{AB} = \Pi(q^*_A, q^*_B) - \epsilon \) with \( \epsilon \to 0 \). To show that consumers also benefit from full-line forcing requires to demonstrate that \( q^*_A > q^*_A \) and \( q^*_B < q^*_B \). Note that to reduce \( R \)'s strategic rent under separate pricing, \( \Pi(q_A, 0) + \Pi(0, q_B) - \Pi(q_A, q_B) \), \( M \) departs from \((q^*_A, q^*_B)\) in the least distortionary way. But since \( q^*_k > q^*_k \), where \( q^*_k = \arg \max_{q_k} \Pi(q_k, q_{-k} = 0) \), this departure necessarily entails moving further away from \((q^*_A, q^*_B)\), that is, \( q^*_A < q^*_A < q^*_A \) and \( q^*_B < q^*_B < q^*_B \). \( \blacksquare \)
We now turn to the problem of M’s optimal contract offer to R in S’s presence. As in the baseline model, we assume that M and S produce the exact same version of product B. Furthermore, let $\Delta$ be S’s production (in)efficiency relative to M’s, so that if B is produced by S while A by M the maximum industry profit becomes

$$\Pi^{MS} = \Pi^{AB} - \Delta \equiv \Pi(q^*_A, q^*_B) - \Delta$$

Note that in our baseline model $\Delta = F_S - F_M < 0$, but here we are also interested in what happens when $\Delta > 0$. Note also that if R carries only product B from S, then

$$\Pi^S = \Pi^B - \Delta \equiv \Pi(0, q^*_B) - \Delta \geq 0$$

If M can use full-line forcing contracts, incentive compatibility requires the transfer $t_{AB}$ to satisfy

$$\Pi^{AB} - t_{AB} \geq \Pi^B - \Delta$$

so, in equilibrium $t_{AB} = \Pi^{AB} - (\Pi^B - \Delta)$. We will now show that this bundling solution strictly applies when $\Delta$ is not too small.

**Lemma 3.4.3.** Full-line forcing contracts strictly emerge when $\Delta \geq \Delta^* > 0$, where

$$\Delta^* = \Pi(q^*_A, q^*_B) - \Pi(q^*_A, 0) + \Pi(0, q^{**}_B) - \Pi(0, q^*_B)$$

**Proof.** Suppose M approaches R with the following separate pricing “direct” mechanism: $(q_A, t_A)$ and $(q_B, t_B)$. M’s problem then is to maximize $t_A + t_B$ subject to R’s incentive compatibility constraint

$$\Pi(q_A, q_B) - t_A - t_B \geq \max \{\Pi(q_A, q_B(q_A)) - t_A - \Delta, \Pi(0, q^{**}_B) - \Delta, \Pi(0, q_B) - t_B, \Pi(q_A, 0) - t_A\}$$

where $q_B(q_A) = \arg \max_{q_B} \Pi(q_A, q_B)$. If $(q^*_A, q^*_B)$ is implementable, (3.20) becomes

$$\Pi(q^*_A, q^*_B) - t_A - t_B \geq \max \{\Pi(q^*_A, q_B) - t_A - \Delta, \Pi(0, q^{**}_B) - \Delta, \Pi(0, q_B) - t_B, \Pi(q^*_A, 0) - t_A\}$$

But from the first term in the curly brackets we have that $t_B \leq \Delta$ which reduces (3.20) further to

$$\Pi(q^*_A, q^*_B) - t_A - t_B \geq \max \{\Pi(0, q^{**}_B) - \Delta, \Pi(0, q^*_B) - t_B, \Pi(q^*_A, 0) - t_A\}$$

Now, for full-line forcing to be strictly needed we require the first term in the curly bracket being the one binding, therefore

$$\Pi(0, q^{**}_B) - \Delta \geq \Pi(0, q^*_B) - t_B$$

$$\Pi(0, q^{**}_B) - \Delta \geq \Pi(q^*_A, 0) - t_A$$

$$\Pi(q^*_A, q^*_B) - t_A - t_B = \Pi(0, q^*_B) - \Delta$$
Since $M$ wants to maximize $t_A + t_B$, the solution is to reduce $t_B$ as much as possible without violating (3.21) and increase $t_A$ accordingly without violating (3.22) and (3.23). From these considerations we arrive that (3.22) holds as long as $\Delta \geq \Delta^*$, where $\Delta^*$ is given by (3.19). ■

In this model full-line forcing emerges in equilibrium as soon as $\Delta \geq \Delta^* > 0$, not only foreclosing a more inefficient rival but also resulting in higher industry profit and consumer surplus. The intuition is the following. Under $S$’s presence, $M$ takes into consideration not only $R$’s strategic rent, as identified above, but also her outside option of signing with $S$. When $\Delta$ is large (i.e., greater than $\Delta^*$), the option of signing with $S$ is little attractive (and completely irrelevant when $\Pi(0, q_B^{**}) = \Delta$), so the strategic rent is relatively more important. Full-line forcing is then employed to keep this rent at a minimum, that is, equal to $R$’s payoff of signing with $S$. As $\Delta$ gets smaller, the strategic rent becomes relatively less important than the threat of $R$ switching to $S$. And when $\Delta \in (0, \Delta^*]$, $M$ only cares about managing this latter threat for which two-part tariffs suffice (provided that $S$ is more inefficient). Finally, when $\Delta < 0$ where are back in Proposition 2a: foreclosure fails to emerge in equilibrium.

This exercise serves to illustrate how our theory can be of great help to a competition authority when assessing a situation in which both wholesale bundling and foreclosure are observed. If in addition to those elements we observe that final consumers are served by monopoly retailers or quite so, our theory (see the figure below and Proposition 2a in particular) would indicate that in this situation wholesale bundling is highly unlikely to be anticompetitive.
References


