Precautionary Saving in a Markovian Earnings Environment*

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July 27, 2017

ABSTRACT:

In an income fluctuation problem, an increase in future income risk (in the sense of second order stochastic dominance) is related to an increase in savings. However, until now there has been no theoretical result that confirms this relationship in Markovian earnings dynamics. This paper aims to fill this gap in the literature.

Keywords: Consumption; Savings; Precautionary savings; Prudence; Income uncertainty; Income fluctuation problem; Comparative statics.

JEL classification: D91, D81, C60

*The author wishes to thank Ehud Lehrer, Stanford seminar participants, two anonymous referees and the editor for their valuable comments.

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1 Introduction

In this paper we theoretically examine the effect of different earnings processes on savings and consumption decisions in a consumption-savings model with income uncertainty (see Ljungqvist and Sargent (2004) for a textbook treatment), sometimes called the income fluctuation problem. In the income fluctuation problem, an infinitely lived agent decides how much to save and how much to consume in each period while his income is fluctuating. The agent receives an income in each period and the income process follows a stochastic process. The income fluctuation problem is an incomplete markets model in the sense that the agent cannot insure himself against any realization of future incomes. The agent can transfer savings from one period to another only by investing in a risk-free bond, and has a liquidity constraint in the form of a borrowing limit that does not allow the agent to default. The income fluctuation problem is at the heart of modern macroeconomics and is used to study many macroeconomic phenomena. For example, Bewley (1977) studies the permanent income hypothesis and Benhabib et al. (2014) study wealth distribution. Carroll (2001) provides a discussion of the development and limitations of the model.

The goal of this paper is to present theoretical results related to precautionary saving in an income fluctuation problem where the income dynamics follow a general Markov process. We give conditions on the Markov process that governs the income dynamics, under which a riskier Markov earnings process (in the sense of second order stochastic dominance, see Definition 4) increases the savings of a prudent agent. We show that the linear Markov earnings model, in particular the AR(1) earnings process that is usually assumed in numerical studies (see De Nardi et al. (2016) and references therein), satisfies those conditions. This implies that convexity of the marginal utility function is enough to guarantee precautionary saving when the earnings follow the AR(1) process, providing a theoretical basis for previous numerical results about precautionary saving (e.g., Huggett (1996)).

The effect of future income uncertainty on savings and consumption decisions when earnings shocks are independent over time has been extensively studied in the theoretical literature on the consumption-savings model. Leland (1968), Sandmo (1970) and Mirman (1971) consider a two-period consumption-savings expected utility maximization model where income in the second period is a random variable. Leland (1968) shows that when the agent’s marginal utility is convex, there is an increase in savings when moving from a deterministic income in the second period to an uncertain one.

1 For example see Schechtman and Escudero (1977), Rabault (2002), Chamberlain and Wilson (2000) and Heathcote et al. (2009).
2 See Aiyagari (1994) for a discussion on borrowing limits.
3 Following Kimball (1990), we call an agent prudent if his marginal utility is convex.
Crainich et al. (2013) and Nocetti (2015) show in a two-period model that risk lovers, like risk averters, engage in precautionary saving too if their utility function has a positive third derivative. Kimball and Weil (2009) and Wang and Li (2016) study precautionary saving in a two-period model with Kreps-Porteus\(^4\) preferences. Hassin and Lieber (1982) analyze a precautionary saving model where the interest rate is not constant over time. Miller (1976), Schechtman (1976), and Mendelson and Amihud (1982) consider a multi-period consumption-savings expected utility maximization model. Miller (1976) proves that if the earnings process is independent over time and the agent’s utility function has a positive third derivative then when the earnings risk increases, in the sense of second order stochastic dominance,\(^5\) the agent’s savings increase. Lehrer and Light (2016) provide a simpler proof of Miller’s result. While our main result (Theorem 2) is a generalization of the results in Miller (1976), the techniques we use are completely different. It is not possible to extend Miller’s proof to a Markovian environment. However, as in Miller’s proof, the first step is to prove certain properties of the value function’s derivative (see Theorem 1). To that end, we overcome some of the technical difficulties involved in proving certain properties of the value function’s derivative (that can be used in other dynamic programs as well). The second step is to prove a comparative statics result that is based on those properties.

Besides the theoretical literature, empirical studies of precautionary saving and wealth accumulation have looked at the individual decision maker, at the household level, and at aggregate savings.\(^6\) There are also a few closed-form solutions of the consumption function reached by relying on certain assumptions regarding the agent’s preferences and the earnings process. For example, Caballero (1991) finds a closed-form solution to the consumption function when the utility function displays constant absolute risk aversion and the earnings dynamics follow the AR(1) process.\(^7\) Caballero finds that savings increase when the earnings dynamics are riskier. Since it is generally hard to find a closed-form solution, we use comparative statics techniques to show that a riskier earnings process leads to an increase in the agent’s savings.

The rest of the paper is organized as follows. Section 2 presents the income fluctuation problem. Section 3 presents some properties of Markov kernels. We introduce the notion of convexity-preserving Markov kernels and provide examples of monotone and convexity-preserving Markov kernels.

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\(^4\)See Kreps and Porteus (1978).

\(^5\)See Rothschild and Stiglitz (1970) and Definition 3.


\(^7\)See also Weil (1993) and Van Der Ploeg (1993) for closed-form solutions for the consumption function in non-expected utility models.
Section 4.1 we discuss the properties of the value function and the value function’s derivative. In Section 4.2 we state our main theorem. In Section 4.3 we consider the important case of unbounded utility functions. In Section 5 we provide a summary, followed by an Appendix containing proofs.

2 The model

We consider a discrete-time stochastic consumption-savings model. In this section we present the main components and assumptions of the model.

2.1 Labor income dynamics

Let $S = [0, \bar{s}]$ be a set of possible incomes. In each period $t = 1, 2, 3, \ldots$ the agent receives an income $s \in S$.

The evolution of the agent’s income is determined by a Markov kernel.\footnote{Let $\Sigma$ be the Borel $\sigma$-algebra on $S$. A Markov kernel on the measurable space $(S, \Sigma)$ is a function $P : S \times \Sigma \to [0, 1]$ such that for all $s \in S$, $P(s, \cdot)$ is a probability measure on $S$ and for each $B \in \Sigma$, $P(\cdot, B)$ is a $\Sigma$-measurable function. If in some period the agent’s income is $s$, the probability that the next period’s income $s'$ will lie in the set $B \in \Sigma$ is denoted by $P(s, B) = \int_B P(s, ds')$.}

We assume that the Markov kernel $P$ has the Feller property, i.e., for any bounded, continuous function $v : S \to \mathbb{R}$, the function $\int_S v(s') P(s, ds')$ is continuous.\footnote{The assumption that $P$ has the Feller property is a standard technical assumption that ensures that the value function is continuous.}

2.2 The agent’s consumption-savings problem

Suppose that before the process starts, the agent has savings of $a(1)$. In each period $t = 1, 2, \ldots$, the agent receives an income $s(t)$. After receiving his income, the agent decides how much to consume in that period, $c(t)$, and how much to save $a(t + 1)$ in a risk-free bond for future consumption. The agent’s savings rate of return is $R$. If the agent’s savings just before time $t$ is $a(t)$, his savings just before time $t + 1$ are

$$a(t + 1) = Ra(t) - c(t) + s(t).$$

We assume that the agent cannot borrow and thus $a(t) \geq 0$ in each period. We denote by $C(a, s) = [0, Ra + s]$ the interval from which the agent may choose his level of consumption when his savings are $a$ and his income is $s$. 

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2.3 The value function

The agent’s utility from consumption in each period is given by a bounded function $u : [0, \infty) \rightarrow [0, M]$. We assume that $u$ is strictly increasing, strictly concave, continuously differentiable and that $u'(0) = \infty$.

Let $0 < \beta < 1$ be the agent’s discount factor and let $A = [0, \infty), Z = A \times S$.

For each initial state $z(1) = (a, s)$, a consumption plan$^{10}$ $\pi$ and a Markov kernel $P$ induce a probability measure over the space of all infinite histories.$^{11}$ We denote the expectation with respect to that probability measure by $E_\pi$.

When the agent follows a consumption plan $\pi$, his expected present discounted value is

$$V_\pi(a, s) = E_\pi\left(\sum_{t=1}^{\infty} \beta^{t-1} u(\pi(z(1), \ldots, z(t)))\right),$$

where $z(1) = (a, s)$ is the initial state. Denote

$$V(a, s) = \sup_\pi V_\pi(a, s).$$

We call $V : Z \rightarrow \mathbb{R}$ the value function and a consumption plan $\pi$ attaining it optimal.

2.4 The Bellman equation

We denote the agent’s savings in the next period by $b$. For any $z = (a, s) \in Z$ and $b \in C(a, s)$ define the following function,

$$h(a, s, b, V) = u(Ra + s - b) + \beta \int_S V(b, s') P(s, ds'). \tag{1}$$

Let $B(Z)$ be the space of all bounded real valued functions defined on $Z$. Define the operator $T : B(Z) \rightarrow B(Z)$ by

$$Tf(a, s) = \max_{b \in C(a, s)} h(a, s, b, f).$$

$^{10}$Let $Z^t := Z \times \ldots \times Z$ be the space of all finite savings-income histories of length $t$. A consumption plan $\pi$ is a function that assigns to every finite history a feasible level of consumption.

$^{11}$The probability measure on the space of all infinite histories $Z^\mathbb{N}$ is uniquely defined (see for example Bertsekas and Shreve (1978)).
Standard dynamic programming arguments (e.g., Blackwell (1965)) show that the value function \( V \) is the unique fixed point of \( T \), i.e., there is a unique function \( V \in B(Z) \) such that \( TV = V \). The equation \( TV = V \) is called the Bellman equation.

### 2.5 The optimal stationary savings policy

Theorem 9.8 in Stokey and Lucas (1989) shows that the value function \( V \) is continuous and concave. Thus \( h \) itself is strictly concave in \( b \), and as such has a unique maximum. Define \( g(a,s) \) to be the savings level of the next period that attains the maximum of \( h(a,s,b,V) \). That is,

\[
g(a,s) = \arg\max_{b \in C(a,s)} h(a,s,b,V).
\] (2)

The argmax function defined in (2) is called the savings policy function. Note that the savings policy function induces an optimal consumption policy function \( \sigma(a,s) \) by the equation \( \sigma(a,s) = Ra + s - g(a,s) \).

### 3 Monotone and convexity-preserving Markov kernels

Let \( f \in B(Z) \). Define the operator \( Mf \) by

\[
Mf(a,s) = \int_S f(a,s') P(s,ds').
\]

Given a class of functions \( \Phi \subseteq B(Z) \) we would like to know whether \( f \in \Phi \) implies \( Mf \in \Phi \). Since \( P \) is a Markov kernel, it is clear that if \( f \in B(Z) \) then \( Mf \in B(Z) \). Furthermore, since \( P \) has the Feller property, if \( f \) is continuous then \( Mf \) is also continuous.\(^{12}\)

**Definition 1** Let \( \Phi_M \subseteq B(Z) \) be the class of all monotone functions. We say that \( P \) is monotone if \( f \in \Phi_M \) implies that \( Mf \in \Phi_M \).

A sufficient condition for the monotonicity of \( P \) is that \( P(s_2, \cdot) \) first order stochastically dominates \( P(s_1, \cdot) \) for all \( s_2 \geq s_1 \). For comprehensive coverage of stochastic orders and their applications, see Shaked and Shanthikumar (2007). The assumption that \( P \) is monotone is natural in the context of

income dynamics. $P$ being monotone implies that moving to high incomes in the next period from a current high income is more likely than from a current low income.

**Definition 2** (i) Let $\Phi_{CX} \subseteq B(Z)$ be the class of all continuous convex functions. We say that $P$ is convexity-preserving (CXP) if $f \in \Phi_{CX}$ implies that $Mf \in \Phi_{CX}$.

(ii) Let $\Phi_{CV} \subseteq B(Z)$ be the class of all continuous concave functions. We say that $P$ is concavity-preserving (CVP) if $f \in \Phi_{CV}$ implies that $Mf \in \Phi_{CV}$.

Our main result (Theorem 2) states that if the agent is prudent (i.e., $u'$ is convex); the Markov kernel $P$ is monotone and CXP or CVP; and the Markov kernel $Q$ is riskier than the Markov kernel $P$ (in the sense of second order stochastic dominance, see Definition 4); then the agent saves more under the riskier Markov kernel $Q$ than under $P$ for every savings-income pair. This theorem means that in a Markovian income environment, convexity of the marginal utility alone does not necessarily imply precautionary saving. We need assumptions on the Markov kernel that governs the income dynamics. This is not surprising, as Huggett (2004) notes that in a Markovian earnings environment a change in the current income can significantly influence the expected present value of all future incomes. In Section 4.2 we will address the question: when is the Markov kernel both monotone and CXP or CVP (implying precautionary saving)?

### 4 Main results

In this section we present our main results regarding precautionary saving in a Markovian income environment. Throughout this section we slightly abuse the notations and when the Markov kernel governing the income dynamics is $P$ we add it as a parameter. For instance, the value function of the parameterized consumption-savings problem presented in Section 2 is denoted by

$$V(a, s, P) = \max_{b \in C(a, s)} h(a, s, b, P, V).$$

Likewise, the savings policy function is denoted by $g(a, s, P)$, the consumption policy function by $\sigma(a, s, P)$, and $h(a, s, b, P, V)$ is the $h$ function associated with the consumption-savings problem with a Markov kernel $P$ as defined in Eq. (1).
4.1 Value function properties

In this section we discuss the properties of the value function and the value function’s derivative. We show that if $P$ is monotone then the value function derivative\(^{13}\) $V'(a, s)$ is decreasing. We also show that if $V$ is CXP or CVP then $V(a, s)$ is (jointly) concave and if, in addition, $u'$ is convex then $V'(a, s)$ is (jointly) convex.

The proof of Theorem 2 elaborates that in order to prove results regarding precautionary saving, it is essential to prove certain properties of the value function’s derivative. Convexity of $V'$ is a crucial property in proving Theorem 2. In general, proving properties of the value function’s derivative is more difficult than proving properties of the value function. The standard way to prove properties of the value function relies on the Banach-fixed point theorem. The idea is to prove that if $f \in D \subseteq B(Z)$ then $Tf \in D$. The Banach-fixed point theorem implies that the sequence $f_n = T^n f$ converges to the value function $V$, so if $D$ is closed under pointwise convergence then $V \in D$. Lemma 1 in the Appendix uses the envelope theorem and the convergence of the policy function to utilize a similar technique to prove certain properties of the value function’s derivative.

Since $V(a, s)$ is concave in $a$, it is immediate that $V'(a, s)$ is decreasing in $a$ for all $s \in S$. Theorem 1 part (ii) shows that if $P$ is monotone then $V'(a, s)$ is decreasing in $s$ for all $a \in A$. This result also implies that the consumption policy function $\sigma(a, s)$ is increasing in the income $s$ for every savings level $a \in A$.

The most important part of Theorem 1 is part (iii). We show that if $P$ is CXP or CVP and $u'$ is convex, then $V'$ is also convex. Convexity of $V'$ is a crucial property in proving Theorem 2.

The next theorem summarizes the above discussion. The proof is in the Appendix.

**Theorem 1**  
(i) Assume that $P$ is CXP or CVP. Then $V(z)$ is concave.

(ii) Assume that $P$ is monotone. Then $V'(z)$ is decreasing and the consumption policy function $\sigma(a, s)$ is increasing in $s$ for all $a \in A$.

(iii) Assume that $u'$ is convex and $P$ is CXP or CVP. Then $V'(z)$ is convex.

We note that in dynamic economies with a framework similar to ours there is a result that proves monotonicity of the policy function. The result is Proposition 2 in Hopenhayn and Prescott (1992) which proves monotonicity using Topkis’s theorem in a more general framework than ours. Hopenhayn and Prescott’s proof requires that $h$ have increasing differences in $(a, s)$, which is not necessarily the case in our model and thus their result cannot be applied to our context.

\(^{13}\) $V'(a, s)$ is the derivative of the value function with respect to $a$. For more details see the Appendix.
4.2 Precautionary saving in a Markovian earnings environment

As noted in the introduction, there are some theoretical results that connect the notion of prudence and the notion of precautionary saving when the earnings process is i.i.d. That is, there is a probability measure \( \theta \) such that \( P(s, B) = \theta(B) \) for all \( s \in S \). Miller (1976) shows that if \( u' \) is convex then the agent’s savings increase if \( \theta \) is riskier than \( \theta' \) in the sense of Definition 3. Thus, prudence implies precautionary saving in the case of i.i.d. earnings.

**Definition 3** We say that a distribution \( \theta \) is riskier than \( \theta' \) if every risk averter prefers \( \theta' \) to \( \theta \). That is, for all bounded, concave and non-decreasing functions \( f \) we have

\[
\int_S f(s') \theta'(ds') \geq \int_S f(s') \theta(ds').
\]

In a Markovian environment, we say that a Markov kernel \( Q \) is riskier than a Markov kernel \( P \) if for any state \( s \in S \), \( P(s, \cdot) \) second order stochastically dominates \( Q(s, \cdot) \).

**Definition 4** We say that a Markov kernel \( Q \) is riskier than a Markov kernel \( P \) if for all bounded, concave and non-decreasing functions \( f \) and for each \( s \in S \) we have

\[
\int_S f(s') P(s, ds') \geq \int_S f(s') Q(s, ds').
\]

The next theorem is our main theorem about precautionary saving in Markovian income dynamics. We show that if \( u' \) is convex and the Markov kernel \( P \) is monotone and CXP or CVP, then the agent’s savings increase in response to higher income uncertainty. The proof is deferred to the Appendix.

**Theorem 2** Assume that \( u' \) is convex and that \( P \) is monotone and CXP or CVP. If \( Q \) is riskier than \( P \) then \( g(a, s, Q) \geq g(a, s, P) \) for all \( (a, s) \in Z \).

In the literature and in applications it is common to decompose the income process into a permanent component and a transitory component. Specifically, the permanent component usually follows a random walk and the transitory component is serially uncorrelated. A natural way to decompose the income process is the following. Say that the Markov kernel \( P \) is decomposed by a function \( m \) if

\[
\int_S f(a, s') P(s, ds') = \int_E f(a, m(s, \epsilon)) \mu(d\epsilon),
\]
where \( m : S \times E \rightarrow S \) is a continuous function and \( \epsilon \) is a random variable that takes values in \( E \) and has a distribution \( \mu \). That is, \( P \) is decomposed if for all \( B \in \Sigma \) and \( s \in S \) we have \( P(s, B) = \mu \{ m(s, \epsilon) \in B \} \). The next period’s income \( s' \) equals \( m(s, \epsilon) \). The function \( m \) decomposes the income process into a permanent component and a transitory component. The current income \( s \in S \) is the permanent component of the income process and \( \epsilon \) is the transitory component of the income process.

Most of the literature on earnings dynamics focuses on linear models (see Meghir and Pistaferri (2011), Guvenen (2009), De Nardi et al. (2016) and references therein).

For example, the earnings dynamics are modeled as an AR(1) process in many applications, i.e., \( s' = ks + \epsilon \) where \( k > 0 \) and \( \epsilon \) is a random variable. To avoid negative incomes we can let \( m(s, \epsilon) = \max\{ks + \epsilon, 0\} \). Then \( m \) is convex and increasing in \( s \) which implies that \( P \) is monotone and CXP. Another example of linear earnings dynamics is a decomposed Markov kernel with \( m(s, \epsilon) = s\epsilon \) where \( \epsilon \) is a positive random variable. More generally, if \( \overline{m}(s, \epsilon) \) is convex and increasing in \( s \), then \( m(s, \epsilon) = \max\{\overline{m}(s, \epsilon), 0\} \) is also convex and increasing in \( s \). This simple fact allows flexibility in choosing a function that decomposes a monotone and CXP Markov kernel.

To see that if \( m \) is increasing and convex in \( s \) then \( P \) is monotone and CXP, assume that \( f \) is convex and increasing. This implies that for all \( \epsilon' \in E \) the function \( f(a, m(s, \epsilon')) \) is also convex and increasing in \( (a, s) \) as a composition of two convex and increasing functions. Since convexity is preserved under integration,

\[
Mf(a, s) = \int_S f(a, s') P(s, ds') = \int_E f(a, m(s, \epsilon)) \mu(d\epsilon)
\]

is also convex and increasing. A similar argument shows that if \( m \) is increasing and concave in \( s \) then \( P \) is monotone and CVP. An important example of a monotone and CVP Markov kernel is a stationary AR(1) in logs. That is, \( m(s, \epsilon) = sk^e \) for some \( k \in (0, 1) \). Thus, we established the following Corollary:

**Corollary 1** Assume that \( u' \) is convex and that the earnings dynamics \( P \) follows a standard AR(1) process or an AR(1) process in logs. If \( Q \) is riskier than \( P \) then \( g(a, s, Q) \geq g(a, s, P) \) for all \( (a, s) \in Z \).
4.3 Unbounded utility function

In many applications it is assumed that the utility function is unbounded. More specifically it is assumed that the utility function is in the HARA class.\textsuperscript{14} In the case of an unbounded utility function, the dynamic programming techniques that are used in this paper cannot be applied unless one shows that the value function is bounded. When $\beta R < 1$, an approach for deriving bounds on the value function which guarantees that savings will remain in a bounded set is developed in Li and Stachurski (2014).\textsuperscript{15} From the discussion in Section 4.2 and from Theorem 2 we have the following:

**Corollary 2** Assume that $u$ is in the HARA class and the agent is prudent,\textsuperscript{16} i.e., $u'''u'/(u'')^2 = k$ for $k \geq 0$ and assume that $P$ is monotone and CXP or CVP. If $Q$ is riskier than $P$ and $\beta R < 1$ then $g(a, s, Q) \geq g(a, s, P)$ for any $(a, s) \in A \times S$.

5 Summary

We examine the effect of earnings uncertainty on consumption and savings decisions in an income fluctuation problem with Markovian earnings dynamics. The conditions we give on the earnings dynamics ensure that when the agent is prudent, his response to riskier future earnings dynamics is to save more. These conditions are trivially satisfied when the distribution of earnings in each period is independent from previous earnings. We extend the known result that prudence implies precautionary saving to Markovian earnings dynamics.

Our results rely on certain properties of the value function’s derivative. We use a simple argument to prove these properties of the value function’s derivative. This argument can be used in other dynamic programs with concavity assumptions (e.g., models like those in Stokey and Lucas (1989) Chapters 9 and 10).

\textsuperscript{14}A utility function $u$ is in the HARA class if the absolute risk aversion is hyperbolic, i.e., $A(c) := -\frac{u''(c)}{u'(c)} = \frac{1}{ac+b}$. Carroll and Kimball (1996) show that the HARA utility functions can be described as the class of functions that satisfy $u'''u'/(u'')^2 = k$ for some $k \in \mathbb{R}$.

\textsuperscript{15}Acikgoz (2016) extends this approach and uses it to prove the existence of equilibrium in Aiyagari’s model with the unbounded CRRA utility function. Light (2017) uses this approach to prove the uniqueness of equilibrium in Aiyagari’s model under certain conditions.

\textsuperscript{16}This includes the important case of a CRRA utility function.
6 Appendix

We first discuss some results concerning the value function’s derivative that will be used in proving Theorem 1 and Theorem 2.

For $f \in B(Z)$ define

$$f'(a, s) := \frac{\partial f(a, s)}{\partial a}.$$ 

In the rest of the Appendix we say that $f \in B(Z)$ is differentiable if it is differentiable in the first argument. Denote by $u'$ the derivative of $u$. Let $g_f(a, s) = \arg\max_{b \in C(a, s)} h(a, s, b, f)$ and $\sigma_f(a, s) = Ra + s - g_f(a, s)$ for all $(a, s) \in Z$.

The envelope theorem (see Benveniste and Scheinkman (1979) for a proof) implies that $Tf$ is differentiable.

In addition, if $\sigma_f(a, s) > 0$ (which will always be the case in this paper since $u'(0) = \infty$, see Aliprantis et al. (2008)), then

$$(Tf)'(a, s) = Ru'(\sigma_f(a, s)).$$

The last equation is called the envelope condition. Berge’s maximum theorem (see Aliprantis and Border (2006) Theorem 17.31) implies that $\sigma_f$ is continuous. Thus, $(Tf)'$ is continuous as a composition of continuous functions. Since $TV = V$, the value function is continuously differentiable and $V'(a, s) = Ru'(\sigma(a, s))$. 

Let $h'(a, s, b, f)$ be the derivative of $h$ with respect to $b$. If $g_f(a, s) \in \text{int} C(a, s) := (0, Ra + s)$ and $f$ is differentiable, then the optimal solution $g_f(a, s)$ must satisfy the first order condition $h'(a, s, g_f(a, s), f) = 0$. That is,

$$-u'(Ra + y(s) - g_f(a, s)) + \beta \int_S f'(g_f(a, s), s') P(s, ds') = 0.$$ 

The last equation and the envelope condition imply

$$(Tf)'(a, s) = \beta R \int_S f'(g_f(a, s), s) P(s, ds').$$

The following Lemma helps to prove properties of the value function’s derivative and will be used in proving Theorem 1 and Theorem 2.

**Lemma 1** Let $B'(Z) \subseteq B(Z)$ be the set of all concave (in the first argument) and continuously
differentiable functions. Let $E$ be a subset of $\mathbb{R}^Z$. Assume that $E$ satisfies the following three properties:

1. If $f' \in E$ and $Tf \in B'(Z)$ then $(Tf)' \in E$.
2. If $f'_n \in E$ for every $n$, $f_n \to f_\infty$, $f_\infty \in B'(Z)$, and $f'_n \to f'_\infty$, then $f'_\infty \in E$.
3. There exists $f : Z \to \mathbb{R}$ such that $f \in B'(Z)$ and $f' \in E$.

Then $V' \in E$.

**Proof.** Let $f \in B'(Z)$ and $f' \in E$. A standard argument shows that $Tf$ is concave in the first argument. The envelope theorem implies that $Tf \in B'(Z)$. From (1), $(Tf)' \in E$. Define $f_n = T^n f := T(T^{n-1} f)$ for $n = 1, 2, \ldots$ where $T^0 f := f$. We conclude that $f'_n \in E$ for every $n$.

From the Banach fixed-point theorem $f_n \to V$. The envelope condition implies that $f'_n = u'(\sigma_{f_n})$ for every $n$. Theorem 3.8 and Theorem 9.9 in Stokey and Lucas (1989) show that $\sigma_{f_n} \to \sigma$. Thus, $f'_n(a, s) = u'(\sigma_{f_n}(a, s)) \to u'(\sigma(a, s)) = V'(a, s)$. From (2) we have $V' \in E$ which proves the Lemma.

**Theorem 1** (i) Assume that $P$ is CXP or CVP. Then $V(z)$ is concave.

(ii) Assume that $P$ is monotone. Then $V'(z)$ is decreasing and the consumption policy function $\sigma(a, s)$ is increasing in $s$ for all $a \in A$.

(iii) Assume that $u'$ is convex and $P$ is CXP or CVP. Then $V'(z)$ is convex.

**Proof.** (i) Assume that $P$ is CXP. A standard argument shows that we only need to prove that if $f \in \Phi_{CV}$ then $Tf \in \Phi_{CV}$ in order to prove that $V \in \Phi_{CV}$, i.e., to prove that the value function is jointly concave in $a$ and $s$.

Assume that $f \in \Phi_{CV}$. The function $f$ is convex if and only if $-f$ is concave. Thus, $P$ being convexity-preserving implies that $Mf \in \Phi_{CV} \cdot u((R, 1) \cdot z - b)$ where $(R, 1) \cdot z := Ra + s$ is jointly concave in $z$ and $b$ as a composition of a concave function with linear function. Thus, $h(z, b, f)$ is jointly concave in $z$ and $b$ as the sum of two concave functions. Proposition 2.3.6 in Bertsekas et al. (2003) implies that $Tf(z) = \max_{w \in C(z)} h(z, b, f)$ is concave. That is, $Tf \in \Phi_{CV}$.

(ii) For part (ii) it will be convenient to change the decision variable in equation (1) from savings to consumption. We denote the consumption of the agent by $c$ and use the relation $b = Ra + s - c$.

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17 $\mathbb{R}^Z$ is the set of all functions from $Z$ to $\mathbb{R}$.
19 Recall that $\Phi_{CV} \subseteq B(Z)$ is the set of all (jointly) concave and continuous functions. Then $\Phi_{CV}$ is a closed subset of the complete metric space $(B(Z), d)$ where $d$ is the metric induced by the sup-norm. Let $f \in \Phi_{CV}$ and define the sequence $f_n = T^n f$ for $n \geq 1$. If $Tf \in \Phi_{CV}$ then $f_n \in \Phi_{CV}$ for every $n$. From the Banach fixed-point theorem $f_n \to V$. $\Phi_{CV}$ being closed implies $V \in \Phi_{CV}$.
20 Recall that $M f(a, s) = \int f(a, s')(P(s, ds')$. 

13
Similarly to Section 2.4, the resulting Bellman operator can be formulated by letting $J(a, s, c, f) := u(c) + \beta \int_S f(Ra + s - c, s')P(s, ds')$ and defining the operator $K : B(Z) \to B(Z)$ by

$$Kf(a, s) := \max_{c \in C(a, s)} J(a, s, c, f)$$

for all $(a, s) \in Z$. Define $\sigma_f(a, s) := \arg\max_{c \in C(a, s)} J(a, s, c, f)$.

Let $f \in B(Z)$ be concave (in the first argument) and continuously differentiable, and assume that $f'(a, s)$ is decreasing in $s$ for all $a \in A$. The envelope theorem and Berge’s maximum theorem imply that $Kf$ is continuously differentiable. From Lemma 1 we only need to show that $(Kf)'(a, s)$ is decreasing in $s$ in order to prove that $V'(a, s)$ is decreasing in $s$.

Let $a \in A, c_2 \geq c_1$ and $s_2 \geq s_1$. Define $x := Ra + s_2 - c_2, t := c_2 - c_1$ and $y := Ra + s_1 - c_2$. Since $x \geq y$ and $t \geq 0$, concavity of $f$ implies that $f(x, s) - f(x + t, s) \geq f(y, s) - f(y + t, s)$ for all $s \in S$. Thus, for all $s \in S$ we have

$$\int_S (f(x, s') - f(x + t, s'))P(s, ds') \geq \int_S (f(y, s') - f(y + t, s'))P(s, ds').$$

(3)

$f'(a, s)$ decreasing in $s$ implies that $f(x, s) - f(x + t, s)$ is increasing in $s$. Since $P$ is monotone, the function

$$\int_S (f(x, s') - f(x + t, s'))P(s, ds')$$

is increasing in $s$. Thus, inequality (3) and the fact that function (4) is increasing in $s$ yield

$$\int_S (f(x, s') - f(x + t, s'))P(s_2, ds')$$

$$\geq \int_S (f(y, s') - f(y + t, s'))P(s_1, ds').$$

Multiplying by $\beta$ and adding $u(c_2) - u(c_1)$ to each side of the last inequality yield

$$J(a, s_2, c_2, f) - J(a, s_1, c_2, f) \geq J(a, s_2, c_1, f) - J(a, s_1, c_1, f).$$

That is, $J$ has increasing differences in $(s, c)$. Theorem 6.1 in Topkis (1978) implies that $\sigma_f(a, s)$ is increasing in $s$. Concavity of $u$ and the envelope condition imply that

$$(Kf)'(a, s_1) = Ru'(\sigma_f(a, s_1)) \geq Ru'(\sigma_f(a, s_2)) = (Kf)'(a, s_2).$$

(5)
Thus, \((Kf)'\) is decreasing in \(s\) which implies that \(V'(a, s)\) is decreasing in \(s\). The same argument as in inequality (9) shows that \(\sigma(a, s)\) is increasing in \(s\) for all \(a \in A\).

(iii) Let \(f \in B(Z)\) be concave (in the first argument) and continuously differentiable with a convex derivative (that is, \(f'(a, s)\) is jointly convex in \(a\) and \(s\)). Since the pointwise limit of a sequence of convex functions is convex, Lemma 1 implies that we only need to show that if \(f'\) is convex then \((Tf)'\) is convex in order to prove that \(V'\) is convex.

Assume in contradiction that \((Tf)'\) is not convex. Then there exists \(z_1 = (a_1, s_1), z_2 = (a_2, s_2)\) and \(\lambda \in [0, 1]\) such that for \(z_\lambda = \lambda z_1 + (1 - \lambda) z_2\) we have

\[
(Tf)'(z_\lambda) > \lambda(Tf)'(z_1) + (1 - \lambda)(Tf)'(z_2). \tag{6}
\]

From the envelope condition we have \((Tf)'(z_i) = Ru'(\sigma_f(z_i))\) for \(i = 1, 2, \lambda\). Thus, inequality (6) and the convexity of \(u'\) imply

\[
u'(\sigma_f(z_\lambda)) > \lambda u'(\sigma_f(z_1)) + (1 - \lambda) u'(\sigma_f(z_2))
\geq u'(\lambda \sigma_f(z_1) + (1 - \lambda) \sigma_f(z_2)).
\]

Since \(u'\) is decreasing, the last inequality implies that \(\lambda \sigma_f(z_1) + (1 - \lambda) \sigma_f(z_2) > \sigma_f(z_\lambda)\). That is,

\[
g_f(z_\lambda) > \lambda g(z_1) + (1 - \lambda) g(z_2). \tag{7}
\]

The envelope theorem implies

\[
(Tf)'(z_i) \geq \beta R \int_S f'(g_f(z_i), s')P(s_i, ds'), \tag{8}
\]

for \(i = 1, 2\). From inequality (7) we have \(g_f(z_\lambda) > 0\). Thus,

\[
(Tf)'(z_\lambda) = \beta R \int_S f'(g_f(z_\lambda), s') P(\lambda s_1 + (1 - \lambda)s_2, ds').
\]
If $P$ is CXP we have
\[
(Tf)' (z_\lambda) > \lambda \beta R \int_S f'(g_f (z_1), s') P(s_1, ds') + (1 - \lambda) \beta R \int_S f'(g_f (z_2), s') P(s_2, ds') \\
\geq \beta R \int_S f'(\lambda g_f (z_1) + (1 - \lambda) g_f (z_2), s') P(\lambda s_1 + (1 - \lambda)s_2, ds') \\
\geq \beta R \int_S f'(g_f (z_\lambda), s') P(\lambda s_1 + (1 - \lambda)s_2, ds') = (Tf)' (z_\lambda),
\]
which is a contradiction. The first inequality follows from combining inequality (6) with inequality (8). The second inequality follows from the convexity of $f'$ and the assumption that $P$ is CXP, which imply the convexity of $\int f'(a, s') P(s, ds')$. The third inequality follows from the fact that $f'$ is decreasing in the first argument and from inequality (7).

If $P$ is CVP we can get a similar contradiction by applying the last inequalities to $-Tf'$.

Thus $(Tf)'$ is convex, which implies that $V'$ is convex. 

**Theorem 2** Assume that $u'$ is convex and that $P$ is monotone and CXP or CVP. If $Q$ is riskier than $P$ then $g(a, s, Q) \geq g(a, s, P)$ for all $(a, s) \in Z$.

**Proof.** Let $f(a, s, P)$ and $f(a, s, Q)$ be bounded, strictly concave and continuously differentiable functions. Assume that $f'(a, s, P)$ is convex and decreasing in $s$ and that $f'(a, s, Q) \geq f'(a, s, P)$ for all $(a, s) \in Z$.

Let $(a, s) \in Z$ and $b \in C(a, s)$. Since $Q$ is riskier than $P$ and $-f'(a, s, P)$ is concave and increasing in $s$ we have
\[
- \int_S f'(b, s', P) P(s, ds') \geq - \int_S f'(b, s', P) Q(s, ds') \\
\geq - \int_S f'(b, s', Q) Q(s, ds').
\]

Multiplying by $\beta$, adding $-u' (Ra + s - b)$ to each side of the last inequality and rearranging yield
\[
 h'(a, s, b, Q, f) \geq h'(a, s, b, P, f).
\]

The last inequality together with Theorem 6.1 in Topkis (1978) imply that $g_f (a, s, Q) \geq g_f (a, s, P)$ for all $(a, s) \in Z$. Thus, $\sigma_f (a, s, P) \geq \sigma_f (a, s, Q)$. Concavity of $u$ and the envelope condition imply that
\[
(Tf)'(a, s, Q) = Ru' (\sigma_f (a, s, Q)) \geq Ru' (\sigma_f (a, s, P)) = (Tf)'(a, s, P). \tag{9}
\]
Thus, \((T_f)'(a, s, Q) \geq (T_f)'(a, s, P)\) for all \((a, s) \in Z\). From Theorem 1 we know that \((T_f)'(a, s, P)\) is convex and decreasing in \(s\). Lemma 1 shows that \(V'(a, s, Q) \geq V'(a, s, P)\) for all \((a, s) \in Z\). Using the same argument as above, it follows that \(g(a, s, Q) \geq g(a, s, P)\) for all \((a, s) \in Z\).

References


