Price cutting and business stealing in imperfect cartels *

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Abstract

Although economists have made substantial progress toward formulating theories of collusion in industrial cartels that account for a variety of fact patterns, important puzzles remain. Standard models of repeated interaction formalize the observation that cartels keep participants in line through the threat of punishment, but they fail to explain two important factual observations: first, apparently deliberate cheating actually occurs; second, it frequently goes unpunished even when it is detected. We propose a theory of equilibrium price cutting and business stealing in cartels to bridge this gap between theory and observation.

1 Introduction

An important objective of theoretical research in Industrial Organization is to achieve a conceptual understanding of the mechanisms through which actual price-fixing cartels arrive at collusive outcomes. Analyses of strategic models involving repeated interaction have

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yielded important insights but also leave significant gaps.\footnote{Leading theories of collusion in price-fixing cartels include Green and Porter (1984), Rotemberg and Saloner (1986), Abreu et al. (1986), Abreu (1988), Bernheim and Whinston (1990), Athey and Bagwell (2001), Athey et al. (2004), Athey and Bagwell (2008), and Harrington and Skrzypacz (2011).} Our object in this paper is to provide a theoretical account of two important but thus far unexplained empirical regularities. First, cartel members sometimes \textit{deliberately} cheat on price-fixing agreements. Second, when cheating is detected, punishment does not always follow. Instead, cartel members often urge each other to recall their common interests and let cooler heads prevail. See Section 2 for a discussion of historical examples of this pattern.

Existing theories of collusion cannot account adequately for these observations. Theories with imperfect monitoring, such as Green and Porter (1984), were originally formulated to explain why cartels tend to break down, giving way to price wars and retaliatory business stealing.\footnote{This tendency has been widely discussed in the literature; see, e.g., Porter (1983), Green and Porter (1984), Genesove and Mullin (1998, 2001), Harrington (2006), and Marshall et al. (2015), as well as Section 2. Standard models with perfect monitoring cannot account for such breakdowns because they imply that neither cheating nor punishment occurs in equilibrium.} Significantly, they attribute the collapse of price fixing solely to events beyond the control of the cartel members, rather than to their intentional choices. Thus they imply that cartel members never \textit{deliberately} cheat on collusive agreements.\footnote{To be clear, the theory of imperfect monitoring can in principle account for \textit{unintended} cheating, such as apparent defections from collusive agreements attributable to “rogue employees” who are not involved in the conspiracy. In particular, one could construct a model with imperfectly controlled sales personnel whom, in equilibrium, each firm would instruct to quote some collusive prices (which means \textit{deliberate} cheating would not occur). However, analogously to Green and Porter (1984), “rogue” salespeople would periodically grant price concessions (e.g., with the object of enhancing their own compensation given their false understanding of their employer’s objectives). Because other firms would be unable to distinguish between actual and rogue defections, all defections would have to occasionally trigger punishments.} In addition, according to these theories, if cheating did occur and was detected, it would \textit{definitely} trigger punishment.\footnote{To be sure, the Folk Theorem tells us that just about anything can happen in standard repeated oligopoly games. Using this ambiguity to contrive an equilibrium that sustains a particular price-quantity sequence – for example, one in which firms occasionally disrupt a stable market allocation without triggering punishment – does not provide a legitimate theoretical explanation for the characteristics of that sequence. By way of analogy, there are also equilibria with alternating periods of high and low prices, but one cannot reasonably characterize that observation as a theory of price wars. Given the vast multiplicity of equilibria, one must impose discipline on the process of equilibrium selection, for example by insisting on optimality within some appropriate class. Generally, the most profitable equilibria \textit{consistently} allocate production among firms according to some principle such as comparative advantage, and therefore do not exhibit patterns interpretable as deliberate business stealing. Similarly, one can construct equilibria in which defections trigger punishments probabilistically rather than with certainty, but that does not explain why cartels would adopt such arrangements. The theory of optimal penal codes in repeated games shows that such arrangements are not generally desirable.}

This gap in the literature has important practical implications. Attorneys for companies accused of collusion often point to evidence of price cuts and business stealing, and to a purported lack of retaliation, as “proof” that a cartel is ineffective (see Section 2). Though
intuitively plausible, the possibility that an imperfect but nevertheless effective real-world cartel might exhibit some degree of deliberate price cutting and business stealing, and that such behavior might sometimes go unpunished despite detection, has as yet found no rigorous theoretical articulation.

In this paper, we attempt to bridge this important gap between theory and observation by constructing a theory of equilibrium price cuts and business stealing in imperfectly effective cartels. We formulate a model in which firms have natural advantages with respect to serving particular market segments and must incur sunk costs (associated with investments required for supplier prequalification and bid preparation) before attempting to do business with any specific customer (or group of customers). For intermediate discount factors, some collusion is feasible but perfect collusion is not. Our agenda is to study the properties of imperfectly collusive equilibria in those settings.

Our main result demonstrates that, under reasonably general conditions, the best collusive equilibria within an important class have several key properties. First, the cartel in effect attempts to divide the market according to the firms’ relative advantages. Second, while the cartel may establish an aspirational price (in our model, the monopoly price), it recognizes that perfect collusion is unsustainable, and consequently also explicitly or implicitly establishes a price floor, at which each firm can be assured of locking up its “home market.” Third, each firm often charges the aspirational price in its home market, but also sometimes cuts its price in an attempt to defend market share against anticipated “raids” by competitors. Fourth, firms sometimes attempt to raid each others’ markets, in all such cases setting prices above the floor, so as to avoid stealing business if the home firm has made the “safe” choice. If, however, the home firm has also set a price above the floor, the rival may successfully steal business. Fifth, whenever such price cuts or business stealing occur, they go unpunished. However, the cartel would punish business stealing by “away” firms at or below the floor. Thus, we demonstrate that deliberate and unpunished price cuts and business stealing (which would appear to observers as “cheating”) can be critical to the healthy functioning of a cartel.5

Our theory has implications for the relationship between conduct and patience that contrast markedly with conventional accounts. In an ideal world, cartels would like to divide the market among their members at the monopoly price. However, sustaining this division requires a certain degree of patience, as in a standard Folk Theorem. In classical cartel models, insufficiently patient firms adapt by lowering the collusive price, without any

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5For a complementary perspective on these issues, see the recent work of Rahman (2015).
further modification to the collusive structure; see Figure 1, Panel A.\(^6\)

Our analysis points to an alternative mode of conduct: rather than coordinate on a lower price, the cartel (optimally) attempts to divide the market at the aspirational monopoly price, but succeeds only imperfectly. Cartel members occasionally “cheat” by raiding their competitors’ markets and cutting prices in their own markets, down to a floor price below which raids are unacceptable (and punished harshly). This floor price reflects the level of collusion achieved by the cartel, and is increasing in the discount factor; see Figure 1, Panel B.\(^7\) The theory thus predicts that, instead of (or possibly in addition to) lowering the aspirational price (which in our particular model is always the monopoly price), impatience may result in greater apparent discord and price-cutting relative to that price, without entirely undermining the efficacy of the cartel.

This theory also has potentially testable implications, for example, that unpunished business stealing occurs at prices above the cartel’s floor, whereas all business stealing at prices below that floor triggers punishment.\(^8\) In practice, this implication may prove difficult

\(^6\)For standard models of Bertrand competition with homogeneous products (which we assume in this paper), the curve in Panel A would exhibit a discontinuous jump from the competitive price level to the monopoly price level at a threshold discount factor.

\(^7\)For the particular Bertrand-style model we examine, the aspirational price is always the monopoly price, as shown in the figure. However, with other models of competition, both the aspirational price and the price floor might be increasing in \(\delta\).

\(^8\)Technically, in our model, business stealing below the price floor never occurs on the equilibrium path. However, in a more general model it could occur for unrelated reasons, such as imperfect control over “rogue”
to test because a cartel’s price floor may not be known with precision (indeed, in practice, price agreements can appear somewhat fuzzy, and understandings concerning the floors may be implicit rather than explicit). For this purpose, one cannot use the average price as a proxy for the floor, because the theory implies that some business stealing will occur at below-average prices. A more robust testable implication of the theory is that, the higher the price at which business is stolen, the lower the likelihood that apparent “cheating” will trigger punishment. Testing this implication falls outside the scope of our current theoretical inquiry.

The rest of this paper is organized as follows. Section 2 reviews historical evidence of deliberate unpunished cheating by members of industrial price-fixing cartels. We present a simple model of industrial competition in section 3. Section 4 characterizes the properties of non-collusive equilibria (i.e., equilibria in one-shot play). Section 5 presents our main results, while section 6 describes an important extension. Some brief concluding remarks appear in section 7. Additional extensions, as well as all proofs, appear in the Appendix.

2 Deliberate cheating in the historical record

Instances of cheating on cartel agreements, such as undercutting agreed-upon price targets to steal customers allocated to competitors, appear in the records of many price-fixing cases. Marshall et al. (2015) provide an excellent survey derived from European Commission decisions on major industrial cartels from 2000 to 2005. Of 22 rulings over that period, they classify nine as “discordant,” a designation indicating “evidence of frequent bargaining problems and deviations by cartel members, occurring throughout the cartel period.” The classify only six as fully concordant, meaning the cartel functioned smoothly throughout the collusive period. These figures likely understate the proportion of discordant cartels, because their sample is inherently selected to include only those price-fixing conspiracies effective enough to draw antitrust scrutiny; see, for example, the discussion of selection issues in Levenstein and Suslow (2006).9

Remarkably, observed price cutting and business stealing often appears to go unpunished, with cartel members instead reminding each other that “our competitors are our friends and our customers are our enemies,”10 and urging one another to recall their common interests

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9Harrington (2006) provides further discussion EC cartel decisions from the early 2000s.
and let cooler heads prevail. Examples of such forbearance litter the record of the discordant cartels surveyed by Marshall et al. (2015). For instance, the global lysine cartel, which operated at least from 1992 through 1995, discussed a persistent and substantial gap between target and actual prices in its European markets at cartel meetings throughout 1993 and 1994. This gap was variously attributed to arbitrage by commodity traders and fluctuating exchange rates, but was eventually blamed on undercutting by particular cartel members. These difficulties culminated in a “reset” by the cartel, during which sales were briefly suspended in order to drive up the price to the collusive target. However, there is no indication in the record that the deviators were punished. A cartel producing nucleotides for use as food flavor enhancers, which operated in Europe from 1988 to 1999, encountered similar difficulties. The cartel attempted to preallocate sales to several large purchasers among its members. During meetings spanning 1991 to 1996, cartel members complained about low prices and requests by their customers to match reduced quotes provided by other cartel members. There is no evidence that punishments for these deviations were contemplated or implemented; rather, in each instance the cartel simply set a new price and urged its members to abide by the latest figure.

11 In what follows, we draw extensively on the text of antitrust decisions by the European Commission, which are published in the Official Journal of the European Union and available online at http://ec.europa.eu/competition/cartels/cases/cases.html

12 Price cuts were often deliberately contrived to skirt the letter of cartel agreements. For example, Sewon Europe, which accused Eurolysine of particularly flagrant deviations, complained that “actual market prices had always dropped after the announcement of price increases, because Eurolysine announced a price increase after securing orders from big customers at the old price,” and that “agreed prices had been rendered meaningless due to the fact that Eurolysine was selling quantities in advance at a price lower than the agreed price.” (EC Decision on Amino Acids (COMP/36.545/F3), paragraphs 138 and 148.)

13 EC Decision on Amino Acids, paragraph 151. It is unclear how successful this intervention proved. No further complaints were lodged publicly against Eurolysine, but the cartel continued to struggle with declining European prices attributed to arbitrage trading and resale by distributors. (Paragraphs 158, 161-2, and 164-5.) Further, a final meeting prior to the discovery of the cartel contemplated an explicit division of European customers between producers. This suggests that the European branch of the cartel may have experienced continued dysfunction following the “reset.”

14 One potential interpretation of this event, and others like it, is that the cartel members opted for renegotiation rather than punishment. However, a theory of unpunished business stealing based on renegotiation is problematic. Going down that path, one must assume either that firms naively fail to anticipate cheating and renegotiation, or that they do anticipate it. The assumption of naiveté is troublesome given that many cartels witness this pattern repeatedly. With the assumption of sophistication, introducing opportunities for renegotiation would lead rational cartel members to consider renegotiation-proof equilibria. (See Bernheim and Ray (1989) and Farrell and Maskin (1989).) There is no reason to think that those equilibria, let alone optimal equilibria within that class, would manifest the properties we are trying to explain.

15 Complaints of undercutting and business stealing were lodged with remarkable regularity. (EC Decision on Food Flavour Enhancers (COMP/C.37.671), paragraphs 94, 104, 109, 111-2, 114, 116, 118, 121, 129, and 148.) In only one case was there any reference to an enforcement mechanism (paragraph 112), and there is no follow-up in the record. On the contrary, in a subsequent meeting the cartel members congratulated themselves on their successful cooperation (paragraph 126), even though they had lodged complaints and
Likewise, a cartel of citric acid producers in western Europe operating from 1991 to 1995 endured persistent suspicions of undercutting by one of its members, Jungbunzlauer, for the latter half of this period. The participants convened special recurring sessions alongside the main cartel meetings to monitor and discuss Jungbunzlauer’s behavior, but apparently never took punitive action prior to the cartel’s collapse. In some instances, one can interpret a collapse as coordinated punishment, but in this instance that interpretation is problematic for two reasons. First, the collapse occurred long after cheating was initially detected. Second, despite Jungbunzlauer’s actions, the cartel members continued to promote mutual forbearance until their efforts proved futile.\footnote{The collapse of the cartel, occurring in the early months of 1995, was chaotic: “[T]hree meetings on 6 January 1995... did not fundamentally change any previous pattern of behaviour. The other undertakings attacked Jungbunzlauer for its ‘almost total lack of adherence to the agreed prices which Jungbunzlauer had reduced’... [W]hile the atmosphere was ‘much less friendly’ and the group was starting to fall apart, monthly sales data continued to be regularly exchanged and all parties were still very much in contact with each other... It was not until the last, unplanned, meeting on 22 May 1995... that it became clear that ‘the cartel was in total disarray and was not working. [Jungbunzlauer] was told that unless [the firm] was seen to do something to repair the damage that they had done the agreement was at an end’.” (EC Decision on Citric Acid (COMP/E-1/36 604), paragraphs 125-6.) It is unclear what actions, if any, Jungbunzlauer was prepared to take, as the cartel was uncovered less than a month later.}

As a final example, the Sugar Institute, which operated throughout the 1920s, experienced frequent episodes of cheating. A typical pattern was for cartel members to secretly skirt the cartel’s price maintenance rules via indirect price cuts or kickback schemes, and then apologize and halt a practice when it was uncovered. The cartel would update its rules to bar that practice explicitly, and members would proceed to devise new schemes to skirt the updated rules. Such tactics appear to have been tolerated without retaliation by cartel members provided misconduct did not continue after it was uncovered, and this habit of forbearance does not seem to have substantially vitiated the efficacy of the cartel (Genesove and Mullin (2001)).

Attorneys for companies accused of collusion have frequently pointed to evidence of price cuts and business stealing, and to a purported lack of retaliation, as “proof” that a cartel is ineffective. This strategy was employed, for instance, by defense attorneys during the criminal prosecution of several top ADM executives following discovery of the lysine cartel. According to news coverage in the September 10, 1998 edition of the Chicago Tribune: “Top executives of Archer Daniels Midland Co. ‘busted up a longstanding Asian cartel,’ introducing ‘fierce competition’ into the market for a livestock-feed additive, a defense attorney said... [ADM] was ‘stealing customers’ and ‘undercutting competitors’ at the time prosecutors say it was carving up the lysine business... [C]ompetitors lied to each other routinely,
he said. ‘This is not Business Ethics 101. This is how you deal with the real world. You have to mislead the competition’.”

Defendants made similar claims in many of the EC cases surveyed in Marshall et al. (2015); see in particular the EC Decisions on Citric Acid (paragraphs 221-3), Food Flavour Enhancers (paragraph 231), Industrial Tubes (paragraph 356), Amino Acids (paragraphs 366-374), and Zinc Phosphate (paragraph 291).

3 The model

A set of firms compete for the business of a collection of customers in an infinite sequence of discrete periods $t = 0, 1, 2, \ldots$. Throughout the main text, we assume that there are two firms $i = 1, 2$ and two markets $m = 1, 2$; we treat cases with additional firms and markets as an extension in the online Appendix. The firms are risk-neutral profit maximizers and share a common discount factor $\delta \in (0, 1)$. They are also differentiated, with firm $i$ holding a comparative advantage in market $m = i$. In practice, comparative advantage can arise either from cost differences associated with factors such as geography and familiarity with the customer’s product requirements, or from differences between a customer’s valuations of competing products. Those valuations often differ significantly across suppliers not only when the products in question are differentiated, but also in commodities industries, where a purchaser concerned with supply and quality assurance may view some suppliers as more reliable or flexible than others. For concreteness and simplicity, we model these advantages as symmetric and pertaining to costs: each firm $i$’s constant marginal cost of production is $c_H$ in market $m = i$, the firm’s “home market,” and $c_A > c_H$ in the other market, its “away market.” We will often refer to the size of the home firm’s cost advantage as $\Delta c \equiv c_A - c_H > 0$. Although we do not formally model demand-side comparative advantage, the analytics are identical to the cost differential case.

In each period, the firms decide whether to compete in each market, and if so at what price. Competing requires the expenditure of a recurring market-specific sunk cost $c > 0$, which is expended whether or not the firm is chosen to supply that market. For the reasons discussed below, these costs are often substantial in markets for intermediate goods. We assume that firms make all of these decisions simultaneously and privately. Thus each firm’s stage-game strategy is a mixture over the action set $(\mathbb{R}_+ \cup \{\emptyset\})^2$, where $\emptyset$ represents a decision not to quote a price in a particular market. For the sake of simplicity, all choices are revealed before the start of the next period.

On the demand side, each market consists of a single customer who views the products of the two firms as perfect substitutes. Each customer demands $D(p) \leq 1$ units of the good when the lowest quoted price is $p$ and purchases from the firm offering the lowest price, splitting demand equally between the two firms in the event of a tie. (Equivalently, one could model the market as comprised of a continuum of infinitesimal consumers with unit demand and heterogeneous valuations for the good.) We assume that customers’ preferences are time-separable and that the good is non-storable, so that demand is independent across periods.\(^{18}\)

We will use $p_i$ for $i = H, A$ to denote the smallest price at which $D(p)(p - c_i) - c = 0$, i.e., the price at which either the home firm ($i = H$) or the away firm ($i = A$) would just break even serving the market by itself. We will similarly use $p^*_i$ to denote the profit-maximizing single-firm prices for the home and away firms. To avoid uninteresting technical complications, we assume that these prices exist and are unique, and that single-firm profits $D(p)(p - c_i) - c$ increase monotonically for prices between $c_i$ and $p^*_i$.\(^{19}\) We also assume that $p_A < p^*_H$, i.e. that $\Delta c$ isn’t too large, to ensure that the home (low-cost) firm cannot simply blockade entry at its monopoly price. (Precise technical restrictions on the demand function are given in the Appendix.)

**Discussion.** Most of our assumptions are reasonably standard and their justifications familiar. That said, two of them require further explanation.

As noted above, we assume that firms must incur market-specific sunk costs to compete credibly for business. While this assumption is not standard, it nevertheless has a solid empirical foundation. In practice, these costs arise from the nature of procurement in markets for intermediate goods, and fall into two categories.

First, purchasers of intermediate goods routinely conduct supplier “qualification screening” to avoid the potentially dire consequences of non-performance; for examples of those consequences, see Lunsford and Glader (2007) or Schmit and Weise (2008). This process, which sorts suppliers into two categories according to whether they are acceptable or unacceptable prior to entertaining bids, is often time-consuming and costly, even for commodity-type parts

\(^{18}\)Our demand specification implicitly assumes that customers are strategically myopic in the sense that they do not consider the potential impact of their current purchasing decisions on future prices. Equivalently, we could permit strategic play by customers but restrict attention to equilibria that condition continuation paths only on past prices, and not on realized demand. Alternatively, we could interpret demand as arising from a continuum of infinitesimal consumers and (following the usual convention) restrict attention to equilibria that do not condition continuations on the realized demand of measure-zero subsets of players. In that case there are no opportunities for consumers to play strategically.

\(^{19}\)Because $D(p)$ is decreasing, a discontinuity on $[c_i, p^*_i)$ would violate monotonicity of $D(p)(p - c_i)$. Our assumptions therefore ensure that $D(p)$ is continuous on $[c_i, p^*_i)$. They do not rule out a (right-)discontinuity at $p^*_i$, which arises in the special case where customers have unit demand at a known reservation value.
such as printed circuit boards (Beil (2009)). Suppliers become “prequalified” for bidding by demonstrating that they have specific operational, technological, financial, and/or organization capabilities (Choi and Hartley (1996)), which they create and maintain through costly investments. To take a leading example, capabilities pertaining to delivery performance play a central role in supplier selection (Verma and Pullman (1998)). Accordingly, purchasers often require potential suppliers to demonstrate that they have sufficient uncommitted production capacity and inventories to meet both baseline demand and unexpected surges (Beil (2009)). Prequalification may therefore require a supplier to put all or part of the requisite incremental capacity into place, and to accumulate needed inventories, prior to bidding events. Purchasers may also insist that suppliers obtain jurisdiction-specific certifications or demonstrate specific technological proficiencies. Requalification is typically less costly than initial qualification, but nevertheless can entail substantial ongoing investments in advance of bidding events, particularly in dynamic industries. For example, latest-generation fabrication plants often permit suppliers to achieve lower costs, ensure greater reliability, and/or produce a wider range of products; see the discussion of the LCD panel industry in Lee et al. (2011). As a result, purchasers may require suppliers to demonstrate that they can meet incremental demand using up-to-date facilities. Likewise, purchasers may seek assurance that suppliers are addressing the emerging implications of jurisdiction-specific regulatory developments or preparing to meet demand growth.

Second, once a supplier is prequalified, it may incur substantial bidding costs. The existence of these costs has been widely recognized in the economics literature and their effects have been studied extensively, beginning with Samuelson (1985). Often bidding costs are mentioned in the context of government procurement contracts, but they can be consequential whenever the product or service involves customized elements, as is often the case in markets for intermediate goods; see, for example, Snir and Hitt’s (2003) analysis of online markets for IT services. One survey of proposal management professionals found that bid costs average 2.0% of contract value overall for large contracts (over £50 million) and 4.2% for small contracts (under £1 million); see Gowing (2014).

We turn next to the second feature of our model that requires further explanation: within each period, both firms make all their decisions - whether to compete for business and what prices to set - simultaneously.\footnote{This timing convention is the same as the one adopted in Samuelson (1985), who considers a one-shot procurement model with entry costs.} In models with endogenous entry, the distinction between simultaneous and sequential choice can be an important one. Here, the assumption of simultaneity leads us to examine mixed strategy equilibria, which are sometimes greeted
with a degree of skepticism. A skeptic might argue that such equilibria emerge only because the assumption of simultaneous choice artificially compresses a competitive dynamic.

To illustrate this potential concern, consider a setting in which two firms must decide whether to enter a market that can profitably accommodate only one of them. If we assume they make their choices simultaneously, there is a symmetric mixed strategy equilibrium in which both enter with positive probability, as well as two asymmetric pure strategy equilibria, each with a single entrant. In contrast, if we assume the firms make their choices sequentially with the second mover observing the choice of the first, only a single-entrant pure strategy (subgame perfect) equilibrium survives. More generally, in settings where entry takes time and entails a sequence of immediately observable actions, small asymmetries will tend to tip the process in favor of entry by a single agent. To the extent we are interested in theorizing about such settings, studying the mixed strategy equilibrium of the simultaneous move game would be inappropriate.

Our model differs in important ways from the one described in the previous paragraph. For example, as we will see in Section 4, the single-market stage game admits a unique mixed strategy equilibrium, which means one cannot point to arguably more plausible pure strategy alternatives. Still, because the particular features of mixed strategy equilibria may hinge on the assumption of simultaneous choice, it is important to evaluate the foundations for that assumption.

In the markets for many intermediate goods, purchasers periodically select suppliers at discrete points in time (procurement events). From our perspective, the relevant question is whether a supplier’s intent to compete for a contract and/or the terms of its bid are revealed to its rivals prior to a given procurement event. If rivals receive no such information in advance of the event, then the assumption of simultaneous choice is appropriate for modeling purposes even if the actual decisions are not perfectly synchronized.

In practice, rivals may draw inferences about each others’ bidding intentions from rumor and/or past experience, but they often have little or no hard evidence. Discussions and negotiations between purchasers and suppliers are usually private and confidential. Nor can rivals usually observe each other’s investments with sufficient granularity to discern the nature or purpose of particular capabilities. For example, when a supplier builds a new fabrication plant, rivals often learn of its existence, but they often have little or no ability to determine its capacity, technological sophistication, or narrow strategic purpose, absent voluntary disclosure. Purchasers may have incentives to advise rival suppliers about each others’ intentions to compete,\textsuperscript{21} but the suppliers typically have the opposite incentives,

\textsuperscript{21}On the other hand, in some settings purchasers may have incentives to maintain the confidentiality of
and the purchaser’s claims have limited credibility without the suppliers’ corroboration. Thus, while rival suppliers may not remain entirely clueless about each others’ competitive intentions prior to specific procurement events, the assumption that they cannot observe hard evidence of those intentions, and hence that choices are effectively simultaneous, is reasonable from a modeling perspective.\textsuperscript{22}

4 The non-collusive outcome

Before studying collusion, we first characterize non-collusive outcomes by examining equilibria of the stage game (a one-shot version of the full model). Because firms’ profits are additively separable across markets, we can consider each market in isolation. The following theorem characterizes equilibrium play in a single market:

**Theorem 1.** There exists a unique Nash equilibrium strategy profile $\tau$ of the one-shot game in a single market. In this equilibrium:

1. The home firm always quotes a price, while the away firm quotes a price with probability strictly between 0 and 1.

2. The home firm makes profits $\Pi_H = D(p_A)\Delta c > 0$, while the away firm makes zero profits.

3. Firms quote prices only in $[p_A, p_H^*]$. Each firm’s price distribution has full support on $[p_A, p_H^*]$, and is atomless except at $p_H^*$, which the home firm chooses with strictly positive probability.

Several features of this equilibrium merit emphasis. First, the consumer always ends up paying strictly more than $p_A$. This is significant because $p_A$ is the lowest price that a high-cost firm would be willing to charge, and hence it is tempting to interpret $p_A$ as the fully competitive price. Notice that the expected profits earned by each firm are nevertheless the same as if both set $p_A$ and the consumer purchased from the home firm. The equilibrium delivers an outcome that is no better than the fully competitive outcome for the firms and strictly worse for consumers because the firms sometimes incur redundant fixed costs and these intentions: if a purchaser manages to devise a credible method for revealing the bidding intentions of potential suppliers, it will also inevitably reveal when rival bidders are absent.\textsuperscript{22}

Additional support for this assumption is found in the literature on auctions with entry costs, which assumes that all potential bidders decide whether to enter simultaneously and in secret. (See, e.g., McAfee and McMillan (1987) and Levin and Smith (1994).) In particular, Levin and Smith (1994) characterize an equilibrium in which bidders randomize over entry.
allocate production inefficiently. Second, because the distribution of prices for both firms has full support on \([p_A, p_H]\), the *ex post* outcome can appear arbitrarily collusive. Third, “business stealing,” which we define as the away firm winning sales, occurs with strictly positive probability.

We construct Nash equilibria of the stage game in the obvious way by pasting together copies of the single-market equilibrium \(\tau\) from Theorem 1. In fact this procedure produces many equilibria, because firms may correlate their play across markets arbitrarily without impacting either firm’s expected profits or best responses within a market. However, the marginal distributions of firms’ equilibrium strategies are unique.\(^{23}\) The following theorem formalizes this discussion.

**Theorem 2.** There exists a Nash equilibrium \(\sigma\) of the stage game in which firms play \(\tau\) in each market and randomize independently across markets. Moreover, given any Nash equilibrium \(\sigma'\) of the stage game, the marginal distribution of \(\sigma\) in each market is \(\tau\).

**Corollary.** There exists a subgame-perfect Nash equilibrium of the repeated game in which firms play \(\sigma\) in each period independent of the history of past play.

Theorem 2 and its corollary highlight a particular Nash equilibrium of the stage game, namely the one in which firms do not correlate their strategies across markets. As we will see shortly, an important class of collusive equilibria share this property.

## 5 Optimal collusion

Industry profits are maximized when each market is monopolized by the low-cost firm at price \(p_H\). This *perfectly collusive* outcome serves as a natural benchmark for gauging the effectiveness of a collusive arrangement.\(^{24}\) We will refer to any equilibrium yielding profits strictly above the stage-game Nash level but below perfect collusion as an *imperfectly collusive* arrangement.

When firms are sufficiently patient, a folk theorem obtains in the sense that perfect collusion is sustainable as a subgame-perfect Nash equilibrium (SPNE). More precisely, as we demonstrate below, there exists a critical discount factor \(\delta^M < 1\) above which perfect collusion is sustainable and below which it is not. We study collusion for discount factors

\(^{23}\)By marginal distributions we mean the distributions of firms’ equilibrium entry and pricing decisions within a single market.

\(^{24}\)To be sure, there exist other points along the firms’ Pareto frontier. But such points are technologically inefficient and not robust to the introduction of side transfers.
In this section we establish that some degree of imperfect collusion is sustainable for a range of discount factors below $\delta^M$, and we characterize optimal collusion within an important class of equilibria.

5.1 Stationary equilibria

Characterizing the Pareto frontier of a repeated game for fixed discount factors is a difficult task. Several notable papers illustrate the inherent complexities. Abreu et al. (1990) demonstrate how to construct the entire set of SPNE payoffs for a repeated game as the largest fixed point of a certain set-valued mapping, but they identify few general properties of the set. Mailath et al. (2002) characterize extremal pure-strategy equilibria of the repeated prisoner’s dilemma, and find that these equilibria may be non-stationary and cyclic with arbitrarily long periodicity. Abreu and Sannikov (2014) examine extremal pure-strategy equilibria of two-player finite-action games. They show that the number of extremal equilibria is finite and bounded by the size of the action set, and that extremal payoffs are characterized by a system of nonlinear equations. We are aware of no work that attempts to derive similar characterizations for mixed-strategy equilibria in games with continuous action spaces.

To retain tractability and provide sharp characterizations of optimal collusion, we restrict attention to SPNEs exhibiting a natural stationary structure. The essence of the stationarity requirement is that, as long as play remains on the equilibrium path, the probability distributions governing each firm’s current choices remain unchanged.

Because we allow for randomization, the formal definition of an equilibrium path, and hence of stationary equilibrium, involve some technical details. Let $\mathcal{H}^\infty$ be the set of all complete histories of the repeated game, with each $h \in \tilde{\mathcal{H}}$ recording an infinite sequence of outcomes in each period and market. Fix any subset of complete histories $\tilde{\mathcal{H}} \subset \mathcal{H}^\infty$. For each period $t$ and every associated partial history $h^t \in \tilde{\mathcal{H}}^t$ recording outcomes up to period $t$, let $A^*(h^t)$ be the set of action profiles that could follow from $h^t$ in $\tilde{\mathcal{H}}$. We call $\tilde{\mathcal{H}}$ rectangular if for all $t$ and $h^t \in \tilde{\mathcal{H}}^t$ the set $A^*(h^t)$ may be written as a Cartesian product of subsets of the firm- and market-specific action sets $A^i_m$.\(^{28}\)

**Definition 1.** An SPNE strategy profile $\sigma$ is a stationary equilibrium if there exists a stage-

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$^{25}$ By contrast, in symmetric Bertrand competition with no fixed costs, the equilibrium set has a “bang-bang” structure with perfect collusion sustainable for $\delta \geq 1/2$ and no collusion sustainable otherwise.

$^{26}$ In what follows, we use standard notation for stage-game action sets and histories in repeated games. For the sake of completeness, we provide the standard definitions in the Appendix.

$^{27}$ That is, $A^*(h^t) \equiv \{a \in A : (h^t, a) \in \tilde{\mathcal{H}}^{t+1}\}$, where $A = \prod_{i=1}^2 \prod_{m=1}^2 A^i_m$ is the set of pure strategies for both firms in both markets.

$^{28}$ More precisely, there must exist $\tilde{A}^i_m \subset A^i_m$ for $i, m = 1, 2$ such that $A^*(h^t) = \prod_{i=1}^2 \prod_{m=1}^2 \tilde{A}^i_m$. 

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game strategy profile \( \tau \) and a rectangular set of complete histories \( H^* \), the equilibrium path, such that: 1) \( H^* \) occurs with probability 1 under \( \sigma \); and 2) for all \( h \in H^* \) and \( t \), \( \sigma(h^t) = \tau \).\(^{29}\)

Consistent with this definition, for any stationary equilibrium \( \sigma \) with equilibrium path \( H^* \), we will speak of a partial history \( h \in \bigcup_{t=0}^{\infty} H^t \) as lying “on the equilibrium path” if there exists a continuation history \( h' \in H^\infty \) such that \( (h, h') \in H^* \). Otherwise \( h \) is an “off-path” history. According to our definition, equilibrium paths are always rectangular. For most stage-game strategy profiles, this feature of the definition is redundant; the exceptions are cases in which firms perfectly correlate certain outcomes across markets.\(^ {30}\)

This definition singles out equilibria in which behavior looks “the same” in all periods so long as no deviations have taken place. It imposes no restrictions on equilibrium structure off-path (beyond subgame perfection), thereby allowing for optimal penal codes, which are generally non-stationary. In practice the equilibrium path of a stationary equilibrium is easy to describe, as it is generated by a set of “acceptable” action profiles that lead to a repetition of the same on-path stage-game strategy profile, \( \tau \), at each stage. The support of \( \tau \) is then necessarily a subset of the allowable action set, though not all acceptable actions need be played in equilibrium.

We restrict attention to stationary equilibria in part for reasons of tractability. We also regard them as potentially more appealing than many non-stationary alternatives due to their comparative simplicity. Indeed, we establish in the Appendix that, within the set of stationary SPNE, one can without loss of generality restrict attention to equilibria in which firms randomize independently across markets along the equilibrium path. Furthermore, as we will show, optimal stationary equilibria have a surprisingly rich structure that matches important features of many real-world cartels. While we cannot entirely rule out the possibility that non-stationarity would improve upon an optimal stationary equilibrium, in Section 6 we establish that stationarity is in fact optimal within a broad class of equilibria that permit fluctuating market shares and pricing strategies over time.

\(^{29}\)For a pure-strategy equilibrium, the equilibrium path is a singleton set consisting of the unique history picked out by \( \sigma \). With mixed strategies and a continuum action space, many or all “on-path” partial histories might occur with probability zero, so we must attend to some technical details. Interpreted as a behavioral strategy profile, \( \sigma \) induces, for each \( t \), a probability measure \( \mu_t \) on the set of partial histories \( H^t \) endowed with the product sigma algebra. By the Kolmogorov extension theorem, we can uniquely extend the collection \( \{ \mu_t \}_{t=1}^{\infty} \) to a measure \( \mu \) on the set of complete histories \( H^\infty \). There then exists an equivalence class of measurable subsets of \( H^\infty \) with measure 1 under \( \mu \). A stationary equilibrium singles out some member \( \hat{H}^\infty \) of this class as the “equilibrium path.”

\(^{30}\)For instance, if a firm’s equilibrium strategy involves randomization between exactly two price price pairs \( (p_1, p_2) \) and \( (p'_1, p'_2) \) in some period, then, according to our definition, histories in which the firm plays \( (p_1, p'_2) \) and \( (p'_1, p_2) \) are also deemed to lie on the equilibrium path.
5.2 Optimal collusion in stationary equilibria

In this section we characterize maximum sustainable profits within the class of stationary equilibria. We further demonstrate that all equilibria achieving this optimum generate behavior interpretable as unpunished business stealing.

We first introduce a bit of useful notation. Let \( \Pi^C \equiv D(p_A)(p_A - c_A) - c \) be the “competitive” profits of the home firm in the unique Nash equilibrium of the stage game. Given that \( p_A \) is the break-even price for the away firm, these profits may also be written \( \Pi^C = \Delta c D(p_A) \).

Similarly, let \( \Pi^M \equiv D(p^*_H)(p^*_H - c_H) - c \) be the stage profits of a monopolist in its home market. We anticipate that for sufficiently low discount factors, optimal stationary equilibria will generate normalized “lifetime” discounted profits for each firm in the interval \( [\Pi^C, \Pi^M) \). At the opposite extreme, let \( \Pi(\delta) \) denote the lowest SPNE-supportable lifetime payoff for a firm when the discount factor is \( \delta \). Note that \( \Pi(\delta) \in [0, \Pi^C] \), as Nash reversion is always a credible continuation promise, and any firm may ensure itself zero lifetime profits simply by never competing.

For \( \Pi \in [\Pi^C, \Pi^M] \), we define \( p^*(\Pi) \) as the price a monopolist would charge in its home market to achieve profits \( \Pi \); formally it is the solution in \([p_A, p^*_H]\) to \( \Pi = D(p)(p - c_H) - c \). Given our assumptions on \( D(\cdot) \), \( p^*(\cdot) \) exists and is single-valued, continuous, and strictly increasing.

Finally, let \( \delta^M \) be the lowest (infimum) discount factor at which perfect collusion is sustainable. We show later that \( \delta^M < 1 \) and that perfect collusion is sustainable for all \( \delta \geq \delta^M \). More importantly, for a range of discount factors below \( \delta^M \), we establish the existence of optimal stationary equilibria with the following structure:

1. In the cooperative phase, the home firm definitely enters the market, and the away firm enters with strictly positive probability. Both set prices between some \( p^* \) and the home firm’s monopoly price, \( p^*_H \). The home firm quotes both \( p^* \) and \( p^*_H \) with strictly positive probability, while the away firm quotes prices strictly between these bounds. As a

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31 As is standard in the literature, we normalize the NPV of a firm’s income stream by \( 1 - \delta \) to obtain lifetime profits, which are then interpretable as a weighted average per-period profitability, with weight \( \delta^t(1 - \delta) \) on the \( t \)th period. For a stationary equilibrium path, lifetime profits are equal to the stage profits earned in each period.

32 Formally, \( \Pi(\delta) \) is the infimum over the set \( \mathcal{E} \) of SPNE-supportable lifetime payoffs, as \( \mathcal{E} \) may not be closed and thus there may be no SPNE actually achieving a minimum. Possible lack of closure of \( \mathcal{E} \) is a purely technical issue and does not affect the substantive results to follow. Throughout the paper we will assume that an SPNE supporting \( \Pi(\delta) \) actually exists. If it does not, then no optimal stationary equilibrium exists, but a sequence of stationary equilibria whose payoffs and stage-game strategies converge uniformly to the claimed optimum do exist. Also, whenever we are able to show that \( \Pi(\delta) = 0 \), we also demonstrate that an SPNE achieving this bound actually exists.
result, customers sometimes make their purchases from the home firm, and sometimes from the away firm. Regardless of which firm prevails, the cooperative phase continues as long as no firm undercuts \( p^* \). Only the away firm has an incentive to price below \( p^* \) (because the home firm can win with certainty by quoting \( p^* \)); any such undercutting triggers the punishment phase.

2. In the punishment phase, the firms engage in a price war yielding negative stage profits to both. If firms participate in the price war for a single period, the equilibrium reverts to the cooperative phase; otherwise the punishment phase starts over.

Plainly, the degree of collusion sustainable in the cooperative phase depends on the harshness of the price war that would follow a deviation. But at the same time, the harshness of the most severe sustainable price war depends on the future reward offered as recompense for participating. Therefore, as is typical in the analysis of repeated games, sustainable cooperation and punishment are determined simultaneously.

The following theorem characterizes the highest cooperative payoff taking as given the harshest punishment payoff \( \Pi(\delta) \), the value of which we leave undetermined for the moment. Because we restrict attention to stationary equilibria, no further information is needed concerning payoffs: all on-path continuations must deliver the cooperative payoff, and (without loss of generality) we can configure the equilibrium so that all deviations trigger the harshest possible punishment.

**Theorem 3.** For each \( \delta \leq \delta^M \), all Pareto-optimal stationary equilibria achieve a unique profit vector \((\Pi^*, \Pi^*)\) satisfying

\[
\Pi^* = (1 - \delta)(2\Pi^* - \Delta cD(p^*(\Pi^*))) + \delta \Pi(\delta).
\]

Further, \( \Pi^* > \Pi^C \) iff \( \Pi(\delta) < \Pi^C \), and \( \Pi^* \) is strictly increasing in \( \delta \) whenever \( \Pi(\cdot) \) is non-increasing in \( \delta \).

To understand this result, consider a collusive arrangement that divides the market along cost lines and prescribes that each home firm satisfy all demand at a price \( p^*(\Pi^*) < p^*_H \). Abiding by this agreement yields lifetime profits of \( \Pi^* \) for each firm. One possible deviation for each firm involves charging \( p^*(\Pi^*) \) in one’s home market while slightly undercutting that price in the away market. The deviating firm then earns (approximately) \( \Pi^* - \Delta cD(p^*(\Pi^*)) \) for a single period before suffering the harshest punishment; as a result, its lifetime (normalized) payoff is \( (1 - \delta)(\Pi^* - \Delta cD(p^*(\Pi^*))) + \delta \Pi(\delta) \). Incentive compatibility requires that
the latter expression is no greater than \( \Pi^* \), and maximizing \( \Pi^* \) subject to that constraint requires equality, as the theorem asserts.

To be clear, the arrangement described in the preceding paragraph is not actually an equilibrium. The proof of the theorem proceeds by constructing equilibria with a similar structure, except that both firms occasionally quote prices above \( p^*(\Pi^*) \) in both their home and away markets. The distributions of choices are selected so that each firm achieves the same expected profits when quoting any price between \( p^*(\Pi^*) \) and \( p^*_H \). Then the deviation described above is the most profitable one for each firm. The proof is completed by showing that all alternative arrangements (in particular, those that share profits between firms in some market) are vulnerable to even more profitable deviations.

Note that the Pareto frontier is degenerate, and both firms agree on the optimal collusive scheme. Intuitively, in any asymmetric stationary equilibrium at most one firm’s IC constraint will bind, namely the firm receiving lower equilibrium profits. This is because it receives strictly less by sticking to the equilibrium and strictly more by undercutting its competitor in its away market. A small increase in this firm’s home-market price (and hence profits) relaxes its IC constraint, which makes deviation relatively less profitable, and does not violate the other firm’s constraint (which was initially slack). Thus no asymmetric stationary equilibrium can be Pareto optimal.

The preceding characterization holds irrespective of the available punishments. Our next theorem precisely characterizes the harshest sustainable punishment when firms are not too impatient.

**Theorem 4.** Suppose \( \delta \geq \tilde{\delta} \equiv 1/(2 + \Delta c D(p^*_H)/c) \). Then there exists an SPNE yielding lifetime profits of zero to both firms, so that \( \Pi(\delta) = 0 \).

The punishment equilibrium supporting zero lifetime profits involves a short-term price war followed by an optimal stationary collusive scheme forever afterward. In the first period, firms drive prices below the break-even level. Both firms enter each market, the home firm sets some \( p^{PW} \leq p^*_H \), and the away firm mixes over a distribution with support on \( (p^{PW}, p^*_H] \). Thus the home firm serves the market at an unprofitable price, while the away firm loses money by incurring fixed costs. As long as both firms participate in the price war for one period, they transition to a phase with lifetime continuation profits of \( \Pi^* \), which we characterized in Theorem 3. Otherwise the punishment restarts. The price \( p^{PW} \) is chosen so that lifetime profits \( (1 - \delta)(D(p^{PW})(p^{PW} - c_H) - 2c) + \delta \Pi^* \) equal zero. The restriction on \( \delta \) in the theorem ensures that \( p^{PW} \leq p^*_H \) and thus that a firm’s best deviation yields zero stage profits (which it can achieve by exiting both markets).
Our next result synthesizes Theorems 3 and 4 by providing a mild sufficiency condition involving the profitability of each market under which $\delta < \delta^M$. Thus there typically exists a range of discount factors below $\delta^M$ for which we can completely characterize optimal collusion.

**Theorem 5.** Suppose $\Pi^M > c + \Delta cD(p^*_H)$. Then $\delta < \delta^M$. Moreover, for each $\delta \in [\delta, \delta^M]$, $\Pi(\delta) = 0$ and the unique Pareto-optimal stationary equilibrium profit vector $(\Pi^*, \Pi^*)$ satisfies

$$\Pi^* = (1 - \delta)(2\Pi^* - \Delta cD(p^*(\Pi^*))).$$

Further, $\Pi^*$ is continuous, strictly greater than $\Pi^C$, and strictly increasing in $\delta$. Finally, $\delta^M$ is characterized by

$$\delta^M = \frac{\Pi^M - \Delta cD(p^*_H)}{2\Pi^M - \Delta cD(p^*_H)},$$

and perfect collusion is sustainable for all $\delta \geq \delta^M$.

For $\delta < \delta$, we cannot guarantee that a continuation payoff of zero is sustainable in an SPNE. Whenever $\Pi(\delta) > 0$, maximum sustainable collusive payoffs fall below the level indicated in Theorem 5. However, Theorem 3 continues to characterize those payoffs even for discount factors below $\delta$, given $\Pi(\delta)$.

Finally, we describe the on-path properties of a stationary equilibrium that supports profits $\Pi^*$ for each firm. This construction is valid regardless of $\Pi(\delta)$.\textsuperscript{33}

**Theorem 6.** When $\delta < \delta^M$, maximal lifetime profits $(\Pi^*, \Pi^*)$ are supported by a stationary equilibrium with the following on-path properties:

1. The home firm’s strategy is the same in each market, as is the away firm’s.
2. The home firm always quotes a price, while the away firm quotes a price with probability strictly between 0 and 1.
3. The home firm makes stage profits $\Pi^*$, while the away firm makes zero profits.
4. Each firm’s price distribution has full support on $[p^*(\Pi^*), p^*_H]$ and is continuous on $(p^*(\Pi^*), p^*_H)$. The home firm places atoms at $p^*(\Pi^*)$ and $p^*_H$, while the away firm’s price distribution is continuous.

\textsuperscript{33}The theorem applies even if $\Pi(\delta) = \Pi^C$, i.e., when Nash reversion is the harshest sustainable punishment. However in this case the equilibrium described by the theorem is simply unconditional repetition of the stage-game Nash equilibrium. This result is consistent with the conclusion of Theorem 3 that supra-competitive profits are sustainable iff punishments harsher than Nash reversion are sustainable.
5. The away firm makes sales with strictly positive probability, which is decreasing in $\Pi^*$. 

6. Deviations by the away firm to prices at or below $p^*(\Pi^*)$ trigger a punishment that provides the deviating firm with a continuation payoff of $\Pi(\delta)$. 

The on-path strategies of this equilibrium are unique among all optimal stationary equilibria for which price distributions have full support on $[p^*(\Pi^*), p^*_H]$ in each market.

Although the proof of Theorem 6 is fairly involved, the intuition is reasonably straightforward. In addition to increasing profits, allocating production disproportionately to home firms relaxes incentive constraints by reducing the potential gains from business stealing and increasing the losses incurred if punishments are triggered. However, with $\delta < \delta^M$, the cartel cannot sustain an agreement that allocates all sales to the home firms at the monopoly price. To limit the incentives for business stealing by away firms, the cartel must instruct home firms to charge lower prices. Unfortunately, this creates incentives for home firms to defect from the agreement by raising those prices. There are two ways to keep the home firms in line. One approach is to treat an increase in the home-market price as a defection that triggers punishment. The downside of this approach is that punishment power is scarce. If the cartel uses it to deter price increases in home markets, less remains to deter business stealing in away markets. The second approach to deterring home-market price increases is to allow the away firms to enter the market with positive probability, charging a price greater than $p^*(\Pi^*)$. The main advantage of this approach is that it conserves on punishment power, but it too has a downside: entry by the away firm is costly, so the cartel must incentivize it. Accomplishing this through the deployment of punishments would defeat the approach’s main advantage. Instead, the cartel incentivizes entry by allowing the away firms to prevail some fraction of the time.

The equilibrium described in Theorem 6 has an appealing interpretation, which we can express in the language used to describe cartel agreements. The firms agree to an efficient division of demand with each firm serving its home market, along with an aspirational collusive price (specifically, the monopoly price), which the home firm often charges. However, because both firms recognize that the aspirational price is not fully sustainable ($\delta < \delta^M$), they anticipate competitive raids, price cutting, and business stealing. To limit the impact of the inevitable episodes of competition, they also reach a common understanding concerning a price floor (or minimum price) that is strictly less than the monopoly price ($p^*(\Pi^*) < p^*_H$).

34 A similar deterrence effect appears in Marshall and Marx (2007), who study bidder collusion in auctions. They find that bidding rings may entail randomized bids just below the winning bid to deter the winner from reducing its bid.
and, if quoted by the home firm, ensures a win. The cartel cannot stop away firms from raiding markets and attempting to steal business at prices strictly between the price floor $p^* \Pi^*$ and the aspirational collusive price $p^*_H$, and consequently they tolerate such activity as part of the optimal collusive agreement. However, they do not tolerate defections to prices below the floor; such defections trigger punishment. Moreover, because home firms anticipate periodic raids, they sometimes set prices below the aspirational level.

For the purposes of interpretation, it is in our view most natural to think of the agreement concerning the price floor as implicit, rather than as an explicit agreement. In other words, experience rather than discussion likely guides firms to a shared understanding of the conditions under which business stealing is tolerated. Accordingly, competitors may express discontentment with, and complain of, business stealing, even though it is an indispensable component of an optimal equilibrium.

Notice that, according to this theory, all business stealing is not created equal. The cartel would harshly punish business stealing at prices below the floor. In contrast, business stealing at prices above the floor is not merely tolerated, but is indeed critical for the cartel’s successful operation. Of course, in our model, firms never trigger punishments by pricing below the floor. At the cost of greater complexity, one could formulate a version of the model in which rogue sales personnel with no direct knowledge of or involvement in the cartel stochastically instigate undercutting at low prices (below the price floor). Assuming rivals cannot verify the cause of business stealing, the equilibrium would have to provide disincentives for firms to mimic such behavior deliberately. Analogously to Green and Porter (1984), the cartel could establish the necessary disincentives by coordinating on an equilibrium in which sufficiently low prices trigger price wars. The analytics governing prices above the floor would be essentially unchanged.

This stark divide between acceptable and unacceptable business stealing would likely prove difficult to test in practice. Still, the theory has a robust testable implication: one should observe a greater tendency for business stealing to trigger punishment when the successful interloper undercuts the cartel’s intended supplier by a larger amount. In contrast, according to other theories, it is the fact of the interloper’s success, rather than degree to which it undercuts the intended supplier, that triggers punishment.

\footnote{An implication of the use of mixed strategies is that home firms nevertheless regard the timing of these raids as unpredictable. This implication is consistent with our reading of the historical record.}

\footnote{This feature of the equilibrium is consistent with the observation that cartel members often complain about needing to cut prices to compete with actual and potential cheaters.}

\footnote{When detailed records of cartel meetings are available, one can potentially distinguish our theory from the alternatives in other ways. For instance, one can look for evidence that cartel members characterized particular instances of business stealing as defections but nevertheless successfully called on each other to}

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An obvious implication of Theorem 5 is that $\delta^M < 1/2$. If one interprets $\delta$ literally, i.e., as reflecting discounting at the market interest rate over the intervals between competitive interactions, then one would typically expect to find $\delta > 1/2$ in practical applications, in which case firms would achieve perfect collusion, and the structure of optimal collusive agreements for $\delta < \delta^M$ would have little bearing on actual cartel behavior. However, one can also interpret $\delta$ more expansively (and less literally) as a reduced-form stand-in for other factors that tend to make firms focus more on present opportunities and less on future consequences. For example, in many simple models of oligopoly, the number of competitors affects the feasibility of collusion through the same channel as discounting (because adding firms increases the potential gains from current deviations and reduces the future benefits of cooperation). In the online appendix to this paper, we show that our central conclusions extend to settings involving more than two firms, albeit with some additional complexities. Further, the threshold $\delta^M$ at which full collusion is possible approaches unity as the number of firms grows large, which shows that the implications of our model are relevant even with very patient firms provided the market is sufficiently crowded. Firms may also effectively discount future profits to a greater extent than market interest rates would imply because of agency problems, leadership turnover, uncertainty about future market conditions, or capital market imperfections that raise internal hurdle rates.

5.3 The inevitability of business stealing in imperfect cartels

We have shown that there exists an optimal (profit-maximizing) stationary equilibrium in which behavior interpretable as business stealing occurs with positive probability and goes unpunished. However, we have not shown that this optimal equilibrium is unique. In fact, there exist many optimal equilibria in which firms’ price distributions do not have full support on $[\Pi^*, p^*_H]$. One can construct such equilibria by “drilling holes” in the full-support distribution of Theorem 6, moving the home firm’s probability mass to an atom at the top of the hole and the away firm’s to an atom at the bottom of the hole. This modification creates an incentive for the away firm to deviate into the hole.\(^{38}\) However, as long as the holes are sufficiently small, the away firm will earn greater current-period profits by undercutting $p^*(\Pi^*)$ than by deviating into the hole, and hence this modification will not tighten the incentive-compatibility constraints.

\(^{38}\)More precisely, if the hole has lower boundary $p_1$ and upper boundary $p_2$, the away firm’s current-period profits are strictly increasing on $[p_1, p_2]$. Thus all posted prices in $(p_1, p_2)$ increase the away firm’s current-period profits relative to their equilibrium level. Its most profitable deviation in the range $(p_1, p_2)$ is to just undercut the home firm’s atom at $p_2$. maintain high prices and refrain from retaliation.
In light of this multiplicity, it is important to determine whether unpunished business stealing is merely a special feature of certain optimal stationary equilibria, such as the one we exhibited in Theorem 6. In fact, it is a general property; indeed, all other optimal stationary equilibria involve strictly more business stealing than the one identified in Theorem 6. The following theorem establishes that result, and shows that other properties of the aforementioned equilibrium are also general.

**Theorem 7.** When $\delta < \delta^M$, all stationary equilibria supporting profits $(\Pi^*, \Pi^*)$ feature the same minimum price, maximum price, and entry probabilities by each firm in each market. In particular, the home firm always enters while the away firm enters with a fixed probability strictly between 0 and 1. Furthermore, among all stationary equilibria supporting profits $(\Pi^*, \Pi^*)$, the equilibrium characterized in Theorem 6 uniquely minimizes the probability that an away firm wins business.

The intuition for this result is essentially the same as that given for Theorem 6: allowing away firms to enter and win business with positive probability is optimal because it deters deviations involving price increases in the home market without deploying scarce punishment power. Note in particular that Theorem 7 rules out the existence of optimal equilibria in which each firm posts a deterministic price $p^*(\Pi^*)$ in its home market and stays out of its away market.

Perhaps surprisingly, an inability to costlessly police overcharging by home firms does not reduce the level of collusive profits achievable by the cartel. In equilibrium, the home firm makes up for business lost to raids by charging higher prices, and ends up just as well off as if the cartel had divided the market perfectly with prices set at the floor, $p^*(\Pi^*)$. Meanwhile, the away firm earns just enough profits to cover its fixed costs. However, this arrangement does impose costs on customers, who end up subsidizing excessive entry by the away firms through higher prices. Thus, ironically, the business stealing that accompanies an imperfect cartel hurts consumers but not firms.

It is natural to wonder whether business stealing is necessary when firms collude but fail to optimize cartel agreements. In fact, any stationary equilibrium supporting profits sufficiently close to $(\Pi^*, \Pi^*)$ must involve business stealing.

**Theorem 8.** Fix $\delta < \delta^M$. Then there exists a $\tilde{\Pi} < \Pi^*$ such that, under any stationary equilibrium supporting profits $(\Pi_1, \Pi_2) > (\tilde{\Pi}, \tilde{\Pi})$, the away firm captures each market with strictly positive probability on the equilibrium path.

The proof of this theorem is essentially a formalization of the intuition provided above for Theorems 6 and 7. When collusive profits are sufficiently close to the level at which
the IC constraint binds for deviations involving undercutting, there is not enough slack punishment power to additionally discourage a firm from simultaneously inflating the price in its home market. To deter this compound deviation, the cartel must reduce its profitability by permitting the away firm to enter with a higher price. Moreover, because there is also little slack in the away firm’s IC constraint, the cartel must compensate that firm for the associated entry costs by allowing it to steal business successfully with positive probability.

5.4 Comparative statics

Our results rely on the presence of both a cost asymmetry $\Delta c$ between home and away firms, and a recurring fixed cost $c$ of attempting to serve a market. To illuminate the roles of both $\Delta c$ and $c$, we examine the limiting behavior of optimal collusion as each becomes small. We first consider the outcome when the cost asymmetry between firms declines to zero. (While we have so far defined $\Pi^*$ only for discount factors below $\delta^M$, we will extend the definition in the obvious way to all $\delta$ for use in the theorems that follow by letting $\Pi^* = \Pi^M$ when $\delta > \delta^M$.)

**Theorem 9.** For all $\delta \in [1/2, 1)$ and any $c_H, c_A, c > 0$, profits $(\Pi^M, \Pi^M)$ are supportable by a stationary equilibrium. For all $\delta \in [0, 1/2)$ and $c_H, c > 0$, $\Pi^* \to 0$ and $\Pi^C \to 0$ as $c_A \downarrow c_H$.

In the limit as $\Delta c$ shrinks to zero, the set of stationary equilibria exhibits a bang-bang structure. No collusion is possible for $\delta$ below the critical threshold $1/2$, while perfect collusion is sustainable for $\delta$ above this threshold. Thus, there is no room for imperfect collusion in the limit.

In contrast, with asymmetric costs, the degree of sustainable collusion rises gradually with the discount factor. As a cartel attempts to achieve a greater degree of collusion, the profits that a firm gains in its away market by undercutting rise relative to the profits it earns in its home market, and this makes the deviation more tempting. Accordingly, sustaining an incrementally higher profit level requires incrementally greater patience.

We can also examine the nature of collusion in the limit as the fixed cost $c$ of attempting to serve a market declines to zero. Note that the maximum sustainable collusive profits, $\Pi^*$, are independent of $c$ so long as $c$ is small enough that $\delta > \delta$ and thus $\Pi(\delta) = 0$. In the following theorem, when we write $\Pi^*$, we mean its value when $c$ is below this threshold.

**Theorem 10.** Fix $\delta \in (0, 1)$. Let $(F_H(\cdot), F_A(\cdot), \pi_A)$ be the home and away firm’s price distributions and the away firm’s entry probability, respectively, for the equilibrium characterized in Theorem 6. As $c \to 0$,
• $F_h(\cdot)$ converges uniformly to $1\{p \leq p_A\}$,

• $\pi_A \to 1 - \frac{\Pi^*}{\Pi M}$,

• $F_A(\cdot)$ converges uniformly to $\frac{1 - \frac{\Pi^*}{\Pi M}}{1 - \frac{\Pi^*}{\Pi M}}$ on $[p_A, p^*_H]$.

The probability of business stealing therefore falls to zero as $c$ vanishes, and in the limit the home firm wins the market at price $p_A$ with probability 1.

This theorem tells us that a nonzero market-specific fixed cost is crucial for generating equilibrium business stealing. Without it, the away firm could costlessly set its price just above $p_A$, thereby deterring the home firm from charging higher prices without actually undercutting it. As we have emphasized previously, in our model these fixed costs give rise to equilibrium business stealing because they preclude the firms from costlessly policing the cartel agreement.

6 Beyond stationarity

Stationarity is a conceptually appealing restriction, the class of stationary equilibria is analytically tractable, and optimal stationary equilibria exhibit qualitative features that resemble those of many real-world cartels. Nonetheless, it is reasonable to ask whether the assumption of stationarity drives our results. This section provides a partial answer: we establish that stationary equilibria are in fact optimal within a broad class that permits market shares and prices to fluctuate over time.

6.1 Balanced equilibria

We study a class of equilibria satisfying two conditions. The first is an invariance condition pertaining to on-path continuation strategies that is weaker than stationarity (in that it applies only within stages rather than both within and across stages). The second is a mild symmetry condition on stage-game strategy profiles. The following definition identifies the invariance requirement.

**Definition 2.** An SPNE $\sigma$ is an invariant equilibrium if there exists a sequence of stage-game strategy profiles $\{\tau^t\}_{t=0}^\infty$ and a rectangular set of complete histories $H^*$, the equilibrium path, such that: 1) $H^*$ occurs with probability 1 under $\sigma$; and 2) for all $h \in H^*$ and $t$, $\sigma(h^t) = \tau^t$. 

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Because we allow for arbitrary sequences of stage-game profiles \( \{\tau_t\}_{t=0}^\infty \), prices and market shares in an invariant equilibrium can in principle vary arbitrarily over time. A cartel can potentially use such variation to redistribute profits systematically among the firms. Significantly, all pure strategy SPNEs are trivially invariant. However, in an invariant mixed strategy equilibrium, the progression of on-path play must be independent of the particular realization of firms’ stage-game choices.

Replacing our stationarity requirement with the weaker invariance condition complicates the analysis of optimal cartels, and therefore necessitates the imposition of a weak symmetry restriction on stage-game strategy profiles:

**Definition 3.** An SPNE \( \sigma \) is a balanced equilibrium if it is invariant, and if, for all \( t \), either 1) \( \Pi_i^m(\tau_t^m) \leq 0 \) for all firms \( i \) and markets \( m \neq i \), or 2) \( \Pi_i^m(\tau_t^m) > 0 \) for all \( i \) and \( m \).

The symmetry restriction embedded in this definition requires either that the home firm captures all profits (if any) in every market, or that both firms profit in both markets. Significant flexibility remains for allocating profits between the firms, and in particular they need not earn the same profits in their roles as home firms, or in their roles as away firms. In effect, the condition requires symmetric access to markets - an appealing feature that cartel members might well demand - but permits asymmetric outcomes.

Because balance requires more than invariance, it should come as no surprise that some stationary equilibria may not be balanced. However, the following proposition establishes that all optimal stationary equilibria are balanced. Thus, when we replace stationarity with balance, we do not rule out the most attractive stationary equilibria.

**Proposition 11.** All stationary equilibria supporting profits \((\Pi^*, \Pi^*)\) are balanced.

This proposition follows directly from the fact that stationary play is trivially invariant, along with Theorem 7, which showed that stage-game play in any optimal stationary equilibria satisfies our weak symmetry condition.

6.2 Optimal balanced equilibria

The following theorem establishes that, within the class of balanced equilibria, optimal equilibria are stationary:

**Theorem 12.** Suppose \( \delta < \delta^M \). Then there exists a unique Pareto-optimal balanced equilibrium payoff vector \((\Pi^{**}, \Pi^{**})\), which is supportable by a stationary equilibrium. Therefore \( \Pi^{**} = \Pi^* \).
This theorem demonstrates that our main results are not driven by the most visible aspect of stationarity - the cartel’s inability to rotate profit shares over time. Within the class of balanced equilibria, firms may reallocate stage profits arbitrarily over time; however when colluding optimally they choose not to do so. Somewhat surprisingly, the Pareto frontier remains degenerate for this expanded class. Thus firms continue to agree on the optimal collusive agreement, even though the cartel could in principle treat them quite differently, for example frontloading profits for one and backloading profits for the other. Any such intertemporal allocation creates such strong incentives for the recipient of back-loaded profits to deviate that there is no scope for improving the overall payoff for the firm with front-loaded profits.

Naturally, one could entertain notions of equilibrium that are even less restrictive than balance. However, it is not obvious to us what, if any, advantage would be gained by such arrangements. Moreover, the additional complexity and subtlety of “unbalanced” equilibria arguably renders their selection implausible.

7 Conclusion

An important empirical feature of many real-world cartels is that deliberate cheating not only occurs, but also goes unpunished. Existing theories of collusion in repeated games cannot account for these observations because they predict that cheating will not occur, and that – if it did occur – it would trigger a price war or some other punishment. We have attempted to bridge this gap between theory and practice by constructing a theory of equilibrium business stealing in imperfect cartels. Our theory concerns settings in which different firms have natural advantages in serving different market segments, and where there is some recurring fixed cost of attempting to serve each segment; moreover, we focus on conditions under which some collusion is sustainable but perfect collusion is not.

We find that, within an important class of equilibria, optimal collusion necessarily entails business stealing. Moreover, the equilibria that sustain optimal collusion look very much like imperfectly enforced cartel agreements with aspirational prices, price floors, and customer allocations. Cartel members attempt to divide the market according to firms’ cost advantages. Each firm often charges the aspirational price in its home market, but also sometimes engages in defensive price cutting down to and including the price floor (at which it is assured of winning the market) to head off “raids” by competitors. Firms sometimes attempt to raid each others’ markets, in all such cases setting prices above the floor, so as to avoid stealing business if the home firm has made the “safe” choice. If, however, the home firm has also set
a price above the floor, the rival may successfully steal business in a raid. Business stealing at prices above the floor occurs in equilibrium and is met with forbearance because all firms understand that they cannot fully sustain the aspirational collusive price.

References


Appendices

Not for Publication.

(If accepted, an abbreviated appendix may be prepared for inclusion in the published version, depending on the editor’s instructions.)

A Assumptions on the demand function

We impose several technical regularity conditions on the demand function $D(p)$ throughout the paper. For $i \in \{H, A\}$, define $p_i \equiv \inf\{p : D(p)(p - c_i) \geq c\}$ and $p_i^* \equiv \sup\{\arg\max_p D(p)(p - c_i)\}$. Then we assume that $D(p)$ satisfies:

(A1) $\sup_p D(p)(p - c_A) > c$ and $\lim_{p \to \infty} D(p)(p - c_H) = 0$.

(A2) $\inf\{p : D(p)(p - c_A) > c\} < \inf\{\arg\max_p D(p)(p - c_H)\}$.

(A3) $D(p)(p - c_H)$ is strictly increasing on $[c_H, p_H^*]$.

Assumptions (A1) through (A3) are jointly sufficient to establish several facts:

(1) The monopoly profits of firm $i \in \{H, A\}$ are positive and strictly increasing on $[p_i, p_H^*]$. Further, they are continuous on $[p_i, p_H^*)$.

(2) The home firm’s monopoly profits are strictly lower above $p_H^*$ than at $p_H^*$.

(3) Away firms’ monopoly profits are strictly negative below $p_A^*$.

(Proof that (A1) through (A3) imply these facts is straightforward and omitted.) These facts will be used freely in the proofs without explicit justification.

B Notation for repeated games

Fix a repeated game $G$ with $I$ players. Player $i$’s stage game action set is denoted $A^i$, and the set of joint actions is written $A \equiv \prod_{i=1}^I A^i$. A complete history $h$ of the repeated game is an element $h \in \mathcal{H}^\infty$, where $\mathcal{H}^\infty \equiv \prod_{t=0}^\infty A$ denotes the set of all complete histories. For every $t \in \mathbb{N}$, the set $\mathcal{H}^t \equiv \prod_{s=0}^{t-1} A$ denotes the set of partial histories up to period $t$. Additionally, $\mathcal{H}^0 = \{h^0\}$ denotes the unique empty history $h^0$ obtaining in the initial period. For any subset of complete histories $\tilde{\mathcal{H}} \subset \mathcal{H}^\infty$, the sets $\tilde{\mathcal{H}}^t$ denote the projections of $\tilde{\mathcal{H}}$ onto each $\mathcal{H}^t$. 
Given a complete history \( h \in \mathcal{H}^\infty \), the vector \( h^t \) denotes the projection of \( h \) onto \( \mathcal{H}^t \). (We will also occasionally abuse notation and use \( h^t \) to refer to an arbitrary element of \( \mathcal{H}^t \); there should be no confusion.)

C Core proofs

This Appendix develops a series of technical results which form the basis for the proofs in the body of the text, as well as the extension to the many-firm case in Appendix E. Most of the results are proven for an extension of the model in the main text to accommodate more than two firms and markets. For a full exposition of this extended model, see Appendix E. The results of this Appendix are explicitly connected to the theorems in the main text in Appendix D.

C.1 The stage game

The following theorem characterizes the set of Nash equilibria for a version of the stage game with \( N \geq 1 \) away firms, each with entry cost \( c \) and marginal cost \( c_A \). As usual, all demand is allocated to the firm posting the lowest price. The home firm will be denoted \( H \), while the set of away firms will be denoted \( \mathcal{I} = \{1, \ldots, N\} \).

**Theorem C.1.** Fix any non-empty subset \( \mathcal{J} \subset \mathcal{I} \) of away firms. Then there exists a unique Nash equilibrium of the stage game in which every firm in \( \mathcal{J} \) enters with positive probability and no firm in \( \mathcal{I} \setminus \mathcal{J} \) ever enters. In this equilibrium:

1. The home firm always enters and makes profits \( \Pi_H = \Delta c D(p_A) \).
2. Each away firm \( i \in \mathcal{J} \) enters with probability strictly less than 1 and makes profits \( \Pi_i = 0 \).
3. Each entering firm’s price distribution has full support on \([p_A, p^*_H] \).
4. All entering away firms play the same strategy.

There are no other Nash equilibria.

**Proof.** Let \( (\pi_H, F_H, \{\pi_i, F_i\}_{i \in \mathcal{I}}) \) be a Nash equilibrium of the stage game. Define \( \mathcal{J} \equiv \{i \in \mathcal{I} : \pi_i > 0\} \). We first establish that at least one away firm must occasionally enter in equilibrium.
Lemma C.1. $\mathcal{J}$ is non-empty.

Proof. If no away firm entered, then each away firm makes zero profits in equilibrium. Meanwhile, the unique profit-maximizing strategy of the home firm is to post price $p_H^\ast$. But then each away firm can make strictly positive profits by pricing just under $p_H^\ast$, a contradiction of equilibrium.

Define $\Pi_i(p)$ to be the expected profits of firm $i \in \{H\} \cup \mathcal{J}$ upon entering and setting price $p$ given the equilibrium strategies of all other firms. We will often overload notation by letting $\Pi_i$ (with no argument) represent the equilibrium profits of firm $i$.

The next lemma establishes that $\Pi_i(p)$ is continuous at $p$ iff no other firm places an atom at $p$, and that when an atom exists the profit function is discontinuous from both directions.

Lemma C.2. $\Pi_i(p−) \geq \Pi_i(p) \geq \Pi_i(p+) \text{ for all } i \in \{H\} \cup \mathcal{J}$ and $p \in [p_A, p_H^\ast]$, with equality for given firm $i$ iff no other firm places an atom at $p$.

Proof. Obvious.

The next lemma establishes that firms set prices only in the interval $[p_A, p_H^\ast]$, that the home firm always enters the market, and that the away firm occasionally enters the market.

Lemma C.3. $F_H([p_A, p_H^\ast]) = F_i([p_A, p_H^\ast]) = 1 \text{ for all } i \in \mathcal{J}$ and $\pi_H = 1$.

Proof. Each $i \in \mathcal{J}$ receives strictly negative profits below $p_A$, no matter the other firms’ strategies. So $F_i(p_A−) = 0$ in equilibrium. Then the home firm is never profit-maximizing below $p_A$ given that his profits are non-positive below $p_H$, zero at $p_H < p_A$, and strictly increasing on $[p_H, p_A]$. Hence $F_H(p_A−) = 0$ as well. Additionally, the home firm achieves strictly positive profits by setting a price just below $p_A$, so his equilibrium profits must be strictly positive and therefore $\pi_H = 1$.

At the other end of the price support, the home firm always makes strictly lower profits setting a price above $p_H^\ast$ than by pricing at $p_H^\ast$, no matter the away firms’ strategies. Then $F_H(p_H^\ast) = 1$. This result, combined with the fact that the home firm always enters the market, means that any away firm pricing above $p_H^\ast$ will make no sale and achieve negative profits. This is less profitable than not entering the market, so $F_i(p_H^\ast) = 1$ for each $i \in \mathcal{J}$ in equilibrium.

We next establish a “no overlapping atoms” result:

Lemma C.4. For each $p \in [p_A, p_H^\ast]$, there exists at most one firm in $\{H\} \cup \mathcal{J}$ whose price distribution is not continuous at $p$. 

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Proof. Suppose some firm \( i \in \{H\} \cup \mathcal{J} \) places an atom at \( p \in (p^*_A, p^*_H] \). Then \( i \) must be profit-maximizing at \( p \). If some other firm also placed an atom at \( p \), then the limit of \( i \)'s profits for prices just below \( p \) would be strictly higher than his profits at \( p \), by the previous lemma. This contradicts the optimality of \( p \) for \( i \), so no other firm can have an atom at \( p \).

Finally, consider placement of an atom at \( p^*_A \) by two firms. At least one of these firms must be an away firm; but as the away firm loses the market with positive probability at that price, he makes strictly negative profits given the definition of \( p^*_A \). This means \( p^*_A \) cannot be optimal for that firm, ruling out the placement of an atom there. Reaching a contradiction, we conclude that at most one firm can place an atom at \( p^*_A \).

Let \( \bar{p}_i \equiv \sup \{ p : F_i(p) < 1 \} \) be the supremum of firm \( i \)'s price support for \( i \in \{H\} \cup \mathcal{J} \), and similarly let \( \underline{p}_i \equiv \inf \{ p : F_i(p) > 0 \} \) be the infimum.

Lemma C.5. \( \bar{p}_i = p^*_H \) for all \( i \in \{H\} \cup \mathcal{J} \).

Proof. We first show that \( \bar{p}_i = \bar{p}_j \) for all \( i, j \in \{H\} \cup \mathcal{J} \). Suppose \( \bar{p}_i > \max_{j \neq i} \bar{p}_j \equiv \bar{p}_{-i} \) for some \( i \in \{H\} \cup \mathcal{J} \). As \( i \) makes non-negative equilibrium profits, he must win with strictly positive probability when playing any price (strictly) above \( \bar{p}_{-i} \). Thus his profits are strictly increasing on \( (\bar{p}_{-i}, p^*_H] \), meaning \( i \) places an atom at \( p^*_H \) and does not set prices in \( (\bar{p}_{-i}, p^*_H) \).

Now, if no atom existed at \( \bar{p}_{-i} \), then some firm \( j \) whose support supremum lies at \( \bar{p}_{-i} \) has continuous profits there. Hence \( \bar{p}_{-i} \) must be profit-maximizing for \( j \). In particular, as \( j \) makes non-negative profits, he wins with positive probability by pricing at \( \bar{p}_{-i} \), so also by setting prices in \( [\bar{p}_{-i}, p^*_H) \). But then \( j \)'s profits are also strictly increasing on \( [\bar{p}_{-i}, p^*_H) \), a contradiction.

Then it must be that some firm, say \( j \) again, places an atom at \( \bar{p}_{-i} \). But by the overlapping atoms result no other firm can place an atom there. Then \( j \)'s profits are strictly increasing on \( [\bar{p}_{-i}, p^*_H) \), contradicting the optimality of \( \bar{p}_{-i} \) for \( j \) implied by his placement of an atom there. We conclude that every firm’s price ceiling is the same, say \( \bar{p} \).

Suppose \( \bar{p} < p^*_H \). If no firm places an atom at \( \bar{p} \), then each firm’s profits are continuous at \( \bar{p} \) and hence this price is profit-maximizing for all firms. For equilibrium profits to be non-negative, each firm must win the market with positive probability at \( \bar{p} \), meaning profits are strictly increasing on \( [\bar{p}, p^*_H] \), a contradiction of optimality. So some firm \( i \) must place an atom at \( \bar{p} \), which is then profit-maximizing for \( i \). Since there can be no overlapping atoms, \( i \)'s profits are continuous at \( \bar{p} \), meaning they are strictly increasing on \( [\bar{p}, p^*_H] \), another contradiction. Hence \( \bar{p} = p^*_H \).

Lemma C.6. \( \underline{p} = p^*_A \) and \( \Pi_i = D(p^*_A)(p^*_A - c_i) - c \) for all \( i \in \{H\} \cup \mathcal{J} \).
Proof. Suppose \( p_i < \min_{j \neq i} p_j \equiv p_{-i} \) for some \( i \). Then \( i \) wins w.p. 1 on \([p_i, p_{-i}]\), meaning his profits are strictly increasing on this interval. This contradicts the optimality of prices strictly less than \( p_{-i} \). Hence \( p_i = p \) for some \( p \in [p_A, p_H^*] \) and all \( i \in \{H \} \cup \mathcal{J} \).

Suppose some firm placed an atom at \( p \). Then every other firm’s profits at \( p \) are strictly higher than at prices just above \( p \), meaning no other firm’s price distribution assigns positive measure to \((p, p + \varepsilon)\) for \( \varepsilon > 0 \) sufficiently small. But given the definition of \( p \), this means every other firm must place an atom at \( p \), contradicting the overlapping atom result.

So there exist no atoms at \( p \), meaning by continuity of the profit function there that each firm’s profits are maximized at \( p \). Hence \( \Pi_i = D(p)(p-c_i) - c \) for all \( i \). Suppose \( p > p_A \). Then \( \Pi_i > 0 \) for all \( i \), meaning \( \pi_i = 1 \) for each \( i \). Now consider the possible existence of an atom at \( p = p_H^* \). At most one firm can place an atom there, say firm \( i \). Then \( i \) is profit-maximizing at \( p_H^* \), but given sure entry by all other participating firms below this price, \( i \) makes \( -c < 0 \) there, a contradiction. So no firm places an atom at \( p_H^* \). But then each firm’s profits are continuous at \( p_H^* \), meaning given the definition of \( p \) that each firm’s profits are maximized there. But their profits are again \( -c < 0 \) there, another contradiction. So we must have \( p = p_A \).

**Lemma C.7.** There can exist at most one atom in equilibrium, by the home firm at \( p_H^* \).

Proof. Suppose some firm \( i \) places an atom at \( p \in [p_A, p_H^*) \). Then there exists an \( \varepsilon > 0 \) such that each firm \( j \neq i \) places no support on \([p, p + \varepsilon)\). Now, we know that \( i \) has profit-maximizing prices arbitrarily close to \( p_H^* \) given that this is the supremum of his price support. Then \( i \) wins with positive probability arbitrarily close to \( p_H^* \), meaning he must win with (constant) positive probability on \([p, p + \varepsilon)\) (assuming \( p + \varepsilon < p_H^* \), which we can assume wlog by taking \( \varepsilon \) small). But then \( i \)'s profits are strictly increasing on \([p, p + \varepsilon)\). This contradicts the fact that \( p \) is profit-maximizing for \( i \) implied by his placement of an atom there. So no such atom exists.

The existence of at most a single atom at \( p_H^* \) follows from the overlapping atoms result. If this atom were placed by an away firm, the fact that \( \pi_H = 1 \) implies that the away firm never wins the market at \( p_H^* \) and thus that his profits are \( -c < 0 \) there. This contradicts the optimality of \( p_H^* \) implied by placement of an atom there. Hence only the home firm can place an atom at \( p_H^* \).

**Lemma C.8.** \( F_H \) has full support on \([p_A, p_H^*) \).

Proof. Suppose not. Then there exists a non-degenerate open interval \( S \subset (p_A, p_H^*) \) assigned zero measure under \( F_H \). Let \( \widehat{F} \equiv F_H(p) \) for any \( p \in S \). Because the infimum and supremum
of the support of \( F_H \) are \( p_A \) and \( p_H \), we must have \( \hat{F} \in (0, 1) \). Expand \( S \) so that \( S = (p_L, p_H) \), where \( p_L \equiv \inf\{p : F_H(p) = \hat{F}\} \) and \( p_H \equiv \sup\{p : F_H(p) = \hat{F}\} \). Because \( \hat{F} \in (0, 1) \), each of \( p_L \) and \( p_H \) is finite and lies in \( [p_A, p_H] \). Also, by assumption \( p_L < p_H \).

Because no other firm can place an atom at \( p_L \) or \( p_H \), the home firm’s profits are continuous at these prices, and therefore both prices are profit-maximizing for him given \( \hat{F} \in (0, 1) \). Then no price in \( S \) can provide higher profits than at one of the endpoints. This implies the inequalities

\[
D(p)(p - c_H) \prod_{j \in J} (1 - \pi_j F_j(p)) \leq D(p_H)(p_H - c_H) \prod_{j \in J} (1 - \pi_j F_j(p_H)) \quad \forall p \in S,
\]

with the lhs and rhs being \( H \)'s profits at \( p \) and \( p_H \) respectively. (Recall that no away firm places an atom in \( S \).) Now multiply each side by \( \frac{p - c_A}{p - c_H} \frac{1 - \pi_H F_H(p)}{1 - \pi_i F_i(p)} \) for \( i \in J \). Then we obtain

\[
D(p)(p - c_A) \prod_{j \neq i} (1 - \pi_j F_j(p)) \leq D(p_H)(p_H - c_H) \frac{p - c_A}{p - c_H} \frac{1 - \pi_H F_H(p)}{1 - \pi_i F_i(p)} \prod_{j \in J} (1 - \pi_j F_j(p_H)) < D(p_H)(p_H - c_A) \frac{p_H - c_A}{p_H - c_H} \frac{1 - \pi_H \hat{F}}{1 - \pi_i \hat{F}} \prod_{j \in J} (1 - \pi_j F_j(p_H)) = D(p_H)(p_H - c_A)(1 - \pi_H \hat{F}) \prod_{j \in J \setminus \{i\}} (1 - \pi_j F_j(p_H)).
\]

The second inequality follows from the fact that \( (p - c_A)/(p - c_H) \) is strictly increasing in \( p \), \( 1 - \pi_H F_H(p) \) is constant, and \( 1 - \pi_i F_i(p) \) is (weakly) decreasing in \( p \) on \( S \). We arrive at the inequalities

\[
D(p)(p - c_A) \prod_{j \neq i} (1 - \pi_j F_j(p)) < D(p_H)(p_H - c_A)(1 - \pi_H \hat{F}) \prod_{j \in J \setminus \{i\}} (1 - \pi_j F_j(p_H)) \quad \forall p \in S.
\]

The lhs are \( i \)'s profits for \( p \in S \) (recall that no firm places on atom in \( S \)). Meanwhile the rhs are \( i \)'s profits in the limit for prices just below \( p_H \). (No away firm places an atom at \( p_H \).) Thus no price in \( S \) can be profit-maximizing for \( i \), as there is always some price very close to \( p_H \) which will do better. So \( i \)'s price distribution assigns zero measure to \( S \) as well.

This reasoning holds for all \( i \), so we conclude that no firm plays in \( S \). As no firm places an atom at \( p_L \), the home firm’s profits are then strictly increasing on \( [p_L, p_H] \), contradicting the optimality of \( p_L \). Hence no such interval \( S \) can exist.  \( \square \)
Lemma C.9. $F_i$ has full support on $[p_A, p_H^*]$ for each $i \in \mathcal{J}$.

Proof. Suppose not. Then for some $i \in \mathcal{J}$ we can construct an $S = (p_L, p_H)$ as in the previous lemma. We know that the home firm’s price distribution has full support in this interval; if no other away firm played in $S$, then the home firm’s profits would be strictly increasing in $S$, a contradiction. We will show, however, that no other away firm will play in $S$, proving the result. If $|\mathcal{J}| = 1$, then the result is trivial, so assume $|\mathcal{J}| \geq 2$.

It can’t be the case that the home firm places an atom at $p_H$, for then neither this price nor any price just above it would be profit-maximizing for $i$. (Recall $\hat{F} < 1$, so $i$ must have profit-maximizing prices arbitrarily close to $p_H$ from above.) Because no other firm places an atom at $p_H$, firm $i$’s profits are continuous there and hence $p_H$ is profit-maximizing for $i$. Then

$$D(p)(p - c_A)\prod_{j \neq i} (1 - \pi_j F_j(p)) \leq D(p_H)(p_H - c_A)\prod_{j \neq i} (1 - \pi_j F_j(p_H)) \forall p \in S.$$ 

Choose $k \in \mathcal{J} \setminus \{i\}$. Multiplying both sides by $\frac{1 - \pi_i F_i(p)}{1 - \pi_k F_k(p)}$ yields

$$D(p)(p - c_A)\prod_{j \neq k} (1 - \pi_j F_j(p)) \leq D(p_H)(p_H - c_A)\frac{1 - \pi_i F_i(p)}{1 - \pi_k F_k(p)} \prod_{j \neq i} (1 - \pi_j F_j(p_H))$$

$$\leq D(p_H)(p_H - c_A)\frac{1 - \pi_i F_i(p_H)}{1 - \pi_k F_k(p_H)} \prod_{j \neq i} (1 - \pi_j F_j(p_H))$$

$$= D(p_H)(p_H - c_A)\prod_{j \neq k} (1 - \pi_j F_j(p_H)).$$

The second inequality is strict whenever $F_k(p_H) > F_k(p)$, in which case the inequality implies $k$’s profits at $p_H$ are strictly higher than at $p$. (Recall no firm places an atom at $p_H$.)

Suppose by way of contradiction that $F_k$ assigns positive measure to some subset of $S$. Then given continuity of $F_k$ there exists a $p \in S$ such that $p$ is profit-maximizing for $k$ and $F_k(p_H) > F_k(p)$. But the latter inequality implies that $k$’s profits at $p$ are strictly lower than at $p_H$, a contradiction of profit-maximization. So $k$ assigns zero measure to $S$. This yields the desired contradiction.

It is an immediate consequence of the previous lemma that each firm’s profits are equal to $\Pi_i = D(p_A)(p_A - c_i) - c$ for all $p \in [p_A, p_H^*)$. For the lack of atoms implies continuity
of profits at all such \( p \), and the full support result implies a sequence of profit-maximizing prices converging to \( p \).

We now construct the unique equilibrium for a given (arbitrary) non-empty subset \( \mathcal{J} \subset \mathcal{J} \) of entering away firms. In light of continuity of each \( F_i \) below \( p_i^* \), the profit-maximization condition on \( [p_A, p_i^*] \) is

\[
D(p_A)(p_A - c_i) - c = D(p)(p - c_i) \prod_{j \neq i}(1 - \pi_j F_j(p)) - c
\]

for all \( i \in \{ H \} \cup \mathcal{J} \) and \( p \in [p_A, p_i^*] \). For \( i = H \) this becomes

\[
\prod_{j \in \mathcal{J}} (1 - \pi_j F_j(p)) = \frac{D(p_A)(p_A - c_H)}{D(p)(p - c_H)}.
\]

Inserting into the condition for \( i \in \mathcal{J} \) yields

\[
1 - \pi_i F_i(p) = \frac{D(p_A)(p_A - c_H)}{c} \frac{p - c_A}{p - c_H} (1 - \pi_H F_H(p)).
\]

Hence, inserting back into the \( i = H \) condition,

\[
\pi_H F_H(p) = 1 - \frac{c}{D(p_A)(p_A - c_H)} \frac{p - c_H}{p - c_A} \left( \frac{D(p_A)(p_A - c_H)}{D(p)(p - c_H)} \right)^{1/|\mathcal{J}|}.
\]

As \( \pi_H = 1 \), this pins down \( F_H(p) \) for \( p < p_i^* \). Note that \( F_H(\cdot) \) is strictly increasing in \( p \), as required. Further, \( F_H(p_i^* -) < 1 \), so \( H \) must place an atom at \( p_i^* \).

We re-write \( H \)'s mixing distribution in final form as

\[
F_H(p) = 1 - \left( \frac{c}{D(p)(p - c_A)} \right) \left( \frac{D(p)(p - c_H)}{D(p_A)(p_A - c_H)} \right)^{1-1/|\mathcal{J}|}
\]

for \( p < p_i^* \), with \( F_H(p_A) = 0 \) and \( F_H(p_i^*) = 1 \).

Next, by inserting \( F_H(\cdot) \) into the relationship between \( F_i \) and \( F_H \), we find that each away firm's mixing distribution satisfies

\[
\pi_i F_i(p) = 1 - \left( \frac{D(p_A)(p_A - c_H)}{D(p)(p - c_H)} \right)^{1/|\mathcal{J}|}.
\]
Because $F_i$ must be continuous and equal to 1 at $p^*_H$, this pins down $\pi_i$ as

$$\pi_i = 1 - \left(\frac{D(p_A)(p_A - c_H)}{D(p^*_H)(p^*_H - c_H)}\right)^{1/|J|}.$$ 

Solving for $F_i(\cdot)$ yields each away firm’s mixing distribution. Note that all participating away firms play an identical strategy.

Finally, we must check that it is optimal for each non-participating away firm not to enter. But each away firm $i \in J$ makes zero profits when playing any $p \in (p_A, p^*_H)$ when faced with $|J| - 1$ other away firms. Then at each price $p > p_A$, all non-participating firms make strictly lower profits than $i$ because they will occasionally lose the sale to $i$. Then no non-participating firm wants to enter above $p_A$. And entering at $p_A$ yields zero profits. So indeed it is optimal for all firms in $I \setminus J$ to refrain from entering. This completely characterizes all Nash equilibria of the stage game.

### C.2 Stationary equilibria

This subsection characterizes basic properties of stationary equilibria for a version of the repeated game with $N + 1$ firms and $N + 1$ markets, where $N \geq 1$ and each firm $i$ plays the role of the home firm in market $m = i$ and is an away firm in all other markets $m \neq i$. The definition of stationarity given in Section 5.1 is extended to the many-firm case in the obvious way.

**Lemma C.10.** Let $\sigma$ be a stationary equilibrium with on-path play $\tau$. Then for each firm $i$ and market $m$, there exists a constant $\Pi^i_m$ such that $\Pi^i_m(a^i_m, \tau^{-i}_m) = \Pi^i_m$ with probability 1 under $\tau^i_m$.

**Proof.** Fix a firm $i$ and a market $m$, and suppose by way of contradiction there existed a $\Pi^*$ such that $\Pi^i_m \leq \Pi^*$ and $\Pi^i_m > \Pi^*$ each occur with strictly positive probability under $\tau_m$. Then there exist actions $a^i, \tilde{a}^i \in A^i$ such that $\Pi^i_m(a^i_m, \tau^{-i}_m) \leq \Pi^*$ and $\Pi^i_m(\tilde{a}^i_m, \tau^{-i}_m) > \Pi^*$ and $a^i, \tilde{a}^i$ are each profit-maximizing for firm $i$ in period 0 under $\sigma$. Further, $a^i$ and $\tilde{a}^i$ may be chosen to lie along the equilibrium path for firm $i$ in period 0 under $\sigma$. But then because the equilibrium path is rectangular, $\tilde{a}^i \equiv (\tilde{a}^i_m, a^i_{-m})$ lies on the equilibrium path as well for $i$. In a stationary equilibrium all actions lying on the equilibrium path yield the same expected continuation payoff. But $\tilde{a}^i$ yields a strictly higher stage-game payoff for $i$ than $a^i$ by construction, thus $a^i$ cannot be profit-maximizing for $i$. This is the desired contradiction. \[\square\]
This lemma gives us a powerful accounting identity for characterizing possible equilibrium strategies: In each market, every firm must receive the same profits for all on-path actions. It is used to prove the following pair of lemmas, which establish that 1) any stationary equilibrium may be replaced by another with independent randomization across markets on-path, and 2) any SPNE featuring the same play in each period and independent randomization across markets on-path can be adapted to produce a stationary equilibrium. Therefore, without loss of generality, we impose stationarity by assuming that firms use the same stage-game strategy profile in all periods and randomize independently across markets on-path.

**Lemma C.11.** Let $\sigma$ be a stationary equilibrium with on-path play $\tau$. Then there exists another stationary equilibrium $\sigma'$ with on-path play $\tau' = \prod_{i=1}^{N+1} \prod_{m=1}^{N+1} \tau^i_m$, where $\tau^i_m$ is the marginal distribution of $\tau^i$ in market $m$. Both $\sigma$ and $\sigma'$ yield the same expected lifetime profits to all firms.

**Proof.** Let $\Pi^i_m$ be the constants whose existence is assured by Lemma C.10. Define $A^i \equiv \{a \in A : \Pi^i_m(a^i_m, \tau^i_m) = \Pi^i_m \forall i, m\}$, and let $\bar{H} \equiv \prod_{t=0}^{\infty} A^i$. Note that $A^i$ is a Cartesian product of the sets $A^i(i, m) \equiv \{a^i_m \in A^i_m : \Pi^i_m(a^i_m, \tau^i_m) = \Pi^i_m\} \subset A^i_m$. Thus $\bar{H}$ is a rectangular set of complete histories, as for each $t$ and $h \in \bar{H}$ we have $A^i(h) = A^i$.

Construct $\sigma'$ by setting $\sigma'(h) = \tau'$ for every $t$ and $h \in \bar{H}$. Also, for $h$ in some $\bar{H}$ and $a \in A$ such that $a^i \notin \prod_{m=1}^{N+1} A^i(i, m)$ for at least two firms $i$, set $\sigma'|_{(h,a)} = \sigma$. Finally, consider $h$ in some $\bar{H}$ and $a \in A$ such that $a^i \notin \prod_{m=1}^{N+1} A^i(i, m)$ for a single firm $i$, while $a^{-i} \in \prod_{j \neq i} \prod_{m=1}^{N+1} A^j(j, m)$. Let $\sigma(i)$ be an SPNE yielding minimal lifetime profits for $i$ among all SPNEs. Set $\sigma'|_{(h,a)} = \sigma(i)$.

We claim that $\sigma'$ is the desired stationary equilibrium. We first demonstrate that $\bar{H}$ is an equilibrium path for $\sigma'$. Note that for all $i$ and $m$, $\tau^i_m = \tau^i_m$ by construction, hence for all $a^i_m \in A^i_m$, we have $\Pi^i_m(a^i_m, \tau^i_m) = \Pi^i_m(a^i_m, \tau^i_m)$. Thus by Lemma C.10 $a^i_m \in A^i(i, m)$ with probability 1 under $\tau^i_m$. We conclude that $a \in A^i$ with probability 1 under $\tau'$. It follows that $\bar{H}$ is an equilibrium path, which is rectangular by construction. Obviously on-path play is $\tau'$ along the equilibrium path.

It remains to check that $\sigma'$ is indeed an SPNE. Following a deviation by one or more firms, continuation play is an SPNE by construction. So we need only confirm that there are no profitable unilateral deviations along the equilibrium path. Suppose there existed $i$ and $a^i \notin \prod_{m=1}^{N+1} A^i(i, m)$ such that $(1 - \delta)\Pi^i(a^i, \tau'^{-i}) + \delta U^i(\sigma(i)) > \Pi^i(\tau')$. Because $\Pi^i(a^i, \tau'^{-i}) =$

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39If such an SPNE does not exist because the equilibrium set is not closed, the following argument goes through by choosing an SPNE yielding profits sufficiently close to the infimum.

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$\Pi^i(a^i, \tau^{-i})$ by summing the market-by-market equivalences derived earlier, it must also be the case that $(1 - \delta)\Pi^i(a^i, \tau^{-i}) + \delta U^i(\sigma(i)) > \Pi^i(\tau)$. But then $\sigma$ is not an equilibrium, as no matter the continuation following play of $a^i$ in period 0 by firm $i$ under $\sigma$, firm $i$ has a profitable one-shot deviation to $a^i$. So no such $a^i$ exists, ruling out profitable deviations on the equilibrium path.

The final claim of the theorem follows simply from noticing that lifetime profits to firm $i$ under $\sigma$ and $\sigma'$ are $\Pi^i(\tau)$ and $\Pi^i(\tau')$, respectively, and recalling that $\Pi^i(\tau) = \Pi^i(\tau')$. \hfill \Box

**Lemma C.12.** Fix an SPNE $\sigma$. Suppose there exists an equilibrium path $H^*$ and mixed strategies $\tau^i_m \in \Delta(A^i_m)$ such that for all $h \in H^*$ and $t$, $\sigma(h^t) = \prod_{i=1}^{N+1} \prod_{m=1}^{N+1} \tau^i_m$. Then there exists a stationary equilibrium $\sigma'$ with on-path play $\prod_{i=1}^{N+1} \prod_{m=1}^{N+1} \tau^i_m$.

**Proof.** Let $\tau = \prod_{m=1}^{N+1} \tau^i_m$. We first establish the existence of constants $\Pi^i_m$ for each $i$ and $m$ such that $\Pi^i_m(a^i_m, \tau^{-i}_m) = \Pi^i_m$ w.p. 1 under $\tau^i_m$. Suppose by way of contradiction that for some $i$ and $m$, there exists a profit level $\Pi^*$ such that $\Pi^i_m(a^i_m, \tau^{-i}_m) \leq \Pi^*$ occurs with probability strictly between 0 and 1 under $\tau^i_m$. Then given the independence of $i$'s actions across markets, the event $E^i = \{\Pi^i(a^i, \tau^{-i}) \leq \Pi^* + \Pi^i_m(\tau^{-i}_m)\}$ must occur with probability strictly between 0 and 1 under $\tau^i$. To see this, note first that

$$\{\Pi^i_m(a^i_m, \tau^{-i}_m) \leq \Pi^* \land \Pi^i_m(a^i_m, \tau^{-i}_m) \leq \Pi^i_m(\tau^{-i}_m)\} \subset E^i,$$

and by independence the probability of the event on the lhs is equal to

$$\mathbb{P}^{\tau^i_m}\{\Pi^i_m(a^i_m, \tau^{-i}_m) \leq \Pi^*\}\mathbb{P}^{\tau^{-i}_m}\{\Pi^i_m(a^i_m, \tau^{-i}_m) \leq \Pi^i_m(\tau^{-i}_m)\},$$

with both terms strictly positive. So $\mathbb{P}^{\tau^i}(E^i) > 0$. Similarly, letting $\overline{E}^i$ be the complementary event to $E^i$, we have

$$\{\Pi^i_m(a^i_m, \tau^{-i}_m) > \Pi^* \land \Pi^i_m(a^i_m, \tau^{-i}_m) \geq \Pi^i_m(\tau^{-i}_m)\} \subset \overline{E}^i,$$

and again the probability of the set on the lhs is strictly positive. So $\mathbb{P}^{\tau^i}(\overline{E}^i) > 0$, or $\mathbb{P}^{\tau^i}(E^i) < 1$.

Now, note that along the equilibrium path $\tau$ is played in every period, thus w.p. 1 under $\sigma$ each firm $j$'s continuation payoff after period 0 must be $\Pi^j(\tau)$. In particular, firm $i$'s expected continuation payoff given $\tau^{-i}$ must be $\Pi^i(\tau)$ w.p. 1 under $\tau^i$. But then $i$'s expected lifetime payoff from playing actions in $E^i$ is strictly lower than from playing actions in $\overline{E}^i$. This is a contradiction of the optimality of $i$'s strategy in period 0. So we conclude that the desired constants $\Pi^i_m$ exist for all $i$ and $m$. 

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Now define \( A^\dagger(i, m) \equiv \{ a^i_m \in A^i_m : \Pi^i_m(a^i_m, \tau^{-i}) = \Pi^\dagger_m \} \) for each firm \( i \) and market \( m \), and let \( A^\dagger = \prod_{i=1}^{N+1} \prod_{m=1}^{N+1} A^\dagger(i, m) \). Consider the rectangular set of complete histories \( \tilde{H} = \prod_{t=0}^{\infty} A^\dagger \). Construct a repeated game strategy profile \( \sigma' \) as follows. For every \( t \) and \( h \in \tilde{H} \), set \( \sigma'(h) = \tau \). Also, for each \( a \in A \) such that \( a^i \notin \prod_{m=1}^{N+1} A^\dagger(i, m) \) for at least two firms, set \( \sigma'_{|_{(h,a)}} = \sigma \). Finally, for each \( a \in A \) such that \( a^i \notin \prod_{m=1}^{N+1} A^\dagger(i, m) \) for some firm \( i \) while \( a^{-i} \in \prod_{j \neq i}^{N+1} A^\dagger(i, m) \), set \( \sigma'_{|_{(h,a)}} = \sigma(i) \), where \( \sigma(i) \) is an SPNE yielding minimal lifetime profits for \( i \) among all SPNEs.

We claim that \( \sigma' \) is a stationary equilibrium with equilibrium path of play \( \tau \). Observe that \( \tilde{H} \) is a rectangular equilibrium path for \( \sigma' \), as \( a \in A^\dagger \) w.p. 1 under \( \tau \) by definition of the \( \Pi^\dagger_m \). And \( \tau \) is the path of play under \( \sigma' \) by construction. It remains only to check that \( \sigma' \) is an SPNE. Off-path play follows an SPNE by construction, so we need only verify that there are no profitable deviations on-path. For each \( i \), all \( a^i \in \prod_{m=1}^{N+1} A^\dagger(i, m) \) are on-path; while all \( a^i \) such that \( \Pi^i(a^i, \tau^{-i}) < \Pi^i(\tau) \) yield lower immediate and continuation profits than on-path play. Finally, the unprofitability of \( a^i \) such that \( \Pi^i(a^i, \tau^{-i}) > \Pi^i(\tau) \) follows from the fact that \( \sigma \) is an equilibrium, as \( \sigma' \) provides continuation payoffs no higher than \( \sigma \) following such actions. Thus \( \sigma' \) is indeed an SPNE.

\[ \Box \]

C.3 Optimal collusion in stationary equilibria

Much of this subsection proves a series of results for a version of the repeated game with \( N + 1 \) firms and \( N + 1 \) markets, where \( N \geq 1 \) and each firm \( i \) plays the role of the home firm in market \( m = i \) and is an away firm in all other markets \( m \neq i \). We will use the notation \( \mathcal{I} = \{1, ..., N + 1\} \) to denote the set of firms in the many-firm case. The Appendix eventually specializes to the case \( N = 1 \) to prove the theorems in the main text, and then considers symmetric equilibria with \( N \geq 2 \) to prove the results of Appendix E. Appendix D completes the exposition by explicitly connecting the results here to the statements of the main text.

Several auxiliary concepts are used in the proofs. Let \( \tilde{\Pi}^M \) be the profit of a high-cost firm acting as a monopolist and setting price \( p^*_H \), which satisfies \( \Pi^M - \tilde{\Pi}^M = \Delta cD(p^*_H) > 0. \) (Note that \( \tilde{\Pi}^M \) is not the monopoly profit of the high-cost provider, as in general \( p^*_A > p^*_H \).) With this notation, define

\[
\delta^M(N) \equiv 1 - \frac{1}{N + 1} \left( \frac{N}{N + 1} \frac{\tilde{\Pi}^M}{\Pi^M} + \frac{1}{N + 1} \right)^{-1}.
\]

Note that \( \Pi^M > \tilde{\Pi}^M \) implies that \( \delta^M(N) < 1 - 1/(N + 1) \). Also, let \( \Pi(\delta; N) \) be the minimum
SPNE-sustainable lifetime profits with discount factor $\delta$ and $N+1$ firms.

Several concepts will be relevant mainly for the many-firm case. A key discount threshold when $N \geq 2$ is $\bar{\delta}(N) \equiv \left(1 - \frac{1}{N}\right) \left(1 + \frac{c}{\Pi^M}\right)$, which is greater than $1 - 1/N$ and may be either larger or smaller than $\delta^M(N)$. The significance of $\bar{\delta}(N)$ is discussed in Appendix E.

**Lemma C.13.** $N \geq \sqrt{1 + \Pi^M/c}$ implies $\delta(N) > \delta^M(N)$.

*Proof.* $\delta(N) = \delta^M(N)$ is equivalent to

$$1 - \frac{(N - 1)c}{\Pi^M} \leq \frac{1}{N+1} - \frac{\Delta_c D(p^*_m)}{\Pi^M}.$$  

As the rhs of this inequality is always greater than $N/(N + 1)$, a sufficient condition is

$$1 - \frac{(N - 1)c}{\Pi^M} \leq 1 - \frac{1}{N+1} \Rightarrow N \geq \sqrt{1 + \Pi^M/c}.$$

Additionally, we restrict attention to the following class of stationary equilibria:

**Definition C.1.** Fix a stationary equilibrium $\sigma$ with associated on-path stage-game strategy profile $\tau$. Then $\sigma$ is market-symmetric if, for each market $m$ and all away firms $i$ and $j$, $\tau^i_m = \tau^j_m$.

This class is weaker than the stationary equilibria studied in Appendix E, as every symmetric equilibrium is also market-symmetric, but not vice versa. Most of the results in this appendix go through in the broader class. In addition, in the two-firm case every stationary equilibrium is automatically market-symmetric. Thus any restriction to market-symmetric equilibrium has no consequences in the two-firm case. Hence focusing on market-symmetric equilibria simplifies the proof structure, as results may be proven all at once for arbitrary $N$.

**Theorem C.2.** Suppose that $\delta < \delta^M(N)$. Then if $N = 1$ or $\delta < \delta(N)$, at most one firm earns positive intra-period profits in each market in every market-symmetric stationary equilibrium.

*Proof.* We first show that the existence of multiple firms making positive profits in a given market implies a minimally profitable deviation for each such firm.

**Lemma C.14.** Fix a stationary equilibrium yielding strictly positive intra-period profits to $2 \leq K \leq N + 1$ firms $i_1, i_2, ..., i_K \in \mathcal{I}$ in market $m$. Then for each $i = i_1, ..., i_K$, stage
profits are bounded above as
\[
\Pi^i_m \leq \frac{1}{K} \left( \Pi^M + c - \Delta c D(p_A^*) 1\{i \neq m\} \right) - c
\]
and there exists a deviation in market \( m \) yielding intra-period profits of at least \( K \Pi^i_m + (K - 1)c \).

Proof. Wlog fix \( m = 1 \). (It will not matter whether the home firm is one of the firms receiving positive profits.) As a first observation, we must have \( \pi^i_k = 1 \) for all \( k = 1, \ldots, K \). For strictly positive intra-period profits in market 1 imply existence of an on-path action \( a \) involving entry which gives strictly positive profits. Failing to enter yields lower intra-period profits and a lower continuation than playing \( a \), so each firm must choose to enter w.p. 1.

Define \( p^i \equiv \sup\{ p : F_i^i(p) < 1 \} \) for \( i \in \mathcal{I} \). Then \( p^i \) is the supremum of the support of firm \( i \)'s price distribution in market 1. \( p^i < \infty \) for \( i = i_1, \ldots, i_k \) given (A1) and the fact that intra-period on-path profits are strictly positive for each such firm in market 1.

We claim that \( \overline{p}^{i_k} = \overline{p}^{i_{k'}} \) for all \( k, k' \in \{1, \ldots, K\} \). Suppose not, say \( \overline{p}^{i_1} > \max_{k=2, \ldots, K} \overline{p}^{i_k} \equiv \overline{p}^{i_{k-1}} \). Then there exists an interval \( [p_A, p_B] \subset (\overline{p}^{i_{k-1}}, \overline{p}^{i_1}] \) such that \( F_i^{i_1}([p_A, p_B]) > 0 \). In particular, there exists a \( p^* \in [p_A, p_B] \) such that \( \Pi_1^{i_1}(p^*) = \Pi_1^{i_1} \). But given \( \pi^{i_k}_1 = 1 \) and \( F_1^{i_k}(p^*) = 1 \) for all \( k \geq 2 \), we have \( \Pi_1^{i_1}(p^*) = -c < 0 \), contradicting the assumption that \( \Pi_1^{i_1} > 0 \).

Let \( \overline{p} \) be the mutual supremum of the price supports for firms \( i_1 \) through \( i_K \). We next argue that each of these firms places an atom at \( \overline{p} \). Suppose that some firm, say \( i_1 \), places no atom at \( \overline{p} \). Then each of \( i_2 \) through \( i_K \) receive intra-period profits of \( -c < 0 \) at \( \overline{p} \) given that \( \pi_1^{i_1} = 1 \). So none of these firms places an atom at \( \overline{p} \) either. But then \( \Pi_1^{i_1}(\overline{p}^-) = -c \), since for prices approaching \( \overline{p} \) firm \( i_1 \) will be underbid by one of \( i_2 \) through \( i_K \) with probability approaching 1. Then \( i_1 \)'s profits are non-positive for prices sufficiently close to \( \overline{p} \), meaning they cannot be profit-maximizing. But then for some \( \varepsilon > 0 \) the interval \( [\overline{p} - \varepsilon, \overline{p}] \) is assigned measure zero by \( F_1^{i_1} (\cdot) \), a contradiction of the definition of \( \overline{p} \). So \( i_1 \) must place an atom at \( \overline{p} \).

Now, the existence of overlapping atoms generates a profitable intra-period deviation for each firm. Consider the case of firm \( i_1 \). His equilibrium intra-period profits in market 1 are equal to his profits at \( \overline{p} \), which are bounded above as
\[
\Pi_1^{i_1} \leq \prod_{j \notin \{i_1, \ldots, i_K\}} (1 - \pi_j F_1^{i_j}(\overline{p}^-)) \prod_{k=2, \ldots, K} \Delta F_1^{i_k}(\overline{p}) \left[ \frac{1}{K} D(\overline{p})(\overline{p} - c_{i_1}) \right] - c.
\]
(The inequality will be strict if some other firm also places an atom at \( \overline{p} \).) This bound is
loosest if no firms enter below $\bar{p}$ and the price ceiling is set to the profit-maximizing value for a given firm. This yields the upper bound in the lemma statement.

Meanwhile, by deviating to just under $\bar{p}$ he can obtain profits of at least

$$\tilde{\Pi}_i^1 = \left[ \prod_{j \in \{i_1, \ldots, i_K\}} (1 - \pi_j F_i^j(\bar{p}^-)) \right] \left[ \prod_{k=2, \ldots, K} \Delta F_i^k(\bar{p}) \right] D(\bar{p})(\bar{p} - c_{i_1}) - c \geq K\Pi_i^1 + (K - 1)c.$$ 

This is the deviation claimed in the lemma statement.\qed

Now, consider an arbitrary market-symmetric stationary equilibrium. Assume that in some market, at least two firms earn positive profits.

The $N = 1$ case: Wlog suppose both firms make positive profits in market 1. There are two possibilities: either both firms earn positive profits in market 2 as well, or some firm $i$ earns non-positive profits. In the former case, firm $i$ has a deviation worth $2\Pi_{i_m}^i + c$ in each market, which when summed imply the IC constraint

$$\Pi_i^i \geq (1 - \delta)(2\Pi_i^i + 2c) + \delta \Pi(\delta; N) \geq (1 - \delta)(2\Pi_i^i + 2c).$$

In the latter case, firm $i$ has a deviation worth at least $2\Pi_1^i + c$ in market 1 and $2\Pi_2^i$ in market 2 (the latter following trivially from $\Pi_2^i \leq 0$), hence the IC constraint

$$\Pi_i^i \geq (1 - \delta)(2\Pi_i^i + c) + \delta \Pi(\delta; N) \geq (1 - \delta)(2\Pi_i^i + c)$$

holds. In either case incentive-compatibility demands $\Pi_i^i \geq (1 - \delta)2\Pi_i^i$. Now, $\Pi_i^i > 0$, else $i$ could deviate and obtain positive profits by exiting market 2 given $\Pi_1^i > 0$. So this inequality implies $\delta \geq 1/2 > \delta^M(1)$.

The $N \geq 2$ case:

Lemma C.15. Suppose $N \geq 2$. Fix a stationary equilibrium in which for some market, at least two firms earn positive intra-period profits in each period on-path. Then there exists a firm $i \in \mathcal{I}$ and an integer $n \in \{1, \ldots, N, N + 1\}$ for which the IC constraint

$$\Pi_i^i \geq (1 - \delta)(N\Pi_i^i + n(N - 1)c)$$

holds, and $0 < \Pi_i^i \leq \frac{n}{N}(\Pi^M - (N - 1)c)$.

Proof. Suppose first that the home firm makes strictly positive profits in every market. Assume multiple firms make positive profits in market 1. Then by market symmetry all
$N + 1$ firms make positive profits in that market, and firm 1 therefore has a deviation worth $(N + 1)\Pi_1 + Nc$ in that market by Lemma C.14. Additionally, in every other market $m \geq 2$ he either makes non-positive profits, and so trivially has a deviation worth $(N + 1)\Pi_1$; or else he makes positive profits and has a deviation worth $(N + 1)\Pi_m + Nc$ in that market as well.

Summing the profits for firm 1 from deviating across all markets, assuming that 1 makes positive profits in $n - 1$ markets other than his own, we obtain the IC constraint

$$\Pi^1 \geq (1 - \delta)((N + 1)\Pi^1 + nNc).$$

As $\Pi^1 > 0$ (else he could deviate by withdrawing from all markets $i \geq 2$ to make positive profits), this IC constraint implies the desired one.

Further, the previous lemma implies that his profits in each positive-profit market $m$ are at most

$$\frac{1}{N + 1}(\Pi^M + c - \Delta D(p^*_A)1\{m \geq 2\}) - c < \frac{\Pi^M}{N + 1} - \frac{N}{N + 1}c.$$  

Hence by making positive profits in $n$ markets, he can make no more than $\frac{n}{N + 1}(\Pi^M - Nc)$, implying the desired bound given that $N/(N + 1) > (N - 1)/N$.

Now suppose that the home firm makes non-positive profits in some market, say market 1. Then firm 1 must make positive profits in some other market $m \geq 2$, else he could deviate to achieve positive profits by undercutting in whatever market yields positive profits to multiple firms. In market $m$ all other away firms also make positive profits by market-symmetry, hence 1 has a deviation worth $N\Pi^1 + (N - 1)c$. The same reasoning holds in all other markets in which he makes positive profits. And when he makes non-positive profits, he trivially has a deviation worth $N$ times his profits. Summing these deviations, assuming 1 makes positive profits in $n$ markets, we obtain the IC constraint

$$\Pi^1 \geq (1 - \delta)(N\Pi^1 + n(N - 1)c),$$

which is the desired one. Reasoning as in the previous case shows that he can make no more than $\frac{n}{N}(\Pi^M - (N - 1)c)$. \qed

By the lemma just proven, for two or more firms to make positive profits in some market there must be some form $i$ and integer $n$ between 1 and $N + 1$ such that

$$\delta \geq 1 - \frac{\Pi^i}{N\Pi^i + n(N - 1)c}.$$
Further $\Pi^i \leq \frac{n}{N}(\Pi^M - (N - 1)c)$, so that

$$\delta \geq 1 - \frac{\frac{n}{N}(\Pi^M - (N - 1)c)}{n(\Pi^M - (N - 1)c) + n(N - 1)c} = \delta(N).$$

\[\square\]

**Theorem C.3.** Suppose $\delta < \delta^M(N)$ and $N = 1$ or $\delta < \delta(N)$. Then payoffs $(\Pi^1, ..., \Pi^{N+1})$ of any market-symmetric stationary equilibrium must satisfy $\Pi^i \in [0, \Pi^M]$ and

$$\Pi^i \geq (1 - \delta) \left( \sum_{j=1}^{N+1} \Pi^j - \Delta c \sum_{j \neq i} D(p^\star(\Pi^j)) \right) + \delta \Pi(\delta; N),$$

for all $i \in I$, where $p^\star(\Pi)$ is the unique solution in $[p^\star_H, p^\star_H]$ to $D(p)(p - c_H) - c = \Pi$ for $\Pi \in [0, \Pi^M]$.

Further, any such equilibrium yields strictly positive profits to each firm in at most one market. When $N \geq 2$, this market is the firm’s home market.

**Proof.** Assume the conditions of the theorem statement, and fix a market-symmetric stationary equilibrium with profits $(\Pi^1, ..., \Pi^{N+1})$. We must have $\Pi^i \geq 0$ for all $i$, else any firm earning negative profits could withdraw from all markets permanently as a profitable deviation.

**Lemma C.16.** Each firm $i \in I$ can receive positive profits in at most one market.

**Proof.** Suppose some firm, say firm 1, received positive profits in two or more markets. Then by the pigeonhole principle, along with the fact that at most one firm can earn positive profits in any one market, there must be some firm $i \geq 2$ which receives non-positive profits in all markets and hence non-positive lifetime profits. But firm $i$ has a deviation yielding positive lifetime profits by undercutting the infimum of firm 1’s price support in some market in which 1 is an away firm and makes positive profits. (Such a market exists by assumption.) So no firm can receive positive profits in multiple markets.

The lemma implies that each $\Pi^i \leq \Pi^M$, as this is the most any firm could make in a single market. So $p^\star(\cdot)$ is well-defined over the range of profits allowable in equilibrium. Also, when $N \geq 2$ market symmetry implies that firm $i$’s positive-profit market must be his home market, else all other away firms would also make positive profits in that market.

Let $m(i)$ be the (unique) market in which firm $i \in I$ receives positive profits. If $i$ receives positive profits in no market, choose $m(i)$ to be some market in which he receives
zero profits. \( m : \mathcal{I} \to \mathcal{I} \) can always be constructed so that it is a bijection, which we will assume in what follows.

Now consider the following deviation by firm \( i \): he follows his equilibrium strategy in market \( i \), while for each \( j \neq i \) firm \( i \) just undercuts the infimum of firm \( j \)'s price support in market \( m(j) \). Because firm \( j \) makes his entire profits \( \Pi^j \) in market \( m(j) \), his price support infimum must be at least \( p^*(\Pi^j) \) there. Hence, since \( D(p^*(\cdot)) \) is a decreasing function, \( i \) makes at least \( \Pi^j - \Delta c D(p^*(\Pi^j)) \) through undercutting in market \( m(j) \). (He makes more from undercutting if the infimum of \( j \)'s price support is larger than \( p^*(\Pi^j) \), or if \( m(j) \) is not \( j \)'s home market, but we will not need to make use of this fact.)

Summing the intra-period profits from this deviation yields

\[
\sum_{j=1}^{N+1} \Pi^j - \Delta c \sum_{j \neq i} D(p^*(\Pi^j)).
\]

Thus, to deter a profitable deviation the IC constraint stated in the problem statement must hold.

\[ \square \]

**Theorem C.4.** Fix a profit vector \((\Pi^1, ..., \Pi^{N+1}) \in [\Delta c D(p_A), \Pi^M]^{N+1}\) satisfying

\[
\Pi^i \geq (1 - \delta) \left( \sum_{j=1}^{N+1} \Pi^j - \Delta c \sum_{j \neq i} D(p^*(\Pi^j)) \right) + \delta \Pi(\delta; N)
\]

for all \( i \in \mathcal{I} \). Then there exists a market-symmetric stationary equilibrium supporting this profit vector with the following properties:

1. If \( \Pi^m = \Pi^{m'} \) for markets \( m \) and \( m' \), then the home and away firms' strategies are the same in both markets.

2. The home firm in market \( m \) enters with probability 1, while all away firms enter with a probability that is strictly between zero and 1 and decreasing in \( \Pi^m \).

3. The home firm earns profits \( \Pi^m \) in market \( m \), while all away firms make zero profits.

4. Each firm posts prices only in \([p^*(\Pi^m), p^*_H]\) in market \( m \), and firms’ price distributions have full support on \((p^*(\Pi^m), p^*_H)\).

5. If \( \Pi^m > \Delta c D(p_A) \), the home firm in market \( m \) plays \( p^*(\Pi^m) \) with some strictly positive probability, which is increasing in \( \Pi^m \).

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6. Each market where $\Pi^m < \Pi^M$ is captured by an away firm with some strictly positive probability, which is strictly decreasing in $\Pi^m$ when $\Pi^m \geq \frac{N-1}{2N-1} \Pi^M - \frac{N}{2N-1} c$.  

7. Any unilateral deviation by an away firm to a price at or below $p^*(\Pi^m)$ in any market $m$ is punished by a continuation payoff of $\Pi(\delta; N)$ to that firm.

Proof. We construct an equilibrium analogous to the stage-game, with a higher price floor. Fix a market $m$. If $\Pi^m = \Pi^M$, then the home firm plays $p^*_H$ w.p. 1 while the away firms do not enter. So assume $\Pi^m < \Pi^M$. The home firm mixes so as to make each away firm indifferent over prices in $(p^*(\Pi^m), p^*_H)$. The indifference condition, assuming symmetric play by all away firms, is

$$0 = D(p)(p - c_A)(1 - \pi_A F_A(p))^{N-1}(1 - F_H(p)) - c.$$  

Meanwhile the away firms play to make the home firm indifferent over all prices in the same interval, yielding

$$\Pi^m = D(p)(p - c_H)(1 - \pi_A F_A(p))^N - c.$$  

Hence

$$\pi_A F_A(p) = 1 - \left( \frac{\Pi^m + c}{D(p)(p - c_H)} \right)^{1/N},$$

which when substituted into the away firm’s indifference condition allows us to solve for $F_H(p)$ on $(p^*(\Pi^m), p^*_H)$:

$$F_H(p) = 1 - \frac{c}{D(p)(p - c_A)} \left( \frac{\Pi^m + c}{D(p)(p - c_H)} \right)^{1-1/N}.$$  

In the limit as $p \searrow p^*(\Pi^m)$, we have

$$F_H(p^*(\Pi^m)+) = 1 - \frac{c}{D(p^*(\Pi^m))(p^*(\Pi^m) - c_A)} = 1 - \frac{c}{\Pi^m + c - \Delta c D(p^*(\Pi^m))}.$$  

Then whenever $\Pi^m > \Delta c D(p_A^*) \geq \Delta c D(p^*(\Pi^m))$, this expression is strictly positive. So the home firm places an atom at $p^*(\Pi^m)$ of size

$$\Delta F_H(p^*(\Pi^m)) = 1 - \frac{c}{\Pi^m + c - \Delta c D(p^*(\Pi^m))}.$$  

Note that $p^*(\cdot)$ is increasing while $D(\cdot)$ is decreasing, and hence $\Pi^m - \Delta c D(p^*(\Pi^m))$ is strictly increasing in $\Pi^m$. Thus $\Delta F_H(p^*(\Pi^m))$ is strictly increasing in $\Pi^m$.  


Further, as \( p \rightarrow p^*_h \) we have

\[
F_H(p^*_H -) = 1 - \frac{c}{\Pi^M + c - \Delta c D(p^*_H)} \left( \frac{\Pi^m + c}{\Pi^M + c} \right)^{1-1/N},
\]

which is strictly less than 1. So the home firm places another atom at \( p^*_H \) of size

\[
\Delta F_H(p^*_H) = \frac{c}{\Pi^M + c - \Delta c D(p^*_H)} \left( \frac{\Pi^m + c}{\Pi^M + c} \right)^{1-1/N}.
\]

Finally, we complete our characterization of the strategy played by all away firms. Note that \( \pi_A F_A(p^*(\Pi^m)+) = 0 \), so the away firm’s price distribution is continuous at its lower end. On the other hand,

\[
\pi_A F_A(p^*_H) = 1 - \left( \frac{\Pi^m + c}{\Pi^M + c} \right)^{1/N},
\]

which is strictly positive. In order to avoid the away firm placing an atom at \( p^*_H \) (which would not be optimal given the atom placed there by the home firm), we set

\[
\pi_A = 1 - \left( \frac{\Pi^m + c}{\Pi^M + c} \right)^{1/N},
\]

which lies strictly between zero and 1 and is decreasing in \( \Pi^m \).

Having exhibited a strategy profile in each market, we must show that it is supportable in equilibrium. The home firm has no profitable deviation in each market, while the away firms’ most profitable deviation is to just below \( p^*(\Pi^m) \), yielding profits \( \Pi^m - \Delta c D(p^*(\Pi^m)) \). So each firm has \( N \) profitable deviations in each market other than his own, and our strategies are an equilibrium iff the IC constraint stated in the theorem holds.

The fact that business-stealing occurs is immediate from the fact that the home and away firms play price distributions with overlapping support. To evaluate when it is decreasing in \( \Pi^m \), we first compute the probability of business-stealing. The probability that some firm undercutts a given price \( p \) is the complement of the probability that no firm does, which is \((1 - \pi_A F_A(p))^N\). Then, taking account of the atom by the home firm at \( p^*_H \), we have

\[
\Pr\{\text{Business stealing}\} = (1 - (1 - \pi_A)^N) \Delta F_H(p^*_H) + \int_{(p^*(\Pi^m), p^*_H)} (1 - (1 - \pi_A F_A(p))^N) dF_H(p).
\]

(The open-set notation for the limits of integration indicates that the atoms at the top and bottom of the support of \( F_H(\cdot) \) are excluded from the integral.)
Let $\tilde{F}_H(p) \equiv \frac{F_H(p)}{(\Pi^m + c)^{1/N}}$. Note that $\tilde{F}_H(p)$ has the property that its increments are independent of $\Pi^m$. Now, inserting the expression for $\pi_A F_A(p)$ derived earlier, we obtain

$$\mathbb{P}\{\text{Business stealing}\} = (\Pi^m + c)^{1-1/N} \left( 1 - \frac{\Pi^m + c}{\Pi^M + c} \right) \Delta \tilde{F}_H(p^*_H) + \int_{(p^*(\Pi^m), p^*_H)} (\Pi^m + c)^{1-1/N} \left( 1 - \frac{\Pi^m + c}{\Pi^M + c} \right) d\tilde{F}_H(p).$$

Now we differentiate wrt $\Pi^m$, assuming that both $\tilde{F}_H$ and $p^*$ are differentiable. (We will show later that this assumption is innocuous.) Differentiating wrt $\Pi^m$ in the limit of integration yields no contribution, as the integrand is zero at the lower limit. The remaining terms are then

$$\frac{d}{d\Pi^m} \mathbb{P}\{\text{Business stealing}\} = (\Pi^m + c)^{-1/N} \left[ \left( (1 - 1/N) - (2 - 1/N) \frac{\Pi^m + c}{\Pi^M + c} \right) \Delta \tilde{F}_H(p^*_H) \right. \left. + \int_{(p^*(\Pi^m), p^*_H)} \left( (1 - 1/N) - (2 - 1/N) \frac{\Pi^m + c}{D(p)(p - c_H)} \right) d\tilde{F}_H(p) \right].$$

(Recall that the increments of $\tilde{F}_H$ are not a function of $\Pi^m$.) Both the leading term and the integrand are negative over the entire integration range when

$$\left( (1 - 1/N) - (2 - 1/N) \frac{\Pi^m + c}{\Pi^M + c} \right) \leq 0,$$

i.e. when

$$\Pi^m \geq \frac{1 - 1/N (\Pi^M + c) - c}{2 - 1/N} = \frac{N - 1}{2N - 1} \Pi^M - \frac{N}{2N - 1} c.$$

Recall that we had assumed sufficient regularity to be able to differentiate wrt the lower limit of integration. In the general case, the increase in business stealing for small increase in $\Pi^m$ will be less than if we had assumed $p^*(\Pi^m)$ to be fixed, as the integrand is non-negative and the integrator is increasing. Thus, an upper bound on the first-order change in business-stealing is the one just derived. Whenever that bound is negative, the change in business-stealing must also be negative for sufficiently small changes in $\Pi^m$. \hfill \Box

**Theorem C.5.** Suppose $N = 1$ and $\delta < \delta^M(1)$. Among profit vectors $(\Pi^1, \Pi^2)$ satisfying the
IC constraints

\[ \Pi^i \geq (1 - \delta) \left( \Pi^1 + \Pi^2 - \Delta cD(p^*(\Pi^{-i})) \right) + \delta \Pi(\delta; 1), \quad i = 1, 2 \]

there is a unique profit vector \((\Pi^*, \Pi^*)\) simultaneously maximizing both \(\Pi^1\) and \(\Pi^2\), where \(\Pi^*\) is the unique solution to

\[ \Pi = (1 - \delta)(2\Pi - \Delta cD(p^*(\Pi))) + \delta \Pi(\delta; 1). \]

Further, \(\Pi^* > \Pi^C\) iff \(\Pi(\delta; 1) < \Pi^C\), and \(\Pi^*\) is strictly increasing in \(\delta\) whenever \(\Pi(\cdot; 1)\) is non-increasing in \(\delta\).

Proof. Fix a profit vector \((\Pi^1, \Pi^2)\) satisfying the IC constraints. Suppose \(\Pi^1\) and \(\Pi^2\) are not equal, say \(\Pi^1 < \Pi^2\). We claim there exists a Pareto-superior profit vector satisfying the IC constraints.

Note that for firm 2, \(D(p^*(\Pi)) \geq D(p^*(\Pi^2))\) and hence the rhs of the IC constraint is smaller for each firm \(i \geq 2\) than it is for firm 1. But the lhs is strictly larger for firm 2, so firm 2’s IC constraint is slack.

Now consider raising \(\Pi^1\) by a small amount. This loosens 1’s IC constraint, as the lhs rises by more than the rhs. Meanwhile, given continuity of \(D(\cdot)\) and \(p^*(\cdot)\) (the latter of which is easy to prove), a small change in \(\Pi^1\) will not violate the slack IC constraints firm 2. Hence there exists another payoff profile, giving slightly more profits to firm 1, which satisfies the IC constraints.

We conclude that any optimal payoff profile satisfying the IC constraints gives equal payoffs to both firms. Such a profile is subject to the single IC constraint

\[ \Pi \geq (1 - \delta)(2\Pi - \Delta cD(p^*(\Pi))) + \delta \Pi(\delta; 1). \]

As \(\delta < \delta^M(1) < 1/2\), raising \(\Pi\) increases the rhs more quickly than the lhs. And at \(\Pi^C\) the rhs is at most \(\Pi^C\) given \(\Pi(\delta; 1) \leq \Pi^C\), while at \(\Pi^M\) the rhs is at least

\[ (1 - \delta)(2\Pi^M - \Delta cD(p^*_H)) = (1 - \delta)(\Pi^M + \tilde{\Pi}^M) > \Pi^M \]

given \(\delta < \delta^M(1)\). Hence, given continuity of the rhs, there exists a unique \(\Pi \in [\Pi^C, \Pi^M]\) for which the IC constraint just binds. This is the largest possible payoff satisfying the IC constraint.
Finally, note that
\[ 2\Pi - \Delta cD(p^*(\Pi)) \geq 2\Pi^C - \Delta cD(p^*(\Pi^C)) = \Pi^C, \]
with equality when \( \Pi = \Pi^C \). Thus \( \Pi^* > \Pi^C \) iff \( \Pi(\delta; 1) < \Pi^C \). And since raising \( \delta \) lowers the rhs of the IC constraint whenever \( \Pi > \Pi^C \) and \( \Pi(\delta; 1) \) is decreasing in \( \delta \), we conclude that \( \Pi^* \) is increasing in \( \delta \) whenever the latter condition holds.

\begin{proof}
Any symmetric stationary equilibrium payoff profile satisfying the IC constraints is subject to the single IC constraint
\[ \Pi \geq (1 - \delta)((N + 1)\Pi - \Delta cND(p^*(\Pi))) + \delta \Pi(\delta; N), \quad i \in \mathcal{I} \]
there is a unique profile \((\Pi^*, ..., \Pi^*)\) simultaneously maximizing both firms’ payoffs, where \( \Pi^* \) is the unique solution to
\[ \Pi = (1 - \delta)((N + 1)\Pi - \Delta cND(p^*(\Pi))) + \delta \Pi(\delta; N). \]
Further, \( \Pi^* > \Pi^C \) iff \( \Pi(\delta; N) < \Pi^C \), and \( \Pi^* \) is strictly increasing in \( \delta \) whenever \( \Pi(\cdot; N) \) is non-increasing in \( \delta \).

\begin{assertion}
Suppose \( N \geq 2 \) and \( \delta < \delta^M(N) \). Among symmetric stationary equilibrium payoff profiles \((\Pi, ..., \Pi)\) satisfying
\[ \Pi \geq (1 - \delta)((N + 1)\Pi - \Delta cND(p^*(\Pi))) + \delta \Pi(\delta; N), \quad i \in \mathcal{I} \]
there is a unique profile \((\Pi^*, ..., \Pi^*)\) simultaneously maximizing both firms’ payoffs, where \( \Pi^* \) is the unique solution to
\[ \Pi = (1 - \delta)((N + 1)\Pi - \Delta cND(p^*(\Pi))) + \delta \Pi(\delta; N). \]
Further, \( \Pi^* > \Pi^C \) iff \( \Pi(\delta; N) < \Pi^C \), and \( \Pi^* \) is strictly increasing in \( \delta \) whenever \( \Pi(\cdot; N) \) is non-increasing in \( \delta \).
\end{assertion}

As \( \delta < \delta^M(N) < 1 - 1/(N + 1) \), raising \( \Pi \) increases the rhs more quickly than the lhs. And at \( \Pi^C \) the rhs is at most \( \Pi^C \) given \( \Pi(\delta; 1) \leq \Pi^C \), while at \( \Pi^M \) the rhs is at least
\[ (1 - \delta)((N + 1)\Pi^M - \Delta cND(p^*_H)) = (1 - \delta)(\Pi^M + N\Pi^M) > \Pi^M \]
given \( \delta < \delta^M(N) \). Hence, given continuity of the rhs, there exists a unique \( \Pi \in [\Pi^C, \Pi^M] \) for which the IC constraint just binds. This is the largest possible payoff satisfying the IC constraint.

Finally, note that
\[ (N + 1)\Pi - \Delta cND(p^*(\Pi)) \geq (N + 1)\Pi^C - \Delta cND(p^*(\Pi^C)) = \Pi^C, \]
with equality when \( \Pi = \Pi^C \). Thus \( \Pi^* > \Pi^C \) iff \( \Pi(\delta; N) < \Pi^C \). And since raising \( \delta \) lowers the
rhs of the IC constraint whenever $\Pi > \Pi^C$ and $\Pi(\delta; N)$ is decreasing in $\delta$, we conclude that $\Pi^*$ is increasing in $\delta$ whenever the latter condition holds. \qed\\

**Theorem C.7.** Suppose that

$$\delta \geq \left(1 + \frac{1}{N} + \frac{\Pi^M - \tilde{\Pi}^M}{c}\right)^{-1}.$$  

Then for each firm $m$ there exists an SPNE supporting lifetime profits of 0 for firm $m$, so that $\Pi(\delta; N) = 0$.

**Proof.** We explicitly construct such an equilibrium strategy profile. Wlog let $m = 1$. The equilibrium consists of two phases: “punishment” and “cooperation.” In the punishment phase, firm 1 enters his home market and prices at some $p^{PW} \leq \tilde{p}_H$, while in market 2 he enters and mixes with any distribution assigning measure 1 to $(p^{PW}, \tilde{p}_H]$. Firm 2 plays symmetrically in the two markets. All other firms stay out of markets 1 and 2, and in all remaining markets all firms play the stage-game NE. In the cooperative phase firms play an SPNE yielding profits $(\Pi^*, \ldots, \Pi^*)$ characterized in Theorem C.5. Players transit from the punishment to the cooperative phase after a single stage, and stay in the cooperative phase forever. All deviations result in a reversion to the punishment phase.

We choose $p^{PW}$ so that lifetime profits for firms 1 and 2 are zero, implying

$$(1 - \delta)(D(p^{PW})(p^{PW} - c_H) - 2c) + \delta \Pi^* = 0.$$  

In order that $p^{PW} \leq \tilde{p}_H$ (crucial to preventing a profitable deviation yielding positive lifetime profits to firms 1 or 2), we therefore require

$$\Pi^* \geq \frac{1 - \delta}{\delta}c.$$  

Now, $\Pi^*$ is characterized by

$$\Pi^* = (1 - \delta)((N + 1)\Pi^* - \Delta c ND(p^*(\Pi^*)))$$

whenever the strategy profile outlined is indeed an equilibrium. Re-arranging yields

$$\Pi^* = (1 - \delta)((N + 1)\Pi^* - \Delta c ND(p^*(\Pi^*))).$$
Combining this expression with the lower bound on $\Pi^*$ derived earlier yields the bound
\[
\delta \geq \left(1 + \frac{1}{N} + \frac{\Delta cD(p^*(\Pi^*))}{c}\right)^{-1}.
\]

Our strategy profile satisfies $p^{PW} \leq p_H$ iff this inequality holds. Now, note that $\Delta cD(p^*(\Pi^*)) \geq \Delta cD(p^*_H) = \Pi^M - \tilde{\Pi}^M$, so the inequality in the problem statement implies this one.

We complete the proof by checking that no firm has any profitable deviations. By construction no such deviations exist in the cooperation phase. As for the punishment phase, no firm has any profitable deviation in markets 3 through $N + 1$ because the stage-game NE is played there. As for markets 1 and 2, firms 3 through $N + 1$ could never win the market at a profitable price by entering, so they have no incentive to enter. As for firms 1 and 2, given $p^{PW} \leq p_H$ their most profitable deviation is to exit both markets at once, yielding 0 in the current stage and a punishment continuation of 0. This is precisely the same as their lifetime payoffs from playing their equilibrium strategies, so they have no profitable deviations.

We conclude that our proposed strategy profile is indeed an equilibrium whenever $p^{PW} \leq p_H$, and in particular whenever the inequality in the theorem statement holds.

**Theorem C.8.** Suppose $N \geq 1 + \sqrt{\frac{\Pi^M - \tilde{\Pi}^M}{c}}$. Then $\bar{\delta}(N) > \bar{\delta}(N)$.

**Proof.** As $\bar{\delta}(N) > 1 - 1/N$, a sufficient condition for $\bar{\delta}(N) > \bar{\delta}(N)$ is
\[
1 - \frac{1}{N} \geq \left(1 + \frac{1}{N} + \frac{\Pi^M - \tilde{\Pi}^M}{c}\right)^{-1}.
\]
Some rearrangement yields the equivalent inequality
\[
N(N - 1) \geq \frac{\Pi^M - \tilde{\Pi}^M}{c},
\]
which is in turn implied by
\[
(N - 1)^2 \geq \frac{\Pi^M - \tilde{\Pi}^M}{c}.
\]
Solving for $N$ yields the inequality in the theorem statement.

For the following pair of theorems, let $\Pi^\dagger$ be defined as the unique solution in $[\Pi^C, \Pi^M]$ to
\[
\Pi^\dagger = (1 - \delta)(N\Pi^\dagger - (N - 1)\Delta cD(p^*(\Pi^\dagger))) + \delta \Pi(\delta; N)
\]
whenever $\delta < \delta^M(N)$, with $\Pi^i = \Pi^M$ whenever $\delta \geq \delta^M(N)$. (See the proof of Theorem C.6 for existence and uniqueness of this $\Pi^i$.)

**Theorem C.9.** Fix $\delta \in (0, 1)$. If $\delta < N/(N+1)$, then $\Pi^i \to 0$ as $c_A \downarrow c_H$. If $\delta \geq N/(N+1)$, then $\Pi^i = \Pi^M$ for all $c_A > c_H$.

**Proof.** Fix $N, \delta, c,$ and $c_H$. If there exists $c_A > c_H$ such that assumptions (A1) through (A3) hold, then these assumptions continue to hold for all smaller $c_A$. We will assume that $c_A$ is sufficiently close to $c_H$ that these assumptions hold everywhere.

Note that $\delta^M(N) < N/(N+1)$. Hence if $\delta \geq N/(N+1)$, $\delta > \delta^M(N)$ and so $\Pi^i = \Pi^*$. So assume $\delta < N/(N+1)$. As $\delta^M(N) \to N/(N+1)$ as $c_A \downarrow c_H$, for $c_A$ sufficiently close to $c_H$ we have $\delta < \delta^M(N)$. Also observe that $\Pi^M$ is independent of $c_A$, while $\Pi^C = \Delta cD(p^*_A) \to 0$ as $c_A \downarrow c_H$ given $D(p) \in [0, 1]$. Then as $c_A \downarrow c_H$, $\Delta cD(p^*(\Pi^*)) \to 0$ and $\Pi(\delta; N) \to 0$, the latter because $\Pi(\delta; N) \in [0, \Pi^C]$. Hence $\Pi^* \to 0$ as well. \hfill \Box

**Theorem C.10.** Fix $\delta \in (0, 1)$. Let $(F^*_H(\cdot), F^*_A(\cdot), \pi^*_A)$ be the home and away firms’ price distributions and the away firm’s entry probability, respectively, for the equilibrium characterized in Theorem C.4 with $\Pi^i = \Pi^1$ for all $i$. As $c \to 0$,

- $F^*_H(\cdot)$ converges uniformly to $1\{p \leq p_A\}$,
- $\pi_A \to 1 - (\Pi^i/\Pi^M)^{1/N}$,
- $F^*_A(\cdot)$ converges uniformly to $\frac{1 - \left(\frac{\Pi^i(p_A - c_H)}{\Pi^i/\Pi^M}\right)^{1/N}}{1 - (\Pi^i/\Pi^M)^{1/N}}$ on $[p_A, p^*_H]$.

The probability of business stealing therefore falls to zero as $c$ vanishes, and in the limit the home firm wins the market at price $p_A$ with probability 1.

**Proof.** Fix $c > 0$ such that (A1) through (A3) hold, and consider a decreasing sequence $c_n \leq c$ with limit 0. Then (A1) through (A3) continue to hold along the sequence. Supposing that $\Pi^{*,n}$ tends to some definite limit independent of the sequence chosen, the claims in the theorem statement follow by inspecting the expressions derived in Theorem C.4. So we need only show that $\Pi^{*,n}$ converges, and to the same limit for all sequences.

Suppose first that

$$\delta \geq \lim_{n \to \infty} \delta^M(N) = 1 - \frac{1}{1 + N(p^*_H - c_A)/(p^*_H - c_H)}.$$

Then as $\delta^M(N)$ is strictly decreasing in $c$, for all $n$ we have $\delta > \delta^M(N)$ and thus $\Pi^{*,n} = \Pi^M$. The limit of this sequence is therefore $D(p^*_H)(p^*_H - c)$ regardless of the sequence chosen.
On the other hand, suppose $\delta$ lies below this threshold. Then for sufficiently large $n$ we have $\delta < \delta^M(N)$. Further, Theorem 16 implies that $\Pi^n(\delta; N) = 0$ for sufficiently large $n$, given $\theta(N) \to 0$. (Note that $\Pi^M - \Pi^M = \Delta c D(p^*_H)$, which is independent of $n$.) Then $\Pi^* n$ is characterized by

$$\Pi^* n = (1 - \delta)((N + 1)\Pi^* n - \Delta c N D(p^* n(\Pi^* n))).$$

Observe that $p^* n(\cdot)$ is decreasing pointwise in $n$, as a lower price is needed to obtain given profits for a smaller fixed cost. Then $\Pi^* n$ is increasing in $n$, and as it is bounded above by $D(p^*_H)(p^*_H - c_H)$ its limit must exist, say $\Pi^* \infty$. Continuity of $D(\cdot)$ then implies that $\Pi^* \infty$ satisfies

$$\Pi^* \infty = (1 - \delta)((N + 1)\Pi^* \infty - \Delta c N D(\lim_{n \to \infty} p^* n(\Pi^* n))).$$

Now, $p^* n(\cdot)$ converges uniformly$^{40}$ to $p^* \infty(\cdot)$, which is a continuous function defined via

$$\Pi = D(p)(p - c_H).$$

An easy argument$^{41}$ then shows that $\lim_{n \to \infty} p^* n(\Pi^* n) = p^* \infty(\Pi^* \infty)$.

We conclude that $\Pi^* \infty$ exists and is defined implicitly by

$$\Pi^* \infty = (1 - \delta)((N + 1)\Pi^* \infty - \Delta c N D(p^* \infty(\Pi^* \infty))),$$

independent of the particular sequence chosen.

C.4 Beyond stationarity

The following three lemmas establish an analogy to the characterization of stationarity in terms of strategy profiles with independent mixing across markets. All proofs are essentially identical to the stationary case and are omitted. They imply that invariant equilibria may be characterized as SPNEs in which each firm $i$ plays the sequence of strategy profiles $\{\tau^i t\}_{t=0}^\infty$ on-path regardless of the history of play, and each $\tau^i t$ mixes independently across markets.

(As in the rest of this Appendix, many of these results are stated for a generalization of the

40There is a difficulty as to the domain over which convergence occurs, as higher profits are possible the lower is $c$. We take the domain to be $[0, D(p^*_H)(p^*_H - c)]$ for all $n$, and simply extend each $p^* n(\cdot)$ continuously to the top of this domain by taking it to be a constant function.

41Take a sequence of functions $f_n$ converging uniformly to a continuous function $f$ on some domain $X$, and a sequence $x_n$ converging to $x$ in $X$. Then $|f_n(x_n) - f(x)| \leq |f_n(x_n) - f(x_n)| + |f(x_n) - f(x)|$ by the triangle inequality, and the first term on the rhs converges to zero given the uniform convergence of the $f_n$, while the second term vanishes by continuity of $f$.
model to many firms and markets. The definitions of invariant and balanced equilibrium generalize in the natural way to this setting.)

**Lemma C.17.** Let $\sigma$ be an invariant equilibrium with on-path play $\{\tau^i t\}_{t=0}^{\infty}$. Then in each period $t$ and for each firm $i$ and market $m$, there exists a constant $\Pi_{m}^{i,t}$ such that $\Pi_{m}^{i}(a_{m}^{i},\tau_{m}^{t-1}) = \Pi_{m}^{i,t}$ with probability 1 under $\tau_{m}^{t,i}$.

**Lemma C.18.** Let $\sigma$ be an invariant equilibrium with on-path play $\{\tau^i t\}_{t=0}^{\infty}$. Then there exists another invariant equilibrium $\sigma'$ with on-path play $\{\tau'^{i} t\}_{t=0}^{\infty}$, where $\tau'^{i} = \prod_{m=1}^{N+1} \prod_{m=1}^{N+1} \tau_{m}^{t,i}$. Both $\sigma$ and $\sigma'$ yield the same expected lifetime profits to all firms.

**Lemma C.19.** Fix an SPNE $\sigma$. Suppose there exists an equilibrium path $H^{*}$ and mixed strategies $\tau_{m}^{i} \in \Delta(A_{m}^{i})$ such that for all $h \in H^{*}$ and $t$, $\sigma(h^{t}) = \prod_{i=1}^{N+1} \prod_{m=1}^{N+1} \tau^{t,i}$, then there exists an invariant equilibrium $\sigma'$ with on-path play $\{\tau'^{i} t\}_{t=0}^{\infty}$, where $\tau'^{i} = \prod_{m=1}^{N+1} \prod_{m=1}^{N+1} \tau_{m}^{t,i}$.

Now we are ready to characterize optimal balanced equilibria. Let $E^B$ be the set of lifetime profit vectors supportable by balanced equilibria.

**Definition C.2.** A balanced equilibrium $\sigma$ with lifetime payoffs $U = (U^{1},...,U^{N+1})$ is B-optimal if, for every balanced equilibrium $\sigma'$ with lifetime payoffs $(\tilde{U}^{1},...,\tilde{U}^{N+1}) \neq U$, $U^{i} > \tilde{U}^{i}$ for some $i$.

**Lemma C.20.** Suppose $\delta \leq 1 - 1/(N + 1)$. Let $\sigma$ be a B-optimal equilibrium $\sigma$ with lifetime payoffs $(U^{1},...,U^{N+1})$. Then there exist constants $\Pi^{1},...,\Pi^{N+1} \in [\Pi^{C},\Pi^{M}]$ and $(\tilde{U}^{1},...,\tilde{U}^{N+1}) \in E^B$ such that for each $i$, $\Pi^{i}$ and $\tilde{U}^{i}$ are firm $i$’s first-period expected stage and continuation profits, respectively, so that $U^{i} = (1 - \delta)\Pi^{i} + \delta \tilde{U}^{i}$; and the IC constraint

$$U^{i} \geq (1 - \delta) \left( \sum_{m=1}^{N+1} \Pi^{m} - \Delta c \sum_{m \neq i} D(p^{*}(\Pi^{m})) \right) + \delta \Pi(\tilde{U}^{i})$$

holds. Conversely, given constants $\Pi^{1},...,\Pi^{N+1} \in [\Pi^{C},\Pi^{M}]$ and payoffs $(\tilde{U}^{1},...,\tilde{U}^{N+1}) \in E^B$ satisfying the above inequalities, there exists a balanced equilibrium with initial-period expected stage payoffs $\Pi^{i}$ and continuation payoffs $\tilde{U}^{i}$ for each firm $i$.

**Proof.** Fix a B-optimal equilibrium $\sigma$ with lifetime payoffs $(U^{1},...,U^{N+1})$ and period-0 stage-game strategy profile $\sigma(h^{0}) = \tau$. Let $\Pi^{i}_{m}$ be the constants whose existence is ensured by Lemma C.17. Let $\Pi^{i} = \sum_{m=1}^{N+1} \Pi^{i}_{m}$ for each $i$. Then we can decompose each $U^{i}$ as

$$U^{i} = (1 - \delta)\Pi^{i} + \delta \tilde{U}^{i}$$

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for some constants \((\tilde{U}^1, \ldots, \tilde{U}^{N+1})\), which are themselves the lifetime payoffs of some balanced equilibrium.

Suppose first that \(\Pi^i_m > 0\) for all \(i\) and \(m\). Then by the argument in the proof of Lemma C.14, there exists a deviation for each firm yielding expected stage profits of at least \((N+1)\Pi^i + Nc\). As the harshest possible punishment continuation following a deviation yields profits \(\Pi(\delta; N)\), the unprofitability of this deviation implies the IC constraint

\[
(1 - \delta)\Pi^i + \delta \tilde{U}^i \geq (1 - \delta)((N+1)\Pi^i + Nc) + \delta \Pi(\delta; N)
\]

for each \(i\). Re-arranging yields

\[
\tilde{U}^i \geq \frac{1 - \delta}{\delta} N(\Pi^i + c) + \Pi(\delta; N).
\]

Now, by definition of B-optimality, for some \(i\) we must have \(U^i \geq \tilde{U}^i\). For this firm we have \(U^i = (1 - \delta)\Pi^i + \delta \tilde{U}^i \geq \tilde{U}^i\), or \(\Pi^i \geq \tilde{U}^i\). Combining this restriction with the IC constraint produces

\[
\tilde{U}^i \geq \frac{1 - \delta}{\delta} N(\tilde{U}^i + c) + \Pi(\delta; N),
\]

which in turn implies \(\delta > N/(N + 1)\), a contradiction.

It must therefore be the case that \(\Pi^i_m > 0\) for each \(i\), with all other stage profits non-positive. Note further that each \(\Pi^i_m \geq \Pi^C\), as otherwise we could construct an equilibrium with strictly higher lifetime profits for some firms by playing the stage-game Nash equilibrium in the first period for all markets \(m\) such that \(\Pi^m_m < \Pi^C\). (This cannot introduce additional profitable deviations and thus must still be supportable as an equilibrium.) Then \(p^*(\Pi^i_m)\) is well-defined for each \(i\), and firm \(i\) must play prices no lower than \(p^*(\Pi^i_m)\) to achieve expected stage profits \(\Pi^i_m\) in his home market. Then each firm \(i\) has a deviation yielding stage-game profits of at least \(\sum_{m=1}^{N} \Pi^m_m - \Delta c \sum_{m \neq i} D(p^*(\Pi^m_m))\), achieved by just undercutting the infimum of the home firm’s price support in all markets other than his own.

In fact, the usual partially collusive structure yields no deviations more profitable than this one, and yields zero profits to all away firms in each market. Thus it must be the case that \(\Pi^i_m = 0\) for all \(i\) and \(m \neq i\) (else we could strictly improve the equilibrium for some firms by playing a partially collusive structure in the first period and then reverting to \(\sigma\)). So \(\Pi^i = \Pi^i_m\) for all \(i\), and the IC constraints

\[
(1 - \delta)\Pi^i + \delta \tilde{U}^i \geq (1 - \delta) \left( \sum_{m=1}^{N+1} \Pi^m_m - \Delta c \sum_{m \neq i} D(p^*(\Pi^m_m)) \right) + \delta \Pi(\delta; N)
\]
must hold for all \( i \). Conversely, given any balanced equilibrium payoffs \((\tilde{U}^1, \ldots, \tilde{U}^{N+1}) \in \mathcal{E}^B\), any constants \( \Pi^i \in [\Pi^C, \Pi^M] \) satisfying the above IC constraints yield a balanced equilibrium with first-period profits \( \Pi^i \) and continuation profits \( \tilde{U}^i \).

\[ \square \]

**Theorem C.11.** Suppose \( N = 1 \) and \( \delta \leq 1/2 \). Then there exists a unique \( B \)-optimal equilibrium payoff vector \((U, U)\), which is supportable by a symmetric stationary equilibrium.

**Proof.** Let \( U^* \in R \) be the supremum of all payoffs \( U_1 \) such that for some \( U_2 \) the payoff vector \((U_1, U_2)\) is supportable by a balanced equilibrium. Let \((U_1^{(n)}, U_2^{(n)})\) be a sequence of balanced equilibrium-supportable payoff vectors such that \( U_1^{(n)} \uparrow U^* \). (If \( U^* \) is itself supportable as a balanced equilibrium, this could be a constant sequence.) For each \( n \), let \( \Pi_1^{(n)}, \Pi_2^{(n)} \) and \((\tilde{U}_1^{(n)}, \tilde{U}_2^{(n)})\) be the corresponding constants whose existence is ensured by lemma C.20. Passing to a subsequence if necessary, suppose that \((\Pi_1^{(n)}, \Pi_2^{(n)})\) and \((\tilde{U}_1^{(n)}, \tilde{U}_2^{(n)})\) both exist. (All of these sequences exist in compact subsets of the real line, so such subsequences exist.)

Because \( U^* \geq \tilde{U}_1^{(n)} \) for all \( n \), we must have \( U^* \geq \tilde{U}_1^{\infty} \). Also given \( U_1^{(n)} = (1-\delta)\Pi_1^{(n)} + \delta \tilde{U}_1^{(n)} \) for all \( n \) we have

\[
U^* = (1-\delta)\Pi_1^{\infty} + \delta U_1^{\infty} \leq (1-\delta)\Pi_1^{\infty} + \delta U^*,
\]

or \( \Pi_1^{\infty} \geq U^* \). And since \( U^* \geq \tilde{U}_2^{(n)} \) for all \( n \) (by symmetry of the game) we must have \( U^* \geq \tilde{U}_2^{\infty} \). Then from the IC constraints

\[
(1-\delta)\Pi_2^{(n)} + \delta \tilde{U}_2^{(n)} \geq (1-\delta) \left( \Pi_1^{(n)} + \Pi_2^{(n)} - \Delta cD(p^*(\Pi_1^{(n)})) \right) + \delta \Pi(\delta; N)
\]

implied by lemma C.20, we conclude that

\[
(1-\delta)\Pi_2^{\infty} + \delta \Pi_1^{\infty} \geq (1-\delta) \left( \Pi_1^{\infty} + \Pi_2^{\infty} - \Delta cD(p^*(\Pi_1^{\infty})) \right) + \delta \Pi(\delta; N),
\]

or equivalently

\[
\Pi_1^{\infty} \geq (1-\delta)(2\Pi_1^{\infty} - \Delta cD(p^*(\Pi_1^{\infty})) ) + \delta \Pi(\delta; N). \tag{1}
\]

Thus there exists a constant \( \Pi_1^{\infty} \) satisfying (1) such that \( \Pi_1^{\infty} \geq U^* \). In particular, \( (\Pi_1^{\infty}, \Pi_2^{\infty}) \geq (U_1, U_2) \) for every balanced equilibrium-supportable payoff vector \((U_1, U_2)\).

Now, we know that every \( \Pi_1^{\infty} \) satisfying (1) yields a symmetric stationary equilibrium with payoffs \((\Pi_1^{\infty}, \Pi_1^{\infty})\) through the usual partially collusive construction. Let \( \Pi^* \) be the maximal such \( \Pi_1^{\infty} \) (which we know exists by continuity of \( D(p^*(\cdot)) \)). Then it must be that \( U^* = \Pi^* \), as there exists a balanced equilibrium supporting this outcome. Further, as there
exists a symmetric stationary equilibrium supporting payoffs \((U^*, U^*)\), this is the unique B-optimal payoff vector. This establishes the claims of the theorem. \(\square\)

## D  Proofs of theorems in the main text

This section provides proofs of all theorems stated for the \(N = 1\) case (i.e. those of sections 4 and 5). The proofs rely heavily on general results developed in Appendix C.

### D.1  Proof of Theorem 1

This is a special case of Theorem C.1.

### D.2  Proof of Theorem 2

This result is an immediate consequence of the fact that firms’ profits are additively separable across markets.

### D.3  Proof of Theorem 3

This result is a consequence of Theorems C.3 and C.5, specialized to the \(N = 1\) case. Theorem C.3 gives a necessary condition for a profit vector to be supportable as a market-symmetric stationary equilibrium, in the form of a set of inequalities. Theorem C.5 shows that these inequalities form a sufficient condition for existence of an equilibrium and characterizes the unique (Pareto-)optimal profit vector within the set of vectors satisfying the inequalities.

### D.4  Proof of Theorem 4

This is a special case of Theorem C.7.

### D.5  Proof of Theorem 5

Take \(\delta^M\) as in the statement of the theorem. By construction, this discount factor satisfies

\[
\Pi^M = (1 - \delta^M)(2\Pi^M - \Delta cD(p^*(\Pi^M))).
\]
Then so long as \( \delta^M > \delta \), the theorem follows from Theorems 3 and 4. To establish this inequality, simply note that our expression for \( \delta^M \) is increasing in \( \Pi^M \), and at \( \Pi^M = c + \Delta c D(p_H^*) \) we have \( \delta^M = \delta \).

### D.6 Proof of Theorem 6

This is a special case of Theorem C.4.

### D.7 Proof of Theorem 7

Recall that by Theorem C.2 at most one firm can earn positive profits in a given market, and by Theorem C.3 each firm makes positive profits in only one market. Then the payoff vector \((\Pi, \Pi)\) (streamlining the notation \(\Pi^*\) to \(\Pi\) for this proof) can only be supported by giving each firm \(\Pi\) in his home market and 0 in his away market. To see this, observe that it cannot be that each firm makes \(\Pi\) in his away market; for this leads to a deviation worth \(\Pi + \Delta c D(p^*(\Pi)) \geq \Pi - \Delta c D(p^*(\Pi))\) in the home market, which violates the IC constraint. And we can rule out making negative profits in the away market, because this also increases the value of a deviation in that market and violates the IC constraint.

Now fix a market, say market 1. As firm 1 makes positive profits in this market, he enters w.p. 1. Further, firm 1 cannot have a profitable deviation in this market, else the IC constraint would be violated.

We first claim that the support of firm 1’s price distribution is contained in \([p^*(\Pi), p_H^*]\). Below \(p^*(\Pi)\) he cannot earn profits \(\Pi\), so such prices can’t be profit-maximizing. Meanwhile above \(p_H^*\) he will earn weakly lower profits than at \(p_H^*\), with profits strictly lower whenever profits at \(p_H^*\) are positive. Thus his profits above \(p_H^*\) are either non-positive, thus not optimal, or else strictly lower than at \(p_H^*\), which would introduce a profitable deviation if 1 did play above \(p_H^*\) in equilibrium.

Let \(p_1\) and \(p_1^*\) be the infimum and supremum of 1’s price support. We claim that \(p_1 = p^*(\Pi)\) and \(p_1^* = p_H^*\). Suppose that \(p_1 > p^*(\Pi)\). Then firm 2 has a deviation worth at least \(D(p_1)(p_1 - c_A) - c > \Pi - \Delta c D(p^*(\Pi))\), violating the IC constraint. So the lower end of 1’s support must be \(p^*(\Pi)\). On the other hand, if \(p_1 < p_H^*\), then 2 must place an atom at \(p_1\) to avoid giving 1 a profitable deviation up to \(p_H^*\). If 1 doesn’t place an atom at \(p_1\), then 2 never wins at \(p_1\) and thus makes negative profits there, which can’t be profit-maximizing. But if he does place an atom at \(p_1\), then he would have a profitable deviation to just below the atom, a contradiction. Hence \(p_1 = p_H^*\).

Now, suppose there exists an interval \([p_L, p_H] \subset [p^*(\Pi), p_H^*]\) such that \(F_1((p_L, p_H)) = 0\).
Let $\hat{F}_1 = F_1^1(p)$ for $p \in (p_L, p_H)$, and expand $[p_L, p_H]$ if necessary so that $p_L = \inf\{p \ : \ F_1^1(p) = \hat{F}_1\}$ and $p_H = \sup\{p \ : \ F_1^1(p) = \hat{F}_1\}$. Given the support of $F_1^1$, we must have $\hat{F}_1 \in (0, 1)$. Thus 1 has profit-maximizing prices arbitrarily close to both $p_L$ and $p_H$.

Consider firm 2’s strategy in $[p_L, p_H]$. He can set at most one price in $(p_L, p_H)$, since stage profits are strictly increasing in the interior of the interval. Say he plays some price $p^* \in (p_L, p_H)$ with positive probability. If he also places an atom at $p_L$, then 1 puts no atom there to avoid a profitable deviation. But then 2’s stage profits at $p^*$ are strictly greater than at $p_L$, a contradiction. So 2 places no atom at $p_L$. But then $p_L$ must be profit-maximizing for 1, a contradiction given that his profits are strictly increasing on $(p_L, p^*)$. So 2 does not play in $(p_L, p_H)$.

Firm 2 does play an atom at $p_L$, else 1 would have a profitable deviation into the gap. It follows that $p_L > p^*(\Pi)$ and 1 plays no atom there and is profit-maximizing in the limit as $p \nearrow p_L$. Conversely, firm 2 does not place an atom at $p_H$, for otherwise 1 would have a profitable deviation just below it. It follows that 1 is profit-maximizing at $p_H$.

From these facts we can pin down the size of firm 2’s atom at $p_L$. Firm 1’s profits from playing just below $p_L$ are

$$\Pi = D(p_L)(p_L - c_H)\left(1 - \pi_1^2 F_1^2(p_L) + \frac{1}{2} \pi_1^2 \Delta F_1^2(p_L)\right) - c,$$

while his profits at $p_H$ are

$$\Pi = D(p_H)(p_H - c_H)(1 - \pi_1^2 F_2^1(p_L)) - c.$$

Using the second equation to eliminate $F_2^1(p_L)$ from the first, we find that

$$\pi_1^2 \Delta F_1^2(p_L) = (\Pi + c) \left(\frac{1}{D(p_L)(p_L - c_H)} - \frac{1}{D(p_H)(p_H - c_H)}\right).$$

This is the probability that 2 enters and plays in $[p_L, p_H]$. It is easy to check that this is equal to the probability that firm 2 plays in $[p_L, p_H]$ under equilibrium characterized in Theorem 6. (We will refer to this equilibrium as the “standard equilibrium” or the “no-gap case” in what follows.)

Similarly, we may calculate the size of firm 1’s atom at $p_H$. Suppose $p_H < p^*_H$. As firm 2 places an atom at $p_L$, he must be profit-maximizing there. Then

$$0 = D(p_L)(p_L - c_A)(1 - F_1^1(p_L)) - c.$$
It is also true that firm 2 must be profit-maximizing arbitrarily close to \( p_H \) from above. For otherwise he would not play in some interval above \( p_H \), and firm 1 would have a profitable deviation upward from \( p_H \). Then

\[
0 = D(p_H)(p_H - c_A) \left( 1 - F_1(p_L) - \frac{1}{2} \Delta F_1(p_H) \right) - c.
\]

So

\[
\Delta F_1(p_H) = c \left( \frac{1}{D(p_L)(p_L - c_A)} - \frac{1}{D(p_H)(p_H - c_A)} \right).
\]

This is the same as the probability that firm 1 plays in \([p_L, p_H] \) under the standard equilibrium. If \( p_H = p'_H \) then we must modify the argument slightly: we still know \( F_1(p_L) \), and now \( F_1(p_H) = 1 \). This again determines the atom, which is easily checked to give the same probability of playing in \([p_L, p_H] \) under the standard equilibrium.

We conclude that, for any gap in firm 1’s mixing distribution, both firms play in the gap with the same frequency as in the no-gap case, except that the away firm concentrates all its support at the bottom of the gap, while the home firm prices only at the top. Hence business-stealing is strictly higher in regions where gaps have been added.

Finally, in any interval with no gap, both firms must play the entry-adjusted mixing distributions of the standard equilibrium. So business-stealing occurs at the same rate in these regions as in the standard equilibrium. Finally, sum the probability of business-stealing across all gap- and no-gap intervals. (Formally: there are at most a countable number of maximally-sized gaps, which can be well-ordered by their upper edges. The no-gap regions are then defined as the intervals between the upper edge of one gap and the lower edge of the next. These are also countable, so can be summed.) This sum is strictly higher than the standard equilibrium when gaps exist, and the standard equilibrium is the unique no-gap equilibrium.

### D.8 Proof of Theorem 8

Fix \( \delta < \delta^M \), and consider a stationary equilibrium supporting profits \( (\Pi^1, \Pi^2) > (\Pi^C, \Pi^C) \). Suppose wlog that in market 1, player 2 never wins the customer’s business. Theorem C.3 says that player 1 earns positive profits only in that market, so \( \Pi_1^1 \geq \Pi^1 \). And as player 2 never wins the business of that market, to earn total profits \( \Pi^2 \) it must be that \( \Pi_2^2 \geq \Pi^2 \).

Now, suppose player 2 does not enter market 1. Then player 1 can deviate upward to \( p_H \) in his own market to earn stage profits \( \Pi^M \), and can undercut player 2 in market 2 to earn
Thus the IC constraint

\[ \Pi_1 \geq (1 - \delta)(\Pi^M + \Pi^2 - \Delta cD(p^*(\Pi^2))) + \delta \Pi(\delta) \]

must hold. (If \( \Pi_2^2 > \Pi^2 \) then an even stricter IC constraint holds.) Meanwhile, the usual IC constraint

\[ \Pi^2 \geq (1 - \delta)(\Pi_1 + \Pi^2 - \Delta cD(p^*(\Pi^1))) + \delta \Pi(\delta) \]

holds for player 2. (\( \Pi_2^2 > \Pi^2 \) would imply that 2 makes negative profits in market 1, which would only increase the profitability of a deviation and tighten the IC constraint.) Because \( \Pi^1 < \Pi^M \), the first constraint is violated at \( (\Pi^1, \Pi^2) = (\Pi^*, \Pi^*) \). But the second constraint would be violated if \( \Pi^2 \) alone were lowered, as the lhs drops faster than the rhs and the constraint is saturated at \( (\Pi^*, \Pi^*) \). Thus \( (\Pi^1, \Pi^2) \) must be bounded below \( (\Pi^M, \Pi^M) \) by continuity of \( D(\cdot) \) and \( p^*(\cdot) \) in order to satisfy both constraints.

On the other hand, suppose player 2 does enter market 1. As he never wins the market by assumption, his stage profits in that market are \(-c\). Then he must enter w.p. 1, else he would not be optimizing by entering. Player 2’s IC constraint is the same no matter what he plays in market 1. Meanwhile, to maximally relax player 1’s IC constraint, 2 may mix just above the single price \( p_1 \) played by player 1 in that market with sufficient density close to \( p_1 \) to deter an upward deviation. (Because 1 always wins, he can be not be willing to mix between multiple prices.) In this case player 1’s IC constraint is the usual

\[ \Pi^1 \geq (1 - \delta)(\Pi^1 + \Pi^2 - \Delta cD(p^*(\Pi^2))) + \delta \Pi(\delta). \]

But now player 2’s deviation to undercut player 1 in market 1 yields additional profits of \( c \), so his IC constraint is tightened to

\[ \Pi^2 \geq (1 - \delta)(\Pi_1 + \Pi^2 + c - \Delta cD(p^*(\Pi^2))) + \delta \Pi(\delta). \]

By a similar argument to the previous case, solutions to this pair of inequalities are bounded below \( (\Pi^M, \Pi^M) \).

D.9 Proof of Theorem 9

This is a direct consequence of Theorem C.9 for the case \( N = 1 \) combined with Theorem C.5, which implies that \( \Pi^i = \Pi^* \).
D.10 Proof of Theorem 10

This is a direct consequence of Theorem C.10 for the case $N = 1$ and Theorem C.5, which implies that $\Pi^\dagger = \Pi^\ast$.

D.11 Proof of Proposition 11

This is a direct consequence of the fact that stationary play is trivially invariant, along with Theorem 7, which establishes that stage-game play in any optimal stationary equilibria yields strictly positive profits to each firm in its home market and zero profits in all away markets in each period along the equilibrium path.

D.12 Proof of Theorem 12

This is a direct consequence of Theorem C.11.

E The case of many competitors

Our simple duopoly model has the implication that perfect collusion is possible even for relatively low values of $\delta$ (in particular, $\delta^M < 1/2$ from Theorem 5). If one interprets $\delta$ literally, i.e., as reflecting discounting at the market interest rate over the intervals between competitive interactions, then one would typically expect to find $\delta > \delta^M$ in practical applications, in which case the firms would achieve perfect collusion, and the structure of optimal collusive agreements for $\delta < \delta^M$ would have little bearing on actual cartel behavior.

However, one can also interpret $\delta$ more expansively (and less literally) as a reduced-form stand-in for other factors that tend to make firms focus more on present opportunities and less on future consequences. For example, in many simple models of oligopoly, the number of competitors affects the feasibility of collusion through the same channel as discounting (because adding firms increases the potential gains from current deviations and reduces the future benefits of cooperation). Firms may also effectively discount future profits to a greater extent than market interest rates would imply because of agency problems, leadership turnover, uncertainty about future market conditions, or capital market imperfections that raise internal hurdle rates.

In this section we explore the implications of multiple competitors explicitly. We show that collusion indeed becomes more difficult to sustain as the cartel size increases, and that perfect collusion is infeasible even with a moderate number of firms and discount factors
close to unity. We also generalize the theorems of Section 4 and, for discount factors below \( \delta^M \), provide a characterization of optimal collusion that is broadly similar to the two-firm case. Our results thus provide some reassurance that our central insights concerning cartels are robust with respect to the introduction of additional factors that make collusion more difficult to sustain.

E.1 Setup

We extend our model to many-firm settings while retaining symmetry across firms: there are now \( N + 1 \) firms and \( N + 1 \) markets, where \( N \geq 2 \). Firm \( i \)'s marginal cost is \( c_H \) for units sold in market \( i \) (its home market), and \( c_A > c_H \) for units sold elsewhere. All other features of the model are unchanged.

E.2 Analyzing the stage game

First consider a single round of the stage game played in isolation. Because payoffs are additively separable across markets, we can focus on play in a single market. Let \( \{H\} \cup \mathcal{I} \) be the set of firms, where \( H \) is the home firm and \( \mathcal{I} = \{1, \ldots, N\} \) includes the away firms. The existence of a two-firm equilibrium implies that there are many Nash equilibria when \( N \geq 2 \). For if \( H \) and any firm \( i \in \mathcal{I} \) play the two-firm equilibrium, no away firm will have an incentive to enter (as it would receive strictly less than \( i \)'s profits, which are zero). Hence there are at least \( N \) Nash equilibria involving competition among pairs of firms.

In fact, for any non-empty subset of away firms, there is a Nash equilibrium in which those firms compete with the home firm. Our main result establishes a limit on the multiplicity of equilibria: once a subset of away firms is chosen, there exists a unique Nash equilibrium involving participation by those firms. The form of this equilibrium is broadly similar to the two-firm equilibrium, with certain entry by the home firm, occasional entry by the away firms, and all firms randomly choosing prices between \( p_A \) and \( p^*_H \). Further, the equilibrium is symmetric in that all away firms play identical strategies. The following theorem summarizes these results:

**Theorem 13.** For every non-empty subset \( \mathcal{I} \subset \mathcal{I} \) of away firms, there exists a unique Nash equilibrium of the stage game in which every firm in \( \mathcal{I} \) enters with positive probability and no firm in \( \mathcal{I} \setminus \mathcal{I} \) ever enters. In this equilibrium:

1. The home firm always enters and makes profits \( \Pi_H = \Delta c D(p_A) \).

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2. Each away firm $i \in J$ enters with probability strictly less than 1 and makes profits $\Pi_i = 0$.

3. Each entering firm’s price distribution has full support on $[p_A, p_H^*]$.

4. All entering away firms play the same strategy.

There exist no Nash equilibria in which no away firms enter with positive probability.

Proof. This is a restatement of Theorem C.1.

E.3 Asymmetric collusion with many firms

When many firms compete, the set of possible collusive arrangements is much richer than with only two firms. For the latter case, we have seen that it is always optimal to allocate production so that each firm earns all of its profits in its home market. In contrast, with three or more firms, it can be worthwhile to spread each firm’s profits across several markets; this reduces the profitability of undercutting in each market and thereby relaxes incentive constraints (in some instances).

In Appendix E.7, we describe an equilibrium which, for particular choices of $c, \Delta c,$ and $\delta$, Pareto-dominates the best equilibrium in which firms earn profits only in their home markets. Table 1 displays the division of profits for the special case of three firms. The table includes a row for each firm and a column for each market; a “+” indicates positive profits while 0 indicates zero profits.

<table>
<thead>
<tr>
<th></th>
<th>M1</th>
<th>M2</th>
<th>M3</th>
</tr>
</thead>
<tbody>
<tr>
<td>F1</td>
<td>+</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>F2</td>
<td>+</td>
<td>+</td>
<td>0</td>
</tr>
<tr>
<td>F3</td>
<td>0</td>
<td>0</td>
<td>+</td>
</tr>
</tbody>
</table>

Table 1: A division of equilibrium profits that is not market-symmetric

With no further restrictions on the structure of the equilibrium, it is difficult to characterize optimal collusion. Note, however, that the equilibrium depicted in Table 1 has a feature that is arguably peculiar: within one of the markets (M1), \textit{ex ante} identical away firms (F2 and F3) do not earn the same profits. It is reasonable to assume that symmetrically situated firms are drawn to symmetric agreements because they are easier to describe, likely simpler to negotiate, and require less coordination than asymmetric ones.
We will therefore impose symmetry going forward to provide sufficient structure for a characterization of optimal collusion with more than two firms.

E.4 Optimal collusive equilibria

We begin our analysis of optimal collusive equilibria by introducing some additional notation. Recall that $\Pi^M$ is the monopoly profit of a low-cost provider in a single market. Let $\tilde{\Pi}^M$ be the profit of a high-cost provider setting the same price $p^*_H$, which satisfies $\Pi^M - \tilde{\Pi}^M = \Delta cD(p^*_H) > 0$. (Note that $\tilde{\Pi}^M$ is not the monopoly profit of the high-cost provider, as in general $p^*_A > p^*_H$.) With this notation, define

$$\delta^M(N) \equiv 1 - \frac{1}{N + 1} \left( \frac{N}{N + 1} \frac{\tilde{\Pi}^M}{\Pi^M} + \frac{1}{N + 1} \right)^{-1}.$$ 

The notation suggests that $\delta^M(N)$ is the minimal discount factor for which perfect collusion is sustainable with $N + 1$ firms, a fact we establish later, under some conditions, in Theorem 17. Note that $\Pi^M > \tilde{\Pi}^M$ implies that $\delta^M(N) < 1 - 1/(N + 1)$.

Another useful discount factor threshold is $\delta(N) \equiv (1 - \frac{1}{N}) (1 + \frac{c}{\Pi^M})$. This expression plays an auxiliary role in our results and we will explain it shortly; for the moment, simply note that it is greater than $1 - 1/N$ and may be either larger or smaller than $\delta^M(N)$. Finally, let $\Pi(\delta; N)$ be the minimum SPNE-sustainable lifetime profits with discount factor $\delta$ and $N + 1$ firms.

Our first theorem is the many-firm analog of Theorem 3:

**Theorem 14.** Suppose $\delta < \delta^M(N)$ and $N \geq \sqrt{1 + \Pi^M/c}$. Then the optimal symmetric equilibrium payoff vector $(\Pi^*, ..., \Pi^*)$ satisfies

$$\Pi^* = (1 - \delta)((N + 1)\Pi^* - \Delta cND(p^*(\Pi^*))) + \delta \Pi(\delta; N).$$

Further, $\Pi^* > \Pi^C$ iff $\Pi(\delta; N) < \Pi^C$, and $\Pi^*$ is strictly increasing in $\delta$ whenever $\Pi(\cdot; N)$ is nonincreasing in $\delta$. Finally, $\Pi^*$ is strictly decreasing in $N$ whenever $\Pi(\delta; \cdot)$ is nonincreasing in $N$.

**Proof.** Note that $N \geq \sqrt{1 + \Pi^M/c}$ implies $\delta(N) > \delta^M(N)$ and thus $\delta < \delta(N)$ by Lemma C.13. Then this result is a consequence of Theorems C.3 through C.5. Theorem C.3 gives a necessary condition for a profit vector to be supportable as a market-symmetric stationary equilibrium (under the conditions of the theorem), in the form of a set of inequalities.
Theorem C.4 shows that these inequalities form a sufficient condition for existence of an equilibrium, while Theorem C.5 characterizes the unique symmetric optimal profit vector within the set of vectors satisfying the inequalities.

The substance of this theorem is identical to that of Theorem 3. In essence it tells us that, provided $\delta$ is not too high, optimal collusion involves allocating markets according to cost advantages. As the theorem shows, the characterization of maximum sustainable profits then depends on the most severe punishment, $\Pi(\delta; N)$, that firms can mete out following a deviation.

In contrast to the case of a duopoly, the optimality of allocating markets according to cost advantages is not guaranteed for all $\delta < \delta^M(N)$. We therefore also require that $N \geq \sqrt{1 + \Pi^M/c}$, which rules out the possibility of achieving higher profits by allocating business only to away firms (while respecting symmetry). In Appendix E.8, we show by way of example that such an arrangement can yield profits exceeding the level indicated in Theorem 14 when this bound is violated. Table 2 depicts the division of profits for this example (in which there are three firms).

<table>
<thead>
<tr>
<th></th>
<th>M1</th>
<th>M2</th>
<th>M3</th>
</tr>
</thead>
<tbody>
<tr>
<td>F1</td>
<td>0</td>
<td>+</td>
<td>+</td>
</tr>
<tr>
<td>F2</td>
<td>+</td>
<td>0</td>
<td>+</td>
</tr>
<tr>
<td>F3</td>
<td>+</td>
<td>+</td>
<td>0</td>
</tr>
</tbody>
</table>

Table 2: A cartel with all profits awarded to away firms

The condition $N \geq \sqrt{1 + \Pi^M/c}$ is a lower bound on the number of competitors given the ratio of monopoly profits to market-specific fixed costs. Because this bound grows slowly in $\Pi^M/c$, it may be satisfied in practice. For instance, $\Pi^M/c = 3$ implies $N \geq 2$ (which is true by assumption), while $\Pi^M/c = 15$ implies $N \geq 4$. Even if fixed costs were a trivial portion of monopoly profits, say 1%, the implied bound on $N$ would be only $N \geq 10$. Consequently, imposing $N \geq \sqrt{1 + \Pi^M/c}$ (and hence $\delta^M(N) < \bar{\delta}(N)$) strikes us as reasonably innocuous.

In fact we can do better. Arrangements in which profits are allocated against cost advantage be shown to be suboptimal whenever the regularity condition $\delta < \bar{\delta}(N)$ holds. (The condition $N \geq \sqrt{1 + \Pi^M/c}$ is merely a sufficient condition for regularity to hold.) And the irregular case $\bar{\delta}(N) \leq \delta < \delta^M(N)$, when the discount factor might be low enough to require partially collusive arrangements but high enough to violate regularity is demonstrably unimportant. In particular:
Theorem 15. \([\delta(N), \delta^M(N)] \subset (1 - 1/N, 1 - 1/(N + 1))\). Therefore if \(\delta \in [\delta(N), \delta^M(N)]\), then \(\delta < \delta(N + 1)\) and \(\delta > \delta^M(N - 1)\).

Proof. The set inclusion follows from the fact that \(\delta^M(N) < 1 - 1/(N + 1)\) while \(\delta(N) > 1 - 1/N\), inequalities which are obvious by inspection of the relevant definitions. The remaining inequalities are immediate corollaries of the fact that the collection of intervals \((1 - 1/N, 1 - 1/(N + 1))\) are pairwise disjoint. 

This theorem tells us that \(\delta \in [\delta(N), \delta^M(N)]\) is a knife-edge case: add one more firm and we will have \(\delta < \delta(N + 1)\), which means we can focus on cartel structures that allocate markets according to cost. Subtract one firm, and the resulting cartel can sustain perfect collusion. The size of the problematic interval \([\delta(N), \delta^M(N)]\) is also at most \(1/(N(N + 1))\), and so collapses rapidly with \(N\). We therefore consider the possibility of alternative collusive structures (ones that allocate profits to away firms) a minor issue that we can safely ignore.

The next result generalizes Theorem 4:

Theorem 16. Whenever

\[
\delta \geq \delta(N) \equiv \left(1 - \frac{1}{N + 1}\right) \left(1 + \frac{N}{N + 1} \frac{\Pi^M - \tilde{\Pi}^M}{c}\right)^{-1},
\]

there exists an SPNE supporting lifetime profits of 0 for each firm, so that \(\Pi(\delta; N) = 0\).

Proof. This is a restatement of Theorem C.7.

The structure of the punishment equilibrium resembles the one used for duopolies, but it has an asymmetric element: the punishment for firm \(i\) consists of a price war between \(i\) and another firm, say \(i + 1\), in their respective markets. All other firms stay out of those markets and play the stage-game Nash equilibrium in the remaining markets. Firms revert to cooperation after one round of a successful price war. Note that \(\delta(N)\) is increasing in \(N\), but is strictly bounded away from 1 given \(\Pi^M > \tilde{\Pi}^M\). Thus even with a large number of somewhat impatient competitors, minmax punishments are feasible.

Next we generalize Theorem 5, and fully characterize optimal collusive payoffs for a range of discount factors below \(\delta^M(N)\) under mildly restrictive conditions:

Theorem 17. Suppose \(\Pi^M > \Delta c D(p^*_H) + \frac{c}{N}\) and \(N \geq \sqrt{1 + \Pi^M/c}\). Then \(\delta(N) < \delta^M(N)\), and for all \(\delta \in [\delta(N), \delta^M(N)]\) the optimal symmetric stationary equilibrium profit vector \((\Pi^*, ..., \Pi^*)\) satisfies

\[
\Pi^* = (1 - \delta)((N + 1)\Pi^* - \Delta c ND(p^*(\Pi^*))).
\]
Further, \( \Pi^* \) is continuous, strictly greater than \( \Pi^C \), and strictly increasing in \( \delta \). Finally, \( \delta^M(N) \) is the minimal discount factor at which perfect collusion is sustainable.

Proof. This result follows from Theorems 14 and 16, once we have established \( \tilde{\delta}(N) < \delta^M(N) \). Write \( \delta^M(N) \) as
\[
\delta^M(N) = \frac{1}{1 + \frac{1}{N}\Pi^M / \tilde{\Pi}^M}
\]
and \( \tilde{\delta}(N) \) as
\[
\tilde{\delta}(N) = \frac{N}{N+1} \frac{1}{1 + \frac{N}{N+1} \frac{\Pi^M - \tilde{\Pi}^M}{c}}.
\]
Then re-arrangement of the inequality yields
\[
1 + \frac{N}{N+1} \frac{\Pi^M - \tilde{\Pi}^M}{c} > \frac{N}{N+1} \left( 1 + \frac{1}{N} \frac{\Pi^M}{\tilde{\Pi}^M} \right).
\]
Multiplying through by \( N + 1 \) and cancelling terms leaves
\[
1 + N \frac{\Pi^M - \tilde{\Pi}^M}{c} > \frac{\Pi^M}{\tilde{\Pi}^M}.
\]
Subtracting both sides by 1 and combining terms on the rhs allows us to cancel a common factor of \( \Pi^M - \tilde{\Pi}^M \). Finally, we are left with \( \tilde{\Pi}^M > c/N \), which is equivalent to the condition \( \Pi^M > \Delta cD(p^*_H) + c/N \) in the theorem statement.

As in Theorem 5, we impose a mild sufficiency condition on \( \Pi^M \) to ensure \( \tilde{\delta}(N) < \delta^M(N) \). This condition grows weaker as \( N \) increases, and is trivially satisfied for sufficiently large \( N \). Note that Theorem 17 does not directly speak to the form of \( \Pi^* \) for any discount factor when \( \tilde{\delta}(N) < \delta^M(N) \). To address this deficiency, in Theorem C.8 of the Appendix we derive a very mild lower bound on \( N \) that ensures \( \tilde{\delta}(N) < \tilde{\delta}(N) \), in which case Theorem 17 continues to characterize optimal profits for discount factors in the range \( [\tilde{\delta}(N), \tilde{\delta}(N)] \).

Finally, we characterize an equilibrium supporting profits \( \Pi^* \) for each firm. As in the case of a duopoly, this construction holds regardless of the value of \( \Pi(\delta; N) \).

**Theorem 18.** Suppose \( \delta < \delta^M(N) \). Then lifetime profits \((\Pi^*, ..., \Pi^*)\) are supported by a symmetric stationary equilibrium with the following properties:

1. The home firm’s strategy is the same in all markets, and all away firms play the same strategy in all markets.
2. The home firm enters with probability 1, while all away firms enter with a probability that is strictly between zero and 1 and decreasing in $\Pi^*$.

3. The home firm earns profits $\Pi^*$, while all away firms make zero profits.

4. Each firm posts prices only in $[p^*(\Pi^*), p^*_H]$, and firms’ price distributions have full support on $(p^*(\Pi^*), p^*_H)$.

5. If $\Pi^* > \Delta_c D(p_A)$, the home firm plays $p^*(\Pi^*)$ with some strictly positive probability, which is increasing in $\Pi^*$.

6. Each market is captured by an away firm with some strictly positive probability, which is strictly decreasing in $\Pi^*$ when $\Pi^* \geq \frac{1}{2} \Pi^M$.

7. Any unilateral deviation by an away firm to a price at or below $p^*(\Pi^*)$ is punished by a continuation payoff of $\Pi(\delta; N)$ to that firm.

Proof. This is a special case of Theorem C.4, with the inequality of property 6 weakened to provide a simpler expression.

This result mirrors our conclusions concerning the optimal collusive structure for a duopoly, and features business-stealing for essentially the same reason.

E.5 Imperfect collusion in large cartels

The following theorem explores how the range of discount factors for which we have characterized optimal collusion changes with cartel size.

Theorem 19. $\delta(N)$ and $\delta^M(N)$ are strictly increasing in $N$, and $\lim_{N \to \infty} \delta(N) < 1$ while $\lim_{N \to \infty} \delta^M(N) = 1$. Further, $\delta^M(N) - \delta(N)$ is strictly increasing in $N$ whenever $\delta^M(N) \geq \delta(N)$.

Proof. Writing $\delta^M(N)$ as

$$\delta^M(N) = 1 - \frac{1}{1 + N\Pi^M/\Pi^M}$$

proves that it is strictly increasing in $N$ and approaches 1 as $N \to \infty$. Similarly, writing $\delta(N)$ as

$$\delta(N) = \left(1 + \frac{1}{N} + \frac{\Pi^M - \tilde{\Pi}^M}{c}\right)^{-1}$$

shows that $\delta(N)$ is strictly increasing but bounded below 1.
To finish the proof, we must show that $\Delta(N) \equiv \delta^M(N) - \delta(N)$ is increasing whenever $\Delta(N) \geq 0$. We showed in the proof of Theorem 17 that the latter inequality holds iff $\Pi^M \geq c/N$. It is then sufficient to verify that $\Delta'(N) > 0$ whenever $N \geq c/\Pi^M$.

Computing the derivative of $\Delta(N)$ yields

$$\Delta'(N) = \frac{\Pi^M/\Pi^M}{(1 + N\Pi^M/\Pi^M)^2} \times \frac{1/N^2}{(1 + 1/N + \Pi^M - \Pi^M/\Pi^M)^2}.$$  

Some re-arrangement shows that $\Delta'(N) > 0$ iff

$$1 + \frac{\Pi^M - \Pi^M/c}{c} > \sqrt{\Pi^M/\Pi^M} + \frac{1}{N} \left( \sqrt{\Pi^M/\Pi^M} - 1 \right).$$

Because $\Pi^M > \Pi^M$, the rhs is largest when $N$ is smallest, i.e. at $N = c/\Pi^M$. It is therefore sufficient to show that

$$1 + \frac{\Pi^M}{c} > \sqrt{\Pi^M/\Pi^M} + \frac{\sqrt{\Pi^M/\Pi^M}}{c}.$$  

But the first term on the lhs is strictly greater than the first term on the rhs, with a similar comparison holding for the second terms. So indeed $\Delta'(N) > 0$ whenever $N \geq c/\Pi^M$, completing the proof.

Because $\delta^M(N)$ goes to 1 as $N$ grows large, large cartels can aspire only to imperfect collusion even when their members are extremely patient. The minimal discount factor required to sustain a price war yielding zero profits also grows with $N$, but more slowly. Thus the range of discount factors for which we completely characterize optimal collusion expands with $N$. Accordingly, this theorem establishes the robustness of our results with respect to cartel size. It also illustrates the point that our analysis of imperfect collusion applies in settings where firms’ discount rates are in line with market interest rates.

### E.6 Comparative statics

Our results rely on the presence of both a cost asymmetry $\Delta c$ between home and away firms, and a recurring fixed cost $c$ of attempting to serve a market. To illuminate the roles of both $\Delta c$ and $c$, we examine the limiting behavior of optimal collusion as each becomes small. We first consider the outcome when the cost asymmetry between firms declines to zero.
Theorem 20. Fix $\delta \in (0, 1)$. If $\delta < N/(N+1)$, then $\Pi^* \to \Pi^C$ as $c_A \downarrow c_H$. If $\delta \geq N/(N+1)$ then $\Pi^* = \Pi^M$ for all $c_A > c_H$.

Proof. This is a direct consequence of Theorem C.9 combined with Theorem C.5, which implies that $\Pi^\dagger = \Pi^*$. \hfill \Box

In the limit as $\Delta c$ shrinks to zero, the set of symmetric stationary equilibria exhibits a bang-bang structure. No collusion is possible for $\delta$ below the critical threshold $N/(N+1)$, while perfect collusion is sustainable for $\delta$ above that threshold. Thus, there is no room for imperfect collusion in the limit.

In contrast, with asymmetric costs, the degree of sustainable collusion rises gradually with the discount factor. As a cartel tries to become more collusive, the ratio of the profits gained by a deviating away firm when it undercuts a home firm rises relative to the profits earned by the home firm. Undercutting therefore becomes more tempting as the cartel tries to sustain higher profits. Accordingly, sustaining a slightly higher profit level requires slightly greater patience.

We can also examine the nature of collusion in the limit as the fixed cost of attempting to serve a market declines to zero.

Theorem 21. Fix $\delta \in (0, 1)$. Let $(F^*_H(\cdot), F^*_A(\cdot), \pi^*_A)$ be the home and away firms’ price distributions and the away firm’s entry probability, respectively, for the equilibrium characterized in Theorem 18. As $c \to 0$,

- $F^*_H(\cdot)$ converges uniformly to $1\{p \leq p^*_A\}$,
- $\pi_A \to 1 - (\Pi^*/\Pi^M)^{1/N}$,
- $F^*_A(\cdot)$ converges uniformly to $\frac{1 - (\Pi^*/\Pi^M)^{1/N}}{1 - (\Pi^*/\Pi^M)^{1/N}}$ on $[p^*_A, p^*_H]$.

The probability of business stealing therefore falls to zero as $c$ vanishes, and in the limit the home firm wins the market at price $p^*_A$ with probability 1.

Proof. This is a direct consequence of Theorem C.10 combined with Theorem C.5, which implies that $\Pi^\dagger = \Pi^*$. \hfill \Box

This theorem tells us that a nonzero market-specific fixed cost is crucial for generating equilibrium business stealing. Without it, away firms could costlessly set their prices just above $p^*_A$, thereby deterring the home firm from charging higher prices without actually undercutting it. As we have discussed previously, in our model these fixed costs give rise
to equilibrium business stealing because they preclude the firms from costlessly policing the cartel agreement.\textsuperscript{42}

\section*{E.7 An equilibrium in asymmetric strategies}

In this subsection we demonstrate parameters under which collusion in symmetric strategies is Pareto-dominated by collusion in more general strategies. Fix $D(p) = 1\{p \leq v\}$, $N = 2$, $\delta = 0.6$, $\Delta c = 0.2$, and $c = 0.1$. $v$ will be assumed to be sufficiently large. The largest symmetric profits which can be supported in this environment by a stationary equilibrium satisfy

$$\Pi^* = (1 - \delta)(N\Pi^* - (N - 1)\Delta c),$$

yielding $\Pi^* = 0.8$.

Now, consider a stationary equilibrium with profits taking the signs indicated in Table 1. Markets 2 and 3 take the standard structure of Theorem C.4.

In market 1, let $p \equiv \Pi_1^* + c + c_H$, $\bar{p}_1 \equiv \frac{1}{2}(v + c_H)$, and $\bar{p}_2 \equiv \frac{1}{2}(v + c_H)$. Firm 3 does not enter. Firms 1 and 2 always enter and play

$$F_1^1(p) = \begin{cases} 
0, & p < p, \\
1 - \frac{p - c_A}{p - c_A}, & p \in [p, \bar{p}_1), \\
1 - \frac{p - c_A}{\bar{p}_2 - c_A}, & p \in [\bar{p}_1, v), \\
1, & p \geq v
\end{cases}$$

and

$$F_1^2(p) = \begin{cases} 
0, & p < \bar{p}, \\
1 - \frac{p - c_H}{p - c_H}, & p \in [p, \bar{p}_1), \\
1 - \frac{p - c_H}{\bar{p}_1 - c_H}, & p \in [\bar{p}_1, v), \\
1, & p \geq v
\end{cases}$$

These two distributions are continuous with full support on $[v, \bar{p}_1]$ and then have a gap on $[\bar{p}_1, v]$.

\textsuperscript{42}Theorem 10 sidesteps the issue of whether whether $\Pi^*$ as characterized by Theorem 17 is indeed the optimal collusive outcome in the limit $c \downarrow 0$. As $\delta(N) \downarrow 0$ in this limit, the only potential issue is whether $\delta(N) \geq \delta^M(N)$. Indeed, the sufficient condition $N \geq \sqrt{1 + \Pi^M/c}$ is clearly not satisfied in the limit. However, a sharper sufficient condition may be derived by evaluating $\delta(N) - \delta^M(N) > 0$ at $c = 0$, which yields the lower bound $N > (p^*_H - c_H)/\Delta c$. So for sufficiently large $N$, or for $N = 1$, Theorem 17 indeed characterizes optimal (stationary, symmetric) profits as $c \downarrow 0$. 

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(\bar{p}_1, v). Firm 1 also places an atom at \(\bar{p}_1\). Finally, both firms place an atom at \(v\), of sizes

\[ \Delta F_1^1 = 2 \left( \frac{\Pi_1 + c}{\Pi^M + c - \Delta c} \right), \quad \Delta F_1^2 = 2 \left( \frac{\Pi_1 + c}{\Pi^M + c} \right). \]

The two firms’ profits are related by \(\Pi_2^1 = \Pi_1^1 - \Delta c\). In order for this construction to be well-defined, we need \( p < \bar{p}_1 \), or equivalently \(\Pi_1^1 < \frac{1}{2}(\Pi^M - c)\), which is satisfied for \(v\) sufficiently large.

The best deviation by each of firms 1 and 2 in market 1 is to undercut the atom at \(p = v\), which yields them profits \(2\Pi_1^1 + c\). Meanwhile firm 3 has two possible maximally profitable deviations, one at \(p = \bar{p}\) and another by undercutting \(p = v\). (It can’t be more profitable to price in the firms’ price support, as this will make firm 3 strictly less than firm 2 would by playing there, and thus strictly less than he would make by playing \(p = \bar{p}\).) His profits at \(\bar{p}\) are \(\Pi_3^2\), while his profits undercutting \(v\) are

\[ (\Pi^M + c - \Delta c)\Delta F_1^1 \Delta F_1^2 - c = 4(\Pi_2 + c)\frac{\Pi_1 + c}{\Pi^M + c} - c. \]

For sufficiently large \(v\), these are lower than his profits at \(\bar{p}\).

Thus, for sufficiently large \(v\) the incentive constraints which need to be satisfied are

\[ \Pi_1^1 \geq (1 - \delta)(2\Pi_1^1 + c + \Pi_2^2 + \Pi_3^3 - 2\Delta c) \]

for firm 1,

\[ \Pi_1^2 + \Pi_1^1 \geq (1 - \delta)(2\Pi_1^2 + c + \Pi_2^2 + \Pi_3^3 - \Delta c) \]

for firm 2, and

\[ \Pi_3^3 \geq (1 - \delta)(\Pi_1^2 + \Pi_2^2 + \Pi_3^3 - \Delta c) \]

for firm 3. In addition, \(\Pi_1^2 = \Pi_1^1 - \Delta c\). It is easily checked that all firms’ profits are simultaneously maximized subject to the IC constraints when \(\Pi_1^1 = 1.4\), \(\Pi_1^2 = 1.2\), \(\Pi_2^2 = 0.2\), and \(\Pi_3^3 = 0.8\). This equilibrium is actually a Pareto-improvement on the best partially collusive one!

### E.8 An equilibrium with no home market profits

In this subsection we demonstrate parameters under which (symmetric) collusion in which profits are won in each firm’s home market is Pareto-dominated by collusion in which profits are won only in firms’ away markets. Fix \(D(p) = \mathbf{1}\{p \leq v\}\), \(N = 2\), \(\delta = 0.62\), \(\Delta c = 0.2\), and
\( c = 0.1. v \) will be assumed to be sufficiently large. The maximal profits supportable by an equilibrium of the type characterized in Theorem C.4 (which is the best that can be done by a symmetric stationary equilibrium when firms earn profits only in their home market) satisfy

\[
\Pi^* = (1 - \delta)(N\Pi^* - (N - 1)\Delta c),
\]

or \( \Pi^* \approx 1.09 \).

Now consider a symmetric equilibrium in which all away firms make positive profits \( \Pi/N \) in each market, while the home firm makes no profits. Thus each firm makes total equilibrium profits \( \Pi \).

The home firm refrains from entering, while the away firms always enter. Let \( \underline{p} \equiv \Pi/N + c + c_A \) and \( \bar{p} \equiv c_A + \frac{1}{N}(v - c_A) \). Each away firm plays

\[
F^A(p) = \begin{cases} 
0, & p < \Pi + c + c_A, \\
1 - \left( \frac{p - c_A}{\bar{p} - c_A} \right)^{1/(N-1)}, & p \in [\underline{p}, \bar{p}], \\
1 - \left( \frac{p - c_A}{\bar{p} - c_A} \right)^{1/(N-1)}, & p \in [\bar{p}, v), \\
1, & p \geq v.
\end{cases}
\]

Each firm’s price distribution is continuous with full support on \([\underline{p}, \bar{p}]\), has a gap on \([\bar{p}, v)\), and places an atom at \( \bar{p} \) of strength

\[
\Delta F^A = \left[ N \left( \frac{\bar{p} - c_A}{v - c_A} \right) \right]^{1/(N-1)}.
\]

For the equilibrium to be well-defined, we need \( \bar{p} > \underline{p} \), i.e. \( \Pi < v - c_A - Nc \), which is possible for \( v \) sufficiently large.

Now, each away firm has a deviation to undercutting \( p = v \), yielding profits \( \Pi + (N - 1)c \). Meanwhile the home firm has two candidate deviations. Setting \( p = \underline{p} \) yields profits \( \Pi/N + \Delta c \), while undercutting \( p = v \) yields profits

\[
(v - c_H)(\Delta F^A)^N - c = N^{N/(N-1)} \frac{v - c_H}{(v - c_A)^{N/(N-1)}} - c.
\]

For \( v \) sufficiently large the home firm’s most profitable deviation is to \( \underline{p} \).

The IC constraint required to support this equilibrium is then

\[
\Pi \geq (1 - \delta) \left[ \left( N + \frac{1}{N} \right) \Pi + N(N - 1)c + \Delta c \right],
\]

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with the additional constraint that $\Pi < v - c_A - N_c$. Given that $(1 - \delta)(N + 1/N) = 0.95 < 1$, any $\Pi \geq 3.04$ will satisfy the IC constraint. So for $v$ sufficiently high, there exist $\Pi > \Pi^*$ supportable in equilibrium.