MAJORITY STABILITY: COMBINING SOCIAL CHOICE AND MATCHING THEORY TO UNDERSTAND INSTITUTIONAL STABILITY

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Abstract. I develop a new concept of institutional stability which is particularly relevant for applications of matching theory in political economy, where the choice of the institution (i.e., the matching mechanism) is chosen and agreed upon by the very people who are assigned by the allocation procedure. In such settings, a natural question arises: when would the choice of the matching mechanism be institutionally stable? In other words, in what environments would the set of agents being assigned prefer or vote in favor of the allocation the existing matching mechanism delivers over any alternative allocation. This is a generalization of social choice to what I call a social allocation choice problem. I discuss a variety of voting rules (plurality, majority, and unanimity) and their stability counterparts in matching theory (popular matching, majority stability, and pareto efficiency). The novel property of majority stability is introduced, which is more relevant in practice, i.e., for considering stability in majoritarian institutions such as the US Senate. I find that certain mechanisms commonly used in practice, such as Serial Dictatorship, are surprisingly robust to highly correlated preferences under majority stability, especially in comparison to popular matching which places highly restrictive requirements on preferences of individuals and details of the assignment procedures. Unlike traditional social choice, where stability is undermined by cycles in preferences, in this application I show that chains of envy undermine majority stability. Chains of envy are crucial to overcome the packing problem which arises in making a majority strictly better off under an alternative allocation. While correlation across agents’ preferences helps ensure stability in traditional social choice sense by making cycles less prevalent, in my setting, higher correlation across agents’ preferences leads to a higher likelihood of chains of envy, thereby undermining institutional stability.

1. Introduction.

Understanding the choice and the stability of an institution—“humanly devised constraints that structure political, economic and social interactions” (North (1991) Institutions)—is important to the history, evolution, longevity, and performance of organizations. One such institutional choice, is that of the matching mechanism used to make allocation and assignment decisions in political economy, e.g. assigning politicians to committees (Thakur 2018a) or assigning civil servants to various parts of the country (Thakur 2018b). A unique feature of these institutional choices, is that the matching mechanism is chosen and agreed upon by the very agents it allocates. This raises an important question unique to political economy: under what conditions is the choice of matching mechanism itself stable? In other words, under what environments would the participants in the matching mechanism...
choose an alternative allocation or mechanism over the allocation generated by their current mechanism?

Addressing this question requires a generalization of social choice. In social choice, a single choice is chosen for a group of agents, each of whom might have varied preferences on what they prefer. This paper generalizes social choice to a social allocation choice over a vector of assignments, where each individual receives their own assignment. Agents have varied preferences over their own allocation, and collectively vote to decide upon whether to use the matching mechanism at hand, i.e., vote in favor of/against the allocation the matching mechanism produces relative to an alternative mechanism/allocation. Moreover, this notion of institutional stability in matching applications in political economy is quite different from the stability notion in canonical matching theory applications such as school choice or residency matching. The notion of stability in canonical matching theory applications, deals with whether the allocation produced by the mechanism has a blocking pair, or a set of agents (and assignments) who would mutually benefit from swapping their assignments. Compared to this more local, near-by allocation notion of stability, this paper considers institutional stability which asks whether the choice of mechanism itself is stable when comparing across all alternative allocations and alternative mechanisms.

A small yet growing literature on Popular Matching and its many variants has been developed in the computer science literature. This literature deals with the plurality rule as the choice rule in discriminating between allocations and considers the existence, characterization, and complexity of finding a Condorcet winner given one-sided and two-sided voting in bipartite graphs and non-bipartite graphs. This literature’s focus on the algorithmic complexity has often rendered it as a normatively appealing, yet abstract theoretical solution concept, that hasn’t received from much attention from the economics and applied market design literature. Moreover, in this paper I show that the existence of Popular matching (i.e., plurality rule) necessitates very stringent requirements on the preferences of individuals and the choice of matching mechanisms. Slight correlation across individuals’ preferences renders popular matching non-existent and any matching mechanism unstable. On the other extreme, Pareto Efficiency, which is akin to unanimity rule, is shown to be quite non-discriminating in many circumstances. In practice, many institutions are majoritarian, using majority or super-majority rules to agree upon or overturn certain group choices. And in fact, I find that many common mechanisms used in practice, such as the Serial Dictatorship, are quite stable and robust for a large parameter space, e.g., despite significant correlation across individuals’ preferences. Finally, considering the stability of the institutional choice itself, shares the spirit of self-stable voting rules and constitutional choice literature.

In this paper, I compare a variety of voting rules (plurality, majority, and unanimity) and their corresponding notions of stability in matching (popular matching, majority stability, and pareto efficiency). The property of Majority stability is introduced. The robustness of the stability of these various voting rules is demonstrated across various levels of correlation across individuals’ preferences through simulations. I then discuss and explain why popular

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1See review article by Manlove (2013) and Cseh (2017) for review articles on the Popular matching.
2Abraham et al. (2007), Sng and Manlove (2010), Manlove (2013), McDermid and Irving (2011), Kavitha and Nasre (2009), McCutchen (2008), Kavitha et al. (2011)
4Chung (2000), Biro et al. (2010), Huang and Kavitha (2017)
matching easily fails to exist, whereas majority stability is robust across many environments and a wide parameter space. The key feature restricting majority overrules is the packing problem which requires chains of envy. This highlights Deep Correlation across preferences of individuals. I then provide a heuristic algorithm for a common mechanism, Serial Dictatorship, which utilizes the idea of Shapley values for each individual which captures how many other individuals a particular individual precludes or blocks from being better off.

2. Voting rules and their corresponding notions of institutional stability.

The criteria for institutional stability, depends on the voting rule chosen to decide between allocations/mechanisms. In this section we introduce three stark notions of voting rules (plurality rule, majority rule and unanimity rule) and their corresponding institutional stability counterparts (Popular matching, majority stability, and pareto efficiency).

Let us consider a set of politicians $P = 1,\ldots,N$ and a set of committees $C = 1,\ldots,C$ indexed by $p$ and $c$. The set of preference relations $\succeq_i$ for all agents $i$ on either side of the market over being matched to the opposite side of the market, are assumed to be complete and transitive. If an agent $i$ is not indifferent between two allocations (i.e., $\sim_i$), then we say he has a strict preference ($\succ_i$). If agent $i$ (on either side of the market) prefers to remain unmatched rather than by matched to $j$, i.e., if $i \succ_j j$, then $k$ is said to be unacceptable to $i$.

An outcome of the game is a matching $M : P \cup C \rightarrow P \cup C$ such that $p = M(c)$ if and only if $M(p) = c$ and for all $p$ and $c$ either $M(p)$ is in $C$ or $M(p) = p$, and either $M(c)$ is in $P$ or $M(c) = c$. I.e., a politician is matched to a committee only if the committee is matched to him, and everyone is matched to a counterpart from the other side of the market or is unmatched (i.e., matched to one’s self).

Agent $i$’s preference over two allocations $M$ and $M'$ is defined by $M \succeq_i M'$.

**Definition 1.** A social allocation choice defines a preference relation $\succeq^S$ between any two matching allocations $M$ and $M'$ which aggregates the preferences of all individuals $p \in P$ given some preference aggregation rule $f$.

In this paper, we make three simplifying assumptions. First, we assume one-sided matching market where only politicians $p \in P$ have preferences over the $C$ committees; however, committees do not have any preference over politicians. Second, we simplify individual $i$’s preference over matchings by assuming $i$ only cares about his own assignment: $M \sim_i M'$ if $M(i) \sim_i M'(i)$ and $M \sim_i M'$ if $M(i) \succ_i M'(i)$. Let us denote by $|M \succ_i M|$, the number of individuals who strictly preferred matching $M$ over matching $M'$. individuals preference. Lastly, we deal with preference aggregation rules $f$, that are simple rules$^6$.

**Definition 2.** A matching $M$ is popular if $\exists M' \text{ s.t. } |M' \succ_i M| > |M \succ_i M'|$

**Definition 3.** Matching $M$ is majority stable$^7$ if $\exists M' \text{ s.t. } |M' \succ_i M| \geq \frac{N+1}{2}$.

**Definition 4.** Matching $M$ is pareto efficient if $\exists M' \text{ s.t. } |M \succ_i M'| = 0 \text{ and } |M' \succ_i M| > 0$

$^6$See Austen-Smith and Banks (2000) chapters 1-3 for a careful discussions and definitions of choice functions, preference aggregation rules, voting rules, decisive coalitions, and simple rules. In this paper, I omit carefully building up these definitions from scratch.

$^7$More generally, we can characterize any $Q$-rule by replacing $\frac{N+1}{2}$ by $Q$. 
With the maintained assumption in the voting rule that all agents who are indifferent between two allocations abstain from voting, Popular matching corresponds to a voting rule \( f \) that is plurality rule and a matching is popular if it is a Condorcet winner with regards to any alternative matching. Majority stability corresponds to a voting rule \( f \) being a majority rule and Pareto efficiency corresponds to a \( f \) being unanimity rule. We can also define a matching \( M \) to be weak pareto efficient if \( \nexists M' \) s.t. \( |M' \succ_i M| = N \).

As shown in Figure 1, popular matchings \( \subset \) majority stable matching and popular matchings \( \subset \) pareto efficient matchings, however, the sets of majority stable and pareto efficient matchings are not fully contained in each other. Appendix B provides more examples and details.

**Figure 1.** Set inclusions for matchings across various institutional stability notions.

It is important to highlight some remarks to clarify i) how the notion of institutional stability differs from canonical notion of stability from matching theory, ii) how social allocation choice differs from traditional social choice and iii) what the key simplifying assumption is relative to the fully general social allocation choice problem.

First, let us juxtapose the standard notion of stability in matching theory.

**Definition 5.** A matching \( M \) is blocked by an individual \( i \) if \( i \succ_i M(i) \), i.e., \( i \) prefers to be unmatched to being matched with \( M(i) \). A matching is blocked by a pair of agents \((c, p)\) if \( c \succ_p M(p) \) and \( p \succ_c M(c) \), i.e., they each prefer each other to \( M \). A matching \( M \) is stable if it is not blocked by any individual or any pair of agents.

Notice that stability is a more ‘local’ concept. Namely, unstable allocations have pairs of individuals (i.e., blocking pairs) who if locally executed swapped their assignments, would lead to an alternative ‘near-by’ allocation in which both would be better off instead of their current assignment. The notion of institutional stability is a more ‘global’ concept, whereby large coalitions choose in favor of some alternative allocations/mechanisms from the entire space of possible alternative allocations/mechanisms.

Second, we underscore the difference between social choice and social allocation choice.
Definition 6. A social choice defines a preference relation $\succeq^S$ between any two choices $a$ and $b$ (from the set of outcomes $X$) which aggregates the preferences of all individuals $p \in P$ given some preference aggregation rule $f$.

In both cases, individuals have potentially different preferences over the choice set. However, in social choice a single choice is chosen for all individuals, whereas in social allocation choice, the choice is a vector of allocations assigning each individual a potentially different assignment. 

Lastly, the major simplification relative to the fully general social allocation choice problem, is that individual $i$ only cares about his own assignment $a_i$, and not about others' assignments $a_{-i}$. Namely, individual $i$ is indifferent across all matchings $M$ which give him the same allocation $a_i$. This simplification is in line with the classical matching theory set up where preference of individual $i$ is independent of allocation outcomes for all $-i$. The further generalization to include preferences of $i$ over allocations depending on $a_{-i}$ is left for future work.


In this section we the environments under which institutionally stable matchings exist given each of the 3 definitions of institutional stability above.

3.1. Unanimity Rule and Pareto Efficiency. Pareto Efficiency easy to guarantee with a Serial Dictatorship. If there are weak preferences, then tie-breaking can matter, as we see from Example 1.

Example 1. Consider two politicians $\{1, 2\}$ and two committees $\{A, B\}$ each with one seat. Suppose $A \sim_1 B$ and $A \succ_2 B$ and suppose 1 is more senior to 2. Then Serial Dictatorship yields assignment $(1 - A, 2 - B)$ which is pareto dominated by the matching $(1 - B, 2 - A)$.

However, with strict preferences, serial dictatorship is pareto efficient as the mechanism cannot make the $n$th senior person in the seniority better off without making some $j \in \{1, \ldots, n\}$ seniorities worse off, otherwise $j$ would have chosen this alternative himself.

An important caveat to this result is when the matching is dynamic instead of static. Then, in order to retain pareto efficiency, the mechanism must be designed to correctly deal with existing tenants (incumbents) who have an existing committee assignment. These concerns, while abstracted from in this paper, are analyzed in Thakur (2018a), when dealing with seniority and property right norms in the US Senate for assigning politicians to committees.

However, perhaps such an unanimous rule is too non-discriminating a rule. In Example 2, we notice how two senior politicians hog committees $A$ and $B$ and prevent a chain of improvements amongst all $N - 2$ other politicians. Allows senior dictators/hoggers (ex: Southern Democrats)

Example 2. Consider $N$ politicians in order of seniority 1 more senior than 2 ... more senior than $N - 1$ more senior than $N$, and $C = N$ committees $\{A, B, \ldots\}$. Notice how politicians 1 and 2 block an entire sequence of potential improvements for all $N - 2$ others.

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8In the case of many:1 (e.g., civil servant allocation to state cadres (Thakur (2018b)) or many:many matching (e.g., as committee assignments in US Senate (Thakur (2018a)) multiple individuals can be assigned the same assignment.)
Historically, this example might parallel the Southern Democrats, a faction of the Democrat party, who were a handful of senior, well-entrenched incumbents from southern states who occupied powerful seats in important committees and through their seniority and committee assignments could control agenda, gate-keep bills from being voted upon, and influence rules, budgets, and appropriations. Perhaps such a crazy queue caused by just a few senior Southern democrats would not be tolerated by others. Historically, perhaps this is why the Democrats use a Boston Mechanism where Committee on Committees breaks ties (Thakur (2018a)).

3.2. Plurality Rule and Popular Matching. At the other extreme, we show in this section that plurality rule requires extremely stringent conditions on preferences and mechanisms to exist.

Abraham et al. (2007) characterize the existence and structure of popular matching with one-sided voting.

**Lemma 1.** *(Abraham et al. (2007) Lemma 2.4)*

A matching $M$ is popular if and only if

1. every post which is somebody’s 1st choice (called a “f-post”) is matched in $M$ and
2. each applicant $i$ in assigned to either his first choice (i.e, $f(i)$) or his most preferred alternative that is not ranked 1st by any other applicant (i.e., $s(i)$) called a “s-post”

Abraham et al. (2007) also consider popular matchings which are not complete (i.e., some politicians or committees are left unmatched) by introducing the “last resort option” $l(i)$ which is added to each $i$’s preference rank order as their least preferred option. This ensures that the s-post for any individual, $s(i)$, is not empty, and it represents remaining unmatched. In a complete matching, no individual $i$ is assigned to $l(i)$.

Consider Example 3 (from Abraham et al. (2007)) which illustrates all the concepts in Lemma 1.

**Example 3.** Politicians $\{1, 2, 3, 4, 5\}$ rank committees $\{A, B, C, D, E\}$. The highlighted committees are the f- and s-posts for each individual. The set of f-posts are $\{A, B, C\}$ and the set of s-posts are $\{D, l(3), E\}$. Hence there are two complete popular matchings $(1-A, 2-D, 3-B, 4-E, 5-C)$ and $(1-D, 2-A, 3-B, 4-E, 5-C)$ and eight not complete popular matchings $(1-A, 2-D, 3-l(3), 4-B, 5-C)$, $(1-D, 2-A, 3-l(3), 4-B, 5-C)$, $(1-A, 2-D, 3-l(3), 4-B, 5-C)$, $(1-D, 2-A, 3-l(3), 4-B, 5-C)$, $(1-A, 2-D, 3-l(3), 4-B, 5-E)$, $(1-D, 2-A, 3-l(3), 4-B, 5-E)$.

The example above highlighted how popular matchings can have different sizes. We now focus on complete popular matchings in a balanced market (i.e., no politician or committee is left unassigned or vacant) and assume all politicians have complete strict preferences over all committees in $C$. 
Next, we provide some corollaries which illustrate how much structure the Lemma 1 characterization, combined with Hall’s Marriage Theorem, places on the preference environment and matching mechanism for a popular matching to exist or be produced by the matching mechanism.

Corollary 1.1. A necessary but not sufficient condition for a complete popular matching to exist is that \( \{f(a), s(a)\} = C \), where \( C \) is the set of distinct committees to be assigned.

Example 4. Politicians \( \{1, 2, 3, 4, 5, 6\} \) rank committees \( \{A, B, C, D, E, F\} \). Notice that committee \( F \) is no one’s f- or s-post. Hence, no complete popular match exists.

\[
\begin{array}{cccccc}
1 & 2 & 3 & 4 & 5 & 6 \\
A & A & B & B & C & \\
C & D & A & C & E & \\
D & B & C & E & A & \\
& l(1) & l(2) & l(3) & l(4) & l(5) \\
\end{array}
\]

Next we define preferences to have blocks \( B_n \) for \( n = 1, 2, ... \) as the finest partition of committees such that all committees in the \( n \)th block \( B_n \) are ranked below all committees in the \( j \)th block \( B_j \) for all \( j < n \) by all \( i \in P \), and are ranked below all committees in the \( k \)th block \( B_k \) for all \( k > n \) by all \( i \in P \).

Corollary 1.2. No complete popular matching exists if 3 or more blocks as \( f(a) \in B_1 \) and \( s(a) \in \{B_1, B_2\} \) \( \forall a \). Therefore posts \( p \in B_3, B_4, ... \) are never \( s(a) \) for any applicant.

Example 5. Consider politicians \( \{1, 2, 3, 4, 5, 6\} \) ranking committees \( \{A, B, C, D, E, F\} \). Notice, that there are three blocks \( B_1 = \{A, B\} \), \( B_2 = \{C, D\} \), and \( B_3 = \{E, F, G, ...\} \). The highlighted f- and s-posts only consist of committees in \( B_1 \) and \( B_2 \), but not \( B_3 \). Hence no complete popular match exists.

Corollary 1.3. If have 2 blocks in preference, then for a complete popular matching to exist, every post \( p \in B_1 \) must be 1st choice of some applicant and every \( p \in B_2 \) must be 1st choice of someone amongst all posts \( p \in B_2 \), else \( \{f(a), s(a)\} \neq P \).

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\(^9\)The graph theoretic formulation of Hall’s Marriage Theorem (Hall (1935)) states that for any finite bipartite graph \( G \) with bipartite sets \( P \) and \( C \) (\( G := (P + C, E) \)). An \( P \)-saturating matching is a matching which covers every vertex in \( P \). For a subset \( W \) of \( P \), let \( N_C(W) \) denote the set of all vertices in \( C \) adjacent to some element of \( W \). Hall’s Marriage Theorem states that there is an \( P \)-saturating matching if and only if for every subset \( W \) of \( P \), \( |W| \leq |N_C(W)| \). In other words, every subset \( W \) of \( P \) has sufficiently many adjacent vertices in \( C \). Proofs of the corollaries use the graph \( G \) of f- and s-posts imposed by Lemma 1 condition 2), along with Hall’s Marriage Theorem.
Example 6. Consider politicians \( \{1,2,3,4,5,6\} \) ranking committees \( \{A,B,C,D,E,F\} \). There are two blocks \( B_1 = \{A, B\} \) and \( B_2 = \{C, D, E\} \), however, committee E is no one’s third choice, and hence no one’s s-post. Hence, there is no complete popular match.

\[
\begin{array}{cccccc}
1 & 2 & 3 & 4 & 5 & 6 \\
A & A & B & B & B & A \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
D & C & C & D & C & D \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
E & F & E & G & G & F \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\end{array}
\]

Thus far, the corollaries have imposed conditions on preferences to guarantee the existence of a popular match given any matching mechanism. Now, we consider a particular matching mechanism, the Serial Dictatorship, and assert the restriction popular matching places on the mechanism details.

Corollary 1.4. A complete popular matching is implemented by a Serial Dictatorship in order of seniority only if the seniority ordering is such that the set of applicants allocated to \( f(a) \), denoted by \( F \), in the popular match, is more senior than set allotted to \( s(a) \), denoted \( S \) in the popular match.

Example 7. Consider politicians \( \{1,2,3\} \) ranking committees \( \{A,B,C\} \). If the seniority is \( 1 > 2 > 3 \), the resulting matching \((1-A, 2-B, 3-C)\) is not one of two popular matchings—which are \((1-A, 2-C, 3-B)\) or \((1-C,2-A,3-B)\)—because politician 3 who must be assigned to his f-post in any complete popular matching was junior to both politicians 1 and 2. Hence, only seniority orderings which have 3 more senior to politician(s) 1 and/or 2 result in Serial Dictatorship yielding a popular match.

\[
\begin{array}{ccc}
1 & 2 & 3 \\
A & A & B \\
B & B & A \\
C & C & C \\
\end{array}
\]

3.3. Majority Rule and Majority Stability. Consider Example 8 where no popular matching exists, but a majority stable matching exists and is implemented by a Serial Dictatorship.

Example 8. Consider politicians \( \{1,2,3,4,5\} \) ranking committees \( P = \{A,B,C,D,E\} \) given a seniority ordering \( 1 > 2 > 3 > 4 > 5 \). Since there are three blocks \( B_1 = A, \)
$B_2 = B$ and $B_3 = \{C, D, E\}$, we know that no popular matching exists. However, Serial Dictatorship in order of seniority produces the underlined allocation (1-A, 2-B, 3-C, 4-D, 5-E) which is majority stable.

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The important question is why this matching is majority stable? Preferences are pretty correlated with all politicians’ first and second choices being committees $A$ and $B$ respectively. However, in this very observation lies the key to majority stability. In this example, 4 politicians (2, 3, 4, and 5) are envious of politician 1’s assignment to committee $A$ and 3 politicians (3, 4, and 5) are envious of politician 2’s assignment to committee $B$. However, there is a packing problem as a majority of $\frac{N+1}{2}$ politicians cannot be re-assigned to less than a majority ($< \frac{N+1}{2}$) of distinct new seats. In this case, at least 3 politicians cannot be assigned to just 2 committee seats.

Hence, to overrule an allocation by majority rule, $\frac{N+1}{2}$ distinct politicians have to be envious of $\frac{N+1}{2}$ distinct committees. And satisfying this condition requires at least 1 chain of envy\(^{10}\) as illustrated in Example 9.

**Example 9.** Consider politicians $\{1, 2, 3, 4, 5\}$ ranking committees $\mathbb{P} = \{A, B, C, D, E\}$ given a seniority ordering $1 \succ 2 \succ 3 \succ 4 \succ 5$. Notice, the slight difference in politician 5’s preference ($D \succ_5 E$) compared to that in Example 8 ($E \succ_5 D$). This example has the same Serial Dictatorship allocation as Example 8. However, politician 5 now being envious of 4’s assigned committee $D$ leads to chain of envy (2 envious of 1, 3 envious of 2, and 5 envious of 3). Thus, this allocation is not majority stable as politicians 2, 3, and 5 would overrule it in favor of matching (1-E, 2-A, 3-B, 4-C, and 5-D).

These graphs of envy, of say politician $i$ of politician $j$’s seat can be represented using directed arrows from politician $i$ to politician $j$. For example, Figure 2 shows the graphs of envy for Examples 8 and 9.

The existence of chains of envy, is not perfectly captured by simple correlation across individuals’ preferences, and highlights a more particular structure of preferences, we call Deep Correlation. Our simulation results in Section 4 further illustrate this distinction.

\(^{10}\)The smallest possible chain of envy needed to break majority stability is of length 2 where the remaining $\frac{N-1}{2}$ politicians are envious of $\frac{N-1}{2}$ distinct committees.
Figure 2. Envy Graph for Examples 8 and 9.
Envy graph for Examples 8 (left) and 9 (right) with the corresponding Shapley Values. The Shapley value for each politician $i$ is thus defined by the length of the maximal chain of envy that politician $i$’s existence blocks. Notice the chain of envy (in red) which allows politicians 2, 4, and 5 to benefit at the expense of 1. Note that this is not the unique alternative allocation a majority would overrule in favor of.

3.3.1. Integer Linear Programming Approach.
Given a set of preferences, checking whether a matching is majority stable and finding the size of the maximal coalition which can benefit from some alternative matching is a hard problem. The number of alternative matchings for a one-to-one matching in a balance market with $N$ candidates and $N$ seats grows exponentially, $N!$. In this section, we use an integer linear program to solve this.

The integer linear programming approach takes the graph of envy in matrix form and finds an alternative one-to-one matching with maximal envy resolved. It turns out that the constraint that the alternative matching should be one-to-one has structure such that the relaxed linear program is guaranteed integer solutions.

Consider the re-allocation matrix $X$, where $X_{ij} = \begin{cases} 1, & \text{if } i \text{ is reallocated to } j \text{'s seat} \\ 0, & \text{otherwise} \end{cases}$

Note that $X_{ii} = 1$ means that $i$ remains in his own seat.

Consider the envy matrix $A$, where $A_{ij} = \begin{cases} 1, & \text{if } i \text{ is envious of } j \text{'s seat} \\ 0, & \text{otherwise} \end{cases}$

Note, $i$ envious of $j$’s seat means that $i$ strictly prefers $j$’s seat to his own allocation.

The goal is thus to find the alternative allocation $X$ such that the maximal envy is realized subject to the alternative allocation being a one-to-one matching of the balanced market:

$$\max_X \sum_{\text{diag}} A'X$$

s.t. $\sum_i X_{ij} = 1 \quad \forall j$

$\sum_j X_{ij} = 1 \quad \forall i$

The first set of constraints guarantee that each seat $j$ is allocated to only one person, while the second set of constraints guarantee that each individual is allocated only one seat. The sum over the diagonal entries of $A'X$ calculates the envy that is satisfied by the re-allocation
X and summing over just the diagonal entries of the matrix ensure no double counting of the envy.

We can now rewrite this problem in vector form:

We stack all the columns of X and of A to form \( \tilde{X} = \begin{bmatrix} X_{11} \\ \vdots \\ X_{N1} \\ X_{12} \\ \vdots \\ \vdots \\ X_{1N} \\ \vdots \\ X_{NN} \end{bmatrix} \) and \( \tilde{A} = \begin{bmatrix} A_{11} \\ \vdots \\ A_{N1} \\ A_{12} \\ \vdots \\ \vdots \\ A_{1N} \\ \vdots \\ A_{NN} \end{bmatrix} \).

Furthermore, consider the matrices 2N-by-1 matrix, 
\[
\begin{bmatrix} 1 \\ \vdots \\ \vdots \\ 1 \end{bmatrix}
\]
and the 2N-by-N^2 matrix,
\[
\begin{bmatrix}
0 0 \\ \vdots \\ 0 0 \\ 1 1 \\ \vdots \\ \vdots \\ \vdots \\ 1 1 \\
\end{bmatrix}
\]

Hence, our integer linear program from 1 can be equivalently expressed as

\[
\begin{aligned}
\max_X & \quad \tilde{A}^T \tilde{X} \\
\text{s.t.} & \quad C \tilde{X} = b
\end{aligned}
\]

We can divide the matrix C into submatrices on either side of the dashed line:

\[
C = \begin{bmatrix}
\begin{bmatrix} \mathbb{I} & \mathbb{I} \end{bmatrix} & \begin{bmatrix} \mathbb{I} & \mathbb{I} \end{bmatrix} & \begin{bmatrix} \mathbb{I} & \mathbb{I} \end{bmatrix} \\
\end{bmatrix}
\]

where \( \begin{bmatrix} \mathbb{I} & \mathbb{I} \end{bmatrix} = b_{N \times N^2} \) ensures that each seat is allotted to only 1 person while \( \begin{bmatrix} \mathbb{I} & \mathbb{I} \end{bmatrix} = b_{N \times N^2} \) ensures that each individual is allotted to only one seat.
Note, that matrices $\tilde{A}$, $C$, and $b$ have all integer entries and $C$ is totally unimodular\textsuperscript{11}. Hence, the relaxed linear program will have integer solutions.

3.4. When does majority stable allocation exist?

In the previous section, we fixed a set of preferences and an allocation, and asked whether the allocation was majority stable. A more general question is whether given a set of preferences, is there a majority stable allocation. There could be two ways to approach this question. First, restricting the set of preferences so as to guarantee the existence of majority stability. Or, we can try to construct an allocation which satisfies majority stability.

3.4.1. Non-existence of Order Restriction from Social Choice in this setting.

If we consider each matching allocation as an object, and consider the social choice problem with majority rule, the restriction that individual preferences are related to their own individual assignment in the matching, puts such structure on the set of preferences, that there can be no order restriction for markets of size $N \geq 3$.

**Lemma 2.** There is no possible order restriction for a market sized $N \geq 3$.

**Proof:**

It suffices to show that there is no order restriction possible for any subset of 3 alternatives and 3 people (i.e., subset of the market of size 3). We consider agents $\{1, 2, 3\}$ with preferences over matching allocations, where a matching $\{A, B, C\}$ implies match 1-A, 2-B, and 3-C. We maintain our restriction that all agents only care about their own allocation and that preferences are strict.

So without loss of generality, consider agent 1’s preference to be $A \succ_1 B \succ_1 C$. Hence in comparing the two allocations $\{A, B, C\}$ and $\{A, C, B\}$, agent 1 is indifferent. This leaves four cases for the possible preference combinations that agents 2 and 3 can have over objects $B$ and $C$.

- **Case 1:** if $B \succ_2 C$ and $B \prec_3 C$.

  Then, the order restriction is 2-1-3 (or reverse) given the comparison of the two allocations above. However, consider the comparison between $\{C, B, A\}$ and $\{B, C, A\}$. Preferences are $\succ_2$, $\prec_1$, and $\sim_3$, which violates the order restriction.

- **Case 2:** if $B \prec_2 C$ and $B \succ_3 C$.

  Then, the order restriction is 2-1-3 (or reverse) given the comparison of the two allocations above. However, consider the comparison between $\{C, A, B\}$ and $\{B, A, C\}$. Preferences are $\sim_2$, $\prec_1$, and $\succ_3$, which violates the order restriction.

- **Case 3:** if $B \succ_2 C$ and $B \succ_3 C$.

  Then, the order restriction is either 1-3-2 (or reverse) or 1-2-3 (or reverse) given the comparison of the two allocations above.

  – **Case 3a:** if order restriction is 1-3-2 (or reverse).

    Consider the comparison between $\{C, A, B\}$ and $\{B, A, C\}$. Preferences are $\prec_1$, $\succ_3$, and $\sim_2$, which violates the order restriction.

  – **Case 3b:** if order restriction is 1-2-3 (or reverse).

    Consider the comparison between $\{C, B, A\}$ and $\{B, C, A\}$. Preferences are $\prec_1$, $\succ_2$, and $\sim_3$, which violates the order restriction.

\textsuperscript{11}Heller and Tompkins show that the coefficient matrix for a bipartite matching is totally unimodular. And $\tilde{A}$ is an unoriented incidence matrix of a bipartite graph. Equivalently, Hoffman and Gale characterize more general sufficiency conditions which $\tilde{A}$ satisfies.
• Case 4: if $B <_2 C$ and $B <_3 C$.
  Then, the order restriction is either 1-3-2 (or reverse) or 1-2-3 (or reverse) given the comparison of the two allocations above.
  - Case 4a: if order restriction is 1-3-2 (or reverse).
    Consider the comparison between $\{C, B, A\}$ and $\{B, C, A\}$. Preferences are $\prec_1$, $\sim_3$, and $\prec_2$, which violates the order restriction.
  - Case 4b: if order restriction is 1-2-3 (or reverse).
    Consider the comparison between $\{C, A, B\}$ and $\{B, A, C\}$. Preferences are $\prec_1$, $\sim_2$, and $\prec_3$, which violates the order restriction.

Thus, order restriction has no bite in our setting.

3.4.2. Non-existence of Weakly Single-Peaked Preferences from Social Choice in this setting.

Similar to the section above, if we consider each matching allocation as an object, and consider the social choice problem with majority rule, the restriction that individual preferences are related to their own individual assignment in the matching, puts such structure on the set of preferences, that there can be no weakly single-peaked preferences for markets of size $N > 3$.

**Lemma 3.** There is no possible weakly single-peaked preferences for a market sized $N > 3$.

**Proof:**

First consider restricting to any agents $\{1, 2, 3\}$ with preferences over matching allocations, where a matching $\{A, B, C\}$ implies match 1-A, 2-B, and 3-C. We maintain our restriction that all agents only care about their own allocation and that preferences are strict. Without loss of generality, consider agent 1’s preference to be $A \succ_1 B \succ_1 C$. We show in appendix C, that restricting to a market of size 3 (any 3 alternatives and 3 players), there is only one possible (ignoring relabeling of objects/agents and reverse order of preferences) ordering of preferences which satisfies weak single-peakedness. This ordering is $\{C, B, A\}$, $\{B, C, A\}$, $\{A, C, B\}$, $\{A, B, C\}$, $\{B, A, C\}$, $\{C, A, B\}$.

Now, consider restricting to any 4 agents $\{1, 2, 3, 4\}$. Agent 1 will be indifferent over all matchings where he gets $D$. For preferences to be single-peaked for agents 2, 3, and 4 over all allocations where 1 gets $D$, we know from above, that the ordering must be $\{D, C, B, A\}$, $\{D, B, C, A\}$, $\{D, A, C, B\}$, $\{D, A, B, C\}$, $\{D, B, A, C\}$, $\{D, C, A, B\}$. Now given that Agent 1 has strict preferences, 1 either has $C \succ_1 D$ or $D \succ_1 C$. Either way, he is indifferent over all allocations where he gets $C$. For weak single-peakedness to hold, allocations where 1 gets $C$ must either be to the right, to the left, or split across both sides of the ordering above. Either way, there will be an allocation where 2 gets $B$. And regardless of where this allocation is placed, weak single-peakedness if violated for 2’s preference.

Thus, single-peakedness also has no bite in our setting.

3.4.3. Constructive Approach to Finding a Majority Stable Allocation (if it exists).

Our simulations consider the application of matching US Senators to committees (Thakur (2018a)). We focus on the seniority-based Republican system. Institutional stability in this mechanism is undermined by i) increased impatience (e.g., close election or retirement which imply a short time horizon or large discounting of utility from committee assignments in the future), ii) a large freshman class since freshmen have lowest priority and no existing tenant allocation with property rights\textsuperscript{12}, and iii) increased correlation across individuals’ preferences, which increases competition over seats and leads to increased likelihood of envy. To calibrate the model, we i) assume maximal impatience, so that all individuals care about the static one-shot matching and ii) calibrate the vacancies to the 97th Congress (1981-1983), which had the largest proportion of 16/53 freshmen amongst the Republican Senators from the 83rd-113th Congresses. The calibration to the 97th Congress with many-to-one matching gives 53 vacancies across 16 standing committees for the 53 Senators ordered by seniority\textsuperscript{13}.

To make the simulations more tractable, we make a few simplifications. First, instead of a many-to-many matching (where each committee is assigned multiple politicians and each politician is assigned multiple committees), we consider a many-to-one matching where each committee is assigned many politicians but each politician is assigned only one committee\textsuperscript{14}. Third, we do not consider existing tenants, i.e., incumbents who have existing committee assignments from previous Congress. Instead, we assume that there is a strict order of seniority, fixed across the senators and the mechanism is a Serial Dictatorship in order of seniority\textsuperscript{15}.

4.1. Fragility of Popular Matching existence.

First, we note that common mechanisms such as Serial Dictatorship in order of seniority and Boston Mechanism (with assumption of truthful revelation) generally fail to guarantee a complete popular matching (see Figure 5).

Second, we consider the necessary but not sufficient for complete Popular matching, that the f- and s-posts span the 16 committees (Figure 3). The figures illustrate that even with a slight correlation (i.e., $\rho \approx 0.2$ which is generated by everyone agreeing on their top choice and then having 15 remaining choices randomly generated), the necessary condition begins to be violated. As preferences become increasingly correlated fewer and fewer of the 16 committees are f- and s-posts. Of course, when preferences are perfectly correlated with $\rho = 1$ and everyone has the same preference, only the first and second choice committees are f- and s-posts.

Third, in these simulations, we find that popular matchings exist for some preference environments with intermediate correlation with 2 blocks, however, for fully random preferences, popular matching existence is rare and for higher levels of correlation with 3 or more blocks, popular matchings do not exist (see Figure 4).

4.2. Existence of Majority Stability.

Under Serial Dictatorship and Boston Mechanisms, allocations satisfy majority stability for

\textsuperscript{12}Since a third of the Senators are up for re-election in each Congress, the largest proportion of freshman can of course be 1, but this is empirically extreme.

\textsuperscript{13}See Appendix A for details regarding the simulation and the calibration.

\textsuperscript{14}Of course we assume that politicians are indifferent across all the seats within a given committee.

\textsuperscript{15}See Thakur (2018a) for why more complicated mechanisms are needed to properly incentivize existing tenants (i.e., incumbents).
environments with low and intermediate correlations across preferences, however, for highly correlated preferences, the allocations are not majority stable (Figures 6 left and 7 left).

The relationship between correlation across individuals’ preferences and the maximal sized coalition trying to overturn the allocation is monotonically increasing (see Figures 6 right and 7 right).

4.3. **Comparing existence of majority stability under Boston Mechanism vs Serial Dictatorship.**

We consider Boston Mechanism with tie-breaks in order of seniority\(^{16}\) and truthful preference revelation\(^{17}\).

Firstly, we see that for almost all preference environments (all but 5/12001 simulations), the maximal sized coalition who can be made better off is weakly larger under Serial Dictatorship than under Boston Mechanism (see Figure 8 right).

Second, Boston Mechanism under truthful revelation is more robust for majority stability and popular matching existence (see Figure 8 left). Out of 12,001 simulated preference environments with varying levels of correlation, there were i) 5154 instances where both were majority stable, ii) 4673 instances where both were not majority stable, iii) 2169 instances where Boston is majority stable but Serial Dictatorship is not, and iv) only 5 instances where Serial Dictatorship was majority stable but Boston mechanism was not.

Here are two examples showing that there are cases where Boston mechanism is majority stable and Serial Dictatorship is not (Example 10), and others where Serial Dictatorship is majority stable but Boston is not (Example 11).

**Example 10.** Consider politicians \(\{1, 2, 3, 4, 5\}\) ranking committees \(\mathcal{P} = \{A, B, C, D, E\}\) given a seniority ordering \(1 \succ 3 \succ 4 \succ 5 \succ 2\). Boston mechanism with truthful preference and same seniority ordering for tie-breaking is majority stable (1-A, 2-B, 3-C, 4-D, 5-E), while the allocation from Serial Dictatorship (1-A, 2-C, 3-B, 4-D, 5-E) is not due to alternative matching (1-C, 2-E, 3-A, 4-D, 5-B) being preferred with a 3:1 vote.

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**Example 11.** Consider politicians \(\{1, 2, 3, 4, 5\}\) ranking committees \(\mathcal{P} = \{A, B, C, D, E\}\) given a seniority ordering \(1 \succ 2 \succ 3 \succ 4 \succ 5\). Serial Dictatorship allocation (1-A, 2-B, 3-E, 4-C, 5-D) is majority stable, while Boston mechanism with truthful preference and same seniority ordering for tie-breaking (1-A, 2-C, 3-B, 4-E, 5-D) is not majority stable as alternative matching (1-E, 2-A, 3-D, 4-C, 5-B) is preferred by 3:2 vote.

4.4. **Summary of Simulations Take-aways.**

Thus, the key takeaways from the simulation are that i) Popular Matchings fail to exist with correlated preferences, ii) Serial Dictatorship and Boston Mechanism often fail to produce

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\(^{16}\)This is for comparability across Serial Dictatorship in order of seniority and Boston Mechanism.

\(^{17}\)Boston mechanism is not strategyproof, so this assumption is just for illustration purposes.
popular matchings, ii) Majority Stable matchings often exist even with a significant amount of correlation across individuals’ preferences, iii) as preferences become more and more correlated across individuals, f- and s-posts don’t span the set of objects and fewer individuals can be assigned their f- and s-posts (i.e., smaller the size of the maximal popular matching), iv) as preferences become more and more correlated across individuals, maximal coalition size wanting to overturn is larger, and v) Boston mechanism with truthful revelation is more robust to majority stability than Serial Dictatorship.

5. DISCUSSION: CORRELATED PREFERENCES IN SOCIAL CHOICE VS. SOCIAL ALLOCATION CHOICE.

Correlation across individuals’ preferences has an opposing effect on stability in social choice as compared to social allocation choice. Cycles in the preference aggregation rule of social choice—whether it be violations of transitivity, acyclicity, or quasi-transitivity—undermine stability in social choice\(^{18}\). However, the more correlated preferences are across individuals, the less likely the social choice aggregation rule has cycles. For social allocation choice however, institutional stability is undermined by chains of envy. And the more correlated preferences are across individuals, the more likely, preferences form chains of envy leading to institutional instability.

This point can be illustrated by two simple examples on the extreme ends of correlation across individuals’ preferences.

**Example 12.** Correlation of preference across individuals is 0. The social choice majority preference relation \(f\) lacks transitivity since \(A \succ f B \succ f C \succ f A\). However, the underlined allocation is majority stable (moreover, institutionally stable by pareto efficiency and popular matching sense as well) because all politicians get their first choice and there is no envy.

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**Example 13.** Correlation of preference across individuals is 1. The social choice majority preference relation \(f\) is stable and \(A\) is the Condorcet winner since \(A \succ f B \succ C\) and \(A \succ f C\). However, perfectly correlated preferences render any social allocation choice institutionally unstable under popular matching and majority stability (however, any allocation is pareto efficient), allotting \(C\) to whoever is assigned committee \(A\), and then moving the other two agents to 1 higher preference gives both a plurality and majority overrule.

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\(^{18}\)Austen-Smith and Banks (2000)
Thus, increased correlation in preferences across individuals bolsters stability of social choice while undermining stability of social allocation choice, as it decreases likelihood of cycles in aggregate choice rule while increasing the likelihood of chains of envy.


In this paper, I have juxtaposed three voting rules (plurality, majority, and unanimity) which correspond to three notions of institutional stability (Popular matching, Majority stability, and pareto efficiency). I show that pareto efficiency with unanimity rule is too non-discriminating, popular matching fails to exist with even a tiny bit of correlation across individuals’ preferences and existence of a popular match imposes stringent requirements on preferences and matching mechanisms. However, majority rule, which is particularly relevant in majoritarian institutions found in political economy such as the US Senate, provides stability of commonly used mechanisms such as Serial Dictatorship until and unless the parameters are quite extreme (i.e., very high correlation across individuals’ preferences). Simulation results calibrated to the US Senate setting illustrate this result. I show how the packing problem which arises when using majority rule in social allocation choice settings, necessitates chains of envy for potential overrule. This particular structure of chains of envy on preferences is labeled as Deep Correlation, which is distinct and imperfectly captured by a simple correlation of individuals’ preferences (a generalization of the Spearman rank order correlation coefficient from Thakur (2018b)). An integer linear programming approach is taken to test whether a given allocation is majority stable under a given preference environment. And comparisons between Serial Dictatorship and Boston Mechanism (assuming truthful preferences) shows that Boston Mechanism is tends to be far more robust in terms of majority stability.

References

### Table 1. Generation of Correlated Preferences for Simulations.

Varying degrees of correlation of preferences were generated by considering 13 different preference block structures across the 16 committees. Fully random preferences generated preferences in the correlation range of \([-0.0139, 0.0404]\), then fixing block of 1 such that all individuals agreed on the top choice and then randomly generating the remaining 15 committees in the next block generated preferences in the correlation range \([0.1625, 0.2109]\), etc. Thus, each row of this table shows the block structure used to generate preferences using randomization within blocks in column 1 and the minimum and maximum correlation \(\rho\) this produced in columns 2 and 3. The different preference block schemes are grouped and numbered such that they correspond to close clusters on the simulation figures on the horizontal axis.

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<th>Min Correl</th>
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Figure 3. Necessary but not sufficient condition for complete Popular Matching across different Correlations in preferences. *Left:* for each simulated set of preferences with a given Correlation on the horizontal axis, the vertical axis plots 1 if f- and s-posts span all 16 committees, 0 otherwise. *Right:* for each simulated set of preferences with a given Correlation on the horizontal axis, the vertical axis plots the number of committees (out of 16 total) which are f- and s-posts.

![Graph](image1)

Figure 4. Existence of Popular Matching. *Left:* Existence of Popular Matching across different Correlations in preferences. *Right:* Maximal number of candidates (out of 53) able to be assigned their f- and s-posts.

![Graph](image2)
Figure 5. Existence of Popular Matching under Serial Dictatorship and Boston Mechanism.

Under Serial Dictatorship (Left) and Boston Mechanism assuming truthful revelation and tie-breaking in order of seniority (Right), these figures plot whether the allocation is popular or not. We see that the allocations produced by these mechanisms are rarely ever complete popular matchings.
Figure 6. Majority Stability Under Serial Dictatorship. Simulating the Serial Dictatorship in order of seniority across various preference environments with varying levels of correlation across individuals’ preferences, Left shows when a majority overturns the allocation produced by the Serial Dictatorship, while Right shows the maximal sized coalition that can improve from each Serial Dictatorship allocation.

Figure 7. Majority Stability Under Boston Mechanism. Simulating the Boston Mechanism (assuming truthful preference revelation and tie-breaks in order of seniority) across various preference environments with varying levels of correlation across individuals’ preferences, Left shows when a majority overturns the allocation produced by the Boston Mechanism, while Right shows the maximal sized coalition that can improve from each Boston Mechanism allocation.
Figure 8. Comparing Majority Stability Under Serial Dictatorship v.s. Boston Mechanism.

Simulating Serial Dictatorship in order of seniority and Boston Mechanism (assuming truthful preference revelation and tie-breaks in order of seniority) across various preference environments with varying levels of correlation across individuals’ preferences, these figures compare the performance of both mechanisms in regards to Majority Stability. **Left:** shows the difference between the sizes of the maximal sized coalitions which can be strictly improved under an alternative matching, across Serial Dictatorship and Boston mechanism. **Right:** shows when allocations are majority stable for both mechanisms (0), under neither mechanism (1), under Boston mechanism but not Serial Dictatorship (-1), and under Serial Dictatorship but not Boston Mechanism (0.5).
This appendix details some of the calibrations and assumptions used for the simulation exercise in Section 4.

Since a larger percentage of freshmen (i.e., those who have no existing tenant allocation and those who have relatively lower seniority) works against stability of the Republican serial dictatorship in order of seniority, I calibrated the simulations to the 97th Congress (1981-1983) which had the largest freshman class by percentage with 16 freshmen out of a total of 53 Republican Senators. Table 2 shows the distribution of Republican freshmen across the 83rd-113th Congresses.

Next, I constructed the distribution of available seats across the 16 standing committees. As shown in Table 3, the 97th Congress had 292 seats across the 16 Standing committees. First, I assumed that Republican seats resembled the 53% of seats in each of these committees (in practice, similar to this assumption, seats on a committee for each party resemble the party ratio in the Senate). Next, since the average number of seats held by senator was around 3, but I wanted to simply the matching to be many-to-one (i.e., consider each politician being assigned to only one committee), I divided each of these seats in each committee by 3. Finally, I rounded to get the total number of vacancies in each of the 16 standing committees.

The final set of vacancies across the 16 committees is shown in Figure 4. However note, that when I generate preferences across committees, I do not take into account any particular correlation structures for these committees (e.g., I ignore for example that Appropriations is an highly sought-after committee by all). I merely consider 16 committees with the allotted number of vacancies in each committee.
Table 2. Proportion of Freshmen amongst Republican Party in the 83rd- 97th Congresses.
First column is the Congress, second is the number of freshmen Republicans, third is the number of Republicans fourth calculates the percentage of republican freshmen, and fifth and 6th split the number of freshmen by number of new Republican seats acquired and by the number of Republicans who replaced existing Republican seats. This simulation calibrates to the 97th Congress (bolded).

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<td>-1</td>
<td>4</td>
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<td>2</td>
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<td>56</td>
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<td>9</td>
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<td>55</td>
<td>12.7</td>
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<td>8</td>
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<tr>
<td>106</td>
<td>4</td>
<td>56</td>
<td>7.1</td>
<td>1</td>
<td>3</td>
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<td>3.8</td>
<td>-4</td>
<td>6</td>
</tr>
<tr>
<td>108</td>
<td>9</td>
<td>51</td>
<td>17.6</td>
<td>-1</td>
<td>10</td>
</tr>
<tr>
<td>109</td>
<td>7</td>
<td>55</td>
<td>12.7</td>
<td>4</td>
<td>3</td>
</tr>
<tr>
<td>110</td>
<td>1</td>
<td>51</td>
<td>2.0</td>
<td>-4</td>
<td>5</td>
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<td>42</td>
<td>7.1</td>
<td>-9</td>
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<td>27.1</td>
<td>6</td>
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</tr>
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<td>113</td>
<td>5</td>
<td>44</td>
<td>11.4</td>
<td>-4</td>
<td>9</td>
</tr>
</tbody>
</table>
Table 3. Derivation of vacancies by committee.
The first column is the committee name. The second column is the total number of seats in the 97th Congress. The third column corrects for 53% Republicans in the Senate in the 97th Congress. The fourth column corrects for each senator having 3 committees on an average. The fifth column rounds the vacancies so that the total number is 53 to match the number of Republicans.

<table>
<thead>
<tr>
<th>Committee Name</th>
<th>Total Seats</th>
<th>97th Congress</th>
<th>53% Rep</th>
<th>3 committees per person</th>
<th>Rounded to match 53 seats</th>
</tr>
</thead>
<tbody>
<tr>
<td>Agriculture</td>
<td>18</td>
<td>10.26</td>
<td>3.42</td>
<td>3</td>
<td></td>
</tr>
<tr>
<td>Appropriations</td>
<td>29</td>
<td>16.53</td>
<td>5.51</td>
<td>6</td>
<td></td>
</tr>
<tr>
<td>Armed</td>
<td>18</td>
<td>10.26</td>
<td>3.42</td>
<td>3</td>
<td></td>
</tr>
<tr>
<td>Banking</td>
<td>18</td>
<td>10.26</td>
<td>3.42</td>
<td>3</td>
<td></td>
</tr>
<tr>
<td>Budget</td>
<td>22</td>
<td>12.54</td>
<td>4.18</td>
<td>4</td>
<td></td>
</tr>
<tr>
<td>Commerce</td>
<td>17</td>
<td>9.69</td>
<td>3.23</td>
<td>3</td>
<td></td>
</tr>
<tr>
<td>Energy</td>
<td>20</td>
<td>11.4</td>
<td>3.8</td>
<td>4</td>
<td></td>
</tr>
<tr>
<td>Environment</td>
<td>16</td>
<td>9.12</td>
<td>3.04</td>
<td>3</td>
<td></td>
</tr>
<tr>
<td>Finance</td>
<td>20</td>
<td>11.4</td>
<td>3.8</td>
<td>4</td>
<td></td>
</tr>
<tr>
<td>Foreign Relations</td>
<td>17</td>
<td>9.69</td>
<td>3.23</td>
<td>3</td>
<td></td>
</tr>
<tr>
<td>Govt Affairs</td>
<td>18</td>
<td>10.26</td>
<td>3.42</td>
<td>3</td>
<td></td>
</tr>
<tr>
<td>Judiciary</td>
<td>18</td>
<td>10.26</td>
<td>3.42</td>
<td>3</td>
<td></td>
</tr>
<tr>
<td>Labor</td>
<td>18</td>
<td>10.26</td>
<td>3.42</td>
<td>3</td>
<td></td>
</tr>
<tr>
<td>Rules</td>
<td>12</td>
<td>6.84</td>
<td>2.28</td>
<td>2</td>
<td></td>
</tr>
<tr>
<td>Small Business</td>
<td>19</td>
<td>10.83</td>
<td>3.61</td>
<td>4</td>
<td></td>
</tr>
<tr>
<td>Veterans Affairs</td>
<td>12</td>
<td>6.84</td>
<td>2.28</td>
<td>2</td>
<td></td>
</tr>
<tr>
<td><strong>TOTAL</strong></td>
<td><strong>292</strong></td>
<td><strong>166.44</strong></td>
<td><strong>55.48</strong></td>
<td><strong>53</strong></td>
<td></td>
</tr>
</tbody>
</table>
Table 4. Final list of vacancies by committee for simulation.
The first column is the type of committee which is commonly used classification in the political science literature on committees. Second column has the name of the committee, third column has the number of seats on the committee.

<table>
<thead>
<tr>
<th>Committee Type</th>
<th>Committee Name</th>
<th># Seats for Simulation</th>
</tr>
</thead>
<tbody>
<tr>
<td>Prestige</td>
<td>Appropriations</td>
<td>6</td>
</tr>
<tr>
<td>Prestige</td>
<td>Budget</td>
<td>4</td>
</tr>
<tr>
<td>Prestige</td>
<td>Rules</td>
<td>2</td>
</tr>
<tr>
<td>Constituency</td>
<td>Agriculture</td>
<td>3</td>
</tr>
<tr>
<td>Constituency</td>
<td>Small Business</td>
<td>4</td>
</tr>
<tr>
<td>Constituency</td>
<td>Veterans Affairs</td>
<td>2</td>
</tr>
<tr>
<td>Constituency</td>
<td>Armed</td>
<td>3</td>
</tr>
<tr>
<td>Constituency</td>
<td>Energy</td>
<td>4</td>
</tr>
<tr>
<td>Policy</td>
<td>Banking</td>
<td>3</td>
</tr>
<tr>
<td>Policy</td>
<td>Commerce</td>
<td>3</td>
</tr>
<tr>
<td>Policy</td>
<td>Environment</td>
<td>3</td>
</tr>
<tr>
<td>Policy</td>
<td>Finance</td>
<td>4</td>
</tr>
<tr>
<td>Policy</td>
<td>Foreign Relations</td>
<td>3</td>
</tr>
<tr>
<td>Policy</td>
<td>Govt Affairs</td>
<td>3</td>
</tr>
<tr>
<td>Policy</td>
<td>Judiciary</td>
<td>3</td>
</tr>
<tr>
<td>Policy</td>
<td>Labor</td>
<td>3</td>
</tr>
<tr>
<td><strong>TOTAL</strong></td>
<td></td>
<td><strong>53</strong></td>
</tr>
</tbody>
</table>
Appendix B. Details: set inclusion for matchings across various Institutional Stability notions.

This appendix provides some examples and proofs of the set inclusion from Figure 1 replicated below:

- Claim: All popular matchings are pareto efficient.
  
  \textit{Proof}: if allocation $M$ is not pareto efficient, then there exists alternative allocation $M'$ s.t. $|M' \succ_i M| > 0$ and $|M \succ_i M'| = 0$. Thus, $|M' \succ_i M| > |M \succ_i M'| = 0$, thus $M$ is not a popular matching

- Claim: All popular matchings are majority stable.
  
  \textit{Proof}: if allocation $M$ is not majority stable, then there exists alternative allocation $M'$ s.t. $|M' \succ_i M| \geq \frac{N+1}{2}$. Thus, $|M' \succ_i M| < \frac{N+1}{2}$. Hence $|M' \succ_i M| > |M \succ_i M'| = 0$, thus $M$ is not a popular matching

- A Pareto efficient matching that is not Popular and no majority stable:
  
  Example: politicians \{1, 2, 3\} ranking committees \{A, B, C\} Consider 1-A, 2-B, 3-C. Since serial dictatorship with seniority $1 > 2 > 3$ implements it, it is pareto efficient. However, 1-C, 2-A, 3-B is strictly preferred by 3 and 2, hence it is not popular or majority stable.

- A Pareto efficient matching that is majority stable, but not Popular:
  
  Example: politicians \{1, 2, 3, 4, 5\} ranking committees $P = \{A, B, C, D, E\}$ given seniority $1 \succ 2 \succ 3 \succ 4 \succ 5$ Consider 1-A, 2-B, 3-C, 4-D, 5-E. Since serial dictatorship

<table>
<thead>
<tr>
<th></th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>A</td>
<td>A</td>
<td>A</td>
<td>A</td>
<td></td>
</tr>
<tr>
<td>B</td>
<td>B</td>
<td>B</td>
<td>B</td>
<td></td>
<td></td>
</tr>
<tr>
<td>C</td>
<td>C</td>
<td>C</td>
<td>D</td>
<td>E</td>
<td></td>
</tr>
<tr>
<td>D</td>
<td>D</td>
<td>D</td>
<td>C</td>
<td>D</td>
<td>E</td>
</tr>
<tr>
<td>E</td>
<td>E</td>
<td>E</td>
<td>C</td>
<td>D</td>
<td>C</td>
</tr>
</tbody>
</table>

with seniority $1 > 2 > 3 > 4 > 5$ implements it, it is pareto efficient. Since envy is
only of committees A and B, it is majority stable due to the packing problem. Since there are 3 blocks, there is no complete popular matching.

- A majority stable matching that is not pareto efficient nor popular

  Example: politicians \{1, 2, 3, 4, 5\} ranking committees \(P = \{A, B, C, D, E\}\) given seniority \(1 \succ 2 \succ 3 \succ 4 \succ 5\). Consider 1-B, 2-A, 3-C, 4-D, 5-E. Only envy is 1 and

  \[
  \begin{array}{cccccc}
  1 & 2 & 3 & 4 & 5 \\
  A & B & C & D & E \\
  C & C & B & B & B \\
  D & D & D & C & C \\
  E & E & E & E & D \\
  \end{array}
  \]

  2 envious of each others’s assignments so majority stable, but fails pareto efficiency and popular matching due to 1 and 2 strictly preferring 1-A, 2-B, 3-C, 4-D, 5-E.

- A matching that is majority stable, popular and pareto efficient

  Example: politicians \{1, 2, 3, 4, 5\} ranking committees \(P = \{A, B, C, D, E\}\) given seniority \(1 \succ 2 \succ 3 \succ 4 \succ 5\). Consider 1-A, 2-B, 3-C, 4-D, 5-E.

  \[
  \begin{array}{cccccc}
  1 & 2 & 3 & 4 & 5 \\
  A & B & C & D & E \\
  C & C & B & B & B \\
  D & D & D & C & C \\
  E & E & E & E & D \\
  \end{array}
  \]
APPENDIX C. Proof of Lemma 3.

We show in this appendix that restricting to a market of size 3 (any 3 alternatives \(\{A, B, C\}\) and 3 players \(\{1, 2, 3\}\)), there is only one possible (ignoring relabeling of objects/agents and reverse order of preferences) ordering of preferences which satisfies weak single-peakedness. This ordering is
\[
\{C, B, A\}, \{B, C, A\}, \{A, C, B\}, \{A, B, C\}, \{B, A, C\}, \{C, A, B\}.
\]

Without loss of generality, consider agent 1’s preference to be \(A \succ_1 B \succ_1 C\). For minimizing notation, we write \(ABC\) for the allocation \(1 - A, 2 - B,\) and \(3 - C\). Consider the following exhaustive cases\(^\text{19}\) for potential weak single-peaked preference orderings (all other possible orderings are either relabeling or reverse orderings which we ignore):

- **Case 1:** Ordering (ABC, ACB, BCA, BAC, CAB, CBA), which satisfies 1’s weak single-peakedness by \(u_1(ABC) = u_1(ACB) > u_1(BCA) = u_1(BAC) > u_1(CAB) = u_1(CBA)\).

Then, neither strict preference of 3’s over B and C can satisfy weak single-peakedness.

- **Case 2:** Ordering (ABC, ACB, BCA, BAC, CBA, CAB), which satisfies 1’s weak single-peakedness by \(u_1(ABC) = u_1(ACB) > u_1(BCA) = u_1(BAC) > u_1(CBA) = u_1(CAB)\).

Then, neither strict preference of 3’s over B and C can satisfy weak single-peakedness.

- **Case 3:** Ordering (ABC, ACB, BAC, BCA, CAB, CBA), which satisfies 1’s weak single-peakedness by \(u_1(ABC) = u_1(ACB) > u_1(BAC) = u_1(BCA) > u_1(CBA) = u_1(CAB)\).

Then, neither strict preference of 3’s over B and C can satisfy weak single-peakedness.

- **Case 4:** Ordering (ABC, ACB, BAC, BCA, CBA, CAB), which satisfies 1’s weak single-peakedness by \(u_1(ABC) = u_1(ACB) > u_1(BAC) = u_1(BCA) > u_1(CBA) = u_1(CAB)\).

Then, neither strict preference of 3’s over B and C can satisfy weak single-peakedness.

- **Case 5:** Ordering (ACB, ABC, BCA, BAC, CAB, CBA), which satisfies 1’s weak single-peakedness by \(u_1(ACB) = u_1(ABC) > u_1(BCA) = u_1(BAC) > u_1(CBA) = u_1(CAB)\).

Then, neither strict preference of 2’s over B and C can satisfy weak single-peakedness.

- **Case 6:** Ordering (ACB, ABC, BCA, BAC, CBA, CAB), which satisfies 1’s weak single-peakedness by \(u_1(ACB) = u_1(ABC) > u_1(BCA) = u_1(BAC) > u_1(CBA) = u_1(CAB)\).

Then, neither strict preference of 2’s over B and C can satisfy weak single-peakedness.

- **Case 7:** Ordering (ACB, ABC, BAC, BCA, CAB, CBA), which satisfies 1’s weak single-peakedness by \(u_1(ACB) = u_1(ABC) > u_1(BAC) = u_1(BCA) > u_1(CBA) = u_1(CAB)\).

Then, neither strict preference of 2’s over B and C can satisfy weak single-peakedness.

- **Case 8:** Ordering (ACB, ABC, BAC, BCA, CBA, CAB), which satisfies 1’s weak single-peakedness by \(u_1(ACB) = u_1(ABC) > u_1(BAC) = u_1(BCA) > u_1(CBA) = u_1(CAB)\).

Then, neither strict preference of 2’s over B and C can satisfy weak single-peakedness.

\(^{19}\) Cases 1-8 are for \(u_1(A) = u_1(B) > u_1(C)\), cases 9-15 are for \(u_1(C) = u_1(C) < u_1(A) = u_1(A) > u_1(B) = u_1(B)\) with \(u_1(B) > u_1(C)\), cases 17-20 are for \(u_1(C) < u_1(B) < u_1(A) = u_1(A) > u_1(B) > u_1(C)\), and cases 21-28 are for \(u_1(C) = u_1(C) < u_1(B) < u_1(A) = u_1(A) > u_1(B)\) with \(u_1(B) > u_1(C)\).
• Case 9: Ordering (CAB, CBA, ACB, ABC, BAC, BCA), which satisfies 1’s weak single-peakedness by $u_1(CAB) = u_1(CBA) < u_1(ACB) = u_1(ABC) > u_1(BAC) = u_1(BCA)$.

Then, neither strict preference of 2’s over A and C can satisfy weak single-peakedness.

• Case 10: Ordering (CBA, CAB, ABC, BAC, BCA), which satisfies 1’s weak single-peakedness by $u_1(CBA) = u_1(CAB) < u_1(ACB) = u_1(ABC) > u_1(BAC) = u_1(BCA)$.

Then, neither strict preference of 2’s over A and C can satisfy weak single-peakedness.

• Case 11: Ordering (CAB, CBA, ABC, ACB, BAC, BCA), which satisfies 1’s weak single-peakedness by $u_1(CAB) = u_1(CBA) < u_1(ACB) = u_1(ABC) > u_1(BAC) = u_1(BCA)$.

Then, neither strict preference of 2’s over A and B can satisfy weak single-peakedness.

• Case 12: Ordering (CBA, CAB, ABC, ACB, BAC, BCA), which satisfies 1’s weak single-peakedness by $u_1(CBA) = u_1(CAB) < u_1(ACB) = u_1(ABC) > u_1(BAC) = u_1(BCA)$.

Then, neither strict preference of 2’s over A and B can satisfy weak single-peakedness.

• Case 13: Ordering (CAB, CBA, ABC, ACB, BCA, BAC), which satisfies 1’s weak single-peakedness by $u_1(CAB) = u_1(CBA) < u_1(ACB) = u_1(ABC) > u_1(BAC) = u_1(BCA)$.

Then, neither strict preference of 3’s over A and B can satisfy weak single-peakedness.

• Case 14: Ordering (CBA, CAB, ABC, ACB, BCA, BAC), which satisfies 1’s weak single-peakedness by $u_1(CBA) = u_1(CAB) < u_1(ACB) = u_1(ABC) > u_1(BAC) = u_1(BCA)$.

Then, neither strict preference of 3’s over A and B can satisfy weak single-peakedness.

• Case 15: Ordering (CAB, CBA, ABC, ACB, BCA, BAC), which satisfies 1’s weak single-peakedness by $u_1(CAB) = u_1(CBA) < u_1(ACB) = u_1(ABC) > u_1(BCA) = u_1(BCA)$.

Then, neither strict preference of 3’s over A and C can satisfy weak single-peakedness.

• Case 16: Ordering (CBA, CAB, ABC, ACB, BCA, BAC), which satisfies 1’s weak single-peakedness by $u_1(CBA) = u_1(CAB) < u_1(ACB) = u_1(ABC) > u_1(BAC) = u_1(BCA)$.

Then, neither strict preference of 3’s over A and C can satisfy weak single-peakedness.

• Case 17: Ordering (CAB, BCA, ABC, ACB, BAC, CBA), which satisfies 1’s weak single-peakedness by $u_1(CAB) < u_1(BCA) < u_1(ABC) = u_1(ACB) > u_1(BAC) > u_1(CBA)$.

Then, neither strict preference of 2’s over B and C can satisfy weak single-peakedness.

• Case 18: Ordering (CBA, BCA, ABC, ACB, BAC, CAB), which satisfies 1’s weak single-peakedness by $u_1(CBA) < u_1(BCA) < u_1(ABC) = u_1(ACB) > u_1(BAC) > u_1(CAB)$.

Then, neither strict preference of 2’s over B and C can satisfy weak single-peakedness.

• Case 19: Ordering (CAB, BCA, ACB, ABC, BAC, CBA), which satisfies 1’s weak single-peakedness by $u_1(CAB) < u_1(BCA) < u_1(ABC) = u_1(ACB) > u_1(BAC) > u_1(CBA)$.

Then, neither strict preference of 3’s over A and B can satisfy weak single-peakedness.
• Case 20: Ordering (CBA, BCA, ACB, ABC, BAC, CAB), which satisfies 1’s weak single-peakedness by $u_1(CBA) < u_1(BCA) < u_1(ACB) = u_1(ABC) > u_1(BAC) > u_1(CAB)$.

  **This is weakly Single Peaked Preference!**

• Case 21: Ordering (CBA, CAB, BCA, ABC, ACB, BAC), which satisfies 1’s weak single-peakedness by $u_1(CBA) = u_1(CAB) < u_1(BCA) < u_1(ABC) = u_1(ACB) > u_1(BCA)$.

  Then, neither strict preference of 2’s over A and B can satisfy weak single-peakedness.

• Case 22: Ordering (CAB, CBA, BCA, ABC, ACB, BAC), which satisfies 1’s weak single-peakedness by $u_1(CAB) = u_1(CBA) < u_1(BCA) < u_1(ABC) = u_1(ACB) > u_1(CAB)$.

  Then, neither strict preference of 2’s over A and B can satisfy weak single-peakedness.

• Case 23: Ordering (CBA, CAB, BCA, ACB, ABC, BAC), which satisfies 1’s weak single-peakedness by $u_1(CBA) = u_1(CAB) < u_1(BCA) < u_1(ABC) = u_1(ACB) > u_1(CAB)$.

  Then, neither strict preference of 2’s over A and B can satisfy weak single-peakedness.

• Case 24: Ordering (CAB, CBA, BCA, ACB, ABC, BAC), which satisfies 1’s weak single-peakedness by $u_1(CAB) = u_1(CBA) < u_1(BCA) < u_1(ABC) = u_1(ACB) > u_1(CAB)$.

  Then, neither strict preference of 2’s over A and B can satisfy weak single-peakedness.

• Case 25: Ordering (CBA, CAB, BAC, ABC, ACB, BCA), which satisfies 1’s weak single-peakedness by $u_1(CBA) = u_1(CAB) < u_1(BAC) < u_1(ABC) = u_1(ACB) > u_1(BCA)$.

  Then, neither strict preference of 3’s over A and B can satisfy weak single-peakedness.

• Case 26: Ordering (CAB, CBA, BAC, ABC, ACB, BCA), which satisfies 1’s weak single-peakedness by $u_1(CAB) = u_1(CBA) < u_1(BAC) < u_1(ABC) = u_1(ACB) > u_1(CBA)$.

  Then, neither strict preference of 3’s over A and B can satisfy weak single-peakedness.

• Case 27: Ordering (CBA, CAB, BAC, ACB, ABC, BCA), which satisfies 1’s weak single-peakedness by $u_1(CBA) = u_1(CAB) < u_1(BAC) < u_1(ABC) = u_1(ACB) > u_1(BCA)$.

  Then, neither strict preference of 3’s over A and B can satisfy weak single-peakedness.

• Case 28: Ordering (CAB, CBA, BAC, ACB, ABC, BCA), which satisfies 1’s weak single-peakedness by $u_1(CAB) = u_1(CBA) < u_1(BAC) < u_1(ABC) = u_1(ACB) > u_1(BCA)$.

  Then, neither strict preference of 3’s over A and B can satisfy weak single-peakedness.