CONTINUOUS-TIME PRINCIPAL MULTI-AGENT PROBLEM: MORAL HAZARD IN TEAMS & FISCAL FEDERALISM

ASHUTOSH THAKUR

Abstract. We analyze continuous time optimal contracting in principal multi-agent moral hazard settings; particularly, what the implications are as the number of agents increases in the model. Using tools from game theory and stochastic optimal control, we derive the optimal, history-contingent contract for the generalized principal n-agent dynamic problem; in the process showing how continuous time framework makes dynamic contracting and analysis tractable. Efficiency gain from specialization and rising disincentives from increased moral hazard counteract one another as we increase the number of agents in our model, thus, we derive the optimal size of a firm/team (microeconomics application) or of a fiscal union (political economy application). Furthermore, our model suggests that development of strong political and economic institutions goes hand-in-hand with greater delegation of responsibility, decentralization, and federalism.

1. Introduction

The topic is motivated by the moral hazard problem which often arises in team-based, collective environments where each team member’s level of effort is imperfectly observed ("hidden action"). The uninformed principal hires informed agents to exert effort and produce output. The principal’s profit is increasing in the amount of output produced by the agents. Output is increasing in the level of effort exerted by each of agents, but also has some random, exogenous noise. Exerting effort is costly for the agents, hence the “conflict of interest”. The output level is publicly observed by all players; however, individual levels of effort are observed only by each agent himself ("private action"). The principal thus commits to a dynamic contract with the agents contingent on the entire history of realized output that he observes.

We want to analyze how the principal’s pay-off is affected from hiring multiple agents (i.e., hiring 2 agents to each do 1/2 of the work, 3 agents to each do 1/3 of the work...) and how the optimal contract is structured. As we increase the number of agents, three complicating interactions arise: a) increased specialization where total cost of effort is less due to agents splitting up the work, b) moral hazard from imperfect monitoring of agents’ actions which incentivizes agents to shirk, and c) repeated prisoner’s dilemma interaction between agents, who have a greater incentive to shirk, hide behind the noise, and free-ride at the expense of the other agent(s). Thus, for a given level of monitoring (i.e. noise) in the output process, these counterbalancing effects allow us to find the optimal size of the team.

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We approach the dynamic contracting problem of moral hazard in teams with two motivating examples (see section 4) in mind: 1) microeconomic theory of the firm where employers want to incentivize workers to exert costly effort by committing to wage contracts which alleviate moral hazard, and 2) political economy application of fiscal federalism where central government uses intra-governmental transfers to discipline state governments to act fiscally responsibly. We find that the optimal incentive compatible and individually rational profit maximizing dynamic contract is not backward looking: the principal makes agent’s payoff volatile with respect to output deviations from the equilibrium path in the current period. Thus, the contract rewards overproduction and punishes underproduction. Quite reasonably, this volatility parameter is equal to the marginal cost of effort for the agent. For a given level of supervision in the firm application or a given institutional arrangement which affects the magnitude of investor confidence response to shocks and macroeconomic volatility, we find the optimal size of the firm and the optimal level of decentralization within a union. In the firm example, we find that investing in supervision technology, depending on its costs, can be more profitable than hiring more agents. And in our fiscal federalism example, we find that development of strong political and economic institutions goes hand-in-hand with greater delegation of responsibility, decentralization, and federalism. Thus, we argue that “the weakness of local government in relation to central government [which] is one of the most striking phenomena of under-developed countries”\(^1\) can be explained by weak institutions.

2. Literature Review

Principal-agent problems are fundamental in classical microeconomic theory. Principal-agent problems, considering the interactions between an uninformed party (called the “principal”) and an informed party (called the “agent”), are divided into adverse selection (hidden information) and moral hazard (hidden action).

Adverse selection arises from asymmetric information between two parties engaging in a transaction.\(^2\) For example, an insurance company (principal) which offers health insurance to its client (agent) cannot perfectly observe and monitor the client’s lifestyle and health which is private information, thus has difficulty figuring out the appropriate insurance contract to offer. The uninformed insurance company can offer its clients a menu of insurance plans, each with different coverage levels and different premiums, so that high risk and low risk clients self-select different insurance plans and can be differentiated (known as “screening mechanisms”).\(^3\)

Another example of adverse selection might involve an uninformed buyer (principal) of a product who does not know the quality of the item the informed seller (agent) is offering him and thus faces difficulty valuing the object. In this case, the seller may signal his product’s quality to the uninformed buyer by offering warranty on the object (known as “signaling mechanisms”).\(^4\) Moral hazard arises when the conflict of interest between principal and agent is heightened by imperfectly observed actions by the agent. For example, consider an insurance provider (principal) who provides a client (agent) with car insurance against collision and

\(^{1}\)Martin, Lewis (1956).
\(^{2}\)Akerlof (1970) is a classic paper on adverse selection.
\(^{3}\)Rothschild and Stiglitz (1976) first proposed the notion of screening.
\(^{4}\)Spence (1973) originally proposed the idea of signaling.
theft, but cannot observe how careful his client is; thus the client has an incentive to drive recklessly and act irresponsibly as his downside is covered. Principal-agent problems are thus contracting, mechanism design problems trying to find optimal contracts or institutions to overcome the conflict of interest between the principal and the agent.

Our model deals with moral hazard, where the principal hires agent(s) to work and produce output, however, each agent’s level of effort exerted is hidden information (i.e. imperfectly monitored by the principal through the noisy output process). Such models of multi-agent and multi-period, repeated principal-agent interactions were developed in the 1980’s. However, these models were analysed in discrete-time environments.

This literature takes one of three forms: 1) deriving optimal incentive structure in multi-agent settings,\(^5\) 2) dealing with implementability of these mechanisms by addressing multiple equilibria problems,\(^6\) and 3) analysing the problem with further signals and added structure to obtain more complex mechanism design/contracting solutions to overcome moral hazard problems.\(^7\) However, analyzing dynamic properties of optimal contracts is hard in discrete-time models and these models are much more computationally intensive to work with.\(^8\)

Starting in mid-2000s, Cvitanic, Sannikov, and others placed principal-agent problems in continuous time, which made computation of optimal contracts much more tractable using stochastic optimal control and differential equation solving machinery developed in physics and mathematics.\(^9\) Sannikov (2007) develops continuous time methodology for analyzing repeated games with imperfect monitoring in continuous time. Sannikov (2008) considers the principal-agent moral hazard problem between a single principal and a single agent; analysing short term and long term incentives of players and designing contracts which incorporate retirement, quitting, replacement, and promotion. We extend Sannikov (2008) to include multiple agents. Usually, such a generalization in complex game theoretic models is often very hard; however, our derivation generalizing to \(n\) agents underscores the computational tractability of using continuous time models.

Qualitative literature on fiscal federalism describes institutional intricacies of intra-governmental transfers, tax allocations,... but often doesn’t consider the underlying contracting problems between the central government and state governments.\(^10\) Whereas, literature modeling the principal-agent contracting problem, is

\(^{5}\) Literature includes Holmstrom (1982), Mookherjee (1984), and Ma, Moore, and Turnbull (1988).

\(^{6}\) Literature includes Ma (1988) and Arya, Glover, and Young (1995).

\(^{7}\) Literature includes Itoh (1991), Malcomson (1986), Arya, Glover, and Hughes (1997), Che and Yoo (2001), and Baldenius and Glover (2010).

\(^{8}\) Some important contributions to the discrete time principal-agent problem include: Spear and Srivastava (1987) who use recursive methods to analyze multi-period principal-agent models; Rogerson (1985) who constructs a simple two-period principal-agent model; Phelan and Townsend (1991) who develop methodology to characterize optimal long-term contracts in discrete time; and Radner (1985) and Fudenberg, Holmstrom and Milgrom (1990) who analyze the limiting case of patient agents whose discount factor approaches 0.


\(^{10}\) Oates (1999).
concerned with static models involving voting structures, taxation, mobility, redistribution, and regulation, but does not consider dynamic contracting which is key, since the political economy application naturally calls for repeated long-term interactions between different levels of government.

3. The Two-Agent Model

Consider a continuous time model with 1 principal and 2 agents \( i \in \{1, 2\} \). Agent \( i \) chooses the level of effort \( a^i_t \geq 0 \) at each time \( t \). We assume that the principal and agent \( -i \) cannot observe \( a^i_t \) directly, however, the output process \((X_t)_{t \geq 0}\) is publicly observed. The output process is noisy and evolves according to

\[
dX_t = (a^1_t + a^2_t)dt + \sigma dZ_t
\]

where \( Z_t \) is a Brownian motion and \( \sigma \) is a known parameter (the noise term \( \sigma dZ_t \) leads to imperfect action monitoring).\(^{12}\) Agent \( i \)'s cost of exerting effort \( a^i_t \) is \( h(a^i_t) \), where \( h : a \mapsto \mathbb{R} \) is continuous, increasing \( h' > 0 \), and convex \( h'' > 0 \). We normalize \( h(0) = 0 \).

Intuitively, we immediately notice three complications which arise in principal multi-agent problems just from considering expression (1) and convex cost function:

1. 'Specialization of sorts' where total cost to produce the same total effort is less due to splitting up of the workload
2. Imperfect monitoring of agents’ effort levels leads to moral hazard where agents have incentive to shirk.
3. Similar to a repeated prisoner’s dilemma interaction, as more agents are added to the team, each agent’s incentive to shirk, hide behind the noise, and free-ride at the expense of the other agent(s) is amplified.

Since the incentive to shirk increases as the number of agents increases, and on the other hand, ‘splitting up the work’ leads to efficiency gains from decreased total cost of effort. These two forces act in opposite directions, creating interesting contractual dynamics which we analyse in this paper.

We let both agents’ outside option be \( \hat{W} \geq 0 \). Thus the agent is willing to accept any wage contract if it yields an expected pay-off of more than \( \hat{W} \).

The principal observes the output process but not the individual effort level of each agent, and offers a contract specifying a consumption schedule \( c_t(X_s; 0 \leq s \leq t) \), which specifies a non-negative cash flow \( c_t \) at each time \( t \) conditional on observations of \( X_s \) for \( s \leq t \). Thus, the principal takes into account observed history of past output levels when specifying the cash flow (‘wage’) at the current time \( t \). We assume that the principal can credibly commit to such a contract and we assume that the principal can choose to not hire the agent(s) and so we are only interested in contracts which yield the principal a non-negative profit.


\(^{12}\)We have specified the game above with \( dX_t = (a^1_t + a^2_t)dt + \sigma dZ_t \) with noise arising from one parameter \( \sigma \) and one Brownian motion \( Z_t \). However, we can also consider a problem with \( dX^1_t = a^1_t dt + \sigma_1 dZ^1_t \) and \( dX^2_t = a^2_t dt + \sigma_2 dZ^2_t \) both of which are publically observed output processes, where the principal cares about total output \( Y_t = X^1_t + X^2_t \). At first, this approach doesn’t seem as interesting since the principal can independently contract with each agent; however, perhaps with correlated Brownian Motions \( Z^1_t \) and \( Z^2_t \), there may be some interesting, more complicated dynamic contracts to analyze.
Both agents’ utilities from consumption are identical $u_1 = u_2 = u$, bounded from below, and normalized to $u(0) = 0$. We assume that the utility function $u : [0, \infty) \to [0, \infty)$ is increasing $u' > 0$, concave $u'' < 0$, and $C^2$ function satisfying $u'(c) \to 0$ as $c \to \infty$. Thus, the agents face decreasing marginal returns from consumption.

For simplicity, we assume a common rate of discounting, $r$, for the principal and for both the agents, such that if a player receives pay-off $z$ at time $t$, his discounted pay-off at time 0 is $ze^{-rt}$.

In short, the two agents are identical and we are interested in how to principal’s optimal contract changes when he hires 1, 2, ..., $n$ agents. The implications for why the agents’ incentives have changed when compared to the one-agent case will become clearer as we set up the agent’s individual rationality and incentive compatibility constraints in the next section. Our equilibrium derivation will closely follow Sannikov (2007), but will be generalized to include multiple identical agents, instead of the single agent case which Sannikov (2007) analyses.

4. Applications

We motivate the dynamic contracting problem of moral hazard in teams with two main applications: 1) microeconomic theory of the firm and 2) political economy application of fiscal federalism.

4.1. Theory of the Firm. The microeconomics application entails an employer (principal) who hires workers (agents) to exert effort and produce output. Individual level of effort exerted is observed only by each worker himself. Output is publicly observed, but is a noisy process, increasing in level of effort exerted by each worker. However, exerting effort is costly for the worker.

$$dX_t = \left(a_1 t + a_2 t + \ldots + a_n t\right) dt + \sigma dZ_t$$

We are interested in how the principal constructs a dynamic optimal wage contract conditional on entire history of realized output? How is the principal’s profitability affected by hiring multiple agents versus investing in supervision technology? And what is the optimal size of the firm?

4.2. Political Economy of Fiscal Federalism. The political economy application consists of a central government (principal) which wants state governments (agents) to act fiscally responsibly—exert ‘fiscal effort’—as this affects investor confidence. Each state’s exertion of fiscal effort is a private action and is costly. Investor confidence/cheapness of borrowing is publicly observed, but is a noisy process (due to market shocks and institutions), increasing in level of fiscal effort exerted by each of agents. As a concrete example, consider the European Union which is a monetary union, under the European Central Bank and European Commission (the principal), but not a fiscal union; hence EU-member countries (agents) take fiscal actions of taxation, budgeting, and spending independently. The central government thus tries to get member states to act fiscally responsibly— since it affects investor confidence— by discipling and rewarding member states via intra-governmental transfers.
Fiscal effort includes enhancements of efficiency of public institutions, stricter monitoring of handing out benefits, structural economic reform in labor and regulation, reductions in perks and privileges of governing party’s constituency or of special interest groups, and reduction in discretionary spending. Fiscal effort is costly for the government: reduces party membership, increases amount of concessions made on other policy issues, and directly affects government’s re-election chance. However, the benefit of decentralization is that fiscal consolidation is easier and more efficient under smaller spheres of influence such as states.

That idiosyncratic micro shocks and aggregate macro-level shocks causes volatility in investor confidence is a given; however, there is a vast literature on how both political and economic institutions affect the magnitude of impact of these shocks on macroeconomic volatility and investor confidence. “Markets acknowledge that better institutions reduce fiscal difficulties rendering the monitoring of annual developments less important.” For example, central bank independence leads to lower real interest rates, “fiscal transparency reduces risk premia,” and “rule of law, strong and independent courts, and protection of property rights have significant positive effects on bond ratings.” “Power-sharing and party system polarization have important effects on long-term interest rates. Where collective responsibility is high and polarization is low, the market perceives a more credible commitment on part of sovereign debtors.” Acemoglu et al. (2002, 2003) suggests that “fundamental cause of post-war instability is institutional,” and many natural experiments and historical juxtapositions also suggest that there exists a “strong relationship between institutions and volatility.” For example, “early modern sovereigns were above the law,” and as Schultz, Weingast (2003) maintain, the “decline in British interest rates occurred after the Glorious Revolution as the institutional innovations associated with regime change came into force”: “crown in Parliament became sovereign,” “crown no longer had unilateral discretion to repudiate its commitments,” “new taxes and efficient tax administration system, largely above corruption and politics.” As a natural juxtaposition, France faced consistently higher borrowing costs as the “crown retained unilateral authority over terms of debt,” “absence of parliamentary oversight allowed the crown to obfuscate its total indebtedness with poor accounting,” and retention of ancien regime where wealthy individuals would lend money in exchange for administrative positions including tax collection lead to loss of control over taxes.
Moreover, Acemoglu et al. (2005) analyse the natural experiment of the splitting of the Koreas, where prior to the split “bad institutions key in place, clearly not for the benefit of society as a whole, but for the benefit of the ruling elite.” They attribute the modern day divergence in economic performance between North and South Korea to institutional design: “the North followed the model of Soviet socialism and the Chinese Revolution in abolishing private property of land and capital. Economic decisions were not mediated by the market, but by the communist state. The South instead maintained a system of private property and the government, especially after the rise to power of Park Chung Hee in 1961, attempted to use markets and private incentives in order to develop the economy.” The Soviet War comparison between the US and Soviet Union also suggests that heightened macroeconomic volatility in USSR was due to a “lack of political constraints in Soviet Unions centrally planned system”\textsuperscript{19} and that “when markets are missing or ignored (as they were in the Soviet Union, for example), gains from trade go unexploited and resources are misallocated.”\textsuperscript{20}

Thus, although we posit this very crude model of investor confidence risk premia or cost of borrowing our primary purpose is to capture the intuition regarding moral hazard in decentralized unions, not price assets, and this vast literature seems to corroborate our modeling assumptions.

We are interested in how the central government can optimally construct intragovernmental transfers contract conditional on entire history of realized cheapness of borrowing? How is investor confidence affected by the amount of fiscal decentralization? What is the optimal size of a union and optimal federalist delegation of power? And how do political and economic institutions affect the level of decentralization and delegation of responsibility?

5. Solving the Principal Multi-Agent Problem

We first consider the two-agent model and explicitly derive the optimal contract. Generalizing our model to include $n$ agents, follows the same derivation as developed below. Thus, instead of re-deriving, we simply state the final equation describing the optimal contract for the generalized $n$-agent case in section 6.

5.1. Setting up the Principal 2-Agent Maximization Problem. Agent $i$’s expected utility if he chooses effort level $a_i$ is

$$
\mathbb{E}^u \left[ r \int_0^\infty e^{-rt} \left( u(c_i) - h(a_i) \right) dt \right]
$$

The principal’s expected utility if agent $i$ chooses effort level $a_i$, for $i \in 1, 2$

$$
\mathbb{E}^{(a^1, a^2)} \left[ r \int_0^\infty e^{-rt} dX_t - r \int_0^\infty e^{-rt} c_t dt \right]
$$

and substituting $dX_t$ term yields

$$
\mathbb{E}^{(a^1, a^2)} \left[ r \int_0^\infty e^{-rt} \left( a_1^2 + a_2^2 - c_t \right) dt \right]
$$

\textsuperscript{19}Schultz, Wiengast (2003).
\textsuperscript{20}Acemoglu, Johnson, Robinson (2005).
Note that the factor $r$ normalizes the total pay-offs to the same scale as flow pay-offs and the superscripts $\mathbb{E}^{(a^1,a^2)}$ and $\mathbb{E}^a$ underscore that the probability distribution over the output paths and hence over compensation is affected by the agents’ strategies.

The agents and the principal enter a non-renegotiable contract committed to by the principal at time $t = 0$, such that the principal maximizes his expected profit subject to each agent’s incentive constraint and individual rationality constraint, by choosing the recommended consumption schedule and suggested effort paths $c_t^*(X_t)$, $a_t^1(X_t)$, $a_t^2(X_t)$ satisfying:

$$\max_{\{c_t,a_t^1,a_t^2\}} \mathbb{E}^{(a^1,a^2)} \left[ r \int_0^\infty e^{-rt} \left( a_t^1 + a_t^2 - c_t \right) dt \right]$$

subject to, for $i = 1, 2$:

$$(IR_i)$$

$$\mathbb{E}^a \left[ r \int_0^\infty e^{-rt} \left( u(c_t^*) - h(a_t^*) \right) dt \right] \geq \hat{W}$$

$$(IC_i)$$

$$\mathbb{E}^a \left[ r \int_0^\infty e^{-rt} \left( u(c_t^*) - h(a_t^*) \right) dt \right] \geq \mathbb{E}^{a_i} \left[ r \int_0^\infty e^{-rt} \left( u(c_t^*) - h(a_t^*) \right) dt \right],$$

where $\hat{a}_t^i \in a_t^i$.

$$(IR_i)$$, for $i = 1, 2$, are ‘individual rationality’ conditions which imply that over the lifetime of the agent, it is profitable in expectation, for agent $i$ to accept the contract and exert effort according to $a_t^*$, given that the principal follows the equilibrium consumption schedule $c_t^*$, $0 \leq t < \infty$ and the other agent $-i$, plays the suggested equilibrium strategy $a_t^{-i*}$. The contract’s expected pay-off is higher than the agent’s outside option value $\hat{W}$. $(IC_i)$, for $i = 1, 2$, are ‘incentive compatibility’ conditions which imply that $a_t^*$ maximizes agent $i$’s expected total pay-off, given that the principal follows the equilibrium consumption schedule $c_t^*$, $0 \leq t < \infty$ and the other agent $-i$, plays the suggested equilibrium strategy $a_t^{-i*}$.\(^{21}\)

5.2. Symmetry & Formulation of the Principal’s Problem. Due to the symmetry in the agents’ preferences and since the principal cannot perfectly observe each agent’s individual effort level, we assume that the principal sets the overall consumption schedule, $c_t^*$, based on observations of the current and past output level, $X_s$, for $s \leq t$, and this consumption is shared equally between the agents, yielding each agent $i$ a consumption of $c_t^* = \frac{c_t^*}{2}$. Thus, on the equilibrium path, effort level exerted by the two agents will be identical: $a_t^* = a_t^1 = a_t^2$. Consequently, the combined effort level by the agents will be $2a_t^*$. However, since each agent’s effort now accounts for only half the output and since the noise obscuring the principal’s monitoring is left unchanged, each agent’s incentives to exert effort have changed, and the principal must account for this change when committing to the optimal dynamic contract. The crux of this paper is that, even though the

\(^{21}\)We note that it is also possible to look at a discrete action space (such as only two effort levels: high and low), however, this does not simplify the problem by much thus we start our analysis directly with the continuous action space described above.
optimal effort levels are symmetric, this problem has tighter ‘incentive compatibility’ constraints compared to a one-agent model since when an agent considers deviating from the prescribed optimal effort path with \( \hat{a}_t \), we consider the other agent fixed under equilibrium effort level \( a^*_t \) (the one-shot deviation principle for subgame perfect Nash Equilibrium).

The principal’s problem becomes choosing \((c^*_t, a^*_t)\) which maximizes:

\[
\max_{\{c_t, a_t\}} \mathbb{E}^a \left[ r \int_0^\infty e^{-rt}(2a_t - c_t)dt \right]
\]

subject to

\( (IR) \)

\[
\mathbb{E}^a \left[ r \int_0^\infty e^{-rt} \left( u\left(\frac{c_t}{2}\right) - h(a_t) \right) dt \right] \geq \hat{W}
\]

\( (IC) \)

\[
\mathbb{E}^* \left[ r \int_0^\infty e^{-rt} \left( u\left(\frac{c^*_t}{2}\right) - h(a^*_t) \right) dt \right] \geq \mathbb{E}^a \left[ r \int_0^\infty e^{-rt} \left( u\left(\frac{c_t}{2}\right) - h(a_t) \right) dt \right].
\]

\( \forall \hat{a}_t \in a_t \).

This is a complicated dynamic optimization problem, in fact, the agent is a dynamic optimizer choosing effort level \( a_t \), and the agent’s maximization problem is embedded in the principal’s optimization problem which involves choosing \( c_t \) to maximize profit. The agent’s pay-off depends directly on the effort level chosen by the agent via the cost of effort \( h(a_t) \), but also indirectly through the effect on the probability distribution over the paths of output \( X_t \). Thus, although the principal’s wage contract only specifies the consumption schedule \( c_t \), the principal optimizes over both the recommended level of effort \( a_t \) and the consumption schedule \( c_t \) to make sure that the agent’s strategy \( a_t \) in response to \( c_t \) is an optimal response to the contract that is offered.

5.3. Reformulating Problem in terms of Expected Continuation Pay-offs.

Despite the vast array of possible contracts making this a complicated dynamic optimization problem, the one-shot deviation principle reduces the set of ‘incentive compatibility’ constraints. Furthermore, recall that when enforcing subgame perfect Nash equilibrium, the equilibrium is protected against any unilateral deviation from equilibrium by a single player. Thus the logic of subgame perfect Nash equilibria and the one-shot deviation principle vastly simplify our problem.

The most convenient way to evaluate the effect of a one-shot deviation on the agent’s pay-off is to consider the agent’s expected continuation pay-off at time \( t \):

\[
W_t = \mathbb{E}_t^a \left[ r \int_t^\infty e^{-r(s-t)} \left( u\left(\frac{c_s}{2}\right) - h(a_s) \right) ds \right]
\]

(2)

\( ^{22} \)Recall that the ‘individual rationality’ constraint will be binding in equilibrium since the principal maximizes profits and thus will offer the optimal contract which leaves the agent just indifferent between accepting and not accepting. Thus we can replace the inequality by an equality in (IR). Once we solve the problem with \( W_o \geq \hat{W} \), we can then optimally choose \( W_o \).
The agent’s expected continuation pay-off $W_t$ given the history of the output process $X_s$ for $s \leq t$, depends on the agent’s future consumption schedule $c_s$ for $s \geq t$ given $X_s$ for $s \leq t$, and on the agent’s future strategy $a_s$ for $s \geq t$.

To derive the evolution of this expected continuation pay-off over time, $dW_t$, let us first define $V_t$, the total benefit an agent receives from following the strategy $a_t$ up to time $t$, given time $-t$ information:

$$V_t = \mathbb{E}_t^a \left[ r \int_0^\infty e^{-r s} \left( u \left( \frac{c_s}{2} \right) - h(a_s) \right) ds \right] = r \int_0^t e^{-r s} \left( u \left( \frac{c_s}{2} \right) - h(a_s) \right) ds + e^{-r t} W_t$$  \hspace{1cm} (3)

We show that $V_t$ is a martingale, since for any $\delta > t$:

$$\mathbb{E}_t[V_\delta] = \mathbb{E}_t \left[ r \int_0^\delta e^{-r s} \left( u \left( \frac{c_s}{2} \right) - h(a_s) \right) ds + e^{-r \delta} W_\delta \right] = \mathbb{E}_t \left[ r \int_0^\delta e^{-r s} \left( u \left( \frac{c_s}{2} \right) - h(a_s) \right) ds \right] + e^{-r \delta} \mathbb{E}_t \left[ \int_\delta^\infty e^{-r(s-\delta)} \left( u \left( \frac{c_s}{2} \right) - h(a_s) \right) ds \right]$$

$$= \mathbb{E}_t \left[ r \int_0^t e^{-r s} \left( u \left( \frac{c_s}{2} \right) - h(a_s) \right) ds + \mathbb{E}_t \left[ r \int_\delta^\infty e^{-r(s-\delta)} \left( u \left( \frac{c_s}{2} \right) - h(a_s) \right) ds \right] \right]$$

$$+ \mathbb{E}_t \left[ r \int_\delta^\infty e^{-r(s-\delta)} \left( u \left( \frac{c_s}{2} \right) - h(a_s) \right) ds \right]$$

$$= \mathbb{E}_t \left[ r \int_0^t e^{-r s} \left( u \left( \frac{c_s}{2} \right) - h(a_s) \right) ds + \mathbb{E}_t \left[ r \int_\delta^\infty e^{-r(s-\delta)} \left( u \left( \frac{c_s}{2} \right) - h(a_s) \right) ds \right] \right]$$

$$= \mathbb{E}_t \left[ r \int_0^t e^{-r s} \left( u \left( \frac{c_s}{2} \right) - h(a_s) \right) ds + e^{-r \delta} \mathbb{E}_t \left[ r \int_\delta^\infty e^{-r(s-\delta)} \left( u \left( \frac{c_s}{2} \right) - h(a_s) \right) ds \right] \right]$$

$$= V_t$$

Since $V_t$ is a martingale, the Martingale Representation Theorem states that there exists a $Y_t$ such that $\mathbb{E} \left[ \int_0^t Y_s^2 ds \right] < \infty$ for all $t$ and

$$dV_t = rY_t e^{-r t} \sigma dZ_t = rY_t e^{-r t} \left( dX_t - (a_t^1 + a_t^2) dt \right)$$  \hspace{1cm} (4)

Differentiating (3) with respect to $t$ yields

$$dV_t = r e^{-r t} \left( u \left( \frac{c_t}{2} \right) - h(a_t) \right) dt - re^{-r t} W_t + e^{-r t} dW_t$$  \hspace{1cm} (5)

Equating (4) and (5), we obtain

$$dW_t = r \left( W_t - u \left( \frac{c_t}{2} \right) + h(a_t) \right) dt + rY_t \left( dX_t - (a_t^1 + a_t^2) dt \right)$$  \hspace{1cm} (6)
We intuitively explain this crucial equation for the law of motion for $W_t$. When the agents take the recommended equilibrium efforts $\{a_t^1, a_t^2\}$, $dX_t = (a_t^1 + a_t^2)dt = \sigma dZ_t$, which has expectation zero. Thus, in equilibrium, $r(W_t - u\left(\frac{c_t}{2}\right) + h(a_t))dt$ is the drift of $W_t$, where $rW_t$ term accounts for what the principal has yet to give in the future and the $u\left(\frac{c_t}{2}\right) - h(a_t)$ term accounts for the pay-off flow which the agent receives. Moreover, the term $rY_t$ represents the volatility of the agent’s continuation pay-off subject to changes in output. This term underscores how the principal’s contract determines the agents’ consumption schedule and effort level subject to the output outcome $X_t$. By making the agents’ pay-off sensitive to the change in output, the principal rewards the agents for positive output shock and punishes the agents for negative output shocks, hoping to prevent shirking and encourage exerting effort.

5.4. Translating the Agent’s Problem. By the one-shot deviation principle, the agent’s strategy $a_t$ is optimal if and only if at each moment in time, the agent maximizes the expected impact of his effort on his continuation value minus the cost of effort: $rY_t a_t$. Thus a strategy $(a^*_t)_{t \geq 0}$ is optimal (i.e., satisfies the ‘incentive compatibility’ condition) if and only if

$$a^*_t = \arg \min_{a_t'} dW_t(a_t') = \arg \min_{a_t'} r h(a_t') - rY_t a_t', \forall t \tag{7}$$

We minimize $dW_t$ since consumption $c_t$ and expected future pay-off $W_t$ are inversely related.

To prove the optimality of $a^*_t$ and prove why this satisfies (IC), let us consider $V_t$, the agent’s expected pay-off from following an alternate strategy $\tilde{a}_t$ until time $t$, and then the strategy $a^*_t$ from time $t$ onwards:

$$V_t = r \int_0^t e^{-rs} \left( u\left(\frac{c_s}{2}\right) - h(\tilde{a}_s) \right) ds + e^{-rt}W_t \tag{8}$$

Differentiating with respect to $t$:

$$d\tilde{V}_t = re^{-rt} \left( u\left(\frac{c_t}{2}\right) - h(\tilde{a}_t) \right) dt - re^{-rt}W_t dt + e^{-rt}dW_t$$

$$= re^{-rt} \left( u\left(\frac{c_t}{2}\right) - h(\tilde{a}_t) \right) dt - re^{-rt}W_t dt + e^{-rt} \left[ r(W_t - u\left(\frac{c_t}{2}\right) + h(a^*_t)) dt 
+ rY_t \left(dX_t - (a^*_t + a^*_t)dt\right)\right] \tag{9}\)

Consider that the agent plays $\tilde{a}_t$ for an additional moment, then using the law of motion of $X_t$ when $\tilde{a}_t$ is played

$$dX_t = (\tilde{a}_t + a^*_t)dt + \sigma dZ_t,$$
we can substitute into (9) to get

$$d\tilde{V}_t = \underbrace{re^{-rt} (h(a^*_t) - h(\tilde{a}_t)) dt}_{\leq 0 \text{ when (7) holds}} - re^{-rt}Y_t (a^*_t - \tilde{a}_t) dt + re^{-rt}Y_t \sigma dZ_t$$
Since the drift term is non-positive, it is a super-martingale, hence for every \( t \geq 0 \), \( \hat{V}_0 \geq E^\hat{a} \left[ \hat{V}_t \right] \). Thus,

\[
E^\hat{a} \left[ r \int_0^t e^{-rs} \left( u \left( \frac{c_s}{2} \right) - h(\hat{a}_s) \right) ds \right] + E^\hat{a} \left[ e^{-rt} W_t \right] = E^\hat{a} \left[ \hat{V}_t \right] \leq \hat{V}_0 = W_0 = E^a \left[ r \int_0^\infty e^{-rs} \left( u \left( \frac{c_s}{2} \right) - h(a^*_s) \right) ds \right]
\]

Taking the limit as \( t \to \infty \), we find that

\[
E^\hat{a} \left[ r \int_0^\infty e^{-rs} \left( u \left( \frac{c_s}{2} \right) - h(\hat{a}_s) \right) ds \right] \leq E^a \left[ r \int_0^\infty e^{-rs} \left( u \left( \frac{c_s}{2} \right) - h(a^*_s) \right) ds \right]
\]

Thus, \( W_0^a \geq W_0^\hat{a} \), proving the optimality of \( a^*_t \) (i.e., satisfying the agent’s (IC)).

5.5. Translating the Principal’s Problem. In designing the contract, the principal cares about the expected future profit \( F(W_t) \) under the optimal policy which specifies the consumption schedule \( c_t \) and the agent’s optimal level of effort \( a_t(Y_t) \), by specifying the volatility of the consumption schedule to output changes by setting \( Y_t \).

\[
F(W_t) = \max_{c_t, Y_t} E^a_t \left[ r \int_t^\infty e^{-rs} \left( 2a(Y_s) - c_s \right) ds \right] W_t \tag{10}
\]

Let \( F(W_t) \) be the principal’s profit from the contract. Then we can derive the corresponding Bellman Equation:

\[
rF(W_t) = \max_{c_t, Y_t} E^a_t \left[ r \left( 2a_t(Y_t) - c_t \right) + dF(W_t) \right] \tag{11}
\]

By Itô Formula,

\[
E \left[ \frac{dF(W_t)}{dt} \right] = E \left[ F'(W_t) dW_t + \frac{1}{2} F''(W_t)(dW_t)^2 \right] \tag{12}
\]

\[
r \left( W_t - u \left( \frac{c_t}{2} \right) + h(a_t) \right) F'(W_t) + \frac{1}{2} r^2 Y_t^2 \sigma^2 F''(W_t) \tag{13}
\]

Substituting into (11) gives the Hamilton-Jacobi-Bellman (HJB) Equation:

\[
rF(W_t) = \max_{c_t, a_t} \left\{ r \left( 2a_t - c_t \right) + r \left( W_t - u \left( \frac{c_t}{2} \right) + h(a_t) \right) F'(W_t) + \frac{1}{2} r^2 (Y_t(a_t))^2 \sigma^2 F''(W_t) \right\} \tag{14}
\]

Rearranging gives us:

\[
F''(W_t) = \min_{c_t, a_t} \frac{F(W_t) - 2a_t + c_t - F'(W_t)(W_t - u \left( \frac{c_t}{2} \right) + h(a_t))}{r \left( Y_t(a_t) \right)^2 \sigma^2} \tag{15}
\]

We also allow the principal to retire the agent at any value \( W \in [0, u(\infty)] \). By retiring the agent with value \( u \left( \frac{c_t}{2} \right) \), the principal promises to pay the agent a
constant $c/2$ in the future and allows the agent to take zero effort. We denote by $F_0$, the principal’s retirement profit:

$$F_0\left(u\left(\frac{c}{2}\right)\right) = -c$$ (16)

Solving (15), such that $F(0) = 0$, and with largest slope $F'(0) > 0$ such that $F(W) = F_0(W)$ and $F'(W) = F'_0(W)$ for some $W > 0$. We assume that such a slope exists, else the boundary conditions $F(0) = F'(0) = 0$ stays weakly above $F_0(W)$, because $F(W) < 0 \forall W > 0$, and the function $F(W)$ is the upper bound on the principal’s profit, so the principal can get no positive profit with any contract.

We note that retirement can occur in two conditions:

- $W = 0$: the agent is worthless in the future and needs to be ‘retired’ (or more precisely laid off or fired) with future consumption 0.
- $W = W$: the agent is ‘retired’ since it is too costly for the principal to continue his employment. This occurs because while the agent’s marginal utility is decreasing with increasing consumption, the principal’s profit from additional output remains constant. This is similar to the real-life scenario where a very senior employee who is rich due to his long service at the firm, needs to be paid a high wage to continue working, and thus, the firm instead, often chooses to give a retirement package and hire a younger employee instead (our model does not consider new hiring).

From (14), we can derive:

- Optimal effort $a^*_t$ maximizes:
  
  $$a^*_t = \arg\max_{a_t} 2ra_t + rh(a_t)F'(W_t) + \frac{1}{2} r^2 \sigma^2 (Y_t(a_t))^2 F''(W_t)$$ (17)

- Optimal consumption schedule $c^*_t$ maximizes:
  
  $$c^*_t = \arg\max_{c_t} -c_t - F'(W_t)u\left(\frac{c_t}{2}\right)$$ (18)

From (7),

$$a^*_t = \arg\min_{a_t} rh(a_t) - rY_t a_t$$ (19)

Thus, the first-order condition is

$$h'(a_t) = Y_t$$ (20)

Note that the second-order condition for the global maximum is satisfied since $h''(a_t) > 0$ since $h$ is convex.

Furthermore, from (18), the first-order condition

$$u'(c_t/2) = -\frac{1}{F'(W_t)}$$ (21)

Thus, the HJB equation simplifies to

$$F''(W_t) = \min_{a_t} \frac{F(W_t) - 2a_t + c_t - F'(W_t)(W_t - u(c_t/2) + h(a_t))}{r(h(a_t))^2 \sigma^2}$$ (22)

such that $F(0) = F_0(0)$, and with largest slope $F'(0) > 0$ such that $F(W) = F_0(W)$ and $F'(W) = F'_0(W)$ for some $W > 0$. 

6. Generalization to n-agent case

All calculations and derivations with 2 agents carry through with \( n \) agents: the \( n \)-agent case gives the HJB equation

\[
F''(W_t) = \min_{a_t} \frac{F(W_t) - na_t + c_t - F'(W_t)(W_t - u(c_t/n) + h(a_t))}{r(Y(a_t))^2\sigma^2} \tag{23}
\]

such that \( F(0) = F_0(0) \), and with largest slope \( F'(0) > 0 \) such that \( F(W) = F_0(W) \) and smooth-pasting condition \( F'(W) = F_0'(W) \) for some \( W > 0 \); where \( u'(c_t/n) = -1/F'(W_t) \), \( h'(a_t) = Y_t \), and \( F_0\left(u\left(\frac{c}{n}\right)\right) = -c \).

7. Numerical Analysis

In our numerical analysis, we use implicit iterative methods such as Runge Kutta, along with a bisection method-style approach, to solve the 2nd order HJB equation with an initial condition and boundary conditions at an endogenously determined boundary.

Rather than analysing the full-fledged HJB equation from (23), we ask the question: what is the optimal team size to reap a total combined effort of \( a_{total} = .3 \) across all agents, given a noise parameter \( \sigma = 0.4 \), rate of discount \( r = 0.1 \), utility function \( u(c) = \sqrt{c} \), and cost function \( h(a) = .5a^2 + .4a \). Thus, in equilibrium, we enforce 1 agent exerting effort \( a_1 = .3 \), versus 2 agents exerting efforts \( a_1 = a_2 = .15 \), etc.

We conclude that as we increase the number of agents in the team, the retirement level \( W \) increases (see figure (1)). Furthermore, for a noise parameter \( \sigma = .4 \), the principal would optimally employ 2 agents to maximize expected profits (see figure (2)). However, if the principal is able to observe the output with less noise, with parameter \( \sigma = .25 \), it is more profitable to hire 3 agents than hire 1, 2, or 4 agents (see figures (3), (4), and (5)).
Figure 1. For $u(c) = \sqrt{c}$, $h(a) = .5a^2 + .4a$, $r = .1$, $\sigma = 0.4$, $a_{total} = .3$, notice the various optimal contracts and respective retirement thresholds.

Figure 2. For $u(c) = \sqrt{c}$, $h(a) = .5a^2 + .4a$, $r = .1$, $\sigma = 0.4$, $a_{total} = .3$, notice that hiring 2 workers yields higher profit than hiring 3 workers. Zoom from figure (1).
Figure 3. For \( u(c) = \sqrt{c} \), \( h(a) = .5a^2 + .4a \), \( r = .1 \), \( \sigma_{\text{total}} = .3 \), \( \sigma = 0.25, 0.4 \), we note the optimal profit-maximizing contracts with respective retirement thresholds and optimal \( W_t \) level.

Figure 4. Optimal contracts for \( u(c) = \sqrt{c} \), \( h(a) = .5a^2 + .4a \), \( r = .1 \), \( \sigma_{\text{total}} = .3 \), \( \sigma = 0.25, 0.4 \), which show the trade-offs between hiring more workers and investing in monitoring technology. Zoom from figure (3).

8. Revisiting Applications & Conclusions

Collectively, these our numerical analysis illustrates the trade-offs involved in increasing size of teams; the optimal team size depends on how imperfectly the
Figure 5. Comparing profits for $n = 1, 2, 3, 4$ for $\sigma = .25$, hiring 3 agents yields higher profit than hiring 2 or 4 agents. Assuming $u(c) = \sqrt{c}$, $h(a) = .5a^2 + .4a$, $r = .1$, $a_{\text{total}} = .3$, $\sigma = 0.25$. Zoom from figure (4).

principal observes the output process. For a decently noisy output process ($\sigma = .4$), there are efficiency gains from hiring 2 agents instead of 1 due to decreased total costs and specialization. However, for that noise level, hiring a 3rd agent would decrease overall profit, as the incentive to shirk, hide behind the noise, and free-ride on other agents’ efforts starts to dominate. However, if the principal is able to better observe ($\sigma = .25$) the output process (i.e., by investing in technology or engaging in a stricter supervisory role as the manager or overseer), then the incentive to shirk is mitigated, and hiring 3 agents is more profitable than hiring 1 or 2 agents; though hiring a 4th agent would decrease overall profit. This is a natural phenomena prevalent in the teamwork environments: if supervision is imperfect, then adding members to the group leads to slacking off; however, if supervision is tighter, then adding members to a team may result in more productivity through specialization and team members sharing work.

Furthermore, note that there are some real trade-offs when it comes to the principal’s investment. Note that not investing in monitoring technology and supervision (i.e., highly noisy output process observation with $\sigma = .4$) while hiring 3 agents, is less profitable than investing in supervision (i.e., less noisy output observation with $\sigma = .25$) but hiring just 1 agent (see figure (4)). This highlights that depending on the cost and gains of increased supervision, sometimes expanding monitoring of the current labor force is more profitable than hiring additional workers.

We argue that countries with under-developed institutions, whether it is a lack of central bank independence, rule of law, or property rights, naturally have more centralized rule with weak local governments and minimally delegated fiscal powers. However, restraining the incentive to be fiscally irresponsible, which increases as the union size increases due to imperfect monitoring of fiscal effort, is not just a
concern for developing countries. Even advanced countries which have developed of strong political and economic institutions and can afford to expand delegation of responsibility, decentralization, and federalism, such as the European Union, face these real trade-offs when balancing political policy (i.e., expansion of the European Union membership) and economic policy (i.e., recent Eurobond example).

As an anecdotal example, consider Eurobonds, or government bonds issued in Euros jointly by the 18 euro-zone nations, which were recently proposed as a possible solution to help alleviate Euro Crisis. However, the EU, being a monetary union and not a fiscal union, had to address the moral hazard concerns. The president of the European Commission, Jose Manuel Barroso, and European Union Commissioner for Economy and Monetary Affairs, Olli Rehn, insisted that, “it is clear that euro bonds, in whatever form they were to be introduced, would have to be accompanied with a substantially reinforced fiscal surveillance and economic policy coordination as an essential counterpart so as to avoid moral hazard and ensure sustainable public finances.” Some submitted proposals attempting to limit moral hazard suggested that Eurozone governments would have to submit their draft national budgets for the following year to the EC by October 15th, the commission would then be able to ask the government to revise the budget if it believed that it was not sound enough to meet its targets for debt and deficit levels as set out in the Euro convergence criteria. This suggests that issues of moral hazard in decentralized unions are present and very much real in all countries, and need to be carefully addressed when setting political and economic policy.

In summary, we have generalized and derived optimal contracts for the principal n-agent problem of moral hazard in teams. By assuming symmetry on the equilibrium path and using the one-shot deviation principle to enforce subgame perfect Nash equilibrium from game theory, we simplify the dynamic constrained optimization problem. Then, by using stochastic optimal control, we are able to derive the Hamiltonian-Jacobi-Bellman Equation for the optimal contract, which we then analyze numerically. First, we find that as we increase the number of agents in our model, two counteracting forces affect the profitability of the principal: a) specialization allows for decreased total cost and efficiency gains (i.e., for quadratic costs, $2 \times (.15)^2 < (.3)^2$), and b) moral hazard is exacerbated as the number of agents increase and agents try to hide behind the noise and free-ride on their team members’ efforts. Second, these counteracting forces allow us to derive the optimal firm size or level of decentralization within a union, given the output noise parameter, cost of effort function, and utility function of the agent(s). Finally, we show the trade-off between investing in monitoring/supervision technology and investing in increased labor force; the trade-off between institutional improvement and further decentralization.

We leave the analysis of more complex incentive structures, mechanisms, and contract spaces in principal multi-agent problems as topics for future research.

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23 Note that the European Union agreed upon the Stability and Growth Pact and the Maastricht Treaty which limits government deficit to 3% of GDP: exactly $a_1 = a_2^* = 3\%$ in our model.
References


Undergraduate, Economics Department, Princeton University
E-mail address: athakur@princeton.edu